Diophantine triples in a Lucas-Lehmer sequence

Krisztián Gueth

Lorand Eötvös University
Savaria Department of Mathematics
Károli Gáspár tér 4
9700 Szombathely
Hungary
guethk@gmail.com

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Abstract
In this paper, we define a Lucas-Lehmer type sequence denoted by \((L_n)_{n=0}^\infty\), and show that there are no integers \(0 < a < b < c\) such that \(ab + 1\), \(ac + 1\), and \(bc + 1\) all are terms of the sequence.

Keywords: Diophantine triples, Lucas-Lehmer sequences

MSC: Primary 11B39; Secondary 11D99

1. Introduction

A diophantine \(m\)-tuple consists of \(m\) distinct positive integers such that the product of any two of them is one less than a square of an integer. Diophantus found the first four, but rational numbers \(1/16, 33/16, 17/4, 105/16\) with this property. Fermat gave 1, 3, 8, 120 as the first integer quadruple. Hoggatt and Bergum [8] provided infinitely many diophantine quadruples by \(F_{2k}, F_{2k+2}, F_{2k+4}, 4F_{2k+1}F_{2k+2}F_{2k+3}\). The most outstanding result is due to Dujella [3], who proved that there are only finitely many quintuples. Recently He, Togbe, and Ziegler submitted a work which solved the longstanding problem of the non-existence of diophantine quintuples [7].

There are several variations of the basic problem, most of them replace the squares by a given infinite set of integers. For instance, Luca and Szalay studied the diophantine triples for the terms of binary recurrences. They proved that there
are no integers $0 < a < b < c$ such that $ab + 1$, $ac + 1$ and $bc + 1$ all are Fibonacci numbers (see [9]), further for the Lucas sequence there is only one such a triple: $a = 1, b = 2, c = 3$ (see [10]). Fuchs, Luca and Szalay [4] gave sufficient and necessary conditions to have infinitely many diophantine triples for a general second order sequence.

For ternary recurrences Fuchs et al. [5] justified that there exist only finitely many triples corresponding to Tribonacci sequence. This paper was generalized by Fuchs et al. [6]. Alp and Irmak were the first who investigated the existence of diophantine triples in a Lucas-Lehmer type sequence (see [2]). They showed that there are no diophantine triples for the so-called pellans sequence.

In this paper, we study another Lucas-Lehmer sequence and prove the non-existence of diophantine triples associated to it. Let $\left( L_n \right)_{n=0}^{\infty}$ be defined by the initial values $L_0 = 0$, $L_1 = 1$, $L_2 = 1$ and $L_3 = 3$, and by the recursive rule

$$L_n = 4L_{n-2} - L_{n-4}. \quad (1.1)$$

Our principal result is the following.

**Theorem 1.1.** There exist no integers $0 < a < b < c$ such that

$$ab + 1 = L_x, \quad ac + 1 = L_y, \quad bc + 1 = L_z \quad (1.2)$$

would hold for any positive integers $x$, $y$ and $z$.

### 2. Preliminaries

The associate sequence of $(L_n)$ is denoted by $(M_n)_{n=0}^{\infty}$, which according to the general theory of Lucas-Lehmer sequences satisfies $M_0 = 2$, $M_1 = 2$, $M_2 = 4$, $M_3 = 10$, and $M_n = 4M_{n-2} - M_{n-4}$. It is easy to see that $L_n$ is divisible by 4 if and only if $4 \mid n$, otherwise $L_n$ is odd. Using the recurrence relation (1.1), for negative subscripts $M_{-n} = (-1)^n M_n$ follows.

The zeros of the common characteristic polynomial $x^4 - 4x^2 + 1$ of $(L_n)$ and $(M_n)$ are $\omega = (\sqrt{3} + 1)/\sqrt{2}$, $\psi = (-\sqrt{3} + 1)/\sqrt{2}$, $-\omega$ and $-\psi$, further the initial values provide the explicit formulae

$$L_n = \frac{1 + \sqrt{2}}{4\sqrt{3}} \left( \omega^n - \psi^n \right) + \frac{1 - \sqrt{2}}{4\sqrt{3}} \left( (-\omega)^n - (-\psi)^n \right),$$

$$M_n = \frac{1 + \sqrt{2}}{2} \left( \omega^n + \psi^n \right) + \frac{1 - \sqrt{2}}{2} \left( (-\omega)^n + (-\psi)^n \right). \quad (2.1)$$

It’s trivial from the recursive rules of both $(L_n)$ and $(M_n)$ that the subsequences of terms with even resp. odd indices form second order sequences by the same coefficients. The zeros of their companion polynomial are $\alpha = \omega^2 = 2 + \sqrt{3}$ and $\beta = \psi^2 = 2 - \sqrt{3}$, and the dominant root is $\alpha$.

Generally the Lucas-Lehmer sequences are union of two binary recursive sequences. Many properties, which are well known for binary sequences with initial
values 0 and 1, hold for Lucas-Lehmer sequences too (may be by a little modification). So the research of Lucas-Lehmer sequences is a new feature in the investigations.

In the sequel, we prove a few lemma which will be useful in proving the main theorem.

**Lemma 2.1.** If \( n = mt \) and \( t \) is odd, then \( M_m | M_n \).

**Proof.** The statement is obvious for \( t = 1 \). Formula (2.1) admits

\[
M_{6k} = M_{2k}(M_{4k} - 1), \quad \text{(2.2)}
\]

\[
M_{6k+3} = M_{2k+1}(M_{4k+2} + 1), \quad \text{(2.3)}
\]

which proves the lemma for \( t = 3 \). It can be seen by induction on \( k \) that

\[
M_{n+k} = \begin{cases} 
\frac{1}{2}M_nM_k + M_{n-k}, & \text{if } n \equiv k \equiv 1 \pmod{2}, \\
M_nM_k - (-1)^kM_{n-k}, & \text{otherwise.} 
\end{cases} \quad \text{(2.4)}
\]

Finally, using (2.4), we can prove the lemma by induction on \( t \).

**Lemma 2.2.** If \( n = mt \) and \( t \) is even, then \( \gcd(M_n, M_m) = 2 \).

**Proof.** Put \( m = 2k \). From (2.1) it follows that

\[
M_{4k} = M_{2k}^2 - 2. \quad \text{(2.5)}
\]

Subsequently, \( \gcd(M_{2k}, M_{4k}) = 2 \). It can be seen that \( M_{2^l k} \) (\( l \geq 3 \)) can be expressed as a polynomial of \( M_{2k} \), where the constant term is always 2. Thus \( \gcd(M_{2k}, M_{2^l k}) = 2 \) (\( l \geq 2 \)).

Now let \( m = 2k + 1 \). Again by (2.1) we see that

\[
M_{4k+2} = M_{2k+1}^2/2 + 2 \quad \text{(2.6)}
\]

holds. Putting \( H_{2k+1} = M_{2k+1}^2/2 \), it is trivial that \( H_{2k+1} \) and \( M_{2k+1} \) are divisible by the same primes, and the exponent of 2 is 1 in both integers. So \( \gcd(H_{2k+1}, N) = 2 \) and \( \gcd(M_{2k+1}, N) = 2 \) are equivalent for an arbitrary integer \( N \). Hence we have \( M_{4k+2} = H_{2k+1} + 2 \), and it implies \( \gcd(M_{4k+2}, H_{2k+1}) = 2 \). By induction and (2.5) we can see that \( M_{2^l (2k+1)} \) can be written as a polynomial of \( H_{2k+1} \) for any positive integer \( l \), with constant term 2. Consequently, \( \gcd(M_{2k+1}, M_{2^l (2k+1)}) = \gcd(H_{2k+1}, M_{2^l (2k+1)}) = 2 \). Together with Lemma 2.1, it shows immediately, that \( \gcd(M_m, M_{tm}) = 2 \) for arbitrary even \( t \).

**Lemma 2.3.** For any \( n \geq 0 \) we have

\[
L_n - 1 = \begin{cases} 
L_{n-1}M_{n+1}, & \text{if } n \equiv 1 \pmod{4}, \\
L_{n+1}M_{n-1}, & \text{if } n \equiv 3 \pmod{4}, \\
\frac{1}{2}L_{n+2}M_{n-2}, & \text{if } n \equiv 0 \pmod{4}, \\
L_{n-2}M_{n+2}, & \text{if } n \equiv 2 \pmod{4}. 
\end{cases} \quad \text{(2.7)}
\]
Proof. To prove the statement one can use the explicit formulae for the terms appearing in (2.7).

Lemma 2.4. The greatest common divisors of the terms of \((L_n)\) and \((M_n)\) satisfy

1. \(\gcd(L_m, L_n) = L_{\gcd(m,n)}\);

2. \(\gcd(M_m, M_n) = \begin{cases} M_{\gcd(m,n)}, & \text{if } \frac{m}{\gcd(m,n)} \equiv 1 \equiv \frac{n}{\gcd(m,n)} \pmod{2}, \\ 2, & \text{otherwise}; \end{cases}\)

3. \(\gcd(L_m, M_n) = \begin{cases} \mu M_{\gcd(m,n)}, & \text{if } \frac{m}{\gcd(m,n)} + 1 \equiv 1 \equiv \frac{n}{\gcd(m,n)} \pmod{2}, \\ 1 \text{ or } 2, & \text{otherwise}, \end{cases}\)

where \(\mu = 1 \text{ or } 1/2.\)

Proof. We omit the proof of the first statement, the easiest part, and start by proving the second one. The main tool is a Euclidean-like algorithm. Assume that \(m = nq + r\), where \(q\) is an odd integer, and \(0 \leq r < 2n\). By (2.4) we have

\[M_m = \mu M_{nq}M_r \pm M_{nq-r}.\]

The terms of \((M_n)\) is even, so \(\mu M_r\) is an integer. Let \(d\) be an integer which divides both \(M_m\) and \(M_n\). Since \(q\) is odd, \(d\) divides \(M_{nq}\), too. Thus \(d \mid M_{nq-r}\) holds. On the other hand, if \(d \mid M_n\) and \(d \mid M_{nq-r}\), then similarly \(d\) divides \(M_m\). Hence \(\gcd(M_m, M_n) = \gcd(M_n, M_{nq-r})\).

Suppose now \(m > n\) and \(n \nmid m\). After the first Euclidean-like division by \(n\), replace \(m\) by \(nq - r\), and continue with this, while the subscript is larger than \(n\). After the last step, \(nq - r\) might be negative. It is obvious that after two steps \(m\) is decreased by \(4n\). The last term of the sequence coming from these steps depends on the residue of the initial value of \(m\) modulo \(4n\). Let \(r_1 \equiv m \pmod{4n}\), \(r_2 \equiv m \pmod{4n}\), and \(0 < r_1 < n\), \(0 < r_2 < 4n\). In particular, for the last subscript \(r'\) we found

\[r' = \begin{cases} r_1, & \text{if } 0 < r < 2n, \\ n - r_1, & \text{if } n < r < 2n, \\ -r_1, & \text{if } 2n < r < 3n, \\ r_1 - n, & \text{if } 3n < r < 4n. \end{cases}\]

Obviously, \(\gcd(n, r_1) = \gcd(n, r')\) and \(0 < |r'| < n\), further if \(d_1 \mid m\) and \(d_1 \mid n\), then \(d_1 \mid nq - r\). Moreover if \(d_1\) divides both \(n\) and \(nq - r\), then it must divide \(r\) and \(m = nq + r\). This shows that \(\gcd(m, n) = \gcd(nq - r, m)\). Thus \(\gcd(m, n) = \gcd(r', n)\). Then apply this approach successively (replace the initial values of \(m\) by \(n\), and \(n\) by \(|r'|\), and continue), and finish when the remainder is zero. The last nonzero remainder is the \(\gcd\).

To complete the proof of the second case, suppose that \(\gcd(m, n) = 1\). By the last division \(n = 1\) follows, and denote the value of \(m\) by \(m_1\). The parities of \(m = nq + r\) and \(nq - r\) coincide in each step. If both \(m\) and \(n\) are odd, then the values of \(nq - r, r'\) are odd, hence so is \(m_1\). If \(m\) is even and \(n\) is odd, then \(r'\) is
even, and then the next division-sequence begins with odd \( m \) and even \( n \). By the last division (where \( n = 1 \)) it follows that \( m_1 \) must be even. Similarly, if the initial value of \( m \) is odd and \( n \) is even, then \( m_1 \) is even, too.

Put \( d_2 = \gcd(m, n) \). It occurs if we multiply all the terms in the last paragraph by \( d_2 \). If both \( m/d_2 \) and \( n/d_2 \) are odd, then the quotient in the last division (that is \( m_1 \)) is odd, and by the algorithm and Lemma 2.1, we have \( \gcd(M_m, M_n) = \gcd(M_{m_1}, d_2, M_{d_2}) = M_{d_2} \). If exactly one of \( m/d_2 \) and \( n/d_2 \) is even, then the last quotient \( (m_1) \) is even, and \( \gcd(M_m, M_n) = \gcd(M_{m_1}, d_2, M_{d_2}) = 2 \) follows by Lemma 2.2.

Now prove the third statement. The explicite formulae provide

\[
2\mu L_{m+n} = L_n M_m + L_m M_n, \tag{2.8}
\]
\[
2\mu M_{m+n} = 12L_n L_m + M_n M_m, \tag{2.9}
\]

where \( \mu = 2 \) if both \( m \) and \( n \) are odd, and \( \mu = 1 \) otherwise.

First we show that \( \gcd(L_k, M_k) = 2 \) if \( 4 \mid k \), and \( \gcd(L_k, M_k) = 1 \) otherwise. It is clear for \( k = 1, 2, 3, 4 \). From (2.8) and (2.9) we obtain

\[
L_{k+4} = \frac{1}{2}(L_k M_4 + L_4 M_k) = 7L_k + 2M_k,
\]
\[
M_{k+4} = \frac{1}{2}(12L_k L_4 + M_k M_4) = 24L_k + 7M_k.
\]

By the Euclidean algorithm we have

\[
\gcd(L_{k+4}, M_{k+4}) = \gcd(7L_k + 2M_k, 24L_k + 7M_k)
\]
\[
= \gcd(7L_k + 2M_k, 3L_k + M_k)
\]
\[
= \gcd(L_k, 3L_k + M_k) = \gcd(L_k, M_k).
\]

An induction implies the assertion for every \( k \).

Now we show \( \gcd(M_{kn}, L_n) = 1 \) or 2, again by induction for \( k \). We have just seen that it is true for \( k = 1 \). Now (2.9) implies

\[
2\mu M_{kn+n} = 12L_{kn} L_n + M_{kn} M_n.
\]

Let \( d \) be an odd integer such that \( d \mid M_{kn+n} \) and \( d \mid L_n \). In this case \( d \mid L_{kn} \), and we have shown that \( \gcd(L_{kn}, M_{kn}) \leq 2 \), so \( d \) is relatively prime to \( M_{kn} \). Thus \( d \mid M_n \). Further \( \gcd(L_n, M_n) \leq 2 \), and \( d \) is odd, so \( d = 1 \). If \( n \) is not divisible by 4, then \( L_n \) is odd, and \( \gcd(M_{kn+n}, L_n) \) is necessarily 1. If \( 4 \mid n \), then \( M_{kn+n} \) is not divisible by 4, but \( L_{kn+n} \) is even, so \( \gcd(M_{kn+n}, L_n) = 2 \).

We will show that if \( k \) is odd, then \( \gcd(M_n, L_{kn}) = 1 \) or 2. Clearly, it is true for \( k = 1 \). Suppose now that it holds for an odd \( k \), and check it for \( k+2 \). It follows from (2.8) that

\[
2\mu L_{kn+2n} = L_{kn} M_{2n} + M_{kn} L_{2n}.
\]

Let \( d \) an odd integer which divides both \( L_{kn+2n} \) and \( M_n \). Then \( d \mid M_{kn} \) holds since \( k \) is odd. But \( d \) is relatively prime to \( M_{2n} \), so \( d \) must divide \( L_{kn} \). We know
that \( \gcd(L_{kn}, M_{kn}) \leq 2 \), henceforward \( d = 1 \). If \( 4 \nmid n \), then odd \( k \) entails odd \( L_{(k+2)n} \), and if \( 4 \mid n \), then \( 4 \nmid M_n \). Hence \( \gcd(M_n, L_{kn+2n}) = 1 \) or 2.

Assuming \( k \) is even, put \( k = 2^l t \), where \( t \) is odd. Then \( M_n \) divides \( M_{tn} \), and we have \( L_{2tn} = \mu L_{tn} M_{tn} \), where \( \mu \) is 1 or 1/2. So \( M_{tn}/2 \mid L_{2tn} \), and by induction, \( M_{tn}/2 \) divides \( L_{2tn} \). Subsequently, \( \gcd(M_n, L_{kn}) = M_n \) or \( M_n/2 \) for even \( k \).

Thus the third statement is proven if one of \( n \) and \( m \) divides the other. For general \( m \) and \( n \), suppose \( m > n \), and let \( m = nq + r \), where \( q \) is odd, \( 0 < r < 2n \). From (2.8), \( 2\mu L_q M_q = L_{nq} M_q + M_{nq} L_r \) follows. It is easy to see that for any odd \( d \) the conditions \((d \mid L_m \text{ and } d \mid M_n)\), and \((d \mid M_q \text{ and } d \mid M_r)\) are equivalent (for odd \( q \) use that \( M_n \) divides \( n_q \) and \( \gcd(M_{nq}, L_{nq}) \) is 1 or 2). So it is enough to determine the greatest odd common divisor of \( M_n \) and \( M_r \), for which we use the second part of this lemma.

Trivially, \( \gcd(n, r) = \gcd(n, m) \). Denote this value by \( c \). If \( m/c \) is even and \( n/c \) is odd, then (because \( q \) is odd) \( r/c \) is odd (say this is case A). By the lemma, \( \gcd(M_n, M_r) = \gcd(n, r) \). If \( m/c \) is odd and \( n/c \) is even, then \( r/c \) is odd. If both \( m/c \) and \( n/c \) are odd, then \( r/c \) is even. In these two cases (we call them case B) \( \gcd(M_n, M_r) = 2 \) hold.

Clearly, \( M_n \) is not divisible by 8, moreover \( L_m \) and \( M_n \) are both divisible by 4 if and only if \( 4 \mid m \) and \( n \equiv 2 \pmod{4} \). In this case the exponent of 2 in \( \gcd(n, m) \) is 1, \( m/c \) is even, and \( n/c \) is odd (this is case A), and \( M_{\gcd(n,m)} \) is divisible by 4. It is easy to see that \( \gcd(L_m, M_n) = M_{\gcd(n,m)} \). In the remaining situations of case A, \( M_{\gcd(m,n)} \) is not divisible by 4. Thus \( \gcd(L_m, M_n) = M_{\gcd(m,n)} \) or one half of it. In case B, 4 does not divide \( L_m \) and \( M_n \) at the same time, so their gcd is 1 or 2.

If \( m < n \), then \( n = mp + r \). Now \( p \) is not necessarily odd, therefore we can suppose \( 0 < r < m \). Then from (2.9) we conclude \( \gcd(L_m, M_n) = \gcd(L_m, M_r) \). To complete the proof we must use the previous case of this lemma.

The next lemma gives lower and upper bounds on the terms of \( (L_n) \) and \( (M_n) \) by powers of dominant root \( \alpha \).

**Lemma 2.5.** Suppose \( n \geq 3 \). We have

\[
\alpha^{n-0.944} < L_{2n} < \alpha^{n-0.943}, \quad \alpha^{n-0.181} < L_{2n+1} < \alpha^{n-0.180},
\]

\[
\alpha^n < M_{2n} < \alpha^{n+0.001}, \quad \alpha^{n+0.763} < M_{2n+1} < \alpha^{n+0.764}.
\]

Further, independently from the parity of the subscript \( k \),

\[
\alpha^{k/2-0.944} < L_k < \alpha^{k/2-0.680} \quad \text{and} \quad \alpha^{k/2} < M_k < \alpha^{k/2+0.264}
\]

hold.

**Proof.** Let \( n_0 \) be a positive integer, and assume \( n \geq n_0 \). The explicit formula (2.1) simplifies \( L_{2n} = (\alpha^n - \beta^n)/(\alpha - \beta) \), which yields

\[
L_{2n} \geq \frac{\alpha^n - \beta^{n_0}}{\alpha - \beta} = \alpha^n \frac{1 - (\beta/\alpha)^{n_0} \alpha^{n-n_0}}{\alpha - \beta} \geq \alpha^n \frac{1 - (\beta/\alpha)^{n_0}}{\alpha - \beta}.
\]
Supposing $n_0 \geq 3$, together with $0 < \beta/\alpha < 1$ it leads to

$$\frac{1 - (\frac{\beta}{\alpha})^{n_0}}{\alpha - \beta} \geq \frac{1 - (\frac{\beta}{\alpha})^{3}}{\alpha - \beta} = 0.28856 \ldots > \alpha^{-0.944}.$$ 

Thus $L_{2n} > \alpha^{n-0.944}$. To get an upper bound is easier, since $\beta > 0$ implies

$$L_{2n} = \frac{\alpha^n - \beta^n}{\alpha - \beta} < \frac{\alpha^n}{\alpha - \beta} = \alpha^n \frac{1}{2\sqrt{3}} < \alpha^{n-0.943}.$$ 

For odd subscripts a similar treatment is available by

$$L_{2n+1} = \frac{1}{\alpha - \beta} \left[ (\sqrt{3} + 1)\alpha^n + (\sqrt{3} - 1)\beta^n \right].$$

First we see

$$L_{2n+1} > \frac{1 + \sqrt{3}}{2\sqrt{3}} \alpha^n > \alpha^{n-0.181}.$$ 

Now assume $n \geq n_0 \geq 3$. Consequently,

$$L_{2n+1} \leq \frac{1}{\alpha - \beta} \left[ (\sqrt{3} + 1)\alpha^n + (\sqrt{3} - 1)\beta^{n_0} \right]
= \alpha^n \left[ \frac{\sqrt{3} + 1}{2\sqrt{3}} + \frac{\sqrt{3} - 1}{2\sqrt{3}} \left( \frac{\beta}{\alpha} \right)^{n_0} \right]^{n_0 - n}
\leq \alpha^n \left[ \frac{\sqrt{3} + 1}{2\sqrt{3}} + \frac{\sqrt{3} - 1}{2\sqrt{3}} \left( \frac{\beta}{\alpha} \right)^{3} \right] = \alpha^n \cdot 0.788753 \ldots < \alpha^{n-0.180}.$$

The bounds for the terms $M_n$ can be shown by an analogous way. 

**Lemma 2.6.** Suppose that $a$, $b$, $z$, and the fractions appearing below are integers. Then

1. if $3a \neq b$, then $\gcd(\frac{z+a}{2}, \frac{3z+b}{8}) \leq \left| \frac{3a-b}{2} \right|$, 
2. if $2a \neq b$, then $\gcd(\frac{z+a}{2}, \frac{2z+b}{6}) \leq \left| \frac{2a-b}{2} \right|$, 
3. if $a \neq b$, then $\gcd(\frac{z+a}{2}, \frac{z+b}{4}) \leq \left| \frac{a-b}{2} \right|$.

**Proof.** The statements follow by a simple use of the Euclidean algorithm.

**Lemma 2.7.** Supposing $z \geq 4$, the following properties are valid.

1. If $z \equiv 1 \pmod{4}$, then $M_{\frac{z+1}{2}}^2 < 2L_z$, further $3L_{\frac{z+1}{2}}^2 < 2L_z$.
2. If $z \equiv 3 \pmod{4}$, then $M_{\frac{z+1}{2}}^2 < 4L_z$.
3. If $z \equiv 2 \pmod{4}$, then $M_{\frac{z+1}{2}}^2 < 2L_z$. 
4. If \( z \equiv 0 \pmod{4} \), then \( M_{z/2}^2 < 4L_z \).

**Proof.** Use (2.5), (2.6), and

\[
M_n = \begin{cases} 
L_{n-1} + L_{n+1}, & \text{if } n \text{ is even}, \\
2(L_{n-1} + L_{n+1}), & \text{if } n \text{ is odd}.
\end{cases}
\]  

(2.10)

Here (2.10) can be proven by induction. \( \Box \)

**Lemma 2.8.** Suppose that \( a \) and \( b \) are positive real numbers and \( u_0 \) is a positive integer. Let \( \kappa = \log_\alpha (a + \frac{b}{\alpha^{u_0}}) \). If \( u \geq u_0 \), then

\[
a^\alpha^u + b \leq \alpha^u + \kappa.
\]

**Proof.** This is obvious by an easy calculation. \( \Box \)

3. Proof of Theorem 1.1

The conditions \( 1 \leq a < b < c \) entail \( 3 \leq x < y < z \). Obviously, \( c \mid L_y - 1 \) and \( c \mid L_z - 1 \). Thus \( c \leq \gcd(L_y - 1, L_z - 1) \). Clearly, \( L_z = bc + 1 < c^2 \), which implies \( \sqrt{L_z} < c \). Combining this with Lemma 2.5, we see

\[
\alpha^{\frac{z}{4} - 0.472} = \alpha^{\frac{y}{2} - 0.944} < \sqrt{L_z} < c < \alpha^{\frac{y}{2} - 0.680},
\]

and then \( z/4 - 0.472 < y/2 - 0.680 \) yields \( z < 2y - 0.832 \). Hence \( z \leq 2y - 1 \).

Now we distinguish two cases.

**Case I:** \( z \geq 117 \).

The key point of this case is to estimate \( G = \gcd(L_y - 1, L_z - 1) \). Assume that \( i, j \in \{\pm 1, \pm 2\} \), and \( \mu_i, \mu_j \in \{1, 1/2\} \). By Lemma 2.3,

\[
G = \gcd(\mu_i^* L_{y-2} M_{y+4}, \mu_j^* L_{z-2} M_{z+4})
\]

\[
\leq \gcd(L_{y-2} M_{y+4}, L_{z-2} M_{z+4})
\]

\[
\leq \gcd(L_{y-2} L_{z-2}) \gcd(L_{y+4}, M_{z+4}) \gcd(M_{y+4}, L_{z+4}) \gcd(M_{y+4}, M_{z+4}).
\]

Let \( Q \) denote the last product. By Lemma 2.4

\[
Q \leq L_{\gcd(y-2, z-2)} M_{\gcd(y+4, z+4)} M_{\gcd(y+4, z+4)} M_{\gcd(y+4, z+4)}
\]

follows. We define \( d_1, d_2, d_3, d_4 \) according to the relations

\[
gcd\left(\frac{y-i}{2}, \frac{z-j}{2}\right) = \frac{z-j}{2d_1}, \quad \gcd\left(\frac{y-i}{2}, \frac{z+j}{2}\right) = \frac{z+j}{2d_2},
\]

\[
gcd\left(\frac{y+i}{2}, \frac{z-j}{2}\right) = \frac{z-j}{2d_3}, \quad \gcd\left(\frac{y+i}{2}, \frac{z+j}{2}\right) = \frac{z+j}{2d_4}.
\]
Let $d = \min\{d_1, d_2, d_3, d_4\}$.

First suppose $d \geq 5$. Now Lemma 2.5, together with $|i|, |j| \leq 2$ implies
\[
\alpha^{\frac{z}{2Q}} - 0.472 < Q \leq L_{\frac{z-j}{2m}} M_{\frac{z+i}{2m}} M_{\frac{z+j}{2m}} \leq L_{\frac{z-j}{2m}} M_{\frac{z+i}{2m}} M_{\frac{z+j}{2m}} < \alpha^{\frac{z+i}{2Q}} - 0.680 \left(\alpha^{\frac{z+i}{2Q}} + 0.264\right)^3 = \alpha^{\frac{z+i}{2Q}} + 0.112.
\]

But $z/4 - 0.472 < (z + 2)/5 + 0.112$ contradicting $z \geq 117$.

Now let $d = 4$, that is one of $d_1, d_2, d_3, d_4$ equals 4. Assume that $\eta_1, \eta_2 \in \{\pm 1\}$. Then $|\eta_1 j|, |\eta_2 i| \leq 2$, and we can assume $z + \eta_1 j \geq y + \eta_2 i$. Contrary, if it does not hold, then by the definition of $d$ the inequality $5/4(z - 2) \leq y + 2$ is true, which together with $z \geq 14$ implies $5z \leq 4y + 18 < 5y + 18$. So $z < 18$, which is not the case. Now we have only two possibilities:
\[
z + \eta_1 j = \frac{y + \eta_2 i}{2} \quad \text{or} \quad z + \eta_1 j = \frac{y + \eta_2 i}{6}.
\]

In the first case we have $z = 4y + (4\eta_2 i - \eta_1 j) \geq 4y - 10$, and by $z \leq 2y - 1$ we get $4y - 10 \leq 2y - 1$, which implies $y \leq 4$, and then $z \leq 7$, a contradiction.

In the second case let $\eta_1', \eta_2' \in \{\pm 1\}$, such that $(\eta_1', \eta_2') \neq (\eta_1, \eta_2)$. Clearly,
\[
y = \frac{3z + 3\eta_1 j - 4\eta_2 i}{4}, \quad \text{and} \quad \frac{y + \eta_1' j}{2} = \frac{3z + 3\eta_1 j + 4(\eta_2' - \eta_2)i}{8}.
\]

Put $t = 4(\eta_2' - \eta_2)$. Thus $t = 0$ or $\pm 8$. Applying the first assertion of Lemma 2.6 with $a = \eta_1 j$ and $b = 3\eta_1 j + ti$, it gives
\[
\gcd\left(\frac{z + \eta_1 j}{2}, \frac{y + \eta_2 i}{2}\right) = \gcd\left(\frac{z + \eta_1' j}{2}, \frac{3z + 3\eta_1 j + ti}{8}\right) \leq \left|\frac{3\eta_1' j - 3\eta_1 j - ti}{2}\right|,
\]

which does not exceed 14. This conclusion is correct if $3a - b \neq 0$, that is if $3\eta_1 - 3\eta_1 j - ti \neq 0$. If $3a - b = 0$, then $3 \mid t$, and then $t = 0$. Thus $\eta_1$ must be equal to $\eta_1$, so $(\eta_1, \eta_2) = (\eta_1, \eta_2)$, which has been excluded. Subsequently, three of the four factors of $Q$ is at most $M_{14} (M_n \geq L_n$ for any index $n)$ and the fourth factor is $L_{\frac{z+j}{4}}$ or $M_{\frac{z+j}{4}}$, none of them exceeding $M_{\frac{z+j}{4}}$. So
\[
Q \leq M_{14}^3 M_{\frac{z+j}{8}} = 10084^3 M_{\frac{z+j}{8}},
\]

and then, by Lemma 2.5, we have
\[
\alpha^{\frac{z}{2Q}} - 0.472 < Q < \alpha^{21.003} \alpha^{\frac{z+j}{2Q}} + 0.264.
\]

Now we conclude $z < 116.7$, and it is a contradiction with $z \geq 117$.

Suppose $d = 3$. We have the two possibilities
\[
z + \eta_1 j = \frac{y + \eta_2 i}{2} \quad \text{and} \quad z + \eta_1 j = \frac{y + \eta_2 i}{4}.
\]
In the first case \(2y - 1 \geq z = 3(y + \eta_2i) - \eta_1j \geq 3y - 8\) implies \(y \leq 7\), and then \(z \leq 13\), which is impossible.

In the second case we repeat the treatment of case \(d = 4\), the variables \(\eta_1'\) and \(\eta_2'\) satisfy the same conditions. Now \(y = (2z + 2\eta_1j - 3\eta_2i)/3\) provides
\[
\frac{y + \eta_2'i}{2} = \frac{2z + 2\eta_1j - 3\eta_2i + 3\eta_2'i}{6} = \frac{2z + 2\eta_1j + 3(\eta_2' - \eta_2)i}{6}.
\]
Let be \(t = 3(\eta_2' - \eta_2)\), with value 0 or \(\pm 6\). Use the second assertion of Lemma 2.6 with \(a = \eta_1'j\), \(b = 2\eta_1j + ti\). If \(2a - b \neq 0\) then
\[
\gcd\left(\frac{z + \eta_1j}{2}, \frac{y + \eta_2'i}{2}\right) = \gcd\left(\frac{z + \eta_1j}{2}, \frac{2z + 2\eta_1j + ti}{6}\right) \leq \frac{|\eta_1'j - 2\eta_1j - ti|}{2},
\]
which is less than or equal to 10. If \(2a - b = 0\), that is if \(2\eta_1'j - 2\eta_1j - ti = 0\), then \(3 | t\) and \(j \div t\) show \(3 | \eta_1' - \eta_1\), which can hold only if \(\eta_1' = \eta_1\). But in this case \(t\) must be zero, too. So \((\eta_1, \eta_2) = (\eta_1, \eta_2)\), which is not allowed. We have
\[
\alpha^{\frac{z}{4} - 0.472} < Q \leq M_1^2 M_3^2 < 724^3 \alpha^{\frac{z}{4} + 0.264}
\]
by using Lemma 2.5. This implies \(z < 96\), again a contradiction.

Now suppose \(d = 2\). The only possibility is
\[
\frac{z + \eta_1j}{4} = \frac{y + \eta_2'i}{2}.
\]
(\(\eta_1'\) and \(\eta_2'\) are the same as in the previous cases.) It leads to \(y = (z + \eta_1j - 2\eta_2i)/2\), and then to
\[
\frac{y + \eta_2'i}{2} = \frac{z + \eta_1j - 2\eta_2i + 2\eta_2'i}{4} = \frac{z + \eta_1j + ti}{4},
\]
where \(t = 2(\eta_2' - \eta_2) \in \{0, \pm 4\}\). Let \(a = \eta_1'j\), \(b = \eta_1j + ti\). If \(a \neq b\), then by the third assertion of Lemma 2.6 we have
\[
\gcd\left(\frac{z + \eta_1j}{2}, \frac{y + \eta_2'i}{2}\right) = \gcd\left(\frac{z + \eta_1j}{2}, \frac{z + \eta_1j + ti}{4}\right) \leq \frac{|\eta_1'j - \eta_1j - ti|}{2} \leq 6.
\]
Thus
\[
\alpha^{\frac{z}{4} - 0.472} < Q \leq M_6^3 M_{\frac{z}{4} + 2} < \alpha^{0.003} \alpha^{\frac{z}{4} + 0.264},
\]
and we arrived at a contradiction via \(z < 80\). If \(a - b = 0\), then \((\eta_1' - \eta_1)j = ti\).

Now, if \(j = \pm 1\), then (because \(t\) is divisible by 4) \(4 | \eta_1' - \eta_1\) must hold. This occurs only if \(\eta_1' = \eta_1\), hence \(t = 0\), so \(\eta_2' = \eta_2\), which has been excluded. Thus we may suppose \(j = \pm 2\) and \(\eta_1' \neq \eta_1\). In this case \(\eta_1' - \eta_1 = \pm 2\), and \(i = \pm 1\). The factors of \(Q\) belong to \((-\eta_1, \eta_2)\) and \((\eta_1, -\eta_2)\) can be estimated by \(M_6\). If \((\eta_1, \eta_2) = (1, 1)\), then this factor is \(\gcd(M_{\frac{z}{4} + 1}, M_{\frac{z}{4} + 1})\), which is 2 via \((z + j)/4 = (y + i)/2\) and
Lemma 2.4. If \((\eta_1, \eta_2) = (1, -1)\), then similarly \(\gcd(L_{y-1}, M_{z+1}) \leq 2\). In these two cases we have
\[
\alpha^{\frac{4}{3} - 0.472} < Q \leq 2M_6^2 M_{z+2} < \alpha^{6.527} \alpha^{\frac{z+2}{8} + 0.264},
\]
and then \(z \leq 60\), a contradiction.

Let \((\eta_1, \eta_2) = (-1, -1)\) or \((-1, 1)\). From \((z + \eta_1 j)/4 = (y + \eta_2 i)/2\) and \(|j| = 2\), \(|i| = 1\), it is easy to see that \((z + \eta_1 j)/2 = 2(y + \eta_2 i)/2\) or \((z + \eta_1 j)/2 = 2(y - \eta_2 i)/2 \pm 4\). If the first case holds, then \(\gcd((z - \eta_1 j)/2, (y - \eta_2 i)/2) = (z - \eta_1 j)/4\). Further if \((\eta_1, \eta_2) = (-1, -1)\), then the factor of \(Q\) belonging to \((\eta_1, \eta_2)\) is \(\gcd(M_{y+1}, M_{z+2}) = 2\) (by Lemma 2.4). If \((\eta_1, \eta_2) = (-1, 1)\), then the factor \(\gcd(L_{y-1}, M_{z+1}) = 1\) or 2. If \((z - \eta_1 j)/2 = 2(y - \eta_2 i)/2 \pm 4\) holds, it can be seen by the Euclidean algorithm that \(\gcd((z - \eta_1 j)/2, (y - \eta_2 i)/2) \leq 4\), and the factor of \(Q\) is at most \(M_{4} = 14\). So in these cases we conclude
\[
\alpha^{\frac{4}{3} - 0.472} < Q \leq M_4 M_6^2 M_{z+2} < \alpha^{8.005} \alpha^{\frac{z+2}{8} + 0.264},
\]
and this implies \(z < 72\).

Assume \(d = 1\). Now
\[
\frac{z + \eta_1 j}{2} = \frac{y + \eta_2 i}{2},
\]
where \(\eta_1, \eta_2 = \pm 1\), and it reduces to \(z \pm j = y \pm i\) with \(i, j \in \{\pm 1, \pm 2\}\). According to Lemma 2.3 the values depend on the residue \(y\) and \(z\) modulo 4. Altogether, it means that we need to verify 16 cases.

1. \(y \equiv z \equiv 1 \pmod{4}\). Clearly, now \(i = j = 1\), so \(z \pm 1 = y \pm 1\). The condition \(y \equiv z \pmod{4}\) leads immediately to \(y = z\), a contradiction.

2. \(y \equiv 1, z \equiv 2 \pmod{4}\). Now \(i = 1, j = 2\). Thus \(z \pm 2 = y \pm 1\), and then \(z = y \pm 3\) or \(z = y \pm 1\). Considering them modulo 4, the only possibility is \(z = y + 1\). By Lemma 2.3, we conclude
\[
L_y - 1 = L_{y-1} M_{y+1} = L_{z-2} M_{z}, \quad \text{and} \quad L_z - 1 = L_{z-2} M_{z+2}.
\]
The common factor \(L_{z+2}\) together with \(\gcd(M_{z}, M_{z+2}) = 2\) and by Lemma 2.5 provides a contradiction again, since
\[
\alpha^{\frac{4}{3} - 0.472} < \gcd(L_y - 1, L_z - 1) = 2L_{z+2} < \alpha^{0.527} \alpha^{\frac{z+2}{4} - 0.680} = \alpha^{\frac{4}{3} - 0.653}.
\]

3. \(y \equiv 1, z \equiv 3 \pmod{4}\). Here \(i = 1, j = -1\), and the only possibility is \(z = y + 2\). It follows that
\[
L_y - 1 = L_{y-1} M_{y+1} = L_{z-3} M_{z+1}, \quad L_z - 1 = L_{z+1} M_{z+1},
\]
where \(\gcd(L_{z+1}, L_{z-3}) = 1\). Now
\[
c | \gcd(L_y - 1, L_z - 1) = M_{z+1} = c_1 c > c_1 \sqrt{L_z}
\]
holds with an appropriate integer \( c_1 \). By Lemma 2.7, \( M_{z+1} < 2\sqrt{L_z} \). So we have \( c_1\sqrt{L_z} < M_{z+1} < 2\sqrt{L_z} \), which implies \( c_1 < 2 \), i.e. \( c_1 = 1 \). Thus \( c = M_{z+1} \), and we can see from the factorization of \( L_y - 1 \) and \( L_z - 1 \) that \( a = L_{z+1}^{-1} \), \( b = L_{z+1} \).Lemma 2.5 shows

\[
\alpha^{\frac{z}{4} - 0.680} > L_x = ab + 1 = L_{z+3} L_{z+1} + 1 > L_{z+3} L_{z+1} > \alpha^{\frac{z+1}{4} - 0.944} \alpha^{\frac{z+1}{4} - 0.944}.
\]

Clearly, \( x > z - 3.416 \), and then \( x \geq z - 3 \). In our case \( x < y = z - 2 \) holds, so \( x = z - 3 \). This implies \( L_{z-3} - 1 = L_x - 1 = L_{z+3} L_{z+1} \), which entails \( L_{z-3} \mid L_{z-3} - 1 \). Combining it with \( L_{z+3} \mid L_{z-3} \), we have \( L_{z+3} = 1 \), and \( z \) is too small.

4. \( y \equiv 1 \), \( z \equiv 0 \ (\text{mod} \ 4) \). In this case \( z = y + 3 \), and

\[
L_y - 1 = L_{z+1} M_{z+1} = L_{z+2} M_{z+2}, \quad L_z - 1 = \frac{1}{2} L_{z+2} M_{z+2}.
\]

The distance of the subscripts of the appropriate terms of \( (L_n) \) is 3, so \( \gcd(L_{z+2}, L_{z+2}^2) \leq \gcd(L_{z+4}, L_{z+2}^2) = 1 \) or 3. So \( \gcd(L_y - 1, L_z - 1) \mid 3M_{z+2} \). Therefore there exist a positive integer \( c_1 \) such that

\[
c \mid \gcd(L_y - 1, L_z - 1) \mid 3M_{z+2} = c_1 c > c_1\sqrt{L_z}.
\]

Lemma 2.7 implies \( M_{z+2} < 2\sqrt{L_z} \), and so \( 6\sqrt{L_z} > 3M_{z+2} > c_1\sqrt{L_z} \) hold. Thus \( c_1 < 6 \). Since \( L_{z+2} \) is odd, \( c_1 \mid M_{z+2} \) does not divide \( L_z - 1 \). So we have \( \gcd(L_y - 1, L_z - 1) = \lambda M_{z+2}/2 \), where \( \lambda = 1 \) or 3.

When \( \lambda = 1 \), \( c \) divides \( M_{z+2}/2 = 3M_{z+2}/6 \), which implies \( c \geq 6 \), a contradiction.

Assuming \( \lambda = 3 \), it yields \( c \mid 3M_{z+2}/2 \). Thus either \( c = 3M_{z+2}/2 \) (\( c_1 = 2 \)) or \( c = 3M_{z+2}/4 \) (\( c_1 = 4 \)) holds. We can exclude the second case, because \( (z - 2)/2 \) is odd, and so \( M_{z+2} \) is not divisible by 4. In the first case \( b = L_{z+2}/3 \) and \( a = 2L_{z+2}/3 \) follow from

\[
bc = L_z - 1 = \frac{1}{2} M_{z+2} L_{z+2} \quad \text{and} \quad ac = L_y - 1 = M_{z+2} L_{z+2},
\]

respectively.

Using the fact that \( L_{2k-2} L_{2k+1} + 1 = L_{2k-1} L_{2k} \) holds for every positive integer \( k \) (this comes from the explicit formula (2.1)), we can write

\[
L_x = ab + 1 = \frac{2}{9} L_{z+3} L_{z+2} + 1 = \frac{2}{9} (L_{z+2} L_{z} - 1) + 1 = \frac{2}{9} L_{z+2} L_{z} + \frac{7}{9}.
\]

By Lemma 2.5 we obtain

\[
\alpha^{\frac{z}{4} - 0.680} > L_x = \frac{2}{9} L_{z+2} L_{z} + \frac{7}{9} > \frac{2}{9} L_{z+2} L_{z} > \alpha^{-1.143} \alpha^{\frac{z+2}{4} - 0.681} \alpha^{\frac{z}{4} - 0.944}
\]

(since \( (z - 2)/2 \) is odd). It implies \( x > z - 5.176 \), so \( x \geq z - 5 \) holds.
We will reach the contradiction by showing $ab + 1 < L_{z-5}$. Knowing that $z$ is even, $L_{z-5} > \alpha^{z-5 - 0.681} = \alpha^{\frac{z}{2} - 3.181}$ follows from Lemma 2.5. Since

$$L_{z-2} L_{\frac{z}{2}} > \alpha^{\frac{z-4}{2} - 0.681} \alpha^{\frac{z}{2} - 0.944} = \alpha^{\frac{z}{2} - 2.125}$$

and $z \geq 16$, the exponent of $\alpha$ is at least 5.875. Applying Lemma 2.8 with $u_0 = 5$, we have $\kappa = \log_{\alpha}((2 + 7\alpha^{-5})/9) < -1.138$, and then

$$ab + 1 = \frac{2}{9} L_{\frac{z}{2}} L_{\frac{z}{2}} + \frac{7}{9} < \alpha^{-1.138} \alpha^{\frac{z-2}{2} - 0.68} \alpha^{\frac{z}{2} - 0.943} = \alpha^{\frac{z}{2} - 3.261}.$$

From these inequalities

$$L_{z-5} > \alpha^{\frac{z}{2} - 3.181} > \alpha^{\frac{z}{2} - 3.261} > ab + 1$$

follows, and the proof of this part is complete.

5. $y \equiv 2, \; z \equiv 1 \pmod{4}$. Now $z = y + 3$, further

$$L_y - 1 = L_{\frac{y}{2}} M_{\frac{y+2}{2}} = L_{\frac{z-5}{2}} M_{\frac{z-1}{2}}, \quad L_z - 1 = L_{\frac{z-1}{2}} M_{\frac{z+1}{2}}.$$

It is easy to see from Lemma 2.4 that $\gcd(L_{\frac{y}{2}}, L_{\frac{z-1}{2}}) = 1$, $\gcd(M_{\frac{y+2}{2}}, M_{\frac{z-1}{2}}) = 2$, $\gcd(L_{\frac{z-5}{2}}, M_{\frac{z+1}{2}}) \leq M_3 = 10$, $\gcd(M_{\frac{z-1}{2}}, L_{\frac{z-1}{2}}) \leq 2$. Consequently,

$$\alpha^{\frac{z}{2} - 0.472} < \gcd(L_y - 1, L_z - 1) \leq 40 < \alpha^{2.802},$$

and then $z < 14$, a contradiction again.

6. $y \equiv z \equiv 2 \pmod{4}$. In this case $i = j = 2$. Then $z = y + 4$ follows. The identities

$$L_y - 1 = L_{\frac{y}{2}} M_{\frac{y+2}{2}} = L_{\frac{z-6}{2}} M_{\frac{z-2}{2}}, \quad L_z - 1 = L_{\frac{z-2}{2}} M_{\frac{z+2}{2}}$$

and $\gcd(L_{\frac{z-6}{2}}, L_{\frac{z-2}{2}}) = 1$, $\gcd(M_{\frac{z-2}{2}}, M_{\frac{z+2}{2}}) = 2$ (because both terms cannot be divisible by 4), $\gcd(L_{\frac{z-6}{2}}, M_{\frac{z+2}{2}}) \leq M_4 = 14$, $\gcd(M_{\frac{z-2}{2}}, L_{\frac{z-2}{2}}) \leq 2$ (see Lemma 2.4) induce

$$\alpha^{\frac{z}{2} - 0.472} < \gcd(L_y - 1, L_z - 1) \leq 56 < \alpha^{3.057},$$

which gives $z < 15$.

7. $y \equiv 2, \; z \equiv 3 \pmod{4}$. Here $z = y + 1$, moreover we have

$$L_y - 1 = L_{\frac{y}{2}} M_{\frac{y+2}{2}} = L_{\frac{z-3}{2}} M_{\frac{z+1}{2}}, \quad L_z - 1 = L_{\frac{z+1}{2}} M_{\frac{z-1}{2}}.$$

Again by Lemma 2.4,

$$\gcd(L_{\frac{z-3}{2}}, L_{\frac{z+1}{2}}) = 1, \quad \gcd(M_{\frac{z+1}{2}}, M_{\frac{z-1}{2}}) = 2,$$

$$\gcd(L_{\frac{z-3}{2}}, M_{\frac{z+1}{2}}) \leq 2, \quad \gcd(M_{\frac{z+1}{2}}, L_{\frac{z+1}{2}}) \leq 2.$$
Thus

\[ \alpha^{\frac{z}{2} - 0.472} < \gcd(L_y - 1, L_z - 1) \leq 8 < \alpha^{1.579} \]

follows, which implies \( z < 9 \).

8. \( y \equiv 2, \ z \equiv 0 \pmod{4} \). Now \( i = 2, j = -2 \), and \( y \pm 2 = j \mp 2 \) cannot hold modulo 4.

9. \( y \equiv 3, \ z \equiv 1 \pmod{4} \). In this case the only possibility is \( z = y + 2 \). Obviously,

\[ L_y - 1 = L_{\frac{y+1}{2}} M_{\frac{y+1}{2}} = L_{\frac{z-1}{2}} M_{\frac{z-3}{2}}, \quad L_z - 1 = L_{\frac{z+1}{2}} M_{\frac{z+1}{2}} \]

hold. Beside the common factor, we get \( \gcd(M_{\frac{z-3}{2}}, M_{\frac{z+1}{2}}) = 2 \) (because the subscripts are odd). Hence \( \gcd(L_y - 1, L_z - 1) = 2L_{\frac{z-1}{2}} \), further we see

\[ c | \gcd(L_y - 1, L_z - 1) = 2L_{\frac{z-1}{2}} = c_1 c > c_1 \sqrt{L_z} \]

with an appropriate \( c_1 \). By the second assertion of case (1) in Lemma 2.7, \( \sqrt{L_z} > \sqrt{3/2L_{\frac{z-1}{2}}} \), subsequently

\[ 2L_{\frac{z-1}{2}} > c_1 \sqrt{L_z} > c_1 \sqrt{\frac{3}{2}L_{\frac{z-1}{2}}} \]

holds, providing \( c_1 < \frac{2\sqrt{3}}{\sqrt{3}} < 2 \). So only \( c_1 = 1 \) is possible. Thus \( c = 2L_{\frac{z-1}{2}} \), and from the factorizations

\[ ac = L_y - 1 = L_{\frac{z-1}{2}} M_{\frac{z-3}{2}}, \quad bc = L_z - 1 = L_{\frac{z-1}{2}} M_{\frac{z+1}{2}} \]

we obtain

\[ a = \frac{1}{2} M_{\frac{z-3}{2}} \quad \text{and} \quad b = \frac{1}{2} M_{\frac{z+1}{2}}. \]

Finally, we show that \( c < b \). (2.10) yields \( M_{2k+1} = 2L_{2k} + 2L_{2k+2} > 4L_{2k} \). Now \( (z - 1)/2 \) is even, so \( 2L_{\frac{z-1}{2}} < \frac{1}{2} M_{\frac{z+1}{2}} \). Thus \( c < b \), contradicting the condition \( a < b < c \).

10. \( y \equiv 3, \ z \equiv 2 \pmod{4} \). We find \( z = y + 3 \), and

\[ L_y - 1 = L_{\frac{y+1}{2}} M_{\frac{y+1}{2}} = L_{\frac{z-2}{2}} M_{\frac{z+4}{2}}, \quad L_z - 1 = L_{\frac{z-2}{2}} M_{\frac{z+2}{2}}. \]

By Lemma 2.4, \( \gcd(M_{\frac{z-4}{2}}, M_{\frac{z+2}{2}}) = 2 \) follows (not \( M_3 = 10 \), because if the subscripts are divisible by 3, dividing them by 3 exactly one of the integers will be odd). Now

\[ \alpha^{\frac{z}{2} - 0.472} < \gcd(L_y - 1, L_z - 1) = 2L_{\frac{z-2}{2}} < \alpha^{0.527} \alpha^{\frac{z+2}{4} - 0.680} \]

leads to a contradiction.
11. \( y \equiv z \equiv 3 \pmod{4} \). In this case, \( i = j = -1 \) implies \( y = z \), which is a contradiction.

12. \( y \equiv 3, \ z \equiv 0 \pmod{4} \). Here \( z = y + 1 \), further

\[
L_y - 1 = L_{y+2} M_{y-1} = L_{\frac{y}{2}} M_{\frac{y-1}{2}}, \quad L_z - 1 = \frac{1}{2} L_{\frac{z+2}{2}} M_{\frac{z-1}{2}}
\]

hold. Lemma 2.4 provides \( \gcd(L_{\frac{y}{2}}, L_{\frac{z+2}{2}}) = 1 \), and we obtain \( \gcd(L_y - 1, L_z - 1) = \frac{1}{2} M_{\frac{y-1}{2}} \) (because \( L_{\frac{z+2}{2}} \) is odd). Hence

\[
c | \gcd(L_y - 1, L_z - 1) = \frac{1}{2} M_{\frac{y-1}{2}} = c_1 c > c_1 \sqrt{L_z}.
\]

By Lemma 2.7 we have \( M_{\frac{z-1}{2}} < 2 \sqrt{L_z} \). Thus \( M_{\frac{z-1}{2}} > 2 c_1 \sqrt{L_z} > c_1 M_{\frac{z-1}{2}} \), which implies \( c_1 < 1 \), an impossibility.

13. \( y \equiv 0, \ z \equiv 1 \pmod{4} \). In this case \( z = y + 1 \), moreover

\[
L_y - 1 = \frac{1}{2} L_{y+2} M_{y-1} = \frac{1}{2} L_{\frac{y+1}{2}} M_{\frac{y-1}{2}}, \quad L_z - 1 = L_{\frac{z-1}{2}} M_{\frac{z-1}{2}}.
\]

By Lemma 2.4, we obtain \( \gcd(L_{\frac{y+1}{2}}, L_{\frac{z-1}{2}}) = 1 \), \( \gcd(M_{\frac{y-1}{2}}, M_{\frac{z+1}{2}}) = 2 \), \( \gcd(L_{\frac{y+1}{2}}, M_{\frac{z+1}{2}}) \leq 1 = 2, \gcd(M_{\frac{y-1}{2}}, L_{\frac{z-1}{2}}) \leq 2 \). Then

\[
\alpha^{\frac{4}{5} - 0.472} < \gcd(L_y - 1, L_z - 1) \leq 8 < \alpha^{1.579}
\]

implies \( z < 9 \).

14. \( y \equiv 0, \ z \equiv 2 \pmod{4} \). Now, by Lemma 2.3, \( i = -2, \ j = 2 \), and \( y \equiv 2 \equiv z \equiv 2 \) follow, which is not possible.

15. \( y \equiv 0, \ z \equiv 3 \pmod{4} \). In this case \( z = y + 3 \), and

\[
L_y - 1 = \frac{1}{2} L_{y+2} M_{y-1} = \frac{1}{2} L_{\frac{y+1}{2}} M_{\frac{y-2}{2}}, \quad L_z - 1 = L_{\frac{z-1}{2}} M_{\frac{z-1}{2}}.
\]

Via Lemma 2.4 we see \( \gcd(L_{\frac{y+1}{2}}, L_{\frac{z-1}{2}}) = 1 \), \( \gcd(M_{\frac{y-2}{2}}, M_{\frac{z+1}{2}}) = 2 \), \( \gcd(L_{\frac{y+1}{2}}, M_{\frac{z+1}{2}}) = 1 \), (because \( z = 1 \), and so \( L_{\frac{z+1}{2}} \) is odd), \( \gcd(M_{\frac{y-2}{2}}, L_{\frac{z-1}{2}}) \leq M_3 = 10 \). These lead to a contradiction via

\[
\alpha^{\frac{4}{5} - 0.472} < \gcd(L_y - 1, L_z - 1) \leq 20 < \alpha^{2.275}.
\]

16. \( y \equiv z \equiv 0 \pmod{4} \). In the last case the only possibility is \( z = y + 4 \). We have

\[
L_y - 1 = \frac{1}{2} L_{y+2} M_{y-2} = \frac{1}{2} L_{\frac{y+2}{2}} M_{\frac{y+1}{2}}, \quad L_z - 1 = \frac{1}{2} L_{\frac{z+2}{2}} M_{\frac{z-2}{2}}.
\]

By Lemma 2.4, we get

\[
\gcd(L_{\frac{z+2}{2}}, L_{\frac{z+2}{2}}) = 1,
\]
\[ \gcd(M_{z-6}, M_{z-2}) = 2, \]
\[ \gcd(L_{z-2}, M_{z-2}) = 1 \text{ (because } (z - 2)/2 \text{ is odd)}, \]
\[ \gcd(M_{z-2}, L_{z-2}) \leq M_4 = 14. \]

Then we obtain \( z < 10 \) from
\[ \alpha^{0.472} < \gcd(L_y - 1, L_z - 1) \leq 14 < \alpha^{2.004}. \]

**Case II:** \( z \leq 116 \). The proof of Theorem 1 will be complete, if we check the finitely many cases \( 3 \leq x < y < z \leq 116 \). It has been done by a computer verification based on the following observation. The equations (1.2) imply
\[ (L_x - 1)(L_y - 1) = a^2bc = a^2(L_z - 1). \]

Thus
\[ \sqrt{\frac{(L_x - 1)(L_y - 1)}{L_z - 1}} \] (3.1)

must be an integer. Checking the given range we found that (3.1) is never an integer.

**References**


