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THE INVESTIGATION OF ESQ GROUP REPRESENTATIONS

Ph.D. thesis

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1 The general theory of ESQ representations

1.1 Notation and definitions

In this section we will introduce the notation and definitions, which will be used throughout the thesis. The definitions of the following algebraic objects and constructions are taken for granted:

- **Groups**
  Cyclic groups of order $n$ will be denoted by $C_n$. The symmetric group and alternating group on $n$ elements will be denoted by $S_n$ and $A_n$ respectively. If $V$ is a vector space, then the classical linear groups will be denoted by the usual notation $GL(V)$, $SL(V)$, $PSL(V)$, $AGL(V)$.

- **Fields**
  The field of order $q$ will be denoted by $F_q$. Field extensions will be denoted by $F \leq E$. The characteristic of field $F$ will be denoted by $\text{char } F$. The multiplicative group of field $F$ will be denoted by $F^\times$.

- **Vector spaces**
  The $d$ dimensional vector space over the field $F$ will be denoted by $F^d$.

- **Group algebras**
  If $F$ is a field and $G$ is a group, then the group algebra of group $G$ over field $F$ will be denoted by $FG$.

- **Modules**
  In our thesis every module $M$ will be a module over some group algebra $FG$. Typically, we will refer to a module $M$ over $FG$ as "$FG$ module $M$". If it does not lead to confusion, we might use one of the shorter terminologies "$G$ module $M$" or simply "$M$".

- **Tensor products**
  If $V_1$ and $V_2$ are vector spaces over the field $F$, then their tensor product will be denoted by $V_1 \otimes_F V_2$. Similarly, if $M_1$ and $M_2$ are modules over the group algebra $FG$, then their tensor product will be denoted by $M_1 \otimes_{FG} M_2$. If it does not lead to confusion, we will denote these tensor products simply by $V_1 \otimes V_2$ and $M_1 \otimes M_2$ respectively.

- **Symmetric and exterior squares**
  If $V$ is a vector space, then the symmetric and exterior square of $V$ will be denoted by $\text{Sym}^2 V$ and $\text{Ext}^2 V$ respectively.
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$\Lambda^2V$ respectively. Similarly, if $M$ is a module over the group algebra $\mathbb{F}G$, then the symmetric and exterior square of $M$ will be denoted by $\text{Sym}^2 M$ and $\Lambda^2 M$ respectively. If $g \in G$, then the action of $g$ on $\Lambda^2 M$ will be denoted by $g \wedge g$.

- Linear representations of groups
  Representations on tensor product spaces, representations on exterior power spaces, restricted and induced representations.

- Characters of groups
  If $\chi$ is a character, then the exterior square of character $\chi$ will be denoted by $\hat{\chi}$. If $\chi_1$ and $\chi_2$ are characters of group $G$, then the scalar product of characters $\chi_1$ and $\chi_2$ will be denoted by $\langle \chi_1, \chi_2 \rangle$.

This list is not exhaustive. Some notations and definitions are needed only in specific parts of the thesis, and these will be defined where they will be relevant.

If a group $G$ is represented on vector space $V$ over the field $\mathbb{F}$ — that is there exists a fixed group homomorphism $\varphi : G \to GL(V)$ —, then $V$ can be considered as an $\mathbb{F}G$ module via the linear actions of group elements defined by $\varphi$. Vice versa, the base set of any $\mathbb{F}G$ module is an $\mathbb{F}$ vector space, and the module multiplication defines a representation of $G$ on this vector space. This means that there is a one-to-one correspondence between group representations and modules over group algebras.

Now we are ready to provide the main definition of the thesis.

**Definition 1.1.1** Let $\mathbb{F}G$ be a group algebra and $M$ be an $\mathbb{F}G$ module. Let us denote the exterior square of $M$ by $\Lambda^2 M$. $M$ is an **ESQ module** (ESQ stands for Exterior Self-Quotient) if and only if there exists an $\mathbb{F}G$ submodule $N \leq \Lambda^2 M$, such that the $\mathbb{F}G$ module isomorphism $\Lambda^2 M/N \cong M$ holds.

We formulate the main definition for group representations also.

**Definition 1.1.2** Let $G$ be a group and $V$ be a vector space over the field $\mathbb{F}$. Let $\varphi$ be a group representation $\varphi : G \to GL(V)$, and let us consider $V$ as a $\mathbb{F}G$ module. $\varphi$ is an **ESQ representation** if and only if $V$ as an $\mathbb{F}G$ module is an ESQ module.

**Remark.** For the sake of simplicity we will use phrases like "module $M$ has the ESQ property" and "representation $\varphi$ has the ESQ property". These statements are equivalent to stating that $M$ is an ESQ module and that $\varphi$ is an ESQ representation respectively.

**Definition 1.1.3** Let $G$ be a group and $V$ be a vector space over the field $\mathbb{F}$. $G$ is an **ESQ subgroup** of $GL(V)$ if and only if there is a faithful representation $\varphi : G \to GL(V)$, which has the ESQ property.
Moreover, \( G \) is an \textbf{irreducible ESQ subgroup of} \( GL(V) \) if and only if there is a faithful irreducible representation \( \varphi : G \to GL(V) \), which has the ESQ property.

\textbf{Remark.} Where it does not lead to confusion, in the topic of ESQ subgroups of \( GL(V) \), we will identify the abstract group \( G \) with its image in \( GL(V) \). This way we will write relations like \( G \leq GL(V) \), and — as \( V \) will be finite dimensional —, we will consider the elements of group \( G \) as matrices.

\subsection*{1.2 Basic observations}

In this section we will provide some straightforward consequences of the definition of ESQ property. Our goal is to give a better overview on the defined concept before going into the details.

\textbf{Theorem 1.2.1} Let \( V \) be a finite dimensional vector space over the field \( \mathbb{F} \), and let \( G \leq GL(V) \) be an ESQ subgroup of \( GL(V) \).

\begin{enumerate}[(i)]
    \item The only scalar matrix (a matrix of form \( \lambda I \)) in \( G \) is the identity.
    \item \( 3 \leq \dim(V) \).
    \item If \( H \leq G \), then \( H \) is also an ESQ subgroup of \( GL(V) \).
\end{enumerate}

\textbf{Proof.} Consider \( V \) as an \( \mathbb{F}G \) module. Let \( g \in G \) be a scalar matrix, that is \( g = \lambda I \), where \( \lambda \in \mathbb{F} \) and \( I \) is the identity of \( V \), or equivalently, \( vg = \lambda v \) for all \( v \in V \). As the wedge product is bilinear, the action of \( g \) on \( \Lambda^2V \) is the following: \( \hat{v}g = \lambda^2 \hat{v} \) for all \( \hat{v} \in \Lambda^2V \). This scaling property is inherited by all the the quotient modules of \( \Lambda^2V \). The ESQ property of \( V \) means that \( V \) is a quotient of \( \Lambda^2V \), hence \( g \) acts on \( V \) as \( \lambda^2 I \). We get that \( \lambda^2 = \lambda \), so \( \lambda \) is either 0 or 1. As \( g \) is invertible, only \( \lambda = 1 \) is possible, which proves the first statement of the theorem.

As \( V \) is a quotient of \( \Lambda^2V \), the inequality \( \dim V \leq \dim \Lambda^2V \) must hold. On the other hand it is known that \( \dim \Lambda^2V = \binom{\dim V}{2} \). From these observations the second statement of the theorem follows.

Finally, for the last statement let us take the faithful ESQ representation of \( G \varphi : G \to GL(V) \). We will show that \( \varphi|_{\mathit{H}} \) is a faithful ESQ representation of \( H \). Only the ESQ property of representation \( \varphi|_{\mathit{H}} \) needs further investigation. As \( G \) is an ESQ subgroup, there exists an \( \mathbb{F}G \) submodule \( U \leq \Lambda^2 V \), such that \( \Lambda^2 V/U \cong V \) as \( \mathbb{F}G \) modules. Since \( H \leq G \), all the modules \( V, \Lambda^2V, U \) can be considered as \( \mathbb{F}H \) modules, and isomorphism \( \Lambda^2V/U \cong V \) still holds for these \( \mathbb{F}H \) modules. This proves the ESQ property of \( \varphi|_{\mathit{H}} \), and the last statement of the theorem. \( \square \)
We will motivate Definition 1.1.1 in section 1.3, and it will be clear that the origins of the ESQ concept lies in modular representation theory. However, in the classical non-modular case — when group \(G\) is finite and the base field \(F = \mathbb{C}\) is the field of complex numbers —, it is possible to reformulate the definition of ESQ representations. In chapters 2 and 3 we will present our results on non-modular irreducible ESQ representations. For understanding these results, it will be enough to know the following reformulations of the ESQ property.

**Theorem 1.2.2** Let \(G\) be a finite group and \(M\) be a finite dimensional irreducible \(\mathbb{C}G\) module. The following statements are equivalent:

- \(M\) is an ESQ module.
- There is a \(\mathbb{C}G\) submodule \(N \leq \Lambda^2 M\), such that \(N \cong M\) as modules.
- If \(\chi\) denotes the irreducible character corresponding to \(M\), then \(\langle \chi, \hat{\chi} \rangle \neq 0\), where \(\hat{\chi}(g) = \frac{\chi(g^2) - \chi(g)}{2}\).

**Proof.** The conditions of Maschke’s theorem are fulfilled, the theorem implies that every \(\mathbb{C}G\) module is semisimple, specially \(\Lambda^2 M\) is semisimple. In a semisimple module every submodule is a direct summand, which proves the equality of the first two statements.

The scalar product \(\langle \chi, \hat{\chi} \rangle\) measures the multiplicity of summands isomorphic to \(M\) in the irreducible decomposition of \(\Lambda^2 M\). The explicit formula for the exterior square character is well-known, the equivalence of the second and third statements follows.

### 1.3 The origins of the ESQ concept

The goal of this section is to give a brief overview on the origins of the ESQ concept. Let us start with a definition in group theory.

**Definition 1.3.1** A finite group is called an UCS group (where UCS stands for Unique proper non-trivial Characteristic Subgroup) if it has exactly one non-trivial characteristic subgroup.

UCS groups were first investigated by Taunt in article [23]. The concept of ESQ representations was introduced by Glasby, Pálfy and Schneider in article [8]. In this article the authors realized that every not abelian UCS \(p\)-group of exponent \(p^2\) gives rise to an ESQ representation. In this section we will present the connection between UCS \(p\)-groups and ESQ representations, this way motivating Definitions 1.1.1 and 1.1.2. Throughout the section we will follow article [8].
Let us state some straightforward properties of UCS \( p \)-groups. The following definition will be useful.

**Definition 1.3.2** Let \( p \) be a prime number. The lower \( p \)-central series of a group \( G \) is defined as follows:

\[
\lambda_1^p(G) = G, \quad \text{and} \quad \lambda_i^p(G) = [\lambda_{i-1}^p(G), G]/(\lambda_{i-1}^p(G))^p \quad \text{for} \quad i \geq 2, \quad \text{where} \quad G^p = \langle g^p \mid g \in G \rangle.
\]

The \( p \)-class of a finite \( p \)-group \( G \) is defined to be the smallest integer \( c \), such that \( \lambda_{c+1}^p(G) = 1 \).

It is clear that for every \( p \)-group \( G \), equation \( \Phi(G) = \lambda_2^p(G) \) holds, where \( \Phi(G) \) denotes the Frattini subgroup of \( G \). Now we state a simple lemma on UCS \( p \)-groups.

**Lemma 1.3.3** Let \( G \) be a UCS \( p \)-group.

(i) \( G \) has exactly three distinct characteristic subgroups, namely, 1, \( \Phi(G) \) and \( G \).

(ii) The \( p \)-class of \( G \) is 2.

(iii) If \( G \) is abelian, then \( G \cong (C_p^r)^r \), where \( r \geq 1 \).

**Proof.** It is clear that \( G \) is not the trivial group, so \( 1 \leq \Phi(G) < G \) holds. For every \( p \)-group it is known that \( G/\Phi(G) \) is an elementary abelian group, and so \( G/\Phi(G) \) is characteristically simple. This implies that \( \Phi(G) \neq 1 \), so the first statement of the lemma follows.

The subgroups in the lower \( p \)-central series are characteristic subgroups, so the \( p \)-class of \( G \) is at most 2. In fact, the \( p \)-class of \( G \) equals 2, as we have already noticed that \( 1 < \Phi(G) = \lambda_2^p(G) \), so the second statement of the lemma follows.

In case \( G \) is abelian, the exponent of \( G \) must be at least \( p^2 \), otherwise \( G \) will be elementary abelian. On the other hand, the exponent must be strictly lower than \( p^3 \), otherwise \( G^p \) and \( G^{p^2} \) would be two not equal proper characteristic subgroups of \( G \). This means that \( G \cong (C_p^r)^r \times (C_p^s)^r \). Considering the characteristic subgroups \( \langle g \mid g^p = 1 \rangle \) and \( G^p, \, q = 0 \) follows.

The following observations on \( p \)-groups of \( p \)-class 2 will be used regularly, so we state them as a lemma. By Lemma 1.3.3, the statements of Lemma 1.3.4 apply to UCS \( p \)-groups, too.

**Lemma 1.3.4** Let \( H \) be a \( p \)-group of \( p \)-class 2.

(i) \( \Phi(H) \) is central and elementary abelian.

(ii) The exponent of \( H \) is either \( p \) or \( p^2 \).
Proof. Equation \( \lambda_p^3(H) = [\Phi(H), H]\Phi(H)^p = 1 \) holds, as group \( H \) is a \( p \)-group of \( p \)-class 2. Now \( [\Phi(H), H] = 1 \) implies that \( \Phi(H) \) is central, and \( \Phi(H)^p = 1 \) implies that \( \Phi(H) \) has exponent \( p \). From these facts statement (i) follows. Furthermore, as \( H^p \leq \Phi(H) \), and \( \Phi(H) \) has exponent \( p \), the exponent of \( H \) is either \( p \) or \( p^2 \), so statement (ii) follows.

For the classification of UCS \( p \)-groups we will examine the variety of \( p \)-groups of \( p \)-class 2.

**Definition 1.3.5** \( H_{p,r} \) denotes the \( r \)-generator free group of the variety of \( p \)-groups of \( p \)-class 2.

**Remark.** Denoting by \( F_r \) the free group of rank \( r \), \( H_{p,r} \) can be expressed as
\[
H_{p,r} = F_r/\lambda_p^3(F_r).
\]

\( H_{p,r} \) is a finite group, as it is nilpotent, finitely generated, and based on Lemma 1.3.4, the generators have finite order. It has exponent \( p^2 \), and \( H_{p,r}/H'_{p,r} \cong (C_{p^2})^r \) due to the free property.

The following lemma shows that \( H_{p,r} \) takes a crucial role in the classification of \( r \)-generated UCS \( p \)-groups.

**Lemma 1.3.6** For any \( r \)-generated UCS \( p \)-group \( G \), there exists a subgroup \( N \leq \Phi(H_{p,r}) \), such that \( G \cong H_{p,r}/N \).

**Proof.** Let us fix some minimal generating sets \( \{x_1, x_2, \ldots, x_r\} \) and \( \{y_1, y_2, \ldots, y_r\} \) of groups \( H_{p,r} \) and \( G \) respectively. Any mapping \( x_i \mapsto y_i \) can be uniquely extended to an epimorphism \( \varphi : H_{p,r} \rightarrow G \), as \( G \) is of \( p \)-class 2, and \( H_{p,r} \) is the \( r \)-generator free group of the variety of \( p \)-groups of \( p \)-class 2. Denoting the kernel of \( \varphi \) by \( N \), the only thing left to prove is that \( N \leq \Phi(H_{p,r}) \). As \( \varphi \) is an epimorphism, \( \varphi \) maps \( H'_{p,r} \) onto \( G' \), and \( H^p_{p,r} \) onto \( G^p \). As a consequence, \( \varphi \) maps \( \Phi(H_{p,r}) \) onto \( \Phi(G) \). According to Burnside’s basis theorem the equation of subgroup indices \( |H_{p,r} : \Phi(H_{p,r})| = |G : \Phi(G)| = p^r \) holds. Using these observations, there is another way to calculate index \( p^r \):

\[
p^r = |G : \varphi(\Phi(H_{p,r}))| = |H_{p,r}/N : \Phi(H_{p,r})N/N| = |H_{p,r} : \Phi(H_{p,r})||N : \Phi(H_{p,r})\cap N| = p^r|N : \Phi(H_{p,r})\cap N|,
\]

hence \( N \leq \Phi(H_{p,r}) \) holds.

**Corollary 1.3.7** Let \( p \) be a prime and let \( r \) be an integer. If \( G = H_{p,r}/N \) with some \( N \leq \Phi(H_{p,r}) \), then
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\[ \Phi(G) = \Phi(H_{p,r})/N. \] Furthermore, the map:

\[ x\Phi(H_{p,r}) \mapsto (xN)\Phi(G), \quad x \in H_{p,r} \]

identifies elementary abelian groups \( H_{p,r}/\Phi(H_{p,r}) \) and \( G/\Phi(G) \).

**Proof.** In the proof of Lemma 1.3.6 we saw that epimorphism \( \varphi \) mapped \( \Phi(H_{p,r}) \) onto \( \Phi(G) \). By the hypothesis \( N \leq \Phi(H_{p,r}) \), so \( \Phi(G) = \Phi(H_{p,r})/N \) holds. Both \( H_{p,r}/\Phi(H_{p,r}) \) and \( G/\Phi(G) \) are rank \( r \) elementary abelian groups, the correctness of the defined identification is assured by \( \Phi(G) = \Phi(H_{p,r})/N \).

Before going on with the investigations on UCS \( p \)-groups, we have to examine the structure of \( \Phi(H_{p,r}) \).

**Lemma 1.3.8** Let \( p \) be an odd prime, \( r \) be an integer, and \( \{x_1, x_2, \ldots, x_r\} \) be a minimal generating set of \( H_{p,r} \). Set \( H = H_{p,r} \). The following statements hold:

- \( H^p \leq \Phi(H) \) is an elementary abelian subgroup minimally generated by \( x_i^p \), where \( 1 \leq i \leq r \).
- \( H' \leq \Phi(H) \) is an elementary abelian subgroup minimally generated by \( [x_i, x_j] \), where \( 1 \leq i < j \leq r \).
- \( \Phi(H) = H^p \oplus H' \).

**Proof.** It is clear that the defined element sets are generator sets of \( H^p \) and \( H' \). The minimality of the generator sets follows from the free property of \( H \). Subgroups \( H^p \) and \( H' \) are elementary abelian due to Lemma 1.3.4, and they generate \( \Phi(H) \). Finally, the free property of \( H \) implies that \( H^p \cap H' = 1 \), hence the direct decomposition stated by the lemma holds.

**Remark.** In Lemma 1.3.8 it is essential that \( p \) is odd, if \( p = 2 \), then \( H' \leq H^2 \).

Our next goal is to define a module structure on elementary abelian groups \( H^p_{p,r} \) and \( H'_{p,r} \). To define this module structure, we will use the natural action of linear group \( GL(H_{p,r}/\Phi(H_{p,r})) \) on elementary abelian group \( H_{p,r}/\Phi(H_{p,r}) \). Let us introduce the following notation.

**Notation 1.3.9** If group \( G \) is a \( p \)-group, then the elementary abelian group \( G/\Phi(G) \) is denoted by \( \overline{G} \).

**Definition 1.3.10** Let \( p \) be an odd prime, let \( r \) be an integer, and set \( H = H_{p,r} \). We define \( GL(\overline{H}) \) actions on the generating sets of elementary abelian groups \( H^p_{p,r} \) and \( H'_{p,r} \) respectively:

- \( x^pg \overset{\text{def}}{=}= \hat{x}^p \), where \( \hat{x} \) is any element of coset \( (x\Phi(H))g \), \( x \in H \) and \( g \in GL(\overline{H}) \).
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- \([x, y]_g \overset{\text{def}}{=} [\widehat{x}, \widehat{y}], \) where \(\widehat{x}\) and \(\widehat{y}\) are any elements of cosets \((x\Phi(H))g\) and \((y\Phi(H))g\) respectively, \(x, y \in H\) and \(g \in GL(\mathcal{P})\).

It is straightforward to check that the defined \(GL(H_{p,r})\) actions are well defined, and they determine module structures on \(H_{p,r}\) and \(H'_{p,r}\). For a rigorous proof we refer Lemma 2.6 in \([16]\). The proof is based on the following three identities of \(H_{p,r}\).

**Lemma 1.3.11** Let \(p\) be an odd prime, let \(r\) be an integer. In \(H_{p,r}\) the following three identities hold:

\[(xy)^p = x^p y^p, \quad [xy, z] = [x, z][y, z] \quad \text{and} \quad [x, yz] = [x, y][x, z].\]

*Proof.* The first identity is a consequence of the Hall-Petresco identity, and the facts that \(p\) is odd, and \(H'_{p,r}\) is central, and elementary abelian. The second and third identities hold, as \(H'_{p,r}\) is central.

Having defined the \(GL(H_{p,r})\) module structures of \(H_{p,r}\) and \(H'_{p,r}\), we can consider \(\Phi(H_{p,r})\) as a \(GL(\mathcal{P})\) module.

**Lemma 1.3.12** Let \(p\) be an odd prime, let \(r\) be an integer, and set \(H = H_{p,r}\).

- \(\Phi(H) = H^p \oplus H'\) as \(GL(\mathcal{P})\) modules.
- \(H^p \cong \mathcal{P}, \ H' \cong \Lambda^2 \mathcal{P}\) as \(GL(\mathcal{P})\) modules.

*Proof.* The first statement is clear based on Definition 1.3.10, and by the third statement of Lemma 1.3.8. To prove the second statement, let us consider the following maps on the module generators:

\[x\Phi(H_{p,r}) \mapsto x^p, \quad x\Phi(H_{p,r}) \wedge y\Phi(H_{p,r}) \mapsto [x, y].\]

It is straightforward to check that the defined maps extend to \(GL(\mathcal{P})\) module isomorphisms using Lemma 1.3.11. For a more detailed argument, see page 26 in \([10]\). \(\square\)

Now, that we know enough about the \(GL(\mathcal{P}_{p,r})\) module structure of \(\Phi(H_{p,r})\), we can continue the investigations on UCS \(p\)-groups. Our main tool will be Lemma 1.3.6. In the standard situation when \(G = H_{p,r}/N\) with some \(N \leq \Phi(H_{p,r})\) (\(G\) is not required to be UCS), we introduce the following linear groups.

**Definition 1.3.13** Let \(p\) be an odd prime, let \(r\) be an integer, and set \(H = H_{p,r}\). Let us assume that \(G = H/N\) with some \(N \leq \Phi(H)\).
• The automorphism group of $G$ acts linearly on elementary abelian group $G$. $\text{Aut}(G)^G$ denotes the image of $\text{Aut}(G)$ as a subgroup of $GL(G)$.

• Lemma 1.3.4 assures that $\Phi(G)$ is an elementary abelian group. The automorphism group of $G$ acts linearly on $\Phi(G)$. $\text{Aut}(G)^{\Phi(G)}$ denotes the image of $\text{Aut}(G)$ as a subgroup of $GL(\Phi(G))$.

• Let us consider $\Phi(H)$ as a $GL(H)$ module. $GL(H)_N$ denotes the subgroup of $GL(H)$ which fixes $N \leq \Phi(H)$ as a set.

• $GL(H)_N$ acts linearly on $\Phi(H)$, and $N$ is a submodule of this action. It follows that $GL(H)_N$ acts linearly on $\Phi(H)/N$. $(GL(H)_N)^{\Phi(H)/N}$ denotes the image of $GL(H)_N$ as a subgroup of $GL(\Phi(H)/N)$.

Let us take $G = H_{p,r}/N$ with some $N \leq \Phi(H_{p,r})$. By Corollary 1.3.7, elementary abelian groups $H_{p,r}$ and $G$ can be naturally identified, and this fact can be used to describe the automorphism group of $G$ in terms of $H_{p,r}$ and $N$.

**Lemma 1.3.14** Let $p$ be an odd prime, let $r$ be an integer, and set $H = H_{p,r}$. Let us assume that $G = H/N$ with some $N \leq \Phi(H)$. Identifying $G$ and $H$, we obtain that

$$\text{Aut}(G)^G = GL(H)_N \text{ and } \text{Aut}(G)^{\Phi(G)} = (GL(H)_N)^{\Phi(H)/N}.$$ 

**Proof.** The proof of the lemma is straightforward, for the details we refer to Theorem 2.10 in [16].

The next theorem gives a condition of representation theory on the UCS property of $p$-groups.

**Theorem 1.3.15** Let $p$ be an odd prime, let $r$ be an integer, and set $H = H_{p,r}$. Let us assume that $G = H/N$ with some $N \leq \Phi(H)$. The following statements are equivalent:

(i) $G$ is a UCS $p$-group.

(ii) $\text{Aut}(G)^G$ module $G$ and $\text{Aut}(G)^{\Phi(G)}$ module $\Phi(G)$ are irreducible.

(iii) $GL(H)_N$ module $H$ and $(GL(H)_N)^{\Phi(H)/N}$ module $\Phi(H)/N$ are irreducible.

**Proof.** Statements (ii) and (iii) are equivalent based on Lemma 1.3.14. Now we prove the equivalence of statements (i) and (ii). The submodules of $\text{Aut}(G)^G$ module $G$ are in one-to-one correspondence to those characteristic subgroups of $G$, which contain $\Phi(G)$ as a subgroup. Similarly, the submodules of
$\text{Aut}(G)^{\Phi(G)}$ module $\Phi(G)$ are in one-to-one correspondence to those characteristic subgroups of $G$, which are contained by $\Phi(G)$. By the first statement of Lemma 1.3.3, implication (i) $\Rightarrow$ (ii) follows.

For the other direction, let us assume that statement (ii) holds, and let $L$ be an arbitrary characteristic subgroup of $G$. Now $L\Phi(G)$ is also a characteristic subgroup, and it contains $\Phi(G)$. Our observations on the submodules of $\overline{G}$, and the irreducibility of $\overline{G}$ implies that either $L\Phi(G) = G$ or $L\Phi(G) = \Phi(G)$ holds. In the first case $L = G$, as $\Phi(G)$ is the subgroup of non-generating elements. In the second case $L \leq \Phi(G)$, and we will further investigate this case. Our observations on the submodules of $\Phi(G)$, and the irreducibility of $\Phi(G)$ imply that either $L = 1$ or $L = \Phi(G)$ holds. We have investigated all the possibilities, and concluded that $L$ equals either 1, or $\Phi(G)$ or $G$. This proves implication (ii) $\Rightarrow$ (i). \qed

We are ready to prove the main theorem of the section.

**Theorem 1.3.16** Let $p$ be an odd prime, let $r$ be an integer, and set $H = H_{p,r}$. Let us assume that $G = H/N$ is an UCS $p$-group with some $N \leq \Phi(H)$. From the following statements exactly one holds:

(i) $H' = N$, and $G$ is abelian.

(ii) $H^p \leq N$, $G$ has exponent $p$ and $G$ is not abelian.

(iii) $H^p \cap N = 1$, $GL(\overline{H})_N$ module isomorphism $\overline{H} \cong H'/(H' \cap N)$ holds, $G$ has exponent $p^2$ and $G$ is not abelian.

**Proof.** Let us assume that $G$ is abelian. It follows that $H' \leq N$, which implies that $G$ is a factor of $H/H' = (C_{pr})^r$. The cyclic decomposition of $G$ is $(C_{pr})^r$ by the third statement of Lemma 1.3.3. As $G$ is $r$-generated, $G = (C_{pr})^r$ holds, hence $H' = N$, so we conclude that statement (i) is fulfilled.

Now let us assume that $G$ is not abelian. $H^p \leq \Phi(H)$ is a $GL(\overline{H})$ submodule by the first statement of Lemma 1.3.12, which implies that $H^p \cap N \leq \Phi(H)$ is a $GL(\overline{H})_N$ submodule. On the other hand, $GL(\overline{H})$ module isomorphism $\overline{H} \cong H^p$ holds by the second statement of Lemma 1.3.12, and $GL(\overline{H})_N$ module $\overline{H}$ is irreducible by the third statement of Lemma 1.3.15. We conclude that $GL(\overline{H})_N$ module $H^p$ is irreducible. It follows that either $N \cap H^p = 1$ or $H^p \leq N$ holds. If $H^p \leq N$, then the exponent of $G$ is $p$, and statement (ii) is fulfilled.

Finally, we have to investigate the case when $N \cap H^p = 1$ holds. $H'/N \leq \Phi(H)/N$ is a $GL(\overline{H})_N$ submodule by the first statement of Lemma 1.3.12. $H'/N \leq \Phi(H)/N$ is also a $GL(\overline{H})_N$ submodule, as $N$ is fixed by $GL(\overline{H})_N$. The third statement of Lemma 1.3.15 implies that $\Phi(H)/N$ is irreducible. It
follows that either $H'N = \Phi(H)$ or $H'N = N$ holds. $H'N = \Phi(H)$ is the only valid possibility, as $G$ is not abelian. Repeating the presented argument of submodule $H'N$ on submodule $H^pN$, we conclude that either $H^pN = \Phi(H)$ or $H^pN = N$ holds. Our assumption $N \cap H^p = 1$ implies that the only valid possibility is $H^pN = \Phi(H)$. Now we get our desired $GL(\overline{H})_N$ module isomorphism:

$$\overline{H} \cong H^p \cong \frac{H^p}{N \cap H^p} \cong \frac{H^pN}{N} = \frac{\Phi(H)}{N} \cong \frac{H'N}{N \cap H'}.$$ 

The exponent of $G$ is $p^2$, since $H^p \cap N = 1$. Statement (iii) is fulfilled.

We interpret the third statement of Theorem 1.3.16. The theorem tells us that if we take the $H_{p,r}/N$ representation of an arbitrary not abelian $r$-generated UCS $p$-group of exponent $p^2$, then the $GL(\overline{H})_N$ module isomorphism $\overline{H} \cong H'/N \cap H'$ holds. The $GL(\overline{H})_N$ module isomorphism $H' \cong \Lambda^2\overline{H}$ is straightforward from the second statement of Lemma 1.3.12. Finally, $\overline{H}$ is irreducible by Lemma 1.3.15. We conclude that $\overline{H}$ is an irreducible ESQ $GL(\overline{H})_N$ module, naturally associated to the initial UCS $p$-group. Based on this interpretation, the motivation behind Definitions 1.1.1 and 1.1.2 is clear. We have achieved the goal of the section.

We give an example how to exploit the correspondence between UCS $p$-groups and ESQ representations. In the next section we will investigate low dimensional irreducible ESQ representations. Theorem 1.4.2 gives a criterion on the existence of irreducible ESQ representations in dimension four. As a direct consequence, we see that there exists no four-generated not abelian UCS 5-group of exponent 25.

Finally, for the sake of completeness, we cite a part of Theorem 5 of article [8]. This theorem characterizes the existence of not abelian UCS $p$-groups of exponent $p^2$ in terms of ESQ representations.

**Theorem 1.3.17** Let $p$ be an odd prime, let $r$ be an integer. The following two assertions are equivalent.

(i) There exists a not abelian UCS $p$-group $G$ with exponent $p^2$, such that $|G| = p^r$.

(ii) There exists an irreducible ESQ-module $V$ over a field $\mathbb{F}_{p^k}$, such that $\dim V = r/k$, and $V$ cannot be written over any proper subfields of $\mathbb{F}_{p^k}$.

### 1.4 The investigation of low dimensional irreducible ESQ groups

In this section we will present the results of article [8] on four- and five-dimensional irreducible ESQ groups. It would be natural to start the investigation on the three-dimensional ESQ groups, however,
on these groups we cannot state as strong assertions as on the four- and five-dimensional ESQ groups.

The reason is the following. Let us fix the three-dimensional vector space bases $v_1, v_2, v_3$ and $v_3 \wedge v_1, v_1 \wedge v_2$ of $V$ and $\Lambda^2 V$ respectively. Taking an arbitrary $g \in SO(3)$, and calculating its action $g \wedge g$ on $\Lambda^2 V$, it can be verified that $g = g \wedge g$ as $3 \times 3$ matrices over the fixed bases. This fact implies that the whole $SO(3)$ group is an ESQ group, so any subgroup of $SO(3)$ determines a three-dimensional ESQ group. The number of three-dimensional ESQ groups is too large.

The two main results of the section are Theorems 1.4.2 and 1.4.10. First, we will focus on Theorem 1.4.2, which classifies the four-dimensional finite irreducible ESQ groups over any field $\mathbb{F}$ with $\text{char } \mathbb{F} \neq 2$.

The following action on the four-dimensional vector space will be important in the classification.

**Definition 1.4.1** Let $L = AGL(\mathbb{F}_1)$ be the one-dimensional affine group over the field of order 5, and $\mathbb{F}$ be any field with $\text{char } \mathbb{F} \neq 5$. $L$ has order 20, and it has a natural permutation action on the set of cardinality 5. Take the five-dimensional permutation representation of $L$ over $\mathbb{F}_5$. We define the following $\mathbb{F}L$ module decomposition: $\mathbb{F}^5 = W \oplus W_1$, where $W = \{(x_0, x_1, \ldots, x_4) | x_0 + x_1 + \ldots + x_4 = 0\}$ and $W_1 = \{(x, x, \ldots, x) | x \in \mathbb{F}\}$.

**Remark.** If $\text{char } \mathbb{F} \neq 5$, then $W$ is the only faithful irreducible $\mathbb{F}L$ module.

**Theorem 1.4.2** Let $\mathbb{F}$ be a field with $\text{char } \mathbb{F} \neq 2$, and let $K \leq \text{GL}(\mathbb{F}^4)$ be a finite irreducible ESQ group. Then $\text{char } \mathbb{F} \neq 5$, and $K$ is isomorphic as a linear group to a subgroup of action $L$ on $W$. (The definition of $L$ and its action on $W$ can be found in Definition 1.4.1.) Moreover, 5 divides $|K|$, and if 5 is a square in $\mathbb{F}$, then $K \cong L$.

Theorem 1.4.2 will be the consequence of the following series of lemmas.

**Lemma 1.4.3** Over a field of characteristic different from 2, the only two-dimensional representation of a not abelian finite simple group is the trivial representation.

**Proof.** For a proof we refer to Proposition 5.5.10 in [12].

**Lemma 1.4.4** Let $\mathbb{F}$ be a field with $\text{char } \mathbb{F} \neq 2$, and let $K \leq \text{GL}(\mathbb{F}^4)$ be a finite irreducible ESQ group. If $M$ is a minimal normal subgroup of $K$, then $M$ is an elementary abelian group.

**Proof.** $M$ is a direct product of isomorphic simple groups, since a minimal normal subgroup is characteristically simple. We will prove that $M$ is abelian by contradiction, so let us assume that $M$ is not abelian. Let us denote by $S$ one of the not abelian simple factors of $M$. Using Lemma 1.4.3 and Clifford’s theorem
on the normal chain \( S \triangleleft M \triangleleft K \), we conclude that \( S \) is irreducible. Let \( V \) be the four-dimensional vector space over \( \mathbb{F} \). By \( S \leq K \) and by the third point of Theorem 1.2.1, \( V \) can be considered as an irreducible ESQ \( FS \) module. It follows that there exists an \( FS \) submodule \( U \leq \Lambda^2V \), such that \( \Lambda^2V/U \cong V \) holds. The dimension of \( U \) is \( 6 - 4 = 2 \), so the action of \( S \) on \( U \) is trivial by Lemma 1.4.3.

Let us fix a basis \( x_1, \ldots, x_4 \) of \( V \) and take its corresponding dual basis \( x_1^*, \ldots, x_4^* \) of \( V^* \). We will define an \( FS \) module structure on vector spaces \( V^* \), \( \text{Hom}(V,V^*) \) and \( V \otimes V \). From now on the elements of \( S \) will be considered as \( 4 \times 4 \) matrices over the fixed basis of \( V \). The action of \( s \in S \) on \( V^* \) is defined to be the \( 4 \times 4 \) matrix \( (s^{-1})^T \) over the dual basis. An \( f \in \text{Hom}(V,V^*) \) can be considered as a \( 4 \times 4 \) matrix over the fixed bases of \( V \) and \( V^* \). We define the action of \( s \in S \) on \( f \in \text{Hom}(V,V^*) \) to be \( fs = s^Tfs \).

Finally, on \( V \otimes V \) we take the usual tensor square action of \( S \).

We define the map \( \varphi : \text{Hom}(V,V^*) \to V \otimes V \) as follows. For every \( f \in \text{Hom}(V,V^*) \), we take \( f \) as an \( (\alpha_{i,j}) \) \( 4 \times 4 \) matrix over the fixed bases, and we define \( \varphi \) by \( \varphi(f) = \sum_{i,j} \alpha_{i,j}(x_i \otimes x_j) \). Having defined the \( FS \) modules \( V, V^*, \text{Hom}(V,V^*) \) and \( V \otimes V \), we can check that \( \varphi \) is an \( FS \) module isomorphism, and that \( f \in \text{Hom}(V,V^*) \) is an \( FS \) module isomorphism if and only if \( \varphi(f) \) is fixed by all \( s \in S \).

The \( S \)-fixed points in \( V \otimes V \) form an at least two-dimensional submodule, since \( S \) acts trivially on \( U \) and \( U \leq \Lambda^2V \leq V \otimes V \). We conclude that the centralizer algebra of \( FS \) module \( V \) is at least two-dimensional, since \( V \cong V^* \). By Schur’s lemma, the centralizer algebra is a quadratic extension field \( \mathbb{E} \) of \( \mathbb{F} \), so the not trivial four-dimensional \( FS \) module \( V \) can be considered as a not trivial two-dimensional \( \mathbb{E}S \) module. This fact contradicts Lemma 1.4.3, so our present lemma is proved.

**Lemma 1.4.5** Let \( \mathbb{F} \) be a field with \( \text{char} \ \mathbb{F} \neq 2 \), and let \( K \leq GL(\mathbb{F}^4) \) be a finite irreducible ESQ group. Let \( M \) be a minimal normal subgroup of \( K \). The following statements hold.

- \( M \) is an elementary abelian \( r \)-group for some prime \( r \), which does not equal the characteristic of \( \mathbb{F} \).
- There exists an element \( g \in M \), such that 1 is not an eigenvalue of \( g \).

**Proof.** \( M \) is an elementary abelian \( r \)-group for some prime \( r \) by Lemma 1.4.4. Let \( V \) be the four-dimensional vector space over \( \mathbb{F} \). If \( M \) fixes an element of \( V \), then \( M \triangleleft K \) implies that \( M \) fixes all the elements of \( V \). This contradicts the faithfulness of \( K \). We conclude that the \( FM \) module \( V \) does not contain the trivial submodule. The orbit counting lemma implies that a representation of a \( p \)-group over characteristic \( p \) contains the trivial module as a submodule. By these observations the first statement of the lemma follows.
We will prove the second statement of the lemma by contradiction, so let us assume that every \( g \in M \) fixes some non-zero vector of \( V \). We denote by \( \mathbb{E} \) the extension field \( F \leq \mathbb{E} \), which contains all the \( r \)th roots of unities. Considering \( V \) as an \( \mathbb{E}M \) module, we can decompose \( V \) into the direct sum of 4 one-dimensional irreducible submodules. By taking the generators of these submodules a basis \( e_1, e_2, e_3, e_4 \) of \( \mathbb{E}^4 \) can be fixed. Over this basis all the elements of \( M \) correspond to \( 4 \times 4 \) diagonal matrices. The \( \mathbb{E}M \) module \( V \) does not contain a trivial submodule, this property is inherited from \( FM \) module \( V \). It follows that at most \(|M|/r\) elements of \( M \) can fix a basis vector \( e_i \), where \( 1 \leq i \leq 4 \). Based on the indirect hypothesis, every \( m \in M \) fixes at least one of the basis vectors \( e_i \), \( 1 \leq i \leq 4 \). We conclude that \(|M| \leq 4|M|/r\), implying that \( r \) is either 2 or 3. \( M \) is not cyclic, otherwise a fixed subspace of a generator would be a trivial \( \mathbb{E}M \) submodule of \( V \). It follows that the determinant map \( det: M \rightarrow \mathbb{F}^* \) is not injective, implying that the intersection \( M \cap SL(\mathbb{F}^4) \) is not trivial. We conclude that \( M \leq SL(\mathbb{F}^4) \), as \( M \) is a minimal normal subgroup. In the following part we will show contradiction in both of the cases \( r = 2 \) and \( r = 3 \).

First, we will investigate \( r = 2 \), in this case \( \mathbb{F} = \mathbb{E} \). We denote by \( D \) the \( 2^3 \) order group of those four-dimensional diagonal matrices with determinant 1, which contain only \( \pm 1 \) in the main diagonal. \( M \leq D \) follows by our previous observations. It is easy to check that any subgroup of \( D \) either contains \(-I\) or stabilizes one of the basis vectors. Neither of these can happen, so \( r = 2 \) cannot hold.

It remains to investigate \( r = 3 \). Let \( \omega \in \mathbb{E} \) be a primitive third root of unity. We take a non-trivial element \( g \in M \leq SL_4(\mathbb{E}) \). If \( g \) has eigenvalue 1 of multiplicity one, then the spectrum of \( g \) is either \( 1, \omega, \omega, \omega \) or \( 1, \omega^2, \omega^2, \omega^2 \). In both cases \( g \land g \) has no eigenvalue 1, and this fact contradicts the ESQ property of \( M \). Hence, for every non-trivial element \( g \in M \), the eigenvalue 1 has at least multiplicity two, implying that the spectrum of \( g \) is \( 1, 1, \omega, \omega^2 \). Now \( M \) is a proper subgroup of the group of diagonal matrices of determinant 1. The order of \( D \) is \( 3^3 \), so \( M \) is two-generated. Let us denote the two generators of \( M \) by \( g_1 \) and \( g_2 \). Generators \( g_1 \) and \( g_2 \) cannot fix the same basis vector from \( e_1, e_2, e_3, e_4 \). It follows that on the eigenvectors fixed by \( g_1 \), the eigenvalues of \( g_2 \) must be \( \omega \) and \( \omega^2 \). Hence, the spectrum of \( g_1g_2 \in M \) is \( \omega, \omega, \omega^2, \omega^2 \), but this fact contradicts the indirect hypothesis. This means that \( r = 3 \) cannot hold. We have derived contradiction in all the possible cases, which means that the indirect assumption is not true. The second statement of the lemma follows.

**Lemma 1.4.6** Let \( \mathbb{F} \) be a field with char \( \mathbb{F} \neq 2 \), and let \( K \leq GL(\mathbb{F}^4) \) be a finite irreducible ESQ group. If \( M \) is a minimal normal subgroup of \( K \), then \( M \) is an elementary abelian 5-group.

**Proof.** \( M \) is an elementary abelian \( r \)-group for some prime \( r \) by Lemma 1.4.4. Let \( \mathbb{E} \) be an extension
field of $F$ containing the $r^{th}$ roots of unities, and let $e_1, e_2, e_3, e_4$ be the eigenbasis of $M$ over $\mathbb{E}^4$. Let us take an element $g \in M$, which has no eigenvalue 1, according to Lemma 1.4.5, this kind of element exists. Let us denote the eigenvalues of $g$ by $\lambda_i$, where $1 \leq i \leq 4$. The eigenvalues of $g \wedge g$ are $\lambda_i \lambda_j$, where $1 \leq i < j \leq 4$. By the ESQ property there exists an injective map $i \mapsto P(i) = \{j, k\}$, such that $\lambda_i = \lambda_j \lambda_k$. Since 1 is not an eigenvalue of $g$, we see that $i \notin P(i)$ holds. We have got an equation system containing 4 equations. There are only two different equation systems up to the permutation of indices, these are the following:

\begin{align*}
\lambda_1 = \lambda_2 \lambda_3, \quad \lambda_2 = \lambda_3 \lambda_4, \quad \lambda_3 = \lambda_4 \lambda_1, \quad \lambda_4 = \lambda_1 \lambda_2, \\
\lambda_1 = \lambda_2 \lambda_3, \quad \lambda_2 = \lambda_1 \lambda_4, \quad \lambda_3 = \lambda_1 \lambda_2, \quad \lambda_4 = \lambda_1 \lambda_3.
\end{align*}

Solving equation system (1.1), we get:

\begin{align*}
\lambda_1 &= \varepsilon^4, \quad \lambda_2 = \varepsilon^3, \quad \lambda_3 = \varepsilon, \quad \lambda_4 = \varepsilon^2,
\end{align*}

where $\varepsilon^5 = 1$. In this case $r = 5$, so the statement of the lemma holds.

Solving equation system (1.2), we get:

\begin{align*}
\lambda_1 &= \varepsilon^4, \quad \lambda_2 = \varepsilon^3, \quad \lambda_3 = \varepsilon, \quad \lambda_4 = \varepsilon^5,
\end{align*}

where $\varepsilon^6 = 1$. We will show that this case cannot occur. Every element of $M$ has prime order, so from $\lambda_3 = \varepsilon$ and $\varepsilon^6 = 1$, it follows that $\varepsilon^2 = 1$ or $\varepsilon^3 = 1$. If $\varepsilon^2 = 1$ holds, then $\lambda_1 = 1$, if $\varepsilon^3 = 1$ holds, then $\lambda_2 = 1$, contrary to our assumption that 1 is not an eigenvalue of $g$. The proof of the lemma is complete.

Now we are ready to prove Theorem 1.4.2.

Proof. Let us take a minimal normal subgroup $M$ of $K$. By Lemma 1.4.6 $M$ is an elementary abelian 5-group. The first statement of Lemma 1.4.5 implies that $\text{char } F \neq 5$. Let $E$ be an extension field of $F$ containing the fifth roots of unities, and let us fix the $e_1, e_2, e_3, e_4$ eigenbasis of $M$ over $\mathbb{E}^4$. We take an element $g \in M$, which has no eigenvalue 1, according to Lemma 1.4.5 this kind of element exists. By the solution of equation system (1.1), we know that $g$ is a diagonal matrix over the fixed basis of $\mathbb{E}^4$ of form
diag(ε^4, ε^3, ε^1, ε^2), where ε is a primitive fifth root of unity.

Now we fix a basis of Λ^2E^4. The basis vectors of Λ^2E^4 will be the following:

\[ e_2 \wedge e_3, \quad e_3 \wedge e_4, \quad e_4 \wedge e_1, \quad e_1 \wedge e_2, \quad e_1 \wedge e_3, \quad e_2 \wedge e_4. \]

Over this basis g \wedge g corresponds to matrix diag(ε^4, ε^3, ε^1, ε^2, 1, 1). Hence, any \langle g \rangle module isomorphism from E^4 into Λ^2E^4 maps e_i to a multiple of e_{i+1} \wedge e_{i+2} (the indices are taken modulo 4).

Let us take an arbitrary element h \in K which centralizes g. Now h is also diagonal, since the eigenvalues of g are pairwise not equal, so h = diag(λ_1, λ_2, λ_3, λ_4). The restriction of h \wedge h to subspace \langle e_2 \wedge e_3, e_3 \wedge e_4, e_4 \wedge e_1, e_1 \wedge e_2 \rangle is diag(λ_2λ_3, λ_3λ_4, λ_4λ_1, λ_1λ_2), and this implies that the eigenvalues λ_i satisfy equation system (1.1). It follows that h \in \langle g \rangle. As M centralizes g, we conclude that M is isomorphic to the 5 element cyclic group, and it is a self-centralizing normal subgroup of K.

Using the fact that M is a self-centralizing normal subgroup of K, we know that K is isomorphic to a subgroup of the holomorph of M. The holomorph of M is isomorphic to L. The subgroup M ≤ GL_4(𝔽) is unique up to conjugacy in GL_4(𝔽), hence, K can be embedded into a subgroup of GL_4(𝔽) isomorphic to L.

L has three subgroups containing M: these are M, D_5 and L. If 5 is a square, then all the irreducible representations of M and D_5 have a degree of at most two, so in this case K ≅ L holds. The proof of Theorem 1.4.2 is now complete.

In the following part, we will investigate five-dimensional ESQ groups. To discuss the topic, we will need two new definitions.

**Definition 1.4.7** A faithful irreducible representation \( ϕ : G \rightarrow GL(V) \) is said to be minimal if and only if — for any proper subgroup \( H < G \) — the restricted representation \( ϕ|_H \) is not irreducible.

The next definition might be well-known, nevertheless, we decided to state it explicitly.

**Definition 1.4.8** A representation over field \( 𝕂 \) is said to be absolutely irreducible, if it is irreducible over any field extension \( 𝕂 \leq 𝕄 \).

**Notation.** Let s be an integer not divisible by prime \( p \). The smallest positive integer \( t \) satisfying equation \( s^t \equiv 1 \pmod{p} \) is denoted by \( \text{ord}_p(s) \).
In the rest of the section our goal will be to present Theorem 1.4.10, which classifies the five-dimensional irreducible minimal ESQ groups over finite fields. In general prime dimensions the following theorem holds.

**Theorem 1.4.9** Let \( p \) be a prime, \( q \) be a power of a prime (possibly distinct from \( p \)), and let \( K \) be a minimal irreducible ESQ subgroup of \( GL(F^p_q) \). Then one of the following holds:

(i) \( K \) is not absolutely irreducible, \( |K| = r \) is a prime, \( \text{ord}_r(q) = p \) holds, and there exist distinct \( \alpha, \beta \in (q) \leq F^r_q \), such that \( \alpha + \beta \equiv 1 \pmod{r} \) holds.

(ii) \( K \) is an absolutely irreducible not abelian simple group.

(iii) \( K \) is absolutely irreducible, \( |K| = pr^s \) where \( r \) is a prime different to \( p \), \( \text{ord}_r(q) = 1 \), and \( \text{ord}_p(r) = s \). Moreover, \( K' \) is an elementary abelian group of order \( r^s \), and \( K/K' \) acts irreducibly on \( K' \).

For a proof of Theorem 1.4.9 we refer to Theorem 14 in [8]. For \( p = 5 \), Theorem 1.4.9 can be strengthened.

**Theorem 1.4.10** Let \( q \) be a power of a prime, and let \( K \) be a minimal irreducible ESQ subgroup of \( GL(F^5_q) \). Then case (ii) of Theorem 1.4.9 does not arise, and more can be said about cases (i) and (iii):

(i') \( K \) is not absolutely irreducible, \( |K| = 11 \) and \( \text{ord}_{11}(q) = 5 \).

(iii') \( K \) is absolutely irreducible, \( |K| = 55 \) and \( \text{ord}_{11}(q) = 1 \).

Furthermore, both of these possibilities occur.

**Proof.** For the proof of the strengthened assertions (i)' and (iii)', and that both of them might occur, we refer to Theorem 15 in [8]. In the following proof we will show that case (ii) of Theorem 1.4.9 cannot hold in five dimensions. We will prove it by contradiction, so let us assume that \( K \) is a not abelian simple group. Let us denote the characteristic of field \( F_q \) by \( c \). \( K \) is not a cyclic group, so the determinant map \( \det: K \to F^*_q \) is not injective, implying that \( K \leq SL(F^5_q) \). As \( K \) is simple \( K \cap Z(SL(F^5_q)) = 1 \) holds, we conclude that \( K \) is isomorphic to a subgroup of \( PSL(F^5_q) \).

Let us investigate the case when \( c = 2 \) holds. The irreducible subgroups of \( PSL(F^5_{2^k}) \) were classified by Wagner in article [24]. This classification states that every irreducible subgroup of \( PSL(F^5_{2^k}) \) is isomorphic to one of the following groups: \( PSL(F^2_{11}) \), \( PSL(F^5_{2^l}) \) (\( l \mid k \)) or \( PSU(F^5_{2^l}) \) (\( 2l \mid k \)). Since containments \( PSL(F^5_{2}) \leq PSL(F^5_{2^k}) \) and \( PSU(F^5_{11}) \leq PSU(F^5_{2^l}) \) hold, the minimality of \( K \) implies that
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$K = PSL(\mathbb{F}_2^2)$ or $K = PSL(\mathbb{F}_1^{11})$. The five-dimensional irreducible representations of these groups over characteristic 2 are known, there are four of them, and we refer to The Atlas for that [25]. Straightforward computation shows that the exterior square of each of these is irreducible. It follows that in case $c = 2$ holds, then $K$ cannot be ESQ.

Now we will investigate the case when $c > 2$ holds. The irreducible subgroups of $PSL(\mathbb{F}_c^5)$ were classified in article [6]. Using this result, it can be checked, that all the irreducible subgroups of $PSL(\mathbb{F}_c^5)$ contain $A_4$ (the alternating group on four elements) as a subgroup. For more details we refer to Theorem 15 in [8].

In the rest of the proof we will show that if $c > 2$, then there is no five-dimensional faithful ESQ $A_4$ module. Indirectly, let us assume that $V$ is a five-dimensional faithful ESQ $A_4$ module. This implies that $V$ is also a faithful ESQ $V_4$ module, where $V_4$ denotes the Klein group $V_4 \triangleleft A_4$. $V_4$ has three non-trivial one-dimensional irreducible representations, since $c > 2$. As $V_4$ is abelian, we can fix an eigenbasis of $V_4$ in $V$. Let us denote by $v$ one of the eigenvectors. If the $V_4$ module $\langle v \rangle$ is not trivial, and $p \in A_4$ has order 3, then calculation shows that subspaces $\langle v \rangle$, $\langle vp \rangle$, $\langle vp^2 \rangle$ are $V_4$ submodules, and they are isomorphic to the three non-trivial one-dimensional representations of $V_4$. It follows that in the decomposition of $V_4$ module $V$ the multiplicities of the different non-trivial one-dimensional representations are equal. Their multiplicity cannot be zero, because in that case $V_4$ acts trivially on $V$, contradicting the faithfulness of $V_4$ module $V$. We conclude that the $V_4$ module $V$ is decomposed into the direct sum of two trivial $V_4$ representations, and to the three non-trivial one-dimensional $V_4$ representations with multiplicity one. This decomposition implies that in the decomposition of $V_4$ module $A^2 V$ the multiplicity of the trivial representation is one. This fact contradicts the ESQ property of $V_4$ module $V$. The proof of Theorem 1.4.10 is now complete.

$\square$
2 The ESQ property of certain representations of metacyclic groups

2.1 Introduction

We presented classification results on low-dimensional irreducible ESQ groups in Section 1.4. The not
abelian groups in Theorems 1.4.2 and 1.4.10 are semidirect products of two cyclic groups, more precisely,
they have the following form: \(F_{p,q} = C_p \rtimes \varphi C_q\), where \(p\) is a prime, \(q | p - 1\) and \(\varphi\) is an injective \(C_q \to C_p^\times\) homomorphism. We denote the fraction \(\frac{p-1}{q}\) by \(r\). This notation remains fixed throughout this chapter. The groups \(F_{p,q}\) are unique up to isomorphism for any given values of \(p\) and \(q\). In the present chapter we will investigate the ESQ property of irreducible \(F_{p,q}\) representations over \(C\). This investigation will produce an infinite number of different irreducible ESQ representations even in dimension six. This chapter is based on article [27] written by the author. The main results of the chapter are the following:

**Theorem 2.1.1** Let \(q\) be fixed. If \(6 \nmid q\), then for any sufficiently large \(p\) with \(q | p - 1\), the metacyclic group \(F_{p,q}\) does not have any irreducible ESQ representation over \(C\). If \(6 | q\), then every \(F_{p,q}\) has an irreducible ESQ representation over \(C\).

**Theorem 2.1.2** Let \(r = \frac{p-1}{q}\) be fixed. For any sufficiently large \(p\), the metacyclic group \(F_{p,q}\) has an irreducible ESQ representation over \(C\).

The character table of an arbitrary \(F_{p,q}\) is well-known. All non-linear irreducible characters of \(F_{p,q}\) are associated with \(q\) dimensional faithful representations. As it will turn out, either all of these representations will have the ESQ property or none of them. The proof of Theorems 2.1.1 and 2.1.2 will be based on the decomposition of the exterior square of a non-linear character of \(F_{p,q}\). Considering this decomposition, we will see that the ESQ property of a \(q\) dimensional irreducible representation of \(F_{p,q}\) will be equivalent to the solvability of a Fermat-type equation in \(F_p^\times\).

2.2 The proof of Theorems 2.1.1 and 2.1.2

To prove Theorems 2.1.1 and 2.1.2, we will make use of the irreducible character values of \(F_{p,q}\). A full description of the irreducible characters of \(F_{p,q}\) can be found in [11], Theorem 25.10, which we cite here.

**Theorem 2.2.1** For the metacyclic group \(F_{p,q} = C_p \rtimes \varphi C_q\) let us introduce the following notation. We set \(r = \frac{p-1}{q}\), and we fix an element \(u\) of order \(q\) in \(F_p^\times\), some coset representatives \(v_1, v_2, \ldots, v_r\) of \((u)\).
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The proof of Theorems 2.1.1 and 2.1.2

in $\mathbb{F}_p^\times$ and $\varepsilon = e^{2\pi i/p}$. Finally, we denote by $a$ and $b$ some fixed generators of cyclic groups $C_p$ and $C_q$ respectively.

Now $F_{p,q}$ has $q + r$ irreducible characters. There are $q$ linear characters and $r$ characters of degree $q$ given by

$$\chi_t((a^x, b^y)) = 0 \text{ if } b^y \neq 1,$$

$$\chi_t((a^x, 1)) = \sum_{s \in v_t(u)} \varepsilon^{sx}$$

for $t = 1, 2, \ldots, r$.

The linear representations of $F_{p,q}$ obviously cannot have the ESQ property. For the non-linear irreducible representations of $F_{p,q}$, the following lemma will provide a reformulation of the ESQ property.

**Lemma 2.2.2** Let $\chi_t$ be a non-linear character of $F_{p,q}$. Then $\chi_t$ is ESQ if and only if there exist natural numbers $k, l$, and an element $z \in \mathbb{F}_p$ satisfying the following:

$$z^k = z^l + 1, \quad z^l \neq 1, \quad z^q = 1.$$

**Proof.** We denote the subgroup $\langle (a, 1) \rangle$ in $F_{p,q}$ by $A$ and the exterior square of the character $\chi_t$ by $\hat{\chi}_t$.

Using Theorem 2.2.1, we can determine the character values of $\hat{\chi}_t$ on $A$ as follows:

$$\hat{\chi}_t((a^x, 1)) = \frac{\chi_t^2((a^x, 1)) - \chi_t((a^{2x}, 1))}{2} = \frac{1}{2} \left( \sum_{s, s' \in v_t(u)} \varepsilon^{(s+s')x} - \sum_{s \in v_t(u)} \varepsilon^{2sx} \right) = \sum_{0 \leq i < j < q} \varepsilon^{v_t(u^i + u^j)x}. (2.1)$$

Let us take any term $\varepsilon^{v_t(u^i + u^j)x}$ of (2.1). If $u^i + u^j = 0$ in $\mathbb{F}_p$, then this term will be 1, which can be interpreted as $1((a^x, 1))$, where 1 denotes the trivial character of $F_{p,q}$. If $u^i + u^j \neq 0$ then we isolate the following partial sum of (2.1):

$$\sum_{k=0}^{q-1} \varepsilon^{v_t(u^{i+k} + u^{j+k})x}.$$

In this case the exponents

$$v_t(u^{i+k} + u^{j+k}) \text{ for } k = 0, 1, \ldots, q - 1$$

form a coset of $\langle u \rangle$ in $\mathbb{F}_p^\times$, so we may assume that this coset corresponds to some coset representative $v_t$. This observation implies that the isolated partial sum equals $\chi_t((a^x, 1))$. By repeating this procedure we
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The proof of Theorems 2.1.1 and 2.1.2 will get a decomposition of the restricted character \( \hat{\chi}_t|_A \) to some non-linear irreducible characters of \( F_{p,q} \) restricted to \( A \) and \( 1|_A \). We denote the set of constituents in this decomposition by \( D \).

The restrictions of non-linear characters of \( F_{p,q} \) to \( A \) and \( 1|_A \) are pairwise orthogonal \( A \) characters, since they are orthogonal in \( F_{p,q} \), and with the exception of \( 1 \) all of them vanish on \( F_{p,q} \setminus A \). This observation implies the following equivalence

\[
\chi_t|_A \in D \iff \langle \chi_t|_A, \hat{\chi}_t|_A \rangle_A \neq 0 \iff \chi_t \text{ is ESQ.}
\]

We have seen from the decomposition procedure that \( \chi_t|_A \in D \) if and only if \( u^i + u^j \in \langle u \rangle \) for some \( 0 \leq i < j < q \). Choosing \( k \) appropriately, we can write this relation as \( u^k = u^l + 1 \) where \( l = j - i \). The lemma follows.

An immediate consequence of Lemma 2.2.2 is that either all of the non-linear representations of \( F_{p,q} \) have the ESQ property or none of them has it. Since the elements which satisfy the equation \( z^q = 1 \) are the \( r \)th powers in \( F_p^\times \), we can characterize the ESQ property by the solvability of a Fermat-type equation as follows.

**Lemma 2.2.3** Let \( \chi_t \) be a non-linear character of \( F_{p,q} \). Then \( \chi_t \) is ESQ if and only if the equation

\[
x^r + y^r = z^r
\]

has a solution in \( F_p^\times \), such that \( x^r \neq y^r \).

Now we prove Theorem 2.1.1 with the help of Lemma 2.2.2.

**Proof.** We start our proof with the assumption that \( 6 \nmid q \). We will show that the equation system \( z^k = z^l + 1, \ z^q = 1 \) has no solution in \( \mathbb{F}_p \), where \( p \) is a sufficiently large prime, \( q \mid p - 1 \), and \( k, l \) are arbitrary non-negative integers. Let us fix some values for \( k \) and \( l \). As \( z^q = 1 \) we may assume that \( 0 \leq k, l < q \).

Our aim is to show that the polynomials \( f(z) = z^k - z^l - 1 \) and \( g(z) = z^q - 1 \) have no common root in \( \mathbb{F}_p \). First we will show that \( f \) and \( g \) have no common root over \( \mathbb{C} \). Assume the contrary that there is a \( t \in \mathbb{C} \) which satisfies \( f(t) = g(t) = 0 \). As \( t^q - 1 = 0 \), it is clear that \( t \) and \( t^k \) are on the complex unit circle centered in 0. On the other hand, \( t^k = t^l + 1 \) implies that \( t^k \) is also on the unit circle centered in 1. These circles have two common points, which are the primitive sixth roots of unity so the order of \( t^k \) is
6. The order of \( t^k \) divides the order of \( t \). Using the equation \( t^q = 1 \), we get \( 6 \mid q \), which contradicts our starting assumption.

The resultant of two polynomials can be computed by a determinant whose entries are either zeroes or coefficients of the polynomials. This value will be zero if and only if the polynomials have a common root. We conclude that the resultant of \( f \) and \( g \) over \( \mathbb{C} \) is a non-zero integer \( R_{kl} \). The resultant of \( f \) and \( g \) over \( \mathbb{F}_p \) is given by the same determinant as in the complex case, so its value will be the residue of \( R_{kl} \) modulo \( p \). This will be non-zero if we choose \( p \) to be greater than \( |R_{kl}| \). This choice ensures that for any prime \( p \) greater than \( C \), the equation system \( z^k = z^l + 1 \), \( z^q = 1 \) will have no solution in \( \mathbb{F}_p \). Hence by Lemma 2.2.2 the first statement of Theorem 2.1.1 follows.

Finally, we prove Theorem 2.1.2. Without the assumption \( x^r \neq y^r \) the solvability of a Fermat-type equation in Lemma 2.2.3 is a well investigated problem. Schur proved in 1916 that a fixed degree Fermat-type equation has a solution in \( \mathbb{F}_p^\times \) for almost every prime \( p \) [21]. Now we briefly give the proof for this statement.

**Lemma 2.2.4** For every positive integer \( c \), there exists \( s(c) \in \mathbb{N}^+ \), so that for an arbitrary coloring of the set \( S(c) = \{1, 2, \ldots, s(c)\} \) by \( c \) colors, there will be a monochromatic solution for the equation \( x + y = z \). (Here \( x \) and \( y \) can be equal.)

**Proof.** For an arbitrary coloring of the set \( T = \{1, 2, \ldots, t\} \) we assign an edge coloring of the complete graph \( K_t \) as follows. The edge \( e_{ij} \) will get the color of \( |i - j| \in T \). As \( c \) is fixed, if \( t \) is greater then some constant \( R(c, 3) \), then the Ramsey theorem ensures that \( K_t \) will have a monochromatic triangle. Let us denote the vertices of such a monochromatic triangle by \( p < q < r \). Now the equation \( (q-p)+(r-q) = r-p \) is a monochromatic solution for \( x + y = z \), hence the lemma is now demonstrated.

For \( s(r) < p \) let us denote a primitive root of \( \mathbb{F}_p^\times \) by \( g \). We color the elements in the coset \( g^i \langle g^r \rangle \subset \mathbb{F}_p^\times \) with color \( i \) where \( i = 1, 2, \ldots, r \). For this coloring Lemma 2.2.4 provides a monochromatic triple in \( \mathbb{F}_p^\times \), which gives us a solution for \( x^r + y^r = z^r \) in \( \mathbb{F}_p^\times \).
We will strengthen the statement of Lemma 2.2.4 in such a way that we will prove the existence of a monochromatic solution of $x + y = z$ with $x \neq y$. After this, Theorem 2.1.2 will follow from the previous argument of Schur and Lemma 2.2.3. The proof of this can be found in [1], nevertheless, we briefly present this result.

**Proof.** From an arbitrary coloring of the set $T = \{1, 2, \ldots, t\}$ we construct the edge coloring of $K_t$ just as before. Now we choose $t$ to be greater than $R(c, 4)$ so there will be an edge monochromatic $K_4$ with vertices $w < x < y < z$. If $x - w = y - x = z - y$ holds, then $(z - y) + (y - w) = z - w$ is a monochromatic solution with $z - y \neq y - w$. If $x - w \neq y - x$, then $(x - w) + (y - x) = y - w$ will be an appropriate monochromatic solution in $T$. Finally, if $y - x \neq z - y$, then $(y - x) + (z - y) = z - x$ shows that the strengthened form of Lemma 2.2.4 is true. 

The proof of Theorem 2.1.2 is now complete.
3 The ESQ property of certain representations of symmetric groups

3.1 Introduction

The present chapter is devoted to the study of the ESQ property of some particular irreducible representations of the symmetric group $S_n$ over the complex field. This chapter is based on article [26] written by the author. A nice introduction to the general theory of $C\mathfrak{S}_n$ modules can be found in [7]. We will refer to this book several times for some fundamental facts on $C\mathfrak{S}_n$ modules. These facts can be taken as black boxes if the reader is unfamiliar with the theory.

A closely related problem to the ESQ property of irreducible $S_n$ representations is the decomposition of tensor products of irreducible $S_n$ representations. It is well-known that the irreducible representations of $\mathfrak{S}_n$ are indexed by the partitions of $n$, see 4.25 in [7]. If $\lambda$ is a partition of $n$ we write $\lambda \vdash n$, and we denote the corresponding irreducible $C\mathfrak{S}_n$ module by $M^\lambda$. With this notation the decomposition of a tensor product is the following:

$$M^\lambda \otimes M^\mu = \bigoplus_{\nu \vdash n} g^{\lambda\mu}_\nu M^\nu.$$

The coefficients $g^{\lambda\mu}_\nu$ arising in this decomposition are called Kronecker coefficients, and they have been extensively studied. It was shown by Bürgisser and Ikenmeyer [4] that the general computation of these coefficients is $\#P$-hard. However, it is conjectured by Mulmuley that deciding the positivity of $g^{\lambda\mu}_\nu$ is in $P$. If the diagonal Kronecker coefficient $g^{\lambda\lambda}_\lambda$ is zero, then $M^\lambda$ is obviously not ESQ. This simple observation establishes the connection between the ESQ property and the Kronecker coefficients.

There are positive results for special tensor products where the authors were able to calculate $g^{\lambda\mu}_\nu$. A partition $\lambda \vdash n$ is called a hook if $\lambda = (n - k, 1, \ldots, 1)$ ($1 \leq k \leq n - 1$) or $\lambda = (n - k, 1^k)$ for short. A partition $\lambda = (n - k, k)$ ($1 \leq k \leq \frac{n}{2}$) is called a height two partition, and if $\lambda = (l_1, \ldots, l_k)$ where $l_i \leq 2$, then $\lambda$ is called a width two partition. These terms come from the shape of the Young diagrams associated with these partitions. Remmel [17] computed the coefficients $g^{\lambda\mu}_\nu$ if $\lambda$ and $\mu$ are hooks. A few years later Remmel and Whitehead [18] determined the coefficients $g^{\lambda\mu}_\nu$ if $\lambda$ and $\mu$ are both height two partitions. In this chapter we will determine whether the ESQ property holds for these particular representations of $\mathfrak{S}_n$. The main results of the chapter are summarized in the following theorem.

**Theorem 3.1.1** Let $M^\lambda$ be an irreducible $C\mathfrak{S}_n$ module.
(a) If $\lambda$ is a height two partition, then $M^\lambda$ is not an ESQ module.

(b) If $\lambda$ is a width two partition, then $M^\lambda$ is not an ESQ module with the exception of $\lambda = (2, 1, 1)$.

(c) If $\lambda$ is a hook with parameters $(n - k, 1^k)$, then $M^\lambda$ is an ESQ module if and only if $\frac{3}{2}k + 1 \leq n$, and $k \equiv 2, 3 \pmod{4}$.

The proof of the first result relies on the Cauchy-Frobenius lemma. In this case the Kronecker coefficients $g^{\lambda \lambda \lambda}$ can be arbitrarily large as $n$ grows, so our result proves that all the $M^\lambda$ constituents of $M^\lambda \otimes M^\lambda$ lie in the symmetric part. The second statement will be a simple consequence of the first. In order to prove the third statement, we will construct an element of the group algebra, which will annihilate one of the submodules $\text{Sym}^2 M^\lambda$ and $\Lambda^2 M^\lambda$, but not $M^\lambda$. The proof uses the result of article [17] on $g^{\lambda \lambda}$. The result on the ESQ property of hook representations will be strengthened in Chapter 4, where we will fully decompose the submodules $\text{Sym}^2 M^\lambda$ and $\Lambda^2 M^\lambda$. The proof presented in the current chapter can be considered as the starting point and the "little brother" of the proof we will show in Chapter 4.

3.2 Height two representations

In this section we will prove Theorem 3.1.1(a). If $G$ is an arbitrary group, which acts on the set $\Omega$, then the function counting the number of fixed points of the elements of $G$ is called a permutation character. The Cauchy-Frobenius lemma can be interpreted as follows: if $G$ acts on $\Omega_1$ with permutation character $\pi_1$, and on $\Omega_2$ with permutation character $\pi_2$, then $\langle \pi_1, \pi_2 \rangle$ is the number of orbits of $G$ acting on the set $\Omega_1 \times \Omega_2$. In the following proof our main tool will be this lemma applied to different permutation characters of $S_n$.

Let us take the natural action of $S_n$ on the set $[n] = \{1, 2, \ldots, n\}$. By $\pi_l$ ($0 \leq l \leq n$) we denote the function $\pi_l : S_n \to \mathbb{N}$, where $\pi_l(g)$ is the number of $l$-element subsets of $[n]$ fixed setwise by $g$. We will denote the set of $l$-element subsets of $[n]$ by $\binom{[n]}{l}$, and refer the elements of it as $l$-subsets in short. The defined $\pi_l$ function equals the permutation character of the natural action of $S_n$ on the set $\binom{[n]}{l}$. In the special case $l = 0$ we get that $\pi_0$ is the trivial character $\chi^{(n)}$ of $S_n$.

**Proof of Theorem 3.1.1(a)** Let us assume that $\lambda = (n - k, k)$, where $1 \leq k \leq \frac{n}{2}$. We denote the irreducible character of the module $M^\lambda = M^{(n - k, k)}$ by $\chi^{(n - k, k)}$. It is well-known that

$$\chi^{(n - k, k)}(g) = \pi_k(g) - \pi_{k-1}(g). \quad (3.1)$$
see for example 7.18.8 in [22], which implies the following decomposition of the permutation characters \( \pi_k \) (\( 0 \leq k \leq \frac{n}{2} \)):

\[
\pi_k = \sum_{i=0}^{k} \chi^{(n-i,i)}.
\]  

From now on let \( n \) and \( k \) be fixed, and we will simply write \( \chi \) instead of \( \chi^{(n-k,k)} \). The character of the module \( \Lambda^2 M^{\lambda} \) is

\[
\hat{\chi}(g) = \chi^2(g) - \chi(g^2).
\]  

To show that \( M^{\lambda} \) is not ESQ, we will prove that the inner product

\[
\langle \chi, \hat{\chi} \rangle = \frac{1}{n!} \sum_{g \in S_n} \chi(g) \hat{\chi}(g)
\]

is 0. Using (3.1), we can see that

\[
\langle \chi, \hat{\chi} \rangle = \langle \pi_k - \pi_{k-1}, \hat{\chi} \rangle = \langle \pi_k, \hat{\chi} \rangle - \langle \pi_{k-1}, \hat{\chi} \rangle.
\]

The main difficulty of the presented proof will be to show \( \langle \pi_k, \hat{\chi} \rangle = 0 \). This equation will imply \( \langle \pi_{k-1}, \hat{\chi} \rangle = 0 \) since the group characters \( \pi_{k-1}, \pi_k \) and \( \hat{\chi} \) are non-negative integer linear combinations of irreducible characters, we have

\[
0 \leq \langle \pi_{k-1}, \hat{\chi} \rangle \leq \langle \pi_{k-1} + \chi, \hat{\chi} \rangle = \langle \pi_k, \hat{\chi} \rangle = 0,
\]

and from this we will eventually conclude that \( \langle \chi, \hat{\chi} \rangle = 0 \). That is why from now on our goal will be to prove \( \langle \pi_k, \hat{\chi} \rangle = 0 \). By (3.1) and (3.3) this is equivalent to

\[
\sum_{g \in S_n} \pi_k(g)(\pi_k(g)^2 + \pi_{k-1}(g)^2 - 2\pi_k(g)\pi_{k-1}(g) - \pi_k(g^2) + \pi_{k-1}(g^2)) = 0.
\]

Next we will express \( \pi_k(g^2) \) through values of characters on \( g \). Considering the action of \( g \) on the \( k \)-subsets of \( [n] \), we denote by \( c_2(g) \) the number of transpositions of this action of \( g \). With this notation \( \pi_k(g^2) = \pi_k(g) + 2c_2(g) \). Now, let \( S_n \) act on the unordered pairs of \( k \)-subsets of \( [n] \), which we denote by \( \binom{\binom{n}{k}}{2} \). The two \( k \)-element subsets of a pair are distinct, so the set of this action has size \( \binom{\binom{n}{k}}{2} \). Writing \( \pi_k^{(2)} \) for the permutation character of this action, \( \pi_k^{(2)}(g) = \binom{\pi_k(g)}{2} + c_2(g) \). From these two equations we get \( \pi_k(g^2) = 2\pi_k^{(2)}(g) - \pi_k(g)^2 + 2\pi_k(g) \). Applying this formula and the analogous one for \( \pi_{k-1}(g^2) \),
we see that our desired identity has the following form

$$\sum_{g \in \mathfrak{S}_n} \pi_k(g)(\pi_k(g)^2 - \pi_k(g)\pi_{k-1}(g) - \pi_k^{(2)}(g) + \pi_k^{(2)}(g) - \pi_k(g) + \pi_{k-1}(g)) = 0.$$  

Now all the function parameters on the left side of the identity are \( g \), so we can use the more compact inner product notation

$$I = \langle \pi_k, \pi_k^2 - \pi_k\pi_{k-1} - \pi_k^{(2)} + \pi_{k-1} \rangle.$$  

It follows from (3.2) that \( \langle \pi_k, \pi_k \rangle = k + 1 \) and \( \langle \pi_k, \pi_{k-1} \rangle = k \), which means that \( I \) can be simplified:

$$I = \langle \pi_k, \pi_k^2 - \pi_k\pi_{k-1} - \pi_k^{(2)} + \pi_{k-1} \rangle = \langle \pi_k, -\pi_k + \pi_{k-1} \rangle = \langle \pi_k, \pi_k^2 - \pi_{k-1} - \pi_k^{(2)} + \pi_{k-1} \rangle - 1.$$  

The following general lemma will show the equation \( I = 0 \).

**Lemma 3.2.1** Let \( 1 \leq k \leq m \leq \frac{n}{2} \). Then

$$\langle \pi_m, \pi_k^2 - \pi_k\pi_{k-1} - \pi_k^{(2)} + \pi_{k-1} \rangle = 1.$$  

**Proof.** We start our proof by expanding the left hand side of (3.4)

$$\langle \pi_m, \pi_k^2 - \pi_k\pi_{k-1} - \pi_k^{(2)} + \pi_{k-1} \rangle = \langle \pi_m, \pi_k^2 \rangle - \langle \pi_m, \pi_k\pi_{k-1} \rangle - \langle \pi_m, \pi_k^{(2)} \rangle + \langle \pi_m, \pi_{k-1} \rangle.$$  

(\text{\((*)\)})

In (\text{\((*)\)}) we have four inner products of various permutation characters. In the following proof a set will be defined for each of these inner products, so that the cardinality of the defined set equals the value of the corresponding inner product. These sets will be \( P \), \( Q \), \( R \) and \( S \) respectively, and we will argue that the right side of (\text{\((*)\)}) is \(|P| - |Q| - |R| + |S| = 1\).

By the Cauchy-Frobenius lemma the inner products in (\text{\((*)\)}) are the number of orbits in different actions of \( \mathfrak{S}_n \). Let us take the inner product \( \langle \pi_m, \pi_k^2 \rangle \). As we have already seen, \( \pi_m \) is the permutation character of the natural \( \mathfrak{S}_n \) action on \( \binom{[n]}{m} \), and \( \pi_k^2 \) is the permutation character of the \( \mathfrak{S}_n \) action on \( \binom{[n]}{k} \times \binom{[n]}{k} \). By the Cauchy-Frobenius lemma \( \langle \pi_m, \pi_k^2 \rangle \) equals the number of orbits of the \( \mathfrak{S}_n \) action on \( \binom{[n]}{m} \times \binom{[n]}{k} \times \binom{[n]}{k} \).

That is, the number of ways an \( m \)-subset and two \( k \)-subsets of \([n]\) can intersect each other, where we do
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not distinguish the elements of set \([n]\). For any ordered triple of sets \((A_1, A_2, A_3)\) \((A_i \subseteq [n])\) we assign an ordered quadruple of integers \((i_{12}, i_{13}, i_{23}, i)\) where

\[
\begin{align*}
  i_{12} &= |A_1 \cap A_2 \cap \overline{A}_3|, \\
i_{13} &= |A_1 \cap \overline{A}_2 \cap A_3|, \\
i_{23} &= |\overline{A}_1 \cap A_2 \cap A_3|, \\
i &= |A_1 \cap A_2 \cap A_3|.
\end{align*}
\]

By the figure below, the quadruple \((i_{12}, i_{13}, i_{23}, i)\) determines the sizes of the eight parts determined by the sets \(A_1, A_2, A_3\). We see that two triples of subsets in \((^{[n]}_m) \times (^{[n]}_k) \times (^{[n]}_k)\) lie in the same orbit if and only if we assign the same quadruple to them.

\[
\begin{align*}
  |A_1 \cap A_2 \cap \overline{A}_3| &= k - i - i_{13} - i_{23}, \\
  |\overline{A}_1 \cap A_2 \cap \overline{A}_3| &= k - i - i_{12} - i_{23}, \\
  |A_1 \cap \overline{A}_2 \cap A_3| &= m - i - i_{12} - i_{13}, \\
  |\overline{A}_1 \cap \overline{A}_2 \cap A_3| &= n - 2k - m + 2i + i_{12} + i_{13} + i_{23}.
\end{align*}
\]

A quadruple of non-negative integers \((i_{12}, i_{13}, i_{23}, i)\) \(\in \mathbb{N}^4\) corresponds to an element of \((^{[n]}_m) \times (^{[n]}_k) \times ^{[n]}_k\).
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\((\binom{n}{k})\) if and only if all the sizes above are non-negative. Our observations can be summarized as

\[
\langle \pi_m, \pi_{2k} \rangle = |P|,
\]

where

\[
P = \{(i_{12}, i_{13}, i_{23}, i) \in \mathbb{N}^4 \mid i + i_{13} + i_{23} \leq k, \ i + i_{12} + i_{23} \leq k, \ i + i_{12} + i_{13} \leq m, \ 2k + m - n \leq 2i + i_{12} + i_{13} + i_{23}\}.
\]

Similarly, by considering the \(S_n\) action on the set \(\binom{n}{m} \times \binom{n}{k} \times \binom{n}{k-1}\), we get that

\[
\langle \pi_m, \pi_{k} \pi_{k-1} \rangle = |Q|,
\]

where

\[
Q = \{(i_{12}, i_{13}, i_{23}, i) \in \mathbb{N}^4 \mid i + i_{13} + i_{23} \leq k - 1, \ i + i_{12} + i_{23} \leq k, \ i + i_{12} + i_{13} \leq m, \ 2k + m - n - 1 \leq 2i + i_{12} + i_{13} + i_{23}\}.
\]

The inner product \(\langle \pi_m, \pi_{2k}^{(2)} \rangle\) is the number of the orbits of the \(S_n\) action on the set \(\binom{n}{m} \times \binom{n}{k} \times \binom{n}{2-k}\).

This is the number of ways an \(m\)-subset \((A_1)\) and two distinct \(k\)-subsets \((A_2\) and \(A_3)\) can intersect each other in an \(n\)-element set, where we do not distinguish the set points, and we do not distinguish the two \(k\)-subsets. We will count the orbits of this action just as before, through intersection quadruples. All the conditions defining the set \(P\) should be satisfied, as they provide the trivial condition that all partition sizes are non-negative. Because of the symmetry of the sets \(A_2\) and \(A_3\), the quadruples \((i_{12}, i_{13}, i_{23}, i)\) and \((i_{13}, i_{12}, i_{23}, i)\) are now assigned to the same \(S_n\) orbit. To count these orbits only once, we will assume \(i_{12} \leq i_{13}\). As \(A_2 \neq A_3\), we have \(|A_2 \cap A_3| = i + i_{23} \leq k - 1\). We conclude that

\[
\langle \pi_m, \pi_{2k}^{(2)} \rangle = |R|,
\]

where

\[
R = \{(i_{12}, i_{13}, i_{23}, i) \in \mathbb{N}^4 \mid i + i_{13} + i_{23} \leq k, i + i_{12} + i_{23} \leq k, i + i_{12} + i_{13} \leq m, \ 2k + m - n \leq 2i + i_{12} + i_{13} + i_{23}, \ i + i_{23} \leq k - 1\}.
\]
The inequalities \( i + i_{13} + i_{23} \leq k \) and \( i_{12} \leq i_{13} \) imply \( i + i_{12} + i_{23} \leq k \), so we get the following, simpler form for the set \( R \):

\[
R = \{ (i_{12}, i_{13}, i_{23}, i) \in \mathbb{N}^4 \mid i + i_{13} + i_{23} \leq k, \ i + i_{12} + i_{13} \leq m, \\
2k + m - n \leq 2i + i_{12} + i_{13} + i_{23}, \ i_{12} \leq i_{13}, \ i + i_{23} \leq k - 1 \}.
\]

Finally, replacing \( k \) with \( k - 1 \) in the preceding formula shows:

\[
\langle \pi_m, \pi_{k-1}^{(2)} \rangle = |S|,
\]

where

\[
S = \{ (i_{12}, i_{13}, i_{23}, i) \in \mathbb{N}^4 \mid i + i_{13} + i_{23} \leq k - 1, \ i + i_{12} + i_{13} \leq m, \\
2k + m - n - 2 \leq 2i + i_{12} + i_{13} + i_{23}, \ i_{12} \leq i_{13}, \ i + i_{23} \leq k - 2 \}.
\]

To calculate \(|P| - |Q| - |R| + |S|\), we will partition these sets into three parts each, \( P_i, Q_i, R_i \) and \( S_i \) \((i \in \{1, 2, 3\})\) so that their respective sizes will be:

\[
|P_1|, |P_2|, k + 1; \ |P_1|, |Q_2|, |Q_3|; \ |P_2|, |R_2|, k - B; \ |R_2|, |Q_2| - B, |Q_3|;
\]

where \( B \) is the following conditional expression:

\[
B = \begin{cases} 
1, & \text{if } k = m = \frac{n}{2} \\
0, & \text{otherwise}.
\end{cases}
\]

Then the equation \(|P| - |Q| - |R| + |S| = 1\) will be a straightforward consequence of the above observations.

First we divide \( P \) into two sets according to the value of \( i + i_{13} + i_{23} \):

\[
P_1 = \{ (i_{12}, i_{13}, i_{23}, i) \in P \mid i + i_{13} + i_{23} \leq k - 1 \},
\]

\[
P \setminus P_1 = \{ (i_{12}, i_{13}, i_{23}, i) \in P \mid i + i_{13} + i_{23} = k \}.
\]

In \( P \setminus P_1 \) the inequality \( i_{12} \leq i_{13} \) holds because of \( i + i_{13} + i_{23} = k \) and \( i + i_{12} + i_{23} \leq k \). We partition
the set $P \setminus P_1$ into two sets according to the value of $i + i_{23}$:

$$P_2 = \{(i_{12}, i_{13}, i_{23}, i) \in P \mid i + i_{13} + i_{23} = k, \ i + i_{23} \leq k - 1\},$$

$$P_3 = \{(i_{12}, i_{13}, i_{23}, i) \in P \mid i + i_{13} + i_{23} = k, \ i + i_{23} = k\}.$$

By the two conditions of $P_3$ we get $i_{13} = 0$, which implies $i_{12} = 0$. The quadruples $(0, 0, i_{23}, i)$, where the equation $i + i_{23} = k$ holds, satisfy all the four inequalities of $P$ (for example, $2k + m - n \leq 2i + i_{12} + i_{13} + i_{23}$, since it is equivalent with $k + m - n \leq i$ and $k \leq m \leq \frac{n}{2}$). So

$$P_3 = \{(0, 0, i_{23}, i) \in \mathbb{N}^4 \mid i + i_{23} = k\},$$

which means

$$|P_3| = k + 1. \quad (3.6)$$

Now we partition $Q$. First we divide $Q$ into two parts according to the value of $2i + i_{12} + i_{13} + i_{23}$:

$$Q_1 = \{(i_{12}, i_{13}, i_{23}, i) \in Q \mid 2k + m - n \leq 2i + i_{12} + i_{13} + i_{23}\},$$

$$Q \setminus Q_1 = \{(i_{12}, i_{13}, i_{23}, i) \in Q \mid 2k + m - n - 1 = 2i + i_{12} + i_{13} + i_{23}\}.$$

$P_1 = Q_1$, because in these sets all the conditions on the quadruples are the same. This means

$$|P_1| = |Q_1|. \quad (3.7)$$

From the equation $2k + m - n - 1 = 2i + i_{12} + i_{13} + i_{23}$, the defining inequalities of $Q$ follow, for example:

$$2k + m - n - 1 = 2i + i_{12} + i_{13} + i_{23} \implies$$

$$i + i_{13} + i_{23} \leq 2k + m - n - 1 = k + (k + m) - n - 1 \leq k - 1,$$

as $k \leq m \leq \frac{n}{2}$. (The rest of the defining inequalities of $Q$ can be checked exactly the same way.) This means that we can simplify the definition of $Q \setminus Q_1$ to

$$Q \setminus Q_1 = \{(i_{12}, i_{13}, i_{23}, i) \in \mathbb{N}^4 \mid 2k + m - n - 1 = 2i + i_{12} + i_{13} + i_{23}\}.$$
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We partition \( Q \setminus Q_1 \) into two sets:

\[
Q_2 = \{(i_{12}, i_{13}, i_{23}, i) \in \mathbb{N}^4 \mid 2k + m - n - 1 = 2i_{12} + i_{13} + i_{23}, i_{12} \leq i_{13}\},
\]

\[
Q_3 = \{(i_{12}, i_{13}, i_{23}, i) \in \mathbb{N}^4 \mid 2k + m - n - 1 = 2i_{12} + i_{13} + i_{23}, i_{13} < i_{12}\}.
\]

Now we partition \( R \) according to the value of \( i + i_{13} + i_{23} \), and then by the value of \( i + i_{23} \):

\[
R_1 = \{(i_{12}, i_{13}, i_{23}, i) \in R \mid i + i_{13} + i_{23} = k\},
\]

\[
R_2 = \{(i_{12}, i_{13}, i_{23}, i) \in R \mid i + i_{13} + i_{23} \leq k - 1, i + i_{23} \leq k - 2\},
\]

\[
R_3 = \{(i_{12}, i_{13}, i_{23}, i) \in R \mid i + i_{13} + i_{23} \leq k - 1, i + i_{23} = k - 1\}.
\]

The defining conditions of the sets \( P_2 \) and \( R_1 \) are equivalent, so

\[|P_2| = |R_1|. \tag{3.8}\]

The definition of \( R_3 \) can be simplified. The two defining conditions show that \( i_{13} = 0 \), which implies \( i_{12} = 0 \) in \( R \). The remaining relevant conditions of \( R_3 \) are the following:

\[
R_3 = \{(0, 0, i_{23}, i) \in \mathbb{N}^4 \mid 2k + m - n \leq 2i + i_{23}, i + i_{23} = k - 1\}.
\]

In \( R_3 \) the inequality \( 2k + m - n \leq 2i + i_{23} \) is equivalent to \( k + m - n + 1 \leq i \). This inequality almost always stands since the right side is typically non-positive due to \( k \leq m \leq n \). The only exception is when \( k = m = \frac{n}{2} \) holds. With the already defined notation of \( B \) in (3.5), we can express the size of \( R_3 \) as

\[|R_3| = k - B. \tag{3.9}\]

We partition \( S \) into three parts according to the value of \( 2i + i_{12} + i_{13} + i_{23} \):

\[
S_1 = \{(i_{12}, i_{13}, i_{23}, i) \in S \mid 2k + m - n \leq 2i + i_{12} + i_{13} + i_{23}\},
\]

\[
S_2 = \{(i_{12}, i_{13}, i_{23}, i) \in S \mid 2k + m - n - 1 = 2i + i_{12} + i_{13} + i_{23}\},
\]

\[
S_3 = \{(i_{12}, i_{13}, i_{23}, i) \in S \mid 2k + m - n - 2 = 2i + i_{12} + i_{13} + i_{23}\}.
\]

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The sets $R_2$ and $S_1$ are equal, so
\[ |R_2| = |S_1|. \tag{3.10} \]

Notice that the sets $S_2$ and $Q_2$ are almost identical. To state precisely:
\[ S_2 = \{(i_{12}, i_{13}, i_{23}, i) \in Q_2 \mid i + i_{23} \leq k - 2\}. \]

Now
\[ 2k + m - n - 1 = 2i + i_{12} + i_{13} + i_{23} \implies i + i_{23} \leq 2k + m - n - 1 = k + (k + m) - n - 1 \leq k - 1, \]
and the equality holds if and only if $i = i_{12} = i_{13} = 0$ (by the first inequality), and $k = m = \frac{n}{2}$ (by the second inequality). In this case $i_{23} = k - 1$. This means that $i + i_{23} \leq k - 2$ holds in every case when $k = m = \frac{n}{2}$ does not hold, and exactly one element, namely $(0, 0, k - 1, 0)$, fails $i + i_{23} \leq k - 2$, when $k = m = \frac{n}{2}$. So
\[ |Q_2| - B = |S_2|. \tag{3.11} \]

We can simplify the conditions of $S_3$ to get
\[ S_3 = \{(i_{12}, i_{13}, i_{23}, i) \in \mathbb{N}^4 \mid 2k + m - n - 2 = 2i + i_{12} + i_{13} + i_{23}, i_{12} \leq i_{13}\}. \]

The map $(i_{12}, i_{13}, i_{23}, i) \mapsto (i_{13}, i_{12} - 1, i_{23}, i)$ establishes a bijection between $Q_3$ and $S_3$, so
\[ |Q_3| = |S_3|. \tag{3.12} \]

By the results (3.6), (3.7), (3.8), (3.10), (3.11) and (3.12) we get $|P| - |Q| - |R| + |S| = 1$, which proves the lemma.

We have seen that Theorem 3.1.1(a) is an immediate consequence of the Lemma 3.2.1, so this part of Theorem 3.1.1 is now demonstrated.
3.3 Width two representations

In this short section we will prove Theorem 3.1.1(b). This result will be a straightforward consequence of the non-ESQ property of height two representations, which we proved in the previous section. By $M^{(1^n)}$ we denote the linear sign representation of $S_n$. It is well-known that for any $\lambda \vdash n$ the tensor product $M^{(1^n)} \otimes M^\lambda$ is an irreducible $\mathbb{C}S_n$ module corresponding to the conjugate partition of $\lambda$, see 4.4 in [7].

Now let $\lambda$ be the width two partition $(2^k, 1^{n-2k})$ $(1 \leq k \leq \frac{n}{2})$. In this case the conjugate partition of $\lambda$ is the height two partition $(n-k, k)$. This means that for the irreducible characters the equality

$$\chi^\lambda(g) = (-1)^g \chi^{(n-k,k)}(g)$$ (3.13)

holds. (In this equation $(-1)^g$ denotes the sign of the permutation $g$). As the squares of $\chi^\lambda$ and $\chi^{(n-k,k)}$ are equal, and $g^2$ is an even permutation for every $g \in S_n$, the exterior square characters of $\chi^\lambda$ and $\chi^{(n-k,k)}$ are also equal. We denote this character by $\hat{\chi}$. In the following proof not every inner product will be defined on $S_n$, so we will indicate the base group throughout the proof.

**Proof of Theorem 3.1.1(b)** We have to investigate the inner product $\langle \chi^\lambda, \hat{\chi} \rangle_{S_n}$. Let us assume first that $9 \leq n$. We will prove that in this case $\langle \chi^\lambda, \hat{\chi} \rangle_{S_n} = 0$. From Theorem 3.1.1(a) we know that $\langle \chi^{(n-k,k)}, \hat{\chi} \rangle_{S_n} = 0$. Adding the two inner products, it will be sufficient to show that

$$\langle \chi^\lambda + \chi^{(n-k,k)}, \hat{\chi} \rangle_{S_n} = 0.$$ (3.14)

Using (3.13), we see that the character $\chi^\lambda + \chi^{(n-k,k)}$ is $2\chi^{(n-k,k)}$ on the even permutations of $S_n$ and 0 on the odd ones. This observation makes it clear that (3.14) is equivalent to

$$\langle \chi^{(n-k,k)}|_{A_n}, \hat{\chi}|_{A_n} \rangle_{A_n} = 0.$$ (3.15)

In the proof of the height two case we have seen that $\chi^{(n-k,k)} = \pi_k - \pi_{k-1}$ and $\hat{\chi} = \pi_k^2 - \pi_k \pi_{k-1} - \pi_{k-1}^2 - \pi_k + \pi_{k-1}$. Using these equations it is possible to express the inner product $\langle \chi^{(n-k,k)}, \hat{\chi} \rangle_{S_n}$ as the linear combination of the inner products of permutation characters of $S_n$. If we restrict these inner products to $A_n$, then we count the number of orbits of the $A_n$ actions on the sets where we counted $S_n$ orbits before, and the result of this restricted linear combination will be $\langle \chi^{(n-k,k)}|_{A_n}, \hat{\chi}|_{A_n} \rangle_{A_n}$. The orbits in the mentioned $S_n$ group actions correspond to the intersection type of two or three ($k$ or $k-1$-element) subsets of $[n]$. Either way, these sets partition $[n]$ into eight parts at most. As $9 \leq n$, one of
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the parts has at least two elements, which implies that two set configurations in the same \( \mathfrak{S}_n \) orbit can be transferred into each other by an element of \( \mathfrak{A}_n \) as well. This proves that the number of \( \mathfrak{A}_n \) and \( \mathfrak{S}_n \) orbits are the same, so (3.15) follows.

It remains only to check the thirteen width two (and height at least three) representations of the groups \( \mathfrak{S}_n \), where \( n \leq 8 \). Direct computation yields that only \( \lambda = (2,1,1) \) gives an ESQ representation of \( \mathfrak{S}_4 \). (Note that in this case the ESQ property of \( M^\lambda \) is clear as the image of this representation lies in \( SO(3) \), which is isomorphic to its own exterior square.)

3.4 Hook representations

In this section we will prove Theorem 3.1.1(c). Let \( \lambda = (n-k,1^k) \) be a hook partition \((1 \leq k \leq n-1)\) with the corresponding irreducible \( \mathbb{C}\mathfrak{S}_n \) module \( M^\lambda \). Our goal will be to decide whether the containment \( M^\lambda \leq \Lambda^2 M^\lambda \) holds or not. In Chapter 4 we will fully decompose \( \Lambda^2 M^\lambda \), hence Corollary 4.1.2 will directly imply the result of this section. Nevertheless, we decided to present the more special argument of this section, as it preceded the general argument, and we consider it interesting on its own.

It is a well-known fact that \( M^\lambda \) is the \( k\text{th} \) exterior power of the standard irreducible representation of \( \mathfrak{S}_n \), see for example 4.6 in [7]. In the following proof we will investigate the action of \( \mathbb{C}\mathfrak{S}_n \) on the modules \( M^\lambda \), \( \text{Sym}^2 M^\lambda \) and \( \Lambda^2 M^\lambda \). To do this, we have to fix some generating sets for these vector spaces, and we need to understand the group action on these generating sets. For this reason we will introduce these generators starting from the standard irreducible representation. Let \( V \) be an \( n-1 \) dimensional vector space with a basis \( v_1, v_2, \ldots, v_{n-1} \). We define \( v_n \in V \) as

\[
v_n = -(v_1 + v_2 + \ldots + v_{n-1}).
\]

By expanding the \( \mathfrak{S}_n \) action on the indices of \( v_i \) \((i = 1, 2, \ldots, n)\) to \( V \), \( V \) will be isomorphic to the standard irreducible representation of \( \mathfrak{S}_n \). As we stated before, \( M^\lambda \cong \Lambda^k V \), which means that if we define

\[
v_{i_1i_2\ldots i_k} = v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k},
\]

then these \( v_{i_1i_2\ldots i_k} \) vectors will span \( M^\lambda \). For the sake of completeness, we remind the reader that the definition of a wedge product (as an element of a tensor space) is the following:

\[
v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k} = \frac{1}{k!} \sum_{g \in \mathfrak{S}_k} (-1)^g v_{i_1g^{-1}} \otimes v_{i_2g^{-1}} \otimes \ldots \otimes v_{i_kg^{-1}},
\]
and similarly, the definition of the symmetric product is

\[ v_{i_1} \odot v_{i_2} \odot \ldots \odot v_{i_k} = \frac{1}{k!} \sum_{g \in S_k} v_{i_1 g^{-1}} \odot v_{i_2 g^{-1}} \odot \ldots \odot v_{i_k g^{-1}}. \]

The \( S_n \) action on the spanning vectors \( v_{i_1 \ldots i_k} \) is inherited from \( V \), which means that this action is simply the \( S_n \) action on the indices. Finally, we define the generators of \( \Lambda^2 M^\lambda \), \( \text{Sym}^2 M^\lambda \) and \( M^\lambda \odot M^\lambda \) by

\[ R_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots j_k} = v_{i_1 i_2 \ldots i_k} \wedge v_{j_1 j_2 \ldots j_k}, \]

\[ S_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots j_k} = v_{i_1 i_2 \ldots i_k} \odot v_{j_1 j_2 \ldots j_k}, \]

\[ T_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots j_k} = v_{i_1 i_2 \ldots i_k} \odot v_{j_1 j_2 \ldots j_k}, \]

respectively. Just as before, \( S_n \) acts on these generators by the natural action on the indices. Having introduced this notation, we can proceed to the proof.

**Proof of Theorem 3.1.1(c)** Theorem 2.1 in [17] tells us that if \( n < \frac{3}{2} k + 1 \), then \( g^\lambda_\lambda \) is 0, and that \( \frac{3}{2} k + 1 \leq n \) implies \( g^\lambda_\lambda = 1 \). As \( g^\lambda_\lambda = 0 \) implies the impossibility of the ESQ property of \( M^\lambda \), we conclude that the condition \( \frac{3}{2} k + 1 \leq n \) is necessary in Theorem 3.1.1. If this inequality holds, then \( g^\lambda_\lambda = 1 \), so it will be sufficient to show that for \( k \equiv 2, 3 \pmod{4} \) \( M^\lambda \) is not a constituent of \( \text{Sym}^2 M^\lambda \), and similarly for \( k \equiv 0, 1 \pmod{4} \) \( M^\lambda \) is not a constituent of \( \Lambda^2 M^\lambda \).

To prove these statements, we will construct an element \( c \) of the group algebra \( \mathbb{C}S_n \), which does not annihilate \( M^\lambda \) but annihilates the appropriate module among \( \text{Sym}^2 M^\lambda \) and \( \Lambda^2 M^\lambda \).

The construction of \( c \) is as follows. For an arbitrary subset \( X \) of \( [n] \), we define the subgroup \( \Pi(X) \leq S_n \) as the pointwise stabilizer of \( [n] \setminus X \). Now let us partition the set \( [n] \) into two subsets \( A \) and \( B \), and having done so we define the following elements of \( \mathbb{C}S_n \):

\[ a = \sum_{g \in \Pi(A)} (-1)^g g, \]

\[ b = \sum_{g \in \Pi(B)} g, \]

\[ c = ab, \]

where \((-1)^g\) denotes the sign of the permutation \( g \). We shall see that if we choose the set \( A \) appropriately,
then \(c\) will have the desired properties mentioned above. However, we postpone the definition of \(A\), as it will depend on the parity of \(k\), and we want to delay the separation of different cases in order to get a compact proof. At this point we only assume that \(A\) has an odd cardinality \(|A| = 2m + 1\), so our following observations on \(c\) will be made in this general case.

As the sets \(A\) and \(B\) are disjoint, the subgroups \(\Pi(A)\) and \(\Pi(B)\) centralize each other, so \(a\) and \(b\) commute. In addition, for every \(g \in \Pi(A)\), we have \(g \cdot a = a \cdot g = (-1)^{q_a}a\), and for \(g \in \Pi(B)\), we have \(g \cdot b = b \cdot g = b\). We conclude that if \(g \in \Pi(A)\), then \(g \cdot c = c \cdot g = (-1)^g c\), and if \(g \in \Pi(B)\) then \(g \cdot c = c \cdot g = c\).

Those generators of \(M^\lambda \otimes M^\lambda\) which are not annihilated by \(c\) will be called relevant from now on. Our next goal is to classify the relevant generators. Let us take an arbitrary generator \(T_{i_1i_2...i_k,j_1j_2...j_k}\) of the tensor product. Whenever it does not lead to confusion, we will refer to this by \(T\). If any two of the \(i\) indices coincide, then the vector \(v_{i_1i_2...i_k}\) is zero, which implies \(T = 0\), so in this case \(Tc = 0\) is trivial. The same is true for the \(j\) indices, therefore we may assume that both sets \(I = \{i_1, i_2, \ldots, i_k\}\) and \(J = \{j_1, j_2, \ldots, j_k\}\) have \(k\) elements. We will use the notation \(I_A = I \cap A, I_B = I \cap B, J_A = J \cap A\) and \(J_B = J \cap B\). (As \(A\) and \(B\) partition \([n]\), we have \(I = I_A \cup I_B\) and \(J = J_A \cup J_B\).

If \(2 \leq |I_A \cap J_A|\), then let us take two different elements \(x\) and \(y\) from the intersection. In this setting \(v_{i_1i_2...i_k}(xy) = -v_{i_1i_2...i_k}\), and \(v_{j_1j_2...j_k}(xy) = -v_{j_1j_2...j_k}\), which means that \((xy)\) fixes \(T\). Now

\[
Tc = (T(xy))c = T((xy)c) = -Tc
\]

as \((xy) \in \Pi(A)\) is an odd permutation. Therefore \(2Tc = 0\), so \(Tc = 0\) and \(T\) is not relevant. This means that for any relevant \(T\)

\[
|I_A \cap J_A| \leq 1
\]

(3.17) holds.

If \(2 \leq |A - (I_A \cup J_A)|\) and \(x, y \in A - (I_A \cup J_A)\) are different elements, then \((xy)\) fixes \(v_{i_1i_2...i_k}\) and \(v_{j_1j_2...j_k}\), so it also fixes \(T\). By this fact we show that \(c\) annihilates \(T\) exactly the same way as before, which means that for any relevant generator the inequality

\[
|A - (I_A \cup J_A)| \leq 1
\]

(3.18) also holds.

If \(I_B - J_B\) or \(J_B - I_B\) have at least two elements, then we show \(Tc = 0\) as follows. Without loss of generality
we may assume that $x, y \in I_B - J_B$ and $x \neq y$. As $v_{i_1i_2...i_k}(xy) = -v_{i_1i_2...i_k}$ and $v_{j_1j_2...j_k}(xy) = v_{j_1j_2...j_k}$, we get $T(xy) = -T$. In this case $(xy) \in \Pi(B)$, so

$$Tc = (T(xy))c = -T((xy)c) = -Tc,$$

which shows that $Tc = 0$. Therefore for relevant generators $T$ the inequalities

$$|I_B - J_B| \leq 1,$$

$$|J_B - I_B| \leq 1$$

hold.

By the inequalities (3.19) and the equation

$$|I_A| + |I_B| = |J_A| + |J_B| = k$$

we get that the set sizes $|I_A|$ and $|J_A|$ can differ by 1 at most. Using this observation together with (3.17), (3.18) and $|A| = 2m + 1$, we get the following classification: the index set of every relevant generator $T$ can be assigned to one of the following types below.

1. $|I_A| = |J_A| = m$, $I_A \cap J_A = \emptyset$.
2. $|I_A| = |J_A| = m + 1$, $|I_A \cap J_A| = 1$.
3. $|I_A| = m + 1$, $|J_A| = m$.
4. $|I_A| = m$, $|J_A| = m + 1$.

Next we will prove the following lemma:

**Lemma 3.4.1** Let $|A| = 2m + 1$. If $m$ is an even number then $c$ annihilates $\Lambda^2M^\lambda$. If $m$ is odd, then $c$ annihilates $\text{Sym}^2M^\lambda$.

**Proof.** First we will deal with the case when $m$ is even. In this case we have to show that $c$ annihilates all the possible $R$ generators. Note that $T_{i_1i_2...i_k,j_1j_2...j_k}c = 0$ implies $R_{i_1i_2...i_k,j_1j_2...j_k}c = 0$, which means that it will be sufficient to show the annihilation of $c$ for those remaining generators only, whose index sets belong to one of the four types above. We will examine these four types separately.
Case I: let us assume that the index set of $R_{i_1 i_2 ... i_k, j_1, j_2, ... j_k}$ belongs to the first type. As a permutation of $I$ and a permutation of $J$ only affect the sign of $R$, we may assume that

$$i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_m \in A.$$ 

This means that the permutation $\pi_1 = (i_1 j_1)(i_2 j_2) \cdots (i_m j_m) \in \Pi(A)$, and similarly, permutation $\pi_2 = (i_{m+1} j_{m+1})(i_{m+2} j_{m+2}) \cdots (i_k j_k) \in \Pi(B)$. As $I_A \cap J_A = \emptyset$, we see that $\pi_1$ is an even permutation. This implies the following equation:

$$Rc = R(\pi_1 \pi_2 c) = R_{j_1, j_2, \ldots, j_k, i_1, i_2, \ldots, i_m} c = -R_{i_1, i_2, \ldots, i_m, j_1, j_2, \ldots, j_k} c.$$ 

In the last equality we used the antisymmetric property of $R$. The annihilation of $R$ follows.

Case II: if the index set of $R$ belongs to the second type, then the annihilation of $R$ follows in a similar way using the assumption

$$i_1, i_2, \ldots, i_{m+1}, j_1, j_2, \ldots, j_m, j_{m+1} \in A$$

and $i_{m+1} = j_{m+1}$.

It remains to check the annihilation of those generators whose index set belongs to the third or fourth type.

Case III: let $R$ be a third type generator. It follows from the inequalities (3.19) that there is exactly one element of $J_B$ which is not in $I_B$. We denote this element by $z$, and we assume that this is the last $j$ index. By (3.16) we can modify $Rc$ as follows:

$$R_{i_1 i_2 \ldots i_k, j_1, j_2, \ldots z} c = - \left( \sum_{t \in [n]-\{z\}} R_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots t} \right) c = - \left( \sum_{t \in A} R_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots t} \right) c = \left( \sum_{t \in B-\{z\}} R_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots t} \right) c.$$ 

In the first sum the occurring non-zero generators belong to the second type, as

$$|I_A| = |\{j_1, j_2, \ldots, j_{k-1}, t\} \cap A| = m + 1.$$ 

We have already seen that these types of generators are annihilated by $c$, so the first sum is zero. The
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generators in the second sum are trivially 0 if \( t \in J_B \). These observations yield

\[
Rc = - \left( \sum_{t \in B - J_B} R_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots t} \right) c.
\]

Let us take a general term from the right side \(-R_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots t'} c\), where \( t' \in B - J_B \). We know that \((t'z) \in \Pi(B)\) and \( t' \notin I_B \) as \( I_B \subset J_B \), so we get the equation

\[
-R_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots t'} c = -R_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots t'} (t'z)c = -R_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots t} c = -Rc,
\]

hence

\[
Rc = - \left( \sum_{t \in B - J_B} R_{i_1 i_2 \ldots i_k, j_1 j_2 \ldots t} \right) c = -(|B - J_B| Rc).
\]

Therefore the annihilation property of \( c \) follows.

**Case IV:** if the index set of \( R \) belongs to the fourth type, then a similar argument shows the annihilation by \( c \). The case when \( m \) is even has been now demonstrated.

We can follow the same procedure for the case when \( m \) is an odd number. If \( S \) belongs to the first or second type we get the annihilation just as before, since in these cases the permutation \( \pi_1 \) is an odd permutation and the \( S \) generator is symmetric. If \( S \) belongs to the third or fourth type, then using the same argument as in the case when \( m \) was even, we get \( Sc = 0 \), so the lemma follows.

Now we specify \( A \). If \( k \) is odd then let \( A = \{1, 2, \ldots, k\} \), and if \( k \) is even, then let \( A = \{1, 2, \ldots, k + 1\} \). After this specification, \( c \in \mathbb{C}S_n \) is determined. In both cases the cardinality of \( A \) is odd, so by the lemma we get that \( c \) annihilates \( \Lambda^2 M^\lambda \) if \( k \equiv 0, 1 \) (mod 4), and it annihilates \( \text{Sym}^2 M^\lambda \) if \( k \equiv 2, 3 \) (mod 4). Still, we have to show that \( c \) does not annihilate \( M^\lambda \).

First we deal with the case when \( k \) is odd, so \( A = \{1, 2, \ldots, k\} \). The elements of the group algebra \((-1)^g (g \in \Pi(A))\) and \( \Pi(B) \) fix \( v = v_{12 \ldots k} \). This implies that the vectors \( va \) and \( vb \) are non-zero constant multiples of \( v \), so \( vc \neq 0 \). Therefore for odd \( k \) values \( M^\lambda \) is not annihilated by \( c \).

Finally, when \( k \) is even we have \( A = \{1, 2, \ldots, k + 1\} \). Let us take the vector \( v = v_{12 \ldots k} \). Then \( vb \) is a non-zero scalar multiple of \( v \), therefore it is sufficient to prove \( va \neq 0 \) (since we have already seen that \( a \) and \( b \) commute). The vectors \( v_{i_1 i_2 \ldots i_k} \) with

\[
1 \leq i_1 < i_2 < \ldots < i_k \leq n - 1
\]

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form a basis of $M^\lambda$. Our $v$ is among these basis vectors. As $k + 1 < n$, each term in the sum

$$va = \sum_{g \in \Pi(A)} (-1)^g vg$$

is a basis vector or its negative. The term $v$ or $-v$ occurs for those permutations $g \in \Pi(A)$ where $g(k + 1) = k + 1$. We have

$$\sum_{g \in \Pi(A), \ g(k+1)=k+1} (-1)^g v_{12\ldots k}g = k! \cdot v_{12\ldots k},$$

because $g$ has the same sign as an element of $\mathfrak{S}_n$, and as an action on the set $\{1, 2, \ldots k\}$. This means that if we write $va$ in the basis defined above, then the coefficient of $v$ is non-zero, hence $vc \neq 0$. The proof of Theorem 3.1.1(c) is now complete. \qed
4 Symmetric and Exterior Squares of Hook Representations

4.1 Introduction

In this chapter we will continue our investigations on hook representations of symmetric groups, which we already examined in Section 3.4. Our goal is to present the irreducible decomposition of symmetric and exterior squares of hook representations based on article [13] written by Szabolcs Mészáros and the author. Let us quickly remind the reader of the notation and basic facts on $\mathbb{C}\mathfrak{S}_n$ modules we already described in Chapter 3.

We will denote by $M^\lambda$ the irreducible representation of the symmetric group $\mathfrak{S}_n$ corresponding to the Young diagram $\lambda \vdash n$ (i.e. $\lambda$ is a finite, non-increasing sequence of positive integers that add up to $n$) over the complex field. The multiplicities of irreducibles in $M^\lambda \otimes M^\mu$ are called Kronecker coefficients, and their study is an active area of research, see [2],[3],[9],[20].

Let $V = M^{(n-1,1)}$ be the $n-1$ dimensional standard irreducible representation of $\mathfrak{S}_n$ over $\mathbb{C}$. In article [17] Remmel determined the multiplicities of irreducible summands of $\Lambda^k V \otimes \Lambda^k V$, for all $n,k \in \mathbb{N}^+$. (The statement is spelled out in detail in Subsection 4.4.2.) The representations of the form $\Lambda^k V$ are called hook representations, since $\Lambda^k V \cong M^{(n-k,1^k)}$, so the Young diagram of $\Lambda^k V$ resembles a hook. The factors appearing in the decomposition are either hook representations themselves (i.e. $\lambda_2 \leq 1$) or "double hook" representations (i.e. $\lambda_3 \leq 2$ but $\lambda_2 > 1$). In this chapter, we refine this decomposition by separating the summands of the symmetric and antisymmetric components. In other words, we determine the multiplicities of irreducible summands in the representations $\text{Sym}^2(\Lambda^k V)$ and $\Lambda^2(\Lambda^k V)$.

Let us consider $\text{Sym}^2(\Lambda^k V)$ and $\Lambda^2(\Lambda^k V)$ as complementary subspaces of the tensor square $(\Lambda^k V)^{\otimes 2}$, and denote the Young diagram $(q, p, 2^{d_2}, 1^{d_1}) \vdash n$ by $(q, p, 2^{d_2}, 1^{d_1})$. We will present the following theorem.

**Theorem 4.1.1** Let $M^\lambda$ be an irreducible summand of $(\Lambda^k V)^{\otimes 2}$ for some $\lambda \vdash n$.

- If $\lambda = (q, p, 2^{d_2}, 1^{d_1})$, where $2 \leq p \leq q$ and
  - $d_1 \equiv 0 \pmod{4}$, then every $M^\lambda$ factor is contained in $\text{Sym}^2(\Lambda^k V)$,
  - $d_1 \equiv 2 \pmod{4}$, then every $M^\lambda$ factor is contained in $\Lambda^2(\Lambda^k V)$,
  - $2 \nmid d_1$, then the multiplicity of $M^\lambda$ is 1 in $\text{Sym}^2(\Lambda^k V)$ and in $\Lambda^2(\Lambda^k V)$.

- If $\lambda = (n-m, 1^m)$, where $0 \leq m \leq n-1$, and

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- If \( m \equiv 0 \) or \( 1 \pmod{4} \), then every \( M^\lambda \) factor is contained in \( \text{Sym}^2(\Lambda^k V) \).

- If \( m \equiv 2 \) or \( 3 \pmod{4} \), then every \( M^\lambda \) factor is contained in \( \Lambda^2(\Lambda^k V) \).

The theorem is expounded with explicit coefficients in Corollary 4.1.2. Note that if \( \lambda \) is a double hook, then the multiplicity of \( M^\lambda \) in \( \text{Sym}^2(\Lambda^k V) \) or \( \Lambda^2(\Lambda^k V) \) depends only on the modulo 4 value of \( d_1 \), the tail of the Young diagram, independently of \( q, p, d_2 \), if the multiplicity of \( M^\lambda \) in \( (\Lambda^k V)^{\otimes 2} \) is given.

The phenomenon is comparable to Murnaghan’s theorem (see [14], [15]) if we rephrase it as follows: for \( \lambda = (n - p - 2d_2 - d_1, p, 2d_2, 1^{d_1}) \) the multiplicity of \( M^\lambda \) in \( \text{Sym}^2(M^{(n-k,1^k)}) \) is independent of \( n \), as long as \( n \) is sufficiently large (more precisely, if \( n \geq 2k + 1 \)).

Our result is motivated by the relative absence of explicit results on the symmetric and exterior Kronecker coefficients, i.e. the multiplicities of irreducibles in the symmetric and exterior squares of irreducible \( \mathfrak{S}_n \)-representations, compared to the well-investigated topic of Kronecker coefficients. The latter is analysed using various tools, such as symmetric functions (see [3]), colored Yamanouchi tableaux (see [2]), or invariant theory of \( GL(V) \)-representations (see [20]). In contrast, the symmetric and exterior squares are considerably less studied. In article [5] the case of \( \mu \) being a rectangle is examined, and finally, let us mention article [26], which we already presented in Chapter 3.

In this chapter we will use a combinatorial approach, by analysing the action of the Young symmetrizers on colored Young tableaux that correspond to a basis of \( \Lambda^k (V \oplus 1) \otimes \Lambda^l (V \oplus 1) \) where \( V \oplus 1 \) is the \( n \) dimensional permutation representation of \( \mathfrak{S}_n \).

### 4.1.1 Example

Let us illustrate Theorem 4.1.1 on an example. Let \( n = 8 \) and \( k = 2 \) (i.e. \( V \) is the 7-dimensional standard irreducible representation of \( \mathfrak{S}_8 \)), and let us consider the tensor square of \( \Lambda^2 V \) (the tensor square is 441-dimensional). The multiplicities of the irreducible factors in the tensor square \( (\Lambda^2 V)^{\otimes 2} \) are known from Remmel’s theorem (first column of Table 1). Using this decomposition and Theorem 4.1.1, we get the decompositions of submodules \( \text{Sym}^2(\Lambda^2 V) \) and \( \Lambda^2(\Lambda^2 V) \) (second and third columns of Table 1). The irreducible constituents (the rows of Table 1) are indexed by their Young diagrams.

The decomposition of \( (\Lambda^2 V)^{\otimes 2} \) into symmetric and antisymmetric components is not isotypic, e.g. \( M^{(5,2,1)} \) appears in both components. Note however, that it may also happen (see e.g. \( M^{(6,2)} \)) that an irreducible factor has multiplicity two in \( (\Lambda^2 V)^{\otimes 2} \), but it is not a factor of the antisymmetric component.
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Introduction

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
& \((\Lambda^kV)^{\otimes 2}\) & \(\text{Sym}^2(\Lambda^kV)\) & \(\Lambda^2(\Lambda^kV)\) \\
\hline
\[6, 2\] & 2 & 2 & 0 \\
\[5, 3\] & 1 & 1 & 0 \\
\[5, 2, 1\] & 2 & 1 & 1 \\
\[4, 2, 2\] & 1 & 1 & 0 \\
\[4, 2, 1^2\] & 1 & 0 & 1 \\
\hline
\[8\] & 1 & 1 & 0 \\
\[7, 1\] & 1 & 1 & 0 \\
\[6, 1^2\] & 1 & 0 & 1 \\
\[5, 1^3\] & 1 & 0 & 1 \\
\[4, 1^4\] & 1 & 1 & 0 \\
\hline
\end{tabular}

\caption{Multiplicities for \(n = 8\) and \(k = 2\)}
\end{table}

4.1.2 Idea of the proof

Let us fix \(\lambda = (q, p, 2d_2, 1^d_1)\). First let us consider the case when \(d_1\) is even. We denote by \(1\) the trivial representation \(M^{(n)}\), and let

\[ f : (\Lambda^k(V \oplus 1))^{\otimes 2} \twoheadrightarrow (\Lambda^kV)^{\otimes 2} \]

be the \(S_n\) module homomorphism induced by the natural projection \(V \oplus 1 \twoheadrightarrow V\). The action of \(S_n\) on \(\Lambda^k(V \oplus 1)\) and on its tensor square is more combinatorial than \(\Lambda^kV\), so we analyse the action of the Young symmetrizers on the covering representation \(\text{Dom}(f)\), and push down the results via \(f\).

More precisely, we will show that there is a subspace \(A \subseteq \text{Dom}(f)\), such that \(f|_A\) is surjective and that the following skew-symmetry holds:

\[ (a_1 \otimes a_2)c_\lambda = (-1)^{m_\lambda}(a_2 \otimes a_1)c_\lambda \quad (\forall a_1 \otimes a_2 \in A, \forall \lambda \vdash n), \quad (4.1) \]

where \(m_\lambda = \frac{d_1}{2}\), and \(c_\lambda\) is the Young symmetrizer corresponding to the canonical Young tableau of \(\lambda\) (i.e. which is filled with 1, 2, \ldots, \(n\) increasingly, from top to bottom, and from left to right in each row).

The existence of such a subspace \(A\) implies that if \(m_\lambda\) is odd, then for any \(b \in \text{Sym}^2(\Lambda^kV)\) by (4.1) we have \(bc_\lambda = (-1)^{m_\lambda}bc_\lambda = (-1)bc_\lambda = 0\), so \(c_\lambda\) vanishes on \(\text{Sym}^2(\Lambda^kV)\), implying that \(M^\lambda\) is not a constituent of \(\text{Sym}^2(\Lambda^kV)\). Similarly, if \(m_\lambda\) is even, then for any \(b \in \Lambda^2(\Lambda^kV)\) we have \(bc_\lambda = (-1)^{m_\lambda+1}bc_\lambda = (-1)bc_\lambda = 0\). So the theorem follows for the case when \(d_1\) is even.

In the case when \(d_1\) is odd, we use the branching rule of \(S_n\) representations, and an induction-restriction argument to derive the result from the even case and from Remmel’s theorem, recalled in Theorem 4.4.3.
Finally, if $\lambda$ is not a double-hook, but a hook, then the statement is a by-product of the lemmas proved for the even case, see Corollary 4.3.13.

4.1.3 Explicit multiplicities

We give the explicit decomposition of submodules $\Sym^2(\Lambda^k V)$ and $\Lambda^2(\Lambda^k V)$ by combining Theorem 4.1.1 with Remmel’s decomposition theorem on $(\Lambda^k V)^{\otimes 2}$.

**Corollary 4.1.2** Let $\lambda \vdash n$ be a Young diagram. The multiplicities of $M^\lambda$ in $\Sym^2(\Lambda^k V)$ (resp. $\Lambda^2(\Lambda^k V)$) are the following:

- if $\lambda = (q,p,2d_2,1d_1)$ is a double hook for some $2 \leq p \leq q$, then
  - $2$, if $|2k + 1 - n| \leq q - p$ and $d_1 \equiv 0 \, (\text{mod } 4)$,
  - $1$, if $|2k + 1 - n| \leq q - p$ and $2 \nmid d_1$,
  - $1$, if $|2k + 1 - n| = q - p + 1$ and $d_1 \equiv 0 \, (\text{mod } 4)$,
- $1$, if $\lambda = (n - m,1^m)$ is a hook where $0 \leq m \leq 2 \min(k,n-k-1)$ and $m \equiv 0$ or $1 \, (\text{mod } 4)$ (resp. $m \equiv 2$ or $3 \, (\text{mod } 4)$),
- $0$ otherwise.

4.1.4 Outline of the proof of Theorem 4.1.1

In Section 4.2 we introduce a notation on bases in the relevant tensor product representations, moreover, we derive some observations on how the tensor-component flipping $T$, and the dualization $P$ act on the subrepresentations of $(\Lambda^k(V \oplus 1))^{\otimes 2}$. We also introduce the notion of proper swaps in Lemma 4.2.5 to simplify calculations in the subsequent sections. In Section 4.3 (resp. 4.4) we prove the case of Theorem 4.1.1 when $d_1$ is even (resp. odd), see Proposition 4.3.1 (resp. 4.4.1). The case of hook representations is a by-product of the argument in Section 4.3 (see Corollary 4.3.13). Finally, in Section 4.5 we assemble the proof of Theorem 4.1.1 based on our results in Sections 4.3 and 4.4.

4.2 Preliminaries

Let us fix $n \in \mathbb{N}^+$, and let us consider the $n$ dimensional permutation representation of the symmetric group $\mathfrak{S}_n$:

$$ U \overset{\text{def}}{=} V \oplus 1, $$
where $V = M^{(n-1,1)}$ is the standard irreducible and $1 = M^{(n)}$ is the trivial representation. The standard basis elements of $\Lambda^k U \otimes \Lambda^l U$ for any $k, l \in \mathbb{N}^+$ are denoted as

$$u_I \otimes u_J = (u_{i_1} \wedge \cdots \wedge u_{i_k}) \otimes (u_{j_1} \wedge \cdots \wedge u_{j_l}),$$  

(4.2)

where $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_l\}$ for some $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_l \leq n$.

The action of $s \in \mathfrak{S}_n$ is defined as

$$(u_I \otimes u_J)s = (u_{i_1s} \wedge \cdots \wedge u_{i_ks}) \otimes (u_{j_1s} \wedge \cdots \wedge u_{j_ls}).$$

Consequently, the basis of $\Lambda^k U \otimes \Lambda^l U$ defined above may be indexed with 4-colorings as follows. Define the set of all 4-colorings as

$$X \overset{\text{def}}{=} \{x : [n] \to \{0, 1, 2, 3\}\},$$

where $[n] = \{1, 2, \ldots, n\}$ and $n$ is assumed to be fixed, hence omitted from the notation. Then let

$$X_{k,l} \overset{\text{def}}{=} \{x \in X \mid |x^{-1}(\{1, 3\})| = k, \ |x^{-1}(\{2, 3\})| = l\}.$$

We claim that there is a bijection between the given basis of $\Lambda^k U \otimes \Lambda^l U$ and $X_{k,l}$, based on the four subsets $I \cap J$, $I \setminus J$, $J \setminus I$ and $[n] \setminus (I \cup J)$. Indeed, for any $x \in X_{k,l}$ let us define

$$w_x \overset{\text{def}}{=} u_{x^{-1}(\{1,3\})} \otimes u_{x^{-1}(\{2,3\})} \in \Lambda^k U \otimes \Lambda^l U.$$ 

Clearly, $\{w_x \mid x \in X_{k,l}\}$ is the standard basis of $\Lambda^k U \otimes \Lambda^l U$ as defined in (4.2).

We define the right action of $\mathfrak{S}_n$ on $X$ as

$$xs = (m \mapsto x(ms^{-1})) \quad (m \in [n])$$

for all $x \in X$. Note that even though there is a bijection on the $\mathfrak{S}_n$-sets $\{w_x \mid x \in X_{k,l}\}$ and $X_{k,l} \subseteq X$, $w_x s$ does not necessarily equal the basis element $w_{xs}$. Instead, for an appropriate choice of $\epsilon_{x,s} \in \{1, -1\}$ for any $x \in X_{k,l}$ and $s \in \mathfrak{S}_n$,

$$w_x s = \epsilon_{x,s} w_{xs}$$  

(4.3)
More explicitly, we may express these signs using inversion numbers as
\[
\varepsilon_{x,s} = (-1)^{N_1(x,s) + N_2(x,s)},
\]
where
\[
N_c(x,s) \overset{\text{def}}{=} \left| \{(p,q) \in [n]^2 \mid p < q, \ ps > qs, \ x(p), x(q) \in \{c,3\} \} \right|
\]  \quad (4.5)
for \(c \in \{1, 2\}\). Indeed, if \(w_x = u_I \otimes u_J\) as above, then
\[
u_{Is} = u_{i_1s} \land \cdots \land u_{i_ks} = (-1)^{N_1(x,s)}u_{Is},
\]
where \(Is = \{is \mid i \in I\}\), since \(N_1(x,s)\) is the inversion number of the permutation required to sort the sequence \((i_1s, \ldots, i_ks)\) increasingly. Similarly, \(u_{Js} = (-1)^{N_2(x,s)}u_{Js}\), hence (4.4) holds.

### 4.2.1. Color-switch \((12)\)

Let us denote by \(t = (12)\) the transposition of 1 and 2 on the set \(\{0, 1, 2, 3\}\). Then \(x \mapsto t \circ x\) gives a bijection \(X \rightarrow X\) that commutes with the \(\mathfrak{S}_n\)-action i.e. \((t \circ x)s = t \circ (xs)\) for any \(s \in \mathfrak{S}_n\).

The following lemma on the elementary properties of \(\varepsilon_{x,s}\) defined in (4.3) will be useful.

**Lemma 4.2.1** Let \(x \in X\) and \(s \in \mathfrak{S}_n\). Then

1. \(\varepsilon_{t0x,s} = \varepsilon_{x,s}\),

2. if \(s\) is a transposition \((ij)\) such that \(x(i) = x(j) \in \{1, 2\}\) then \(\varepsilon_{x,s} = -1\),

3. if \(s\) is a transposition \((ij)\) such that \(x(i) = x(j) \in \{0, 3\}\) then \(\varepsilon_{x,s} = 1\).

**Proof.** Let \(N_1(x,s)\) and \(N_2(x,s)\) as in (4.5). Then \(N_2(t \circ x,s) = N_1(x,s)\) and \(N_1(t \circ x,s) = N_2(x,s)\) by definition, hence \(\varepsilon_{t0x} = \varepsilon_{x,s}\) holds.

For the second statement, it is enough to prove the case of \(x(i) = x(j) = 1\) and \(i < j\) by symmetry. Then we may note that \(N_2(x,s) = 0\) and

\[
N_1(x,s) = 1 + 2\left| \{m \in [n] \mid i < m < j, \ x(m) \in \{1, 3\} \} \right|
\]
hence \(\varepsilon_{x,s} = -1\). The proof of the last statement follows similarly. □
**Lemma 4.2.2** The linear extension of the map \( w_x \mapsto w_{\top x} \) \((x \in X_{k,l})\)

\[
T : \Lambda^k U \otimes \Lambda^l U \rightarrow \Lambda^l U \otimes \Lambda^k U
\]

is a \(C\mathfrak{S}_n\) module isomorphism.

**Proof.** Indeed, as \( t \) commutes with the group action, Lemma 4.2.1/1 implies

\[
T(w_x)s = w_{\top x}w_{\top (\top z)s} = \varepsilon_{x,s}w_{\top (\top z)s} = \varepsilon_{x,s}T(w_z)s = T(w_zs),
\]

so the claim follows. \(\square\)

### 4.2.2 Color-switch (03)(12)

In the previous subsection we introduced the isomorphism \( T \), that can be interpreted combinatorially as switching the colors 1 and 2 for the elements of \( X_{k,l} \), which parametrize the standard basis of \( \Lambda^k U \otimes \Lambda^l U \).

Now we define a similar isomorphism, switching color 1 with 2 and color 0 with 3, at the cost of an extra sign.

Let us define \( p : \{0, 1, 2, 3\} \rightarrow \{0, 1, 2, 3\} \) as \( p(c) = 3 - c \). Clearly, \( p \circ (xs) = (p \circ x)s \) for any \( x \in X_{k,l} \).

By definition we may write \( w_x = u_I \otimes u_J \) for some \( I, J \subseteq [n] \). Then

\[
w_{p(x)} = u_J \otimes u_I,
\]

where \( I^c = [n] \setminus I \). We define the already mentioned extra sign for \( x \in X_{k,l} \) as follows.

**Definition 4.2.3** For every \( x \in X_{k,l} \) let us take the uniquely defined basis vector \( w_x = u_I \otimes u_J \), where \( I = \{i_1, \ldots, i_k\} \) and \( J = \{j_1, \ldots, j_l\} \) for some \( 1 \leq i_1 < \cdots < i_k \leq n \) and \( 1 \leq j_1 < \cdots < j_l \leq n \). We define the sign corresponding to coloring \( x \) to be

\[
h(x) \overset{\text{def}}{=} (-1)^{\sum_{\alpha=1}^{k}(i_\alpha - \alpha) + \sum_{\alpha=1}^{l}(j_\alpha - \alpha)}.
\]

Now we can formulate the \( p \) analogue of Lemma 4.2.2.

**Lemma 4.2.4** The linear extension of the map \( w_x \mapsto h(x)w_{p(x)} \) \((x \in X_{k,l})\)

\[
P : \Lambda^k U \otimes \Lambda^l U \rightarrow \Lambda^{n-k} U \otimes \Lambda^{n-l} U
\]
is a $\mathbb{C}S_n$ module isomorphism.

**Proof.** First we show that the linear extension of the map $u_I \mapsto u_I \otimes (u_I \wedge u_I \circ) \ (I \in \binom{[n]}{k})$

$$\hat{P}_k : \Lambda^k U \rightarrow \Lambda^{n-k} U \otimes \Lambda^n U$$

is a $\mathbb{C}S_n$ module isomorphism, where $(u_I \wedge u_I \circ)$ denotes the following wedge product of $n$ elements: $u_{i_1} \wedge \cdots \wedge u_{i_k} \wedge u_{j_1} \wedge \cdots \wedge u_{j_{n-k}} \in \Lambda^n U$ defined by (ordered) sets $I = \{i_1, \ldots, i_k\}$ and $I^c = \{j_1, \ldots, j_{n-k}\}$.

As vectors $u_{I \circ}$ and $u_{I \circ \circ}$ are $\pm 1$ multiplies of each other, we define $\text{sign}_I(s) \in \{1, -1\}$ through the equation

$$u_{I \circ} = \text{sign}_I(s) u_{I \circ \circ}.$$ 

On the other hand,

$$\hat{P}_k(u_{I \circ}) = \hat{P}_k(\text{sign}_I(s) u_{I \circ \circ}) = \text{sign}_I(s) u_{I \circ \circ \circ} \otimes (u_{I \circ} \wedge u_{I \circ \circ \circ}).$$

Therefore, $\hat{P}_k$ is indeed a $\mathbb{C}S_n$ module isomorphism. Note that we may express $\hat{P}_k(u_I)$ equivalently as

$$\hat{P}_k(u_I) = (-1)^{\sum_{\alpha=1}^{k}(i_{\alpha} - \alpha)} u_I \otimes (u_{1 \wedge 2 \wedge \cdots \wedge n})$$

because we may sort the components of $u_I \wedge u_I \circ$ using $\sum_{\alpha=1}^{k}(i_{\alpha} - \alpha)$ transpositions. Now let us consider the tensor product of $\mathbb{C}S_n$ module isomorphisms $\hat{P}_k$ and $\hat{P}_l$:

$$\hat{P}_k \otimes \hat{P}_l : \Lambda^k U \otimes \Lambda^l U \rightarrow \Lambda^{n-k} U \otimes \Lambda^{n-l} U \otimes \Lambda^n U$$

acting on the natural basis as

$$u_I \otimes u_J \mapsto (-1)^{\sum_{\alpha=1}^{k}(i_{\alpha} - \alpha) + \sum_{\alpha=1}^{l}(j_{\alpha} - \alpha)} u_{I \circ} \otimes \kappa \otimes u_{J \circ} \otimes \kappa,$$

where $\kappa = u_1 \wedge u_2 \wedge \cdots \wedge u_n$. However, $\Lambda^n U$ spanned by $\kappa$ is the sign representation of $S_n$, hence its square is the identity. Therefore, $\hat{P}_k \otimes \hat{P}_l$ is the same as $P$, in particular, $P$ is a $\mathbb{C}S_n$ module isomorphism. □
4.2.3 Proper swaps

We will need another statement about $\varepsilon_{x,s}$ in Section 4.3. First, let us illustrate it on an example. Let $n = 9$ and

\[ x = (1, 2, 2, 1, 2, 3, 2, 1, 1) \in X_{5,5}, \quad s = (12)(38)(79) \in \mathfrak{S}_9. \]

Permutation $s$ is chosen so, that it is a product of disjoint transpositions switching elements of color 1 with elements of color 2 in such a way, that $s$ preserves the ordering of any two moved (not fixed) elements of the same color. Moreover, if we take any interval defined by the transpositions of $s$, the number of fixed elements of color 1, and the number of fixed elements of color 2 in the interval equals. (In intervals $[1, 2]$ and $[7, 9]$ there are no fixed elements of color 1 or 2, in interval $[3, 8]$ elements 4 and 5 are fixed elements of color 1 and 2 respectively.) Now we check that $\varepsilon_{x,s} = 1$ holds. Indeed,

\[ w_x s = u_{14689,23567}(12)(38)(79) = u_{24637,18569} = u_{23467,15689} = w_{xs}. \]

In the next lemma we generalize this example. For integers $i, j$ let us use the following notation

\[ [[i,j]] \overset{\text{def}}{=} \begin{cases} [i,j] \cap \mathbb{Z} & \text{if } i < j, \\ [j,i] \cap \mathbb{Z} & \text{otherwise.} \end{cases} \]

**Lemma 4.2.5** (Proper Swap Lemma) Let $x \in X$ and $s \in \mathfrak{S}_n$, such that

- $s = \prod_{\ell=1}^m (i_\ell j_\ell)$ is a product of $m$ disjoint transpositions for some $1 \leq i_1 < \cdots < i_m \leq n$ and $1 \leq j_1 < \cdots < j_m \leq n$,
- for all $\ell \in [m]$, $x(i_\ell) = 1$ and $x(j_\ell) = 2$,
- for all $\ell \in [m]$,

\[ |\{ \nu \in [[i_\ell,j_\ell]] \mid \nu s = \nu, x(\nu) = 1 \}| \equiv |\{ \nu \in [[i_\ell,j_\ell]] \mid \nu s = \nu, x(\nu) = 2 \}| \pmod{2}. \]

Then $\varepsilon_{x,s} = 1$.

We call $s$ a proper swap with respect to $x$ if the assumptions of Lemma 4.2.5 hold.

**Proof.** To determine $N_1(x, s)$, let us investigate some $p < q$, $p, q \in [n]$, such that $x(p), x(q) \in \{1, 3\}$. If $p$ and $q$ are both fixed by $s$, then they clearly do not contribute to $N_1(x, s)$. Similarly, if they are non-fixed.
elements, then $p = i_{\ell_1}$ and $q = i_{\ell_2}$ for some $1 \leq \ell_1 < \ell_2 \leq m$, hence $ps = j_{\ell_1} < j_{\ell_2} = qs$, and again they do not contribute.

Now let $p$ be a non-fixed element and $q$ be a fixed element. Then $p = i_{\ell}$ for some $\ell$, and the pair contributes to $N_1(x, s)$ if and only if $p = i_{\ell} < qs = q < ps = j_{\ell}$ i.e. if $q \in [i_{\ell}, j_{\ell}]$. Similarly, if $p$ is a fixed element and $q$ is a non-fixed element, then they contribute to $N_1(x, s)$ if and only if $j_{\ell} = qs < ps = p < q = i_{\ell}$ for some $\ell$. In short,

$$N_1(x, s) = \sum_{\ell} \left( \left| \{ \nu \in [i_{\ell}, j_{\ell}] \mid \nu s = \nu, x(\nu) = 1 \} \right| + \left| \{ \nu \in [i_{\ell}, j_{\ell}] \mid x(\nu) = 3 \} \right| \right).$$

The same holds for $N_2(x, s)$ if we replace $x(\nu) = 1$ by $x(\nu) = 2$. The claim follows by the third assumption.

4.2.4 Canonical Young symmetrizers

Let $\lambda \vdash n$ be a Young diagram with rows of length $(\lambda_1, \ldots, \lambda_h)$ for some height $h \in \mathbb{N}^+$ and let us consider the subgroup of row-preserving permutations

$$R_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_h} \subseteq \mathfrak{S}_n.$$ 

Similarly, we denote by $C_\lambda = R_\lambda^\top$ the subgroup of column-preserving permutations, where $\lambda^\top$ denotes the conjugate (transpose) of the diagram $\lambda$.

We define the Young symmetrizer corresponding to (the canonical Young tableau of) $\lambda$ as

$$c_\lambda = \sum_{a \in R_\lambda} a \sum_{b \in C_\lambda} \text{sign}(b) b \in \mathbb{C}\mathfrak{S}_n.$$ 

Given a fixed Young diagram, e.g. $\lambda = (5, 3, 2)$, we may visualize a coloring $x = (0, 1, 3, 0, 3, 2, 0, 1, 0, 2)$ as a coloring of the Young diagram using the set of colors $\{0, 1, 2, 3\}$:

$$\begin{array}{cccccc}
0 & 1 & 3 & 0 & 3 \\
2 & 0 & 1 \\
0 & 2
\end{array}.$$ 

This terminology implied by the visualization makes it easier to formulate statements such as "there are two 3's in the first row" as a shorthand for $\exists i, j \leq \lambda_1 : x(i) = x(j) = 3$. 

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4.3 Double Hooks with Even Tail

In this section we prove the case of Theorem 4.1.1, where \( \lambda \) is a double hook \((q, p, 2d_2, 1d_1)\) and the length of its 'tail' \(d_1\) is even.

**Proposition 4.3.1** Let \( \lambda \vdash n \) be a Young diagram of the form \( \lambda = (q, p, 2d_2, 1d_1) \) for some \( 2 \leq p \leq q \). If \( d_1 \equiv 2 \pmod{4} \), then the multiplicity of \( M^\lambda \) in \( \text{Sym}^2(\Lambda^k V) \) is zero. Similarly, if \( d_1 \equiv 0 \pmod{4} \), then the multiplicity of \( M^\lambda \) in \( \Lambda^2(\Lambda^k V) \) is zero.

Equivalently, we prove that \( \text{Sym}^2(\Lambda^k V)c_\lambda = 0 \) if \( d_1 \equiv 2 \pmod{4} \), where \( c_\lambda \) is the (canonical) Young-symmetrizer corresponding to \( \lambda \), hence \( M^\lambda \) is not a summand of \( \text{Sym}^2(\Lambda^k V) \), and similarly for the exterior square. The steps of the proof are the following. First, in Lemma 4.3.3 we show a skew-symmetry relation for \( w_x c_\lambda \) in the case of \( n = 6 \), using the observations of Lemma 4.3.2. Then we prove Lemma 4.3.5 so we may induce these skew-symmetries for larger diagrams. This induction is carried out in Proposition 4.3.9, where we show that the crucial identity

\[ w_x c_\lambda = (-1)^{d_2} w_{tox} c_\lambda \]

holds under the assumption that the first row of the colored Young diagram corresponding to coloring \( x \) does not contain 1’s or 2’s (the subspace spanned by the basis vectors \( w_x \), where the corresponding colored Young diagrams do not contain 1’s or 2’s in the first row, was mentioned as \( A \) in Subsection 4.1.2). These skew-symmetries are important for us, as the symmetric and exterior squares \( \text{Sym}^2(\Lambda^k U) \) and \( \Lambda^2(\Lambda^k U) \) are generated by the vectors \( (w_x + w_{t ox}) \) and \( (w_x - w_{t ox}) \) respectively (where \( x \in X_{k,k} \)). Let us note that the cases where \( n \leq 5 \) are covered in Proposition 4.3.9 Case III. Until this point of the proof all of our statements will investigate the action of \( c_\lambda \) on \( (\Lambda^k U)^{\otimes 2} \). In the rest of the proof we will prove Lemma 4.3.10 using the natural projection \( U \to U/\left( \sum_{i=1}^n u_i \right) \cong V \). Finally, Proposition 4.3.9 combined with Lemma 4.3.10 will imply Proposition 4.3.1.

4.3.1 Base case

Let us make some simple observations on the annihilation of \( c_\lambda \) on the basis vectors. Recall from Subsection 4.2.4 that for a given \( \lambda \) we may visualize \( x \) as a 4-colored Young diagram.

**Lemma 4.3.2** Let \( x \in X \) and \( \lambda \vdash n \) a Young diagram.

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1. If there are two 1’s or two 2’s in the same row, then \( w_x c_\lambda = 0 \).

\[
\begin{array}{|c|c|}
\hline
1 & 1 \\
\hline
\end{array}
\]

2. If there are two 0’s or 3’s in the same column, then \( w_x \sum_{b \in C_\lambda} \text{sign}(b)b = 0 \).

\[
\begin{array}{|c|}
\hline
3 \\
\hline
3 \\
\hline
\end{array}
\]

3. If for every \( a \in R_\lambda \) there are three (resp. five) of 0’s and 3’s in the first (resp. first two) columns of \( xa \) in total, then \( w_x c_\lambda = 0 \).

\[
\begin{array}{|c|c|}
\hline
0 & 3 & 0 \\
3 & 3 & 0 \\
\hline
\end{array}
\]

Proof. To prove the first statement of the lemma we may assume without the loss of generality that there are two 1’s in the same row of \( \lambda \) on positions \( i \) and \( j \). Let \( a_0 \in R_\lambda \) be the transposition \((ij)\). Equation \( c_\lambda = a_0 c_\lambda \) holds by the definition of \( c_\lambda \). Moreover, \( w_x a_0 = -w_x a_0 = -w_x \) by Lemma 4.2.1/2. This means that \( w_x c_\lambda = w_x (a_0 c_\lambda) = (w_x a_0) c_\lambda = -w_x c_\lambda \) holds, and this implies the first statement of the lemma.

To prove the second statement of the lemma we may assume without the loss of generality that there are two 0’s in the same column of \( \lambda \) on positions \( i \) and \( j \). Let us introduce \( s = \sum_{b \in C_\lambda} \text{sign}(b)b \) and let \( b_0 \in C_\lambda \) be the transposition \((ij)\). Then we have \( s = -b_0 s \) and \( w_x b_0 = w_x b_0 = w_x \) by Lemma 4.2.1/3. Now \( w_x s = -w_x (b_0 s) = -(w_x b_0) s = -w_x s \) holds, so the second statement of the lemma follows.

The third statement of the lemma follows from the second statement directly.

\[\square\]

Lemma 4.3.3 Let \( \lambda \vdash 6 \) be a Young diagram with \( \lambda_1 = 2 \). Let \( x \in X_{k,l} \) be a coloring such that in the first row of \( \lambda \) there are no elements of color 1 or 2, and in every other row with length at least two, there is at least one element of color 0 or 3.

1. If \( \lambda \) is \( (2,2,2) \), then

\[ w_x c_\lambda = w_{\text{toe}} c_\lambda. \]  \hspace{1cm} (4.6)

In particular, if \( k \neq l \), then both sides of (4.6) are zero.
2. If \( \lambda \) is \((2, 2, 1, 1)\), \( k = l \) and at least one of the elements on the tail is of color 0 or 3, then
\[
w_x c_\lambda = - w_{102} c_\lambda.
\] (4.7)

Remark. Equations (4.6) and (4.7) can be checked one by one for the finitely many basis vectors, thus we could safely ignore the proof of Lemma 4.3.3. Nonetheless, we have decided to present a proof in detail, so that we can provide some explicit calculations using the notation introduced in Section 4.2, which might prove useful later in following the general argument.

Proof. We start the proof of the lemma with four general observations on the action \( w_x c_\lambda \) to reduce the number of cases where (4.6) and (4.7) are needed to be checked.

First, \( w_x c_\lambda = 0 \) implies \( w_{t0x} c_\lambda = 0 \), since \( T : \Lambda^k U \otimes \Lambda^l U \rightarrow \Lambda^l U \otimes \Lambda^k U \) is a \( C \Sigma_n \) module isomorphism by Lemma 4.2.2.

Second, if \( w_x c_\lambda = \rho w_{t0x} c_\lambda \) for some \( \rho \in \{1, -1\} \) and \( a \in R_\lambda \) is an arbitrary row permutation, then \( w_{xa} c_\lambda = \rho w_{t0xa} c_\lambda \). Indeed
\[
w_{xa} c_\lambda = \varepsilon_{x,a} w_{xa} c_\lambda = \varepsilon_{x,a} w_x c_\lambda = \varepsilon_{x,a} \rho w_{t0x} c_\lambda = \varepsilon_{x,a} \rho w_{t0xa} c_\lambda = \rho w_{t0xa} c_\lambda
\]
where equation \( ac_\lambda = c_\lambda \) was used in the second and fourth step and Lemma 4.2.1/1 in the last step. Our second observation means that if the statement of the lemma holds for some \( w_x \) corresponding to a coloring \( x \), then it will also hold for any other \( w_x', \) where \( x' \) is derived from \( x \) by rearranging the colors of \( x \) in the rows of \( \lambda \).

Third, using the fact that \( w_{t0x} c_\lambda = w_x c_\lambda \), it can be assumed that the coloring \( x \) contains at least as many elements of color 1 as of color 2. Moreover, in case of non-zero equality, the first element of color 1 is smaller than the first element of color 2 (that is \( \min(x^{-1}(\{1\})) < \min(x^{-1}(\{2\})) \) holds).

Fourth, if Lemma 4.3.3 holds for some basis vector \( w_x \), then it also holds for \( w_{p0} \) i.e. when we swap the colors 0 and 3. Recall that \( p(c) = 3 - c \) for any \( c \in \{0, 1, 2, 3\} \).
\[
w_x c_\lambda = \rho w_{t0x} c_\lambda \implies w_{p0x} c_\lambda = \rho w_{p0} c_\lambda \implies w_{p0} c_\lambda = \rho w_{tp0} c_\lambda,
\]
where \( \rho \in \{1, -1\} \). Indeed, by Lemma 4.2.4, \( w_x \mapsto h(x) w_{p0} \) is a \( C \Sigma_n \) module isomorphism and by Definition 4.2.3 equation \( h(x) = h(t \circ x) \) holds, hence the first implication follows. For the second
implication we used the fact that actions $p$ and $t$ are commuting on the colorings.

Now we will prove the first statement of the lemma. Let us determine those $w_x$ vectors which shall be investigated in order to prove (4.6). By the hypotheses, it is clear that coloring $x$ contains at least four elements of color 0 or 3. Combining Lemma 4.3.2/3 with the first observation, we get that it will be enough to investigate those colorings $x$, which satisfy $|x^{-1}([0])| = 2$ and $|x^{-1}([3])| = 2$. By the second observation, we may assume that the elements of color 0 or 3 are on the following positions of $\lambda$:

\[
\begin{array}{cc}
\ast & \ast \\
\ast & \ast \\
\end{array}
\]

We show in the next paragraph that it is sufficient to check (4.6) for the following colorings:

\[
x_1 = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}, x_2 = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 2 & 3 \end{bmatrix}, x_3 = \begin{bmatrix} 0 & 3 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}, x_4 = \begin{bmatrix} 0 & 3 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}.
\]

Two cases are distinguished based on whether the colors in the first row are equal or not. If they equal, then by the fourth observation we may assume that $x(1) = x(2) = 0$ and $x(4) = x(6) = 3$. If they are not equal, then with the help of the second and fourth observations we might assume that $x(1) = 0$, $x(2) = 3$, $x(4) = 0$ and $x(6) = 3$. The two remaining entries are of color 1 and 2. They might have identical colors or different colors. Using the third observation two cases shall be investigated $x(3) = x(5) = 1$, and $x(3) = 1$, $x(5) = 2$. Let us list the basis vectors corresponding to the identified critical colorings:

\[
w_{x_1} = u_{3456} \otimes u_{46}, w_{x_2} = u_{346} \otimes u_{456}, w_{x_3} = u_{2356} \otimes u_{26}, w_{x_4} = u_{236} \otimes u_{256}.
\]

We will show in details how to handle the actions $w_{x_1}c_\lambda$ and $w_{x_4}c_\lambda$, as the investigation of these actions contain all the necessary type of computational steps needed for the remaining two cases. We start with expanding the Young symmetrizer $c_\lambda = a_\lambda b_\lambda$. Here $a_\lambda = (1 + (12))(1 + (34))(1 + (56))$ by definition, so

\[
w_{x_1}c_\lambda = w_{x_1}a_\lambda b_\lambda = 2w_{x_1}(1 + (34))(1 + (56))b_\lambda.
\]

Now we check the action of $(1 + (34))(1 + (56))$ on the coloring $x_1$:

\[
x_1 = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}, x_1(34) = \begin{bmatrix} 0 & 0 \\ 3 & 1 \\ 1 & 3 \end{bmatrix}, x_1(56) = \begin{bmatrix} 0 & 0 \\ 1 & 3 \\ 3 & 1 \end{bmatrix}, x_1(34)(56) = \begin{bmatrix} 0 & 0 \\ 3 & 1 \\ 3 & 1 \end{bmatrix}.
\]
By Lemma 4.3.2/2, $b_\lambda$ will annihilate the terms coming from $w_{x_1}(1 + (34)) + (56))b_\lambda = 2w_{x_1}((34) + (56))b_\lambda =$

\[
2u_{3456} \otimes u_{46}(34)b_\lambda + 2u_{3456} \otimes u_{46}(56)(35)(46)b_\lambda =
\]

\[
2u_{3456} \otimes u_{36}b_\lambda + 2u_{5643} \otimes u_{63}b_\lambda = -2u_{3456} \otimes u_{36}b_\lambda + 2u_{3456} \otimes u_{36}b_\lambda = 0.
\]

By the first observation (4.6) follows for $w_{x_1}$.

Now let us consider $w_{x_4}c_\lambda$. The annihilation property of $b_\lambda$ stated in Lemma 4.3.2/2 allows us to consider only two summands of $a_\lambda$, $(56)$ and $(12)(34)$. By this fact, and $(135)(264)b_\lambda = (153)(246)b_\lambda = b_\lambda$ we get:

\[
w_{x_4}c_\lambda = w_{x_4}(1 + (12))(1 + (34))(1 + (56))b_\lambda = w_{x_4}((56) + (12)(34))b_\lambda =
\]

\[
u_{236} \otimes u_{256}((56) + (12)(34))b_\lambda = u_{235} \otimes u_{265}b_\lambda + u_{146} \otimes u_{156}b_\lambda =
\]

\[
u_{235} \otimes u_{265}(135)(264)b_\lambda + u_{146} \otimes u_{156}(153)(246)b_\lambda =
\]

\[
u_{u51} \otimes u_{641}b_\lambda + u_{562} \otimes u_{332}b_\lambda = u_{156} \otimes u_{146}b_\lambda - u_{256} \otimes u_{235}b_\lambda.
\]

On the other hand

\[
t \circ x_4 = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ 1 & 3 \end{bmatrix}
\]

and $w_{t \circ x_4} = u_{256} \otimes u_{236}$, so

\[
w_{t \circ x_4}c_\lambda = w_{t \circ x_4}(1 + (12))(1 + (34))(1 + (56))b_\lambda = w_{t \circ x_4}((56) + (12)(34))b_\lambda =
\]

\[
u_{u265} \otimes u_{235}b_\lambda + u_{156} \otimes u_{146}b_\lambda = -u_{256} \otimes u_{235}b_\lambda + u_{156} \otimes u_{146}b_\lambda,
\]

which means that (4.6) holds for $w_{x_4}$. For the remaining two basis vectors we provide the raw computations.

\[
w_{x_2}c_\lambda = 2u_{346} \otimes u_{456}(34) + (56))((35)(46)b_\lambda = 2(-u_{456} \otimes u_{345} - u_{346} \otimes u_{345})b_\lambda.
\]

\[
w_{t \circ x_2}c_\lambda = 2u_{456} \otimes u_{346}(34) + (56))b_\lambda = 2(-u_{345} \otimes u_{346} - u_{456} \otimes u_{345})b_\lambda.
\]

\[
w_{x_3}c_\lambda = u_{2356} \otimes u_{26}(12)(34)b_\lambda + u_{2356} \otimes u_{26}(56)((135)(264))b_\lambda = 0.
\]
The first statement of the lemma is now proved.

For the second statement of the lemma, let us take $\lambda = (2, 2, 1, 1)$. The hypotheses combined with Lemma 4.3.2/3 and the first observation give, that it is enough to investigate those colorings $x$, which satisfy $|x^{-1}({0})| = |x^{-1}({3})| = 2$ and $|x^{-1}({1})| = |x^{-1}({2})| = 1$. Using the second observation, there are two possible configurations for the location of elements of color 0 and 3:

$$
\begin{array}{ccc}
* & * & *\\
* & * & *\\
* & * & *
\end{array}
\quad \quad
\begin{array}{ccc}
* & * & *\\
* & * & *\\
* & * & *
\end{array}
$$

However, it will be sufficient to prove (4.7) for the first configuration, as we can act with transposition (56) on (4.7), providing a proof for the second type of configurations. Here we used the fact that (56) commutes with $c_\lambda$. Now we can list the critical colorings that are needed to be checked, just as in the first part of the lemma:

$$
\begin{array}{ccc}
0 & 0 & 1 \\
1 & 3 & 2
\end{array},
\quad
\begin{array}{ccc}
0 & 3 & 1 \\
1 & 0 & 3 \\
2 & 3 & 2
\end{array}
$$

The corresponding basis vectors are the following:

$$
w_{x_1} = u_{345} \otimes u_{456}, \quad w_{x_2} = u_{235} \otimes u_{256}.
$$

The computations are a bit simpler than the ones we have already seen in the first part of the lemma.

$$
w_{x_1}c_\lambda = -2u_{345} \otimes u_{456}((36))b_\lambda = -2u_{456} \otimes u_{345}b_\lambda.
$$

$$
w_{t_{02}x_1}c_\lambda = 2u_{456} \otimes u_{345}b_\lambda.
$$

$$
w_{x_2}c_\lambda = -u_{123} \otimes u_{126}((36))b_\lambda = -u_{126} \otimes u_{123}b_\lambda.
$$

$$
w_{t_{02}x_2}c_\lambda = u_{126} \otimes u_{123}b_\lambda.
$$

The proof of Lemma 4.3.3 is complete. \qed

### 4.3.2 Technical lemmas for the induction step

In this subsection we will present technical lemmas, which will be useful in the inductive step of the proof of Proposition 4.3.1. The most important lemma of the subsection will be Lemma 4.3.5. We recommend...
the reader to skip the proofs of this subsection at first reading, as we think it is better to see the key steps of the proof of Proposition 4.3.1 before going into the technical details.

First, let us consider the following simplified version of Lemma 4.3.5, and recall the definition of proper swaps from Subsection 4.2.3.

**Lemma 4.3.4 (Simplified Induction Lemma)** Let $\lambda \vdash n$ be a Young diagram, $x \in X$, $a_0 \in R_\lambda$ and $b_0 \in C_\lambda$ such that

1. $a_0 b_0$ is a proper swap with respect to $x$,
2. $t \circ x = xa_0 b_0$,
3. $b_0$ centralizes $R_\lambda$.

Then $w_x c_\lambda = \text{sign}(b_0) w_{t \circ x} c_\lambda$.

In more colorful language, the lemma says that if we may mimic the action of $t$ on $x$ by a proper swap $a_0 b_0 \in R_\lambda C_\lambda$ where $b_0$ only moves the tail of $\lambda$, then a skew-symmetry relation holds. For example, if $\lambda$ and $x$ are visualized as

\[
\begin{array}{cccc}
3 & 1 & 3 & 2 \\
0 & 1 & 2 \\
2 \\
1 \\
3
\end{array}
\]

then $a_0 = (24)(67)$, $b_0 = (89)$ satisfies the assumptions of the lemma and hence $w_x c_\lambda = -w_{t \circ x} c_\lambda$. Note that this may happen only if $x \in X_{k,k}$ for some $k \in \mathbb{N}^+$. 

**Proof.** Let us denote $\text{sign}(b_0)$ by $\delta$. As $b_0$ centralizes $R_\lambda$ we obtain

$$w_x c_\lambda = \delta w_x \sum_{a \in R_\lambda} a_0 a \sum_{b \in C_\lambda} \text{sign}(b) b_0 b = \delta w_x a_0 b_0 c_\lambda.$$ 

We may apply Lemma 4.2.5 to $a_0 b_0$, so $\varepsilon_{x,a_0 b_0} = 1$. Hence, we may continue as

$$\delta w_x a_0 b_0 c_\lambda = \delta w_{x a_0 b_0} c_\lambda = \delta w_{t \circ x} c_\lambda,$$

our claim follows. \qed
For the generalization of Lemma 4.3.4, let us define restrictions of Young symmetrizers. Let \( H \subseteq [n] \), \( \lambda \vdash n \) a Young diagram and \( x \in X \). Recall that \( T_\lambda \) denotes the canonical Young tableau i.e. \( T_\lambda \) is \( \lambda \) filled with \( 1, 2, \ldots, n \) row-continuously from left to right, from top to bottom. We will say that \( H \) is compatible with \( \lambda \) if the subset of \( T_\lambda \) determined by \( H \) is left-aligned and has non-increasing row lengths. For example, the cells marked with \( H \) in diagram

\[
\begin{array}{ccc}
\hline
H & H & H \\
\hline
H & H \\
\end{array}
\]

form a subset compatible with \( \lambda \).

For any \( H \subseteq [n] \) denote by \( R_\lambda(H) \) (resp. \( C_\lambda(H) \)) the pointwise stabilizer of \( H^c = [n] \setminus H \) in \( R_\lambda \) (resp. \( C_\lambda \)), and similarly let

\[
R_\lambda(H^c) = \text{Stab}_{R_\lambda}(H) \quad C_\lambda(H^c) = \text{Stab}_{C_\lambda}(H)
\]

be the pointwise stabilizers of \( H \) in \( R_\lambda \) and \( C_\lambda \) respectively. Define the \( H \)-restricted Young symmetizer of \( \lambda \) as

\[
c_{\lambda,H} \overset{\text{def}}{=} \sum_{a \in R_\lambda(H)} a \sum_{b \in C_\lambda(H)} \text{sign}(b)b.
\]

It is clear from the definition that \( c_{\lambda,H} \in \mathbb{C}\text{Sym}(H) \subseteq \mathbb{C}\mathfrak{S}_n \). We will also need an \( H \)-restricted version of the color-swap \( x \mapsto t \circ x \) defined in Subsection 4.2.1. We denote by \( t_H : X \to X \) the map

\[
t_H(x)(i) = \begin{cases} 
1 & \text{if } i \in H \text{ and } x(i) = 2, \\
2 & \text{if } i \in H \text{ and } x(i) = 1, \\
x(i) & \text{otherwise}.
\end{cases}
\]

Note that while \((t \circ x)s = t \circ (xs)\) for any \( s \in \mathfrak{S}_n \), the same does not hold for \( t_H \). Let us illustrate the action of \( t_H \) using the example for \( H \) given above and some \( x \in X_{\lambda,5} \):

\[
\begin{array}{cccc}
0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 \\
2 & 2 & 0 & \end{array} \xrightarrow{t_H} \begin{array}{cccc}
0 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & \end{array}
\]

The following lemma helps us to deduce relations of the form \( w_x c_\lambda = \pm w_{t_2c_\lambda} \) if we know similar
relations with $c_{\lambda,H}$.

**Lemma 4.3.5** (Induction Lemma) Let $\lambda \vdash n$ be a Young diagram, $x \in X$ and $\rho, \delta \in \{1, -1\}$. Assume that $H \subseteq [n]$ is compatible with $\lambda$. Moreover, assume that for all $r \in R_\lambda$ if $w_x c_{\lambda,H}$ is non-zero, then there exists $a_r \in R_\lambda(H^c)$ and $b_r \in C_\lambda(H^c)$ such that

1. $w_x c_{\lambda,H} = \rho w \cdot t(x) c_{\lambda,H}$,
2. $a_r b_r$ is a proper swap with respect to $t(x)$,
3. $t(x) a_r b_r = t \cdot x r$,
4. $b_r$ centralizes $R_\lambda(H^c)$ and $\text{sign}(b_r) = \delta$.

Then $w_x c_{\lambda} = (\rho \delta) w \cdot t(x) c_{\lambda}$.

In short, if we may supplement the restricted color-swap $t(x)$ with an element $a_r b_r \in R_\lambda(H^c) C_\lambda(H^c)$ such that they together mimic the action of $t$ on $x r$, where $b_r$ only moves the tail of $\lambda$ outside of $H$ (and this holds for all $r \in R_\lambda$), then we may lift the skew-symmetry relation from $H$ to $[n]$.

**Proof.** Let $(r_i)_{i \in I}$ resp. $(s_j)_{j \in J}$ be a set of representatives for the left cosets resp. right cosets of the subgroups $R_\lambda(H) \times R_\lambda(H^c) \subseteq R_\lambda$ and $C_\lambda(H) \times C_\lambda(H^c) \subseteq C_\lambda$.

In particular, we may write

$$\sum_{r \in R_\lambda} r = \sum_{i \in I} \sum_{a \in R_{\lambda}(H)} \sum_{a' \in R_{\lambda}(H^c)} r_i a a'$$

and

$$\sum_{c \in C_\lambda} \text{sign}(c)c = \sum_{j \in J} \sum_{b \in C_{\lambda}(H)} \sum_{b' \in C_{\lambda}(H^c)} \text{sign}(b b' s_j) b b' s_j.$$ 

Let’s start to compute $w_x c_{\lambda}$. Since $R_\lambda(H^c)$ centralizes $C_\lambda(H)$ we have

$$w_x c_{\lambda} = w_x \sum_{a,a',i} r_i a a' \sum_{b,b',j} \text{sign}(bb' s_j) b b' s_j = \sum_{i \in I} w_x r_i \sum_{a,b} \text{sign}(b) a b \sum_{a',b',j} \text{sign}(b' s_j) a' b' s_j.$$ 

Let us denote the terms

$$c_{\lambda,H} = \sum_{a,b} \text{sign}(b) a b$$

and

$$s_0 = \sum_{a',b',j} \text{sign}(b' s_j) a' b' s_j.$$ 


Clearly, if we denote by $I'$ the set of $i \in I$ such that $w_{xr_i}c_{\lambda,H} \neq 0$, then (4.8) holds with summation index $i \in I'$ as well.

For each $i \in I'$ we may choose $a_i \in R_{\lambda}(H)\mathcal{c}$ and $b_i \in C_{\lambda}(H)\mathcal{c}$ corresponding to $r = r_i$ as in the conditions of the lemma. Note that for $a_i$ and $b_i$ we have

$$s_0 = \delta a_i b_i s_0$$

for any $i \in I'$, using Condition 4. Note also that

$$c_{\lambda,H} a_i b_i = a_i b_i c_{\lambda,H}$$

as $\text{Sym}(H)$ centralizes $\text{Sym}(H\mathcal{c})$. Recall that Condition 2 of the lemma implies $w_{tH(x)}a_r b_r = w_{tH(x)}a_r b_r$ by Lemma 4.2.5, therefore we obtain

Finally, note that if $w_{xr_i}c_{\lambda,H} = 0$ then $w_{t_0 x r_i}c_{\lambda,H} = 0$ holds as well by Lemma 4.2.2 and 4.2.1/1. Hence we may apply (4.8) for $t \circ x$, that gives

$$w_{t_0 x}c_{\lambda} = \sum_{i \in I'} \varepsilon_{t_0 x,r_i} w_{t_0 x r_i}c_{\lambda,H} s_0,$$

where $I'$ is still defined as above. The claimed equality $w_x c_{\lambda} = (\rho \delta) w_{t_0 x}c_{\lambda}$ follows. \qed

The following lemma helps to check Condition 1 of Lemma 4.3.5.

**Lemma 4.3.6** Let $\lambda \vdash n$ be a Young diagram, $x \in X_{k,l}$ be a coloring of $[n]$ and let us assume that $H \subseteq [n]$ of size $m$ is compatible with $\lambda$. We denote by $\lambda'$ the partition corresponding to $H$, and by $E$ the
unique monotonically increasing $[m] \to H$ function. We define the restricted coloring of $H$ as

$$x' = (x|_H \circ E) : [m] \to \{0, 1, 2, 3\}.$$  

Let us assume that for each $h_1, h_2 \in H$ such that $\{x(h_1), x(h_2)\} = \{1, 2\}$ we have

$$\left| \{ \nu \in H^c \mid h_1 < \nu < h_2, \ x(\nu) = 1 \} \right| = \left| \{ \nu \in H^c \mid h_1 < \nu < h_2, \ x(\nu) = 2 \} \right|. \quad (4.11)$$

Then

$$w_{x'} c_{\lambda'} = \rho w_{t_{x'}} c_{\lambda'} \implies w_x c_{\lambda, H} = \rho w_{t_H(x)} c_{\lambda, H}. \quad (4.12)$$

Remark. The implication

$$w_{x'} c_{\lambda'} = \rho w_{t_{x'}} c_{\lambda'} \implies w_x c_{\lambda, H} = \pm \rho w_{t_H(x)} c_{\lambda, H}$$

is always true because of the linearity of exterior power spaces. The point of the lemma is to prove that Condition (4.11) assures that this additional sign in (4.12) does not appear.

Proof. We denote by $f_H$ the composition of the group homomorphisms

$$\mathfrak{S}_m \cong \text{Sym}(H) \to \mathfrak{S}_n,$$

where the first is induced by $E$ and the second by the inclusion $H \hookrightarrow [n]$. The map $f_H$ induces a $\mathbb{C}\mathfrak{S}_m$ module structure on $\Lambda^k U \otimes \Lambda^l U$, in this $\mathbb{C}\mathfrak{S}_m$ module element $r \in \mathfrak{S}_m$ will act as $v \ast r = v f_H(r)$. This definition assures that

$$v \ast c_{\lambda'} = v c_{\lambda, H} \quad (4.13)$$

holds for any $v \in \Lambda^k U \otimes \Lambda^l U$.

Let us denote by $U'$ the $m$ dimensional permutation representation of $\mathfrak{S}_m$, using this notation we may assume that $w_{x'} \in \Lambda^k U' \otimes \Lambda^l U'$. For the fixed colorings $x$ and $x'$ we can take the map $w_{x'} \mapsto w_x \ast r$ for all $r \in \mathfrak{S}_m$. As $w_{x'}$ generates the cyclic module $\Lambda^k U' \otimes \Lambda^l U'$, this map extends to a well defined

$$F : \Lambda^k U' \otimes \Lambda^l U' \to \Lambda^k U \otimes \Lambda^l U$$

$\mathbb{C}\mathfrak{S}_m$ module homomorphism.
First, let us assume that \( k' \neq l' \). In this case \( \omega_{tox'} c_{\lambda'} \notin \Lambda^k U \otimes \Lambda^l U \), so on the left side of implication (4.12) \( \omega_{tox'} c_{\lambda} = \omega_{tox'} c_{\lambda'} = 0 \) must hold. But \( \omega_{tox'} c_{\lambda} = 0 \) implies \( \omega_{tox'} c_{\lambda, H} = 0 \) because of the linearity of exterior power spaces, and similarly \( \omega_{tox'} c_{\lambda'} = 0 \) implies \( \omega_{t_H(x)} c_{\lambda, H} = 0 \). We conclude that if \( k' \neq l' \), then implication (4.12) is correct.

Now we assume that \( k' = l' \). By the definition of \( F \), it is clear that \( F(\omega_{tox'}) \) is either \( \omega_{t_H(x)} \) or \( -\omega_{t_H(x)} \). We prove that it is always the former. (Without Condition (4.11) the negative case could also happen, see Example 4.3.7.) This will be sufficient, as then

\[
\omega_{t_H(x)} c_{\lambda, H} = \omega_{t_H(x)} * c_{\lambda'} = F(\omega_{tox'}) c_{\lambda'} \overset{\text{(4.12)}}{=} F(\rho \omega_{tox'} c_{\lambda}) = \\
\rho F(\omega_{tox'}) c_{\lambda'} = \rho \omega_{t_H(x)} * c_{\lambda'} = \rho \omega_{t_H(x)} c_{\lambda, H},
\]

so implication (4.12) will follow.

To show that \( F(\omega_{tox'}) = \omega_{t_H(x)} \), note that there exists a proper swap \( s \) with respect to \( x' \) that satisfies \( t \circ x' = x's \) (simply swap the first 1 with the first 2, the second 1 with the second 2, etc.). By Lemma 4.2.5, \( \omega_{tox'} = \omega_{x's} = \omega_{x's} \). Applying \( F \) gives \( F(\omega_{tox'}) = F(\omega_{x's}) = \omega_{x} * s = \omega_{x} f_H(s) \). Notice that \( f_H(s) \) is a proper swap with respect to \( x \). Indeed, \( f_H(s) \) fulfills the first two conditions of proper swaps by the inheritance of \( s \), and fulfills the third condition of proper swaps by Condition (4.11) of the present lemma. Applying Lemma 4.2.5 on \( f_H(s) \) gives \( \omega_{x} f_H(s) = \omega_{x} f_H(s) = \omega_{t_H(x)} \), so \( F(\omega_{tox'}) = \omega_{t_H(x)} \) is proved.

**Example 4.3.7.** Without Condition (4.11) it may happen that \( F(\omega_{tox'}) = -\omega_{t_H(x)} \), which leads to a false implication in (4.12). Define \( \lambda \) and \( x \) as

\[
x = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
2 & 2
\end{bmatrix},
\]

and let \( H = \{1, 3, 5\} \). Then, \( \omega_{x} = u_{14} \otimes u_{56}, \omega_{t_H(x)} = u_{45} \otimes u_{16} \) and

\[
F(\omega_{tox'}) = F(\omega_{x}(13)) = \omega_{x} * (13) = \omega_{x}(15) = u_{14} \otimes u_{56}(15) = u_{54} \otimes u_{16} = -u_{45} \otimes u_{16}.
\]

We will also need an \( H \)-restricted version of Lemma 4.3.2/3 in the next subsection.

**Lemma 4.3.8** Let \( \lambda \vdash n \) be a Young diagram, \( x \in X \) be a coloring of \([n]\) and let us assume that \( H \subseteq [n] \).
is a subset of the first two columns of $\lambda$ that is compatible with $\lambda$. If there are at least five elements of $H$ that are of color 0 or 3, then $w_x c_{\lambda,H} = 0$.

Proof. Let us denote the term $\sum_{b \in C_\lambda(H)} \text{sign}(b)b$ by $b_0$. It is enough to show that $w_x b_0 = 0$. Indeed, then

$$w_x c_{\lambda,H} = w_x \sum_{a \in R_\lambda(H)} ab_0 = \sum_{a \in R_\lambda} \varepsilon_{x,a}w_x a b_0 = 0,$$

as we may apply $w_x b_0 = 0$ for $xa$ instead of $x$ as the assumptions of the lemma are invariant under $x \mapsto xa$ by $a \in R_\lambda(H)$.

To prove $w_x b_0 = 0$, we take a transposition $s = (ij) \in C_\lambda(H)$ such that $x(i) = x(j) \in \{0,3\}$ (such transposition exists by the assumptions). Then $xs = x$, and by Lemma 4.2.1/3,

$$w_x b_0 = w_x (-sb_0) = -\varepsilon_{x,s}w_x xs b_0 = -w_x b_0,$$

hence $w_x b_0 = 0$ is proved. \qed

4.3.3 Inductive step

Now that all the necessary technical machinery is available, we can take a direct step toward the proof of Proposition 4.3.1. In this subsection we apply Lemma 4.3.5 (Induction Lemma) on the double hook Young diagrams with no 1’s or 2’s in the first row.

**Proposition 4.3.9** Let $\lambda \vdash n$ be a Young diagram of the form $\lambda = (q,p,2^{d_2},1^{d_1})$ for some $2 \leq p \leq q$, $0 \leq d_1,d_2$. Let $x \in X_{k,k}$ be a coloring of $[n]$ such that $x(\{1,2,\ldots,q\}) \subseteq \{0,3\}$. If $d_1$ is even, then

$$w_x c_{\lambda} = (-1)^{d_1} w_{102x} c_{\lambda} \quad (4.14)$$

holds.

Proof. We may assume that $w_x c_{\lambda} \neq 0$. Indeed, otherwise $w_{102x} c_{\lambda} = 0$ also holds by Lemma 4.2.2. We may also assume that there is at most one element of color 1 and at most one element of color 2 in each row in $\lambda$ for the given coloring $x$, by Lemma 4.3.2/1.

Let us call $m \in [n]$ unpaired if it is the only element in its row of color 1 or 2, and the row is of length at least two. Our tactic will be to choose an $H \subseteq [n]$ that covers as many unpaired elements as possible, and satisfies the assumptions of Lemma 4.3.5 (Induction Lemma).
There are at most two unpaired elements in total, by Lemma 4.3.2/3 and our assumption on the first row, so we may distinguish three cases based on the number of unpaired elements.

**Case I:** Let us assume that there are two unpaired elements. For example,

\[
\begin{array}{cccc}
0 & 0 & 0 & 3 \\
1 & 3 & 0 & \\
1 & 2 &  & \\
2 & 3 & & \\
\end{array}
\]

Let their rows be the \( m \)-th and the \( m' \)-th row of \( \lambda \). Then we may define \( H = \{1, 2, m_1, m_2, m'_1, m'_2\} \) where \( m_1 \) and \( m_2 \) denotes the first and second elements of the \( m \)-th row, and similarly for \( m' \).

Let us check the assumptions of Lemma 4.3.5. Let \( r \in R_\lambda \) and assume that \( w_{xr}c_{\lambda, H} \neq 0 \). If \( H \) contains at least five elements of color 0 or 3 in \( xr \), then we would have \( w_{xr}c_{\lambda, H} = 0 \) by Lemma 4.3.8. Consequently, all unpaired elements of \( xr \) are contained in \( H \). Observe that for each \( h_1, h_2 \in H \subseteq [n] \) we have

\[
|\{\nu \in H^c \mid h_1 < \nu < h_2, (xr)(\nu) = 1\}| = |\{\nu \in H^c \mid h_1 < \nu < h_2, (xr)(\nu) = 2\}|.
\]

Indeed, all the unpaired elements are contained in \( H \) so every other row between the elements of \( H \) either doesn’t contain any 1’s or 2’s, or it does contain one of each. This equation and Lemma 4.6/1 assures that Lemma 4.3.6 can be applied to the coloring \( xr \) and the given choice of \( H \). The lemma gives \( w_{xr}c_{\lambda, H} = w_{xt}(xr)c_{\lambda, H} \), hence Condition 1 of Lemma 4.3.5 is verified with \( \rho = 1 \).

Let us check the remaining assumptions of Lemma 4.3.5. If there are different number of 1’s as 2’s in \( H \) then \( w_{(xr)c_{\lambda}} \) would be zero by Lemma 4.3.3/1, hence \( w_{xr}c_{\lambda, H} = 0 \) too, by Lemma 4.3.6. As we assumed this is not the case, we may define \( a_r \in R_\lambda(H^c) \) as the product of disjoint transpositions swapping the elements of color 1 and 2 that are in the same row, and similarly \( b_r \in C_\lambda(H^c) \) as the product of disjoint transpositions swapping the 1’s and 2’s on the tail of \( \lambda \). The permutation \( b_r \) has to be chosen in a monotonic way so \( a_r b_r \) is a proper swap. Then it is straightforward to check the Conditions 2, 3 and 4 of Lemma 4.3.5 with \( \delta = \text{sign}(b_r) = (-1)^{\frac{d_2}{2}} \). The proof of this case follows from Lemma 4.3.5.

**Case II:** Let us assume that there is exactly one unpaired element and it is of color 1 (the case of color 2 is analogous). For example,

\[
\begin{array}{cccc}
0 & 0 & 0 & 3 \\
1 & 3 & 0 & \\
1 & 2 & & \\
2 & 3 & & \\
\end{array}
\]
Let the row of the unpaired element be the $m$-th row of $\lambda$. Then by Lemma 4.3.2/3 and that $d_1$ is even, there is exactly one element $l$ on the tail (i.e. $l > n - d_1$) that is of color 0 or 3. Moreover, as the number of 1’s and 2’s in $x$ agree (i.e. $x \in X_{k,k}$ for some $k$), there is one more element of color 2 on the tail, than of color 1.

We define $j$ as the least element such that it is on the tail (i.e. $n - d_1 < j \leq n$), is of color 2 and there are the same number of elements of color 1 as of color 2 strictly between $n - d_1$ and $j$. Then we may define $H = \{1, 2, m_1, m_2, j, l\}$ where $m_1$ and $m_2$ denotes the first and second elements of the $m$-th row.

Let us repeat the argument of the previous case. Let $r \in R_\lambda$ and assume that $w_{xr}c_{\lambda,H} \neq 0$. Then $H$ contains at most four elements of color 0 or 3 by the same argument using Lemma 4.3.8. In particular, the unpaired element in row $m$ is contained in $H$. The definition of $j$ assures that for each $h_1, h_2 \in H \subseteq [n]$ such that $\{x(h_1), x(h_2)\} = \{1, 2\}$ we have

$$|\{\nu \in H^c \mid h_1 < \nu < h_2, (xr)(\nu) = 1\}| = |\{\nu \in H^c \mid h_1 < \nu < h_2, (xr)(\nu) = 2\}|.$$ 

Just as in the previous case, this equation and Lemma 4.6/2 assures that Lemma 4.3.6 can be applied to coloring $xr$ and restriction $H$. It gives $w_{xr}c_{\lambda,H} = -w_{r,\tau(xr)}c_{\lambda,H}$, hence Condition 1 of Lemma 4.3.5 (Induction Lemma) is verified with $\rho = -1$.

We define $a_r \in R_\lambda(H^c)$ (resp. $b_r \in C_\lambda(H^c)$) exactly the same way as in the previous case, in particular $b_r$ is defined in a monotonic way. Then it is straightforward to check the Conditions 2, 3 and 4 of Lemma 4.3.5 with $\delta = \text{sign}(b_r) = (-1)^{d_1/2}$, hence $\rho \delta = (-1)^{d_1/2}$. The proof of this case follows analogously.

**Case III:** Let us assume that there are no unpaired elements, e.g.

$$\begin{bmatrix} 0 & 0 & 0 & 3 & 3 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

Then we may simply define $a_0 \in R_\lambda$ (resp. $b_0 \in C_\lambda$) as the product of disjoint transpositions swapping the elements of color 1 and 2 that are in the same row (resp. on the tail of $\lambda$), where $b_0$ (in fact both) are chosen in an order-preserving way. Then the claim follows by Lemma 4.3.4. \qed
4.3.4 Proof of Proposition 4.3.1

In this subsection we show that it was enough to prove (4.14) under the assumption that the first row of the colored Young diagram corresponding to coloring $x$ is not containing 1’s or 2’s, by finishing the proof of Proposition 4.3.1. For a given $\lambda \vdash n$ Young diagram and $x \in X$ coloring, denote by $\ell(x)$ the number of 1’s and 2’s in total appearing in the first row i.e. $\ell(x) = |x^{-1}([1, 2]) \cap [\lambda_1]|$, and let

$$W_{\ell(x)} = \text{Span}\{w_y \mid y \in X_{k, k}, \ell(y) < \ell(x)\} \leq (\Lambda^k U)^{\otimes 2}.$$ 

The plan is to prove Proposition 4.3.1 for $x$ by induction on $\ell(x)$.

Consider the natural $C\mathfrak{S}_n$ module projection

$$U \to U / \left(\sum_{i=1}^{n} u_i\right) \cong V.$$ 

This projection induces $C\mathfrak{S}_n$ module projections $\Lambda^k U \to \Lambda^k V$ and $(\Lambda^k U)^{\otimes 2} \to (\Lambda^k V)^{\otimes 2}$ respectively. Let us denote the kernel of the latter projection by $K$. For the inductive step, we prove the following lemma.

**Lemma 4.3.10** Let $\lambda \vdash n$ be a Young diagram, and $x \in X_{k, k}$ be a coloring of $[n]$. If $1 \leq \ell(x)$ then

$$w_x c_\lambda \in W_{\ell(x)} c_\lambda + K.$$ 

Note that the statement is interesting only for $\ell(x) \leq 2$ by Lemma 4.3.2/1.

**Proof.** Let $m \in [\lambda_1]$ be an element such that $x(m) = 1$ (the case of $x(m) = 2$ is analogous). Let us write

$$w_x = \varepsilon \cdot (u_m \wedge u_I) \otimes u_J,$$

where $\varepsilon \in \{1, -1\}$ is depending on the position of $m$ and $m \notin I \cup J \subseteq [n]$ by $x(m) = 1$. Let us take the following element of $K$

$$\varepsilon \left(\sum_{i=1}^{n} u_i \wedge u_I\right) \otimes u_J = w_x + \varepsilon \sum_{i \neq m} (u_i \wedge u_J) \otimes u_J.$$ 

(4.15)

We will show the for any given $i \neq m$ the tensor product element $((u_i \wedge u_J) \otimes u_J)c_\lambda$ either equals $\varepsilon w_x c_\lambda$ or is contained in $W_{\ell(x)} c_\lambda$. The lemma will follow by this fact, because acting with $c_\lambda$ on the $K$ element
defined by (4.15) gives
\[ zw_xc_\lambda \in W_{\ell(x)c_\lambda} + K \]
for some positive integer \( z \). Here we used the fact that \( K \) is a \( \mathbb{C} \mathfrak{S}_n \) submodule of \( (\Lambda^k U)^{\otimes 2} \).

Let us investigate the tensor product element \( ((u_i \wedge u_I) \otimes u_J)c_\lambda \), where \( i \neq m \). If \( x(i) \in \{1, 3\} \) then \( u_i \wedge u_I = 0 \), so \( ((u_i \wedge u_I) \otimes u_J)c_\lambda = 0 \in W_{\ell(x)c_\lambda} \). If \( x(i) = 2 \), then \( (u_i \wedge u_I) \otimes u_J = \pm w_y \), where \( y \in X_{k,k} \) is defined as
\[
y(j) = \begin{cases} 
0 & \text{if } j = m, \\
3 & \text{if } j = i, \\
x(j) & \text{otherwise.}
\end{cases}
\]
Notice that \( \ell(y) = \ell(x) - 1 \), in particular \( (u_i \wedge u_I) \otimes u_J \in W_{\ell(x)} \), so \( ((u_i \wedge u_I) \otimes u_J)c_\lambda \in W_{\ell(x)c_\lambda} \) holds. Finally, we investigate \( x(i) = 0 \). If \( i > \lambda_1 \) then \( (u_i \wedge u_I) \otimes u_J \in W_{\ell(x)} \) and our claim follows. If \( i \leq \lambda_1 \), then \( ((u_i \wedge u_I) \otimes u_J)s = (u_m \wedge u_I) \otimes u_J \) where \( s = (im) \in R_\lambda \). Consequently,
\[
((u_i \wedge u_I) \otimes u_J)c_\lambda = ((u_i \wedge u_I) \otimes u_J)s c_\lambda = ((u_m \wedge u_I) \otimes u_J)c_\lambda = w_x c_\lambda.
\]
We checked all the possibilities, our lemma is now proved.

**Lemma 4.3.11** Let \( \lambda \vdash n \) be a Young diagram of the form \( \lambda = (q, p, 2d_2, 1^{d_1}) \) for some \( 2 \leq p \leq q, \) \( 0 \leq d_1, d_2 \), where \( d_1 \) is even. For any \( x \in X_{k,k} \) the congruence
\[
w_xc_\lambda \equiv (-1)^{\frac{n-m}{2}}w_{c_\lambda} \pmod{K} \tag{4.16}
\]
holds.

**Proof.** Applying \( \ell(x) \) times Lemma 4.3.10, we get that \( w_x c_\lambda \in W_1 c_\lambda + K \). Using the fact that the generators of subspace \( W_1 \) satisfy (4.16) by Proposition 4.3.9, and Lemma 4.2.2, we conclude that \( w_x \) also satisfies (4.16).

Before going on, let us prove the analogue of the previous lemma when \( \lambda \vdash n \) is an arbitrary hook partition.

**Corollary 4.3.12** Let \( \lambda \vdash n \) be a Young diagram of the form \( \lambda = (n - m, 1^m) \) and \( x \in X_{k,k} \). Then
\[
w_x c_\lambda \equiv (-1)^{\frac{n-m}{2}}w_{c_\lambda} \pmod{K}.
\]
Proof. Based on Lemma 4.3.10, it is enough to prove for the case when every element in the first row of \( \lambda \) is of color 0 or 3. Now the number of elements of color 1 and 2 on the tail of \( \lambda \) are the same, as \( \lambda \) is a hook and \( x \in X_{k,k} \). Therefore, we may apply Lemma 4.3.4 with \( a_0 = \text{id} \) and \( b_0 \) being the product of disjoint transpositions swapping the elements of color 1 with the elements of color 2 in a monotonic way. To determine \( \text{sign}(b_0) \), we note that by Lemma 4.3.2/2, there are at most two elements in the first column of \( \lambda \) that are of color 0 or 3, but at least one, by the assumption on the first row. Therefore, \( \text{sign}(b_0) = (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} \) and the claim follows.

Now we are ready to prove Proposition 4.3.1.

Proof. Let us assume that \( d_1 \equiv 2 \pmod{4} \) holds, in this case we will show that \( \text{Sym}^2(\Lambda^k V)c_\lambda = 0 \). It will be sufficient to prove that \( \text{Sym}^2(\Lambda^k U)c_\lambda \subseteq K \), since by restricting the projection \( (\Lambda^k U)^{\otimes 2} \to (\Lambda^k V)^{\otimes 2} \) to \( \text{Sym}^2(\Lambda^k U) \), we get a surjective \( \mathbb{C}S_n \) module homomorphism \( \text{Sym}^2(\Lambda^k U) \to \text{Sym}^2(\Lambda^k V) \).

We prove that \( \text{Sym}^2(\Lambda^k U)c_\lambda \subseteq K \). Indeed, the natural generator vectors of \( \text{Sym}^2(\Lambda^k U) \leq (\Lambda^k U)^{\otimes 2} \) have the form \( (w_x + w_{1ox}) \) for some \( x \in X_{k,k} \), and Lemma 4.3.11 implies that \( (w_x + w_{1ox})c_\lambda \in K \), as \( d_1 \) is odd. By this argument the statement of the proposition in case \( d_1 \equiv 2 \pmod{4} \) is proved.

The case when \( d_1 \equiv 0 \pmod{4} \) holds is analogous: \( \Lambda^2(\Lambda^k V)c_\lambda = 0 \) follows by the application of Lemma 4.3.11 on the covering generators \( (w_x - w_{1ox}) \).

Corollary 4.3.13 Let \( \lambda \vdash n \) be a Young diagram of the form \( \lambda = (n - m, 1^m) \) for some \( 0 \leq m \leq n - 1 \). If \( m \equiv 0 \) or 1 \( \pmod{4} \) then the multiplicity of \( M^\lambda \) in \( \Lambda^2(\Lambda^k V) \) is zero. Similarly, if \( m \equiv 2 \) or \( 3 \pmod{4} \), then the multiplicity of \( M^\lambda \) in \( \text{Sym}^2(\Lambda^k V) \) is zero.

Proof. The argument is the same as in the previous proof of Proposition 4.3.1, only now we have to apply Corollary 4.3.12 instead of Lemma 4.3.11.

4.4 Double Hooks with Odd Tail

In this section we prove the case of Theorem 4.1.1 where \( \lambda \) is a double hook \( (q,p,2^{d_2},1^{d_1}) \) and the length of its tail \( d_1 \) is odd. We will show the following proposition.

Proposition 4.4.1 Let \( \lambda \vdash n \) be a Young diagram of the form \( \lambda = (q,p,2^{d_2},1^{d_1}) \) for some \( 2 \leq p \leq q \). If \( d_1 \) is odd then the multiplicity of \( M^\lambda \) in \( \Lambda^2(\Lambda^k V) \) equals the multiplicity of \( M^\lambda \) in \( \text{Sym}^2(\Lambda^k V) \).

The proof is based on Frobenius reciprocity, the branching rule, the fact that we already proved the case of even length tails, and that the exact multiplicities of \( (\Lambda^k V)^{\otimes 2} \) are known by Remmel’s theorem.
4.4.1 Branching Argument

Let $\mu \vdash n - 1$ be a Young diagram such that the number of rows of length one is even, i.e. it has even length tail. Let us denote by $\text{Ind}$ the induction operator $\text{Ind}_{S_n}^{S_{n-1}}$, and similarly let us write $\text{Res}$ for the restriction operator $\text{Res}_{S_n}^{S_{n-1}}$, finally, we introduce the notation $\langle M, N \rangle$ to be the usual inner product of $S_n$ representations, i.e. $\langle M, N \rangle = \dim \text{Hom}_{S_n}(M, N)$. By Frobenius reciprocity we have

$$\langle \text{Ind} M^\mu, F(\Lambda^k V) \rangle = \langle M^\mu, \text{Res} F(\Lambda^k V) \rangle \quad F \in \{\text{Sym}^2, \Lambda^2\}. \quad (4.17)$$

By the branching rule of $S_n$ representations, we may decompose the left hand side of (4.17) as follows. For two Young diagrams $\mu \vdash (n - 1)$ and $\lambda \vdash n$ let us write $\mu \nearrow \lambda$ if and only if $\lambda$ may be obtained from $\mu$ by adding a single cell to it. With this notation the branching rule states that

$$\text{Ind} M^\mu = \sum_{\lambda \mu \nearrow \lambda} M^\lambda. \quad (4.18)$$

Let us denote by $\mu[i]$ the diagram obtained from $\mu$ by adding a cell to the $i$-th column. It might happen that $\mu[i]$ is not a Young diagram any more, in this case we define $M^\mu[i]$ to be the zero module. In the followings we will investigate Young diagrams

$$\lambda = (q, p, 2d_2, 1d_1) \quad \text{and} \quad \mu = (q, p, 2d_2, 1d_1 - 1) \quad (4.19)$$

for some odd $d_1$ and $2 \leq p \leq q$. Then we have $\mu[1] = \lambda$ and $\mu[2] = (q, p, 2d_2 + 1, 1d_1 - 2)$. By (4.18) we get

$$\text{Ind} M^\mu = M^\lambda \oplus (M^\mu[2] \text{ if } 1 < d_1) \oplus (M^\mu[3] \text{ if } 2 < p) \oplus (M^\mu[p+1] \text{ if } p < q) \oplus M^\mu[q+1]. \quad (4.20)$$

The conditional terms are defined to be zero if the condition fails.

**Lemma 4.4.2** If $V_{n-1}$ denotes the $n - 2$ dimensional standard irreducible representation of $S_{n-1}$, then

$$\text{Res} F(\Lambda^k V) \cong F(\Lambda^k V_{n-1}) \oplus F(\Lambda^{k-1} V_{n-1}) \oplus (\Lambda^k V_{n-1} \otimes \Lambda^{k-1} V_{n-1})$$

holds for $F \in \{\text{Sym}^2, \Lambda^2\}$.

It is at least plausible that (4.17) together with (4.20) and Lemma 4.4.2 completely determines the multiplicities for double hooks $\lambda$ with odd length tail. We will prove this in the next subsection.
Proof. As Res commutes with $F$ and $\Lambda^k$ we have

$$\text{Res } F(\Lambda^k V) = F(\Lambda^k \text{Res } V).$$

We denote by $1_{n-1}$ the trivial representation of $S_{n-1}$. By definition $\text{Res } V \cong V_{n-1} \oplus 1_{n-1}$. Moreover, $\Lambda^k(N \oplus 1) \cong \Lambda^k N \oplus \Lambda^{k-1} N$ for any $N$, hence

$$F(\Lambda^k \text{Res } V) \cong F(\Lambda^k(V_{n-1} \oplus 1_{n-1})) \cong F(\Lambda^k V_{n-1} \oplus \Lambda^{k-1} V_{n-1}).$$

Finally, one can observe that $F(N_1 \oplus N_2) = F(N_1) \oplus F(N_2) \oplus (N_1 \otimes N_2)$ for any $N_1, N_2$ and for any $F \in \{\text{Sym}^2, \Lambda^2\}$, hence the claim of the lemma follows.

Remark. The argument given above is not dependent on the parity of $d_1$ i.e. with induction-restriction we may get similar equations for $d_1$ even. In the end, one could combine this argument with a simultaneous induction on 4 variables ($n, q, p, d_1$), descending on $q$ and $p$) and derive some parts of Proposition 4.3.1 too. This approach would have two serious drawbacks: on one hand it wouldn’t solve the case of $\lambda = (q, p, 2d_2)$, where we would need a proof similar to the one given in Section 4.3. Moreover, it wouldn’t explain why the mod 4 value of $d_1$ appears in the answer, while we think that Lemmas 4.3.3, 4.3.4 and 4.3.11 are more insightful in this regard.

4.4.2 Application of Remmel’s theorem

First let us recall Remmel’s theorem.

Theorem 4.4.3 (Remmel [17], Rosas [19]) Let $n, k, l \in \mathbb{N}^+$ and $\lambda \vdash n$ be a Young diagram. Then the multiplicities of $M^\lambda$ in $\Lambda^k V \otimes \Lambda^l V$ are the following:

- if $\lambda = (q, p, 2d_2, 1^{d_1})$, $2 \leq p \leq q$ is a double hook then
  - $2$, if $|k - l| \leq d_1$ and $|k + l + 1 - n| \leq q - p$,
  - $1$, if $|k - l| \leq d_1$ and $|k + l + 1 - n| = q - p + 1$,
  - $1$, if $|k - l| = d_1 + 1$ and $|k + l + 1 - n| \leq q - p$,

- $1$, if $\lambda = (n-m, 1^m)$ is a hook, where $|k' - l'| \leq m^{k,l} \leq k' + l'$, using the notation $u' = \min(u, n-u-1)$
and
\[ m^{k,l} = \begin{cases} 
m & \text{if } (k = k' \text{ and } l = l') \text{ or } (k \neq k' \text{ and } l \neq l'), \\
n - m - 1 & \text{otherwise},
\end{cases} \]

- 0 otherwise.

**Remark.** The notation of the statement is an alternative version of the one, which was used by M. H. Rosas in Theorem 3 of article [19], where she characterized the case of multiplicity 2 as

\[ |k - l| \leq d_1 \quad \text{and} \quad 2p - 1 \leq k + l - 2d_2 - d_1 \leq 2q - 1. \]

The latter is equivalent to \(|k+l+1-n| \leq q-p\) by \(q+p+2d_2+d_1 = n\). Note also that Remmel’s formulation in Theorem 2.1(b) of article [17] contains a mathematical typo in the case of \(\lambda = (r, 1^{n-r})\), as he writes \(c_\lambda = \chi(s + t - n - 1 \leq r \leq s + n - t)\) instead of the correct equation \(c_\lambda = \chi(s + t - n \leq r \leq s + n - t)\), where the characteristic function \(\chi\) is defined below.

Let us apply Theorem 4.4.3 for some special cases. For any statement \(P\) let us introduce the notation

\[ \chi(P) = 1 \text{ if } P \text{ is true, and } 0 \text{ otherwise, in particular } \chi(a \leq b) = 1 \text{ if and only if } a \leq b. \]

Moreover, denote

\[ \psi(a, b) = \begin{cases} 
2 & \text{if } |a| < b, \\
1 & \text{if } |a| = b, \\
0 & \text{otherwise.}
\end{cases} \]

Recall the definition of \(\lambda\) and \(\mu\) from (4.19). Using the notation of the previous paragraph, by Theorem 4.4.3, we have

\[ \langle M^{\lambda}, (\Lambda^k V)^{\otimes 2} \rangle = \psi(2k + 1 - n, q - p + 1). \quad (4.21) \]

Moreover,

\[ \langle M^{\mu}, \Lambda^k V_{n-1} \otimes \Lambda^{k-1} V_{n-1} \rangle = \begin{cases} 
\psi(2k + 1 - n, q - p + 1) & \text{if } 1 < d_1,
\chi(2k + 1 - n \leq q - p) & \text{if } d_1 = 1,
\end{cases} \quad (4.22) \]

where \(V_{n-1}\) is the \((n-2)\) dimensional standard irreducible representation of \(\mathfrak{S}_{n-1}\).

**Corollary 4.4.4** Let \(\mu = (q, p, 2d_2, 1^{d_1-1})\) for some \(2 \leq p \leq q\), \(d_1\) odd. If \(p < q\) then

\[ \langle M^{\mu[q+1]} \otimes M^{\mu[p+1]}, F(\Lambda^k V) \rangle = \langle M^{\mu}, F(\Lambda^{k-1} V_{n-1}) \oplus F(\Lambda^k V_{n-1}) \rangle. \quad (4.23) \]
Moreover, if $p = q$ then
\[
\langle M^{\mu[q+1]}, F(\Lambda^k V) \rangle = \langle M^{\mu}, F(\Lambda^{k-1} V_{n-1}) \oplus F(\Lambda^k V_{n-1}) \rangle.
\] (4.24)

**Proof.** First, we investigate the cases when either $F = \text{Sym}^2$ and $(d_1 - 1) \equiv 2 \pmod{4}$ or $F = \Lambda^2$ and $(d_1 - 1) \equiv 0 \pmod{4}$. As $\mu$ has even tail we may apply Proposition 4.3.1, and see that both sides of (4.23) and (4.24) are zero.

Now we prove (4.23), so let us assume that $p < q$ holds, and that $F$ and $d_1$ are not as above. By Proposition 4.3.1, for any $k \in \mathbb{N}^+$ we have
\[
\langle M^{\mu}, F(\Lambda^k V_{n-1}) \rangle = \langle M^{\mu}, (\Lambda^k V_{n-1})^{\otimes 2} \rangle,
\]
\[
\langle M^{\mu'}, F(\Lambda^k V) \rangle = \langle M^{\mu'}, (\Lambda^k V)^{\otimes 2} \rangle \quad \mu' \in \{\mu[p+1], \mu[q+1]\}.
\]
Therefore, by using Theorem 4.4.3 in the form of (4.21), we get equations
\[
\langle M^{\mu}, F(\Lambda^{k-1} V_{n-1}) \rangle = \psi(2(k-1) + 1 - (n-1), q - p + 1),
\]
\[
\langle M^{\mu}, F(\Lambda^k V_{n-1}) \rangle = \psi(2k + 1 - (n-1), q - p + 1),
\]
\[
\langle M^{\mu[p+1]}, F(\Lambda^k V) \rangle = \psi(2k + 1 - n, q - (p+1) + 1),
\]
\[
\langle M^{\mu[q+1]}, F(\Lambda^k V) \rangle = \psi(2k + 1 - n, (q+1) - p + 1).
\]
It is easy to check that if $a, b$ are integers such that $b \geq 2$ then
\[
\psi(a, b) - \psi(a - 1, b - 1) = \psi(a + b - 1, 1).
\]
Hence, we get
\[
\langle M^{\mu[p+1]}, F(\Lambda^k V) \rangle - \langle M^{\mu}, F(\Lambda^k V_{n-1}) \rangle = -\psi(2k - n + q - p + 2, 1),
\]
and
\[
\langle M^{\mu[q+1]}, F(\Lambda^k V) \rangle - \langle M^{\mu}, F(\Lambda^{k-1} V_{n-1}) \rangle = \psi(2k - n + q - p + 2, 1),
\]
so the first statement of the lemma follows.
We can prove (4.24) in the same way. Using \( p=q \), we get the equation
\[
\langle M^{[q+1]}(\Lambda^k V), F(\Lambda^{k-1} V_{n-1}) \rangle = \psi(2k+1-n, 2) - \psi(2k-n, 1) - \psi(2k+2-n, 1).
\]
By the identity \( \psi(x, 2) - \psi(x-1, 1) - \psi(x+1, 1) = 0 \), (4.24) follows.

Now we are ready to prove Proposition 4.4.1.

**Proof.** Let us derive recursive equations on the multiplicities. We consider (4.17) provided by the Frobenius reciprocity
\[
\langle \text{Ind} M^\mu(\Lambda^k V), F(\Lambda^k V) \rangle = \langle M^\mu, \text{Res} F(\Lambda^k V) \rangle.
\]
Now we expand the left hand side by (4.20) and the right hand side by Lemma 4.4.2, and subtract the appropriate terms appearing in Corollary 4.4.4 (depending on whether \( p=q \) holds or not) to get the equation
\[
\langle M^\lambda \oplus M^\mu[2], F(\Lambda^k V) \rangle = \langle M^\mu, \Lambda^k V_{n-1} \otimes \Lambda^{k-1} V_{n-1} \rangle.
\] (4.25)
We used the facts that \( \mu[1] = \lambda \) and that \( \langle M^{[3]}(\Lambda^k V) \rangle = 0 \) by Theorem 4.4.3, moreover \( M^{[2]} \) on the left side of (4.25) is not zero only if \( 1 < d_1 \) holds. (If \( d_1 = 1 \), then \( \mu[2] \) is not a Young diagram.) By (4.22) we conclude that if \( d_1 = 1 \), then we have
\[
\langle M^\lambda, F(\Lambda^k V) \rangle = \chi(|2k+1-n| \leq q-p),
\] (4.26)
and if \( 1 < d_1 \), then
\[
\langle M^\lambda \oplus M^\mu[2], F(\Lambda^k V) \rangle = \psi(2k+1-n, q-p+1).
\] (4.27)
Note that the right hand sides of (4.26) and (4.27) are independent of whether \( F = \text{Sym}^2 \) or \( F = \Lambda^2 \). As these equations uniquely determine each multiplicity by induction on \( d_1 \), the proposition follows.

### 4.5 Proof of the main theorem

Now we can assemble the proof of Theorem 4.1.1.

**Proof.** Let \( \lambda \vdash n \) be a Young diagram of the form \( \lambda = (q,p,2^{d_2},1^{d_1}) \). If \( d_1 \) is even, then the statement on \( \lambda \) follows from Proposition 4.3.1.
If $d_1$ is odd, then based on Theorem 4.4.3 we know that $\langle M^\lambda, (A^kV)^\otimes 2 \rangle \leq 2$. On the other hand, by Proposition 4.4.1, the multiplicity of the symmetric and the exterior part are the same, so the multiplicity is either zero or one in both.

If $\lambda = (n - m, 1^m)$, then the statement on $\lambda$ follows by Corollary 4.3.13. We covered all the cases, hence the main theorem of the section is proved.

Finally, Corollary 4.1.2 is directly implied by Theorem 4.1.1 and Remmel's Theorem 4.4.3.
References


Summary

This thesis provides a short introduction into the general theory of ESQ representations, and then presents three results of the author, which were achieved by the investigation of ESQ representations. The key concept of the thesis is: a module is said to be an ESQ module, if it is isomorphic to a factor of its exterior square. This concept of representation theory was introduced in article [8].

In the first chapter we give the definition of ESQ representations, and introduce the notation and prerequisites that are needed to understand the thesis. Then, we provide some straightforward consequences of the definition of ESQ property. In the rest of the chapter, based on article [8], we motivate the investigation of ESQ representations through a question of group theory, and classify all the four-dimensional irreducible faithful ESQ representations, and those five-dimensional irreducible faithful ESQ representations, which are minimal.

In the second chapter, which is based on article [27] written by the author, we investigate the ESQ property of $F_{p,q}$ representations, where $p$ is a prime, $q | p - 1$, and $F_{p,q} = C_p \rtimes C_q$. The investigation of $F_{p,q}$ groups is motivated by the four- and five-dimensional classification theorems on ESQ representations. We show that there exists an irreducible $F_{p,q}$ representation with the ESQ property if and only if a Fermat-type equation is solvable over $F_p$. Using this equivalence, we prove two asymptotic theorems on the ESQ property of irreducible $F_{p,q}$ representations.

In the third chapter, which is based on article [26] written by the author, we investigate the ESQ property of the irreducible representations of the symmetric group $\mathfrak{S}_n$. It is well-known, that the irreducible representations of $\mathfrak{S}_n$ can be indexed by the partitions of $n$. A partition $\lambda \vdash n$ is called a height two partition if $\lambda = (n - k, k)$, where $1 \leq k \leq \frac{n}{2}$ holds, $\lambda$ is called a width two partition if $\lambda = (l_1, l_2, \ldots, l_k)$, where $l_i \leq 2$ holds. Finally, $\lambda$ is called a hook partition if $\lambda = (n - k, 1^k)$, where $0 \leq k \leq n - 1$ holds. We prove that none of the $\mathfrak{S}_n$ representations corresponding to two height partitions have the ESQ property, and exactly one $\mathfrak{S}_n$ representation corresponding to width two partition has the ESQ property (namely, the representation corresponding to partition $\lambda = (2, 1, 1)$). These statements are proved by the multiple application of Cauchy-Frobenius lemma on various permutation characters. Finally, we investigate the $\mathfrak{S}_n$ representations corresponding to hook partitions, and we prove that in this case the ESQ property of the representation depends on the modulo 4 value of $k$.

In the fourth chapter, which is based on article [13] written by Szabolcs Mészáros and the author, we strengthen our results on hook representations. In article [17] Remmel determined the irreducible decomposition of any tensor product of two hook representations. In this chapter we will refine Remmel’s decomposition in the case of hook tensor squares by determining the irreducible decomposition of the symmetric and antisymmetric parts of the tensor square. To get this finer decomposition, we examine the action of Young symmetrizers on the natural generator set of the tensor square. This result is not related closely to the topic of ESQ representations, however, the investigated question, and the fundamental idea of the proof were found during the investigation of the ESQ property of hook representations.
Összefoglalás

A doktori disszertáció rövid bevezetést nyújt az ESQ reprezentációk általános elméletébe, majd a szerző három eredményét mutatja be, melyeket ESQ reprezentációk tanulmányozása során ért el. A disszertáció kulcsfogalma a következő: egy modulust akkor nevezünk ESQ modulusnak, ha a külső négyzetének van a modulussal izomorf faktora. Ezt a reprezentációelméleti fogalmat a [8] cikkben vezették be először.

Az első fejezetben definiáljuk az ESQ reprezentációk fogalmát, tisztázzuk a jelölésbeli konvenciókat és a disszertáció megértéséhez szükséges előismereteket. Ezután ismertetünk néhány ESQ reprezentációkkal kapcsolatos észrevételt, amelyek közvetlenül a definícióból adódnak. A fejezet további részében a [8] cikk alapján egy tisztán csoportelméleti kérdésen keresztül motiváljuk az ESQ reprezentációk vizsgálatát, és leírjuk a négydimenziós hú irreducibilis ESQ reprezentációkat, illetve az ötdimenziós hú irreducibilis ESQ reprezentációk közül a minimálisakat.

A második fejezetben, ami a szerző [27] cikkén alapul, az $F_{p,q}$ metaciklikus csoportok reprezentációinak ESQ tulajdonságát vizsgáljuk, ahol $p$ prím, $q | p - 1$, és $F_{p,q} = C_p \rtimes C_q$. Ezen csoportok vizsgálatát az ESQ reprezentációkra vonatkozó négy- és ötdimenziós klasszikációs eredmények motiválták. Bebizonyítjuk, hogy akkor és csak akkor létezik irreducibilis $F_{p,q}$ reprezentáció az ESQ tulajdonsággal, ha egy Fermat-tétel szerű egyenlet megoldható $F_p$-ben. Ezt az ekvivalenciát felhasználva bebizonyítjuk két aszimptotikus tételt irreducibilis $F_{p,q}$ reprezentációk ESQ tulajdonságával kapcsolatban.

A harmadik fejezetben, ami a szerző [26] cikkén alapul, az $S_n$ szimmetrikus csoport irreducibilis reprezentációinak ESQ tulajdonságát vizsgáljuk. Ismert, hogy az $S_n$ csoport irreducibilis reprezentációi megfelelőtethetők az $n$ szám partícióinak. Egy $\lambda \vdash n$ partíción kettő magasan nevezzünk, ha $\lambda = (n - k, k)$, ahol $1 \leq k \leq \frac{n}{2}$, kettő szélesnek nevezzünk, ha $\lambda = (l_1, l_2, \ldots, l_k)$, ahol $l_i \leq 2$, és kampónak nevezzünk, ha $\lambda = (n - k, 1^k)$, ahol $0 \leq k \leq n - 1$. Bebizonyítjuk, hogy a kettő magas partíciókhoz tartozó $S_n$ reprezentációk egyikének sincs ESQ tulajdonsága, és pontosan egy kettő széles partíciónhoz tartozó $S_n$ reprezentáció rendelkezik az ESQ tulajdonsággal (nevezetesen a $\lambda = (2, 1, 1)$ partíciónhoz tartozó). Ezeket az állításokat a Cauchy-Frobenius lema különböző permutációkarakterereken való többszörös alkalmazásával bizonyítjuk. Végül a kampópartíciónhoz tartozó $S_n$ reprezentációkat vizsgáljuk, és bebizonyítjuk, hogy az ESQ tulajdonság $k$ négyes osztási maradékától függ.

ADATLAP
a doktori értekezés nyilvánosságra hozatalához*

I. A doktori értekezés adatai

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MTMT-azonosító: 10063798  
A doktori értekezés címé és alcímé: The Investigation of ESQ Group Representations  
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A doktori iskola neve: Matematika Doktori Iskola  
A doktori iskolán belüli doktori program neve: Elméleti Matematika Doktori Program  
A témavezető neve és tudományos fokozata: Pálfy Péter Pál egyetemi tanár  
A témavezető munkahelye: Eötvös Loránd Tudományegyetem, Algebra és Szármelmélet Tanszék

II. Nyilatkozatok

1. A doktori értekezés szerzőjeként
   a) hozzájárulok, hogy a doktori fokozat megszerzését követően a doktori értekezésem és a tézisek nyilvánosságra kerüljenek az ELTE Digitális Intézményi Tudástárban. Felhatalmazom a Természettudományi kar Dékáni Hivatal Doktori, Habilitációs és Nemzetközi Ügyek Csoportjának ügyintézőjét, hogy az értekezést és a téziseket feltöltse az ELTE Digitális Intézményi Tudástárba, és ennek során kitöltse a feltöltéshez szükséges nyilatkozatokat.  
   b) kérem, hogy a mellékelt kérelemben részletezett szabadalmi, illetőleg oltalmi bejelentés közzétételeig a doktori értekezést ne bocsássák nyilvánosságra az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban;  
   c) kérem, hogy a nemzetbiztonsági okból minősített adatot tartalmazó doktori értekezést a minősítés (dátum)-ig tartó időtartama alatt ne bocsássák nyilvánosságra az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban;  
   d) kérem, hogy a mű kiadására vonatkozó mellékelt kiadó szerződésre tekintettel a doktori értekezést a könyv megjelenéséig ne bocsássák nyilvánosságra az Egyetemi Könyvtárban, és az ELTE Digitális Intézményi Tudástárban csak a könyv bibliográfiai adatait tegyék közzé. Ha a könyv a fokozatszerzést követően egy évig nem jelenik meg, hozzájárulok, hogy a doktori értekezésem és a tézisek nyilvánosságra kerüljenek az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban.

   2. A doktori értekezés szerzőjeként kijelentem, hogy
   a) az ELTE Digitális Intézményi Tudástárba feltöltendő doktori értekezés és a tézisek saját eredeti, önálló szellemi munkám és legjobb tudomásom szerint nem sértem vele senki szerzői jogait;  
   b) a doktori értekezés és a tézisek nyomtatott változatai és az elektronikus adathordozón benyújtott tartalmak (szöveg és ábrák) mindenben megegyeznek.

3. A doktori értekezés szerzőjeként hozzájárulok a doktori értekezés és a tézisek szövegének plágiumkereső adatbázisba helyezéséhez és plágiumellenőrző vizsgálatok lefuttatásához.


a doktori értekezés szerzőjének aláírása

*ELTE SZMSZ SZMR 12. sz. melléklet