Ph.D. thesis

Haar null and Haar meager sets

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Chapter 1

Introduction

This thesis presents various notions that allow us to formally state that a subset of a Polish group (a separable, completely metrizable topological group) is negligibly small. These are very helpful when one studies these “large” structures and wants to formalize a fact that a certain property is satisfied by “almost all” elements (or the “typical” element).

In the Euclidean space $\mathbb{R}^n$ there is a generally accepted, natural notion of smallness: a set is considered to be small if it has Lebesgue measure zero. The Haar measure (which was introduced by Alfréd Haar in 1933) generalizes the Lebesgue measure for arbitrary locally compact topological groups (see section 2.2 for a brief introduction). Although “the” Haar measure is not completely unique (except in compact groups, where there exists a natural choice), the system of sets that have Haar measure zero is well-defined. This yields a very useful notion of smallness in locally compact groups, but says nothing in groups that are not locally compact, because there it is possible to prove that no Haar measure exists.

In the paper [23] (which was published in 1972) Christensen introduced the notion of Haar null sets, which is equivalent to having Haar measure zero in locally compact groups and defined in every abelian Polish group. Twenty years later this concept was introduced again (under the name “shy sets”) in [64] by Hunt, Sauer and Yorke, who were apparently unaware of Christensen’s work. Since then, many papers studied this area, including [78], where Mycielski generalized the notion for groups that are not abelian.

**Definition 1.1.** Assume that $(G, \cdot)$ is a Polish group. A set $A \subseteq G$ is said to be *Haar null* if there are a Borel set $B \supseteq A$ and a Borel probability measure $\mu$ on $G$ such that $\mu(gBh) = 0$ for every $g, h \in G$. A measure $\mu$ satisfying this is called a *witness measure* for $A$. The system of Haar null subsets of $G$ is denoted by $\mathcal{HN} = \mathcal{HN}(G)$. 
Remark 1.2. Using the terminology introduced in \[31\], a set \( A \subseteq G \) is called shy if it is Haar null, and prevalent if \( G \setminus A \) is Haar null.

There is another widely used notion of smallness, the notion of meager sets (also known as sets of the first category). Meager sets can be defined in any topological space; a set is said to be meager if it is the countable union of nowhere dense sets (i.e. sets which are not dense in any open set). In so-called Baire spaces (in particular, in all the completely metrizable spaces) the nonempty open sets are non-meager and therefore we can consider the meager sets to be negligibly small.

In 2013, Darji defined the notion of Haar meager sets in the paper \[29\] to provide a better analog of Haar null sets in Polish groups that are not locally compact. (Meagerness remains a useful notion in these groups, but this new notion is a closer analogue to Haar null sets.) This original definition only considered the case of abelian groups, but it was generalized in \[38\] to work in arbitrary Polish groups.

Definition 1.3. Assume that \((G, \cdot)\) is a Polish group. A set \( A \subseteq G \) is said to be Haar meager if there are a Borel set \( B \supseteq A \), a (nonempty) compact metric space \( K \) and a continuous function \( f : K \to G \) such that \( f^{-1}(gBh) \) is meager in \( K \) for every \( g, h \in G \).

A function \( f \) satisfying this is called a witness function for \( A \). The system of Haar meager subsets of \( G \) is denoted by \( \mathcal{HM} = \mathcal{HM}(G) \).

We note that according to Theorem 2.4.3, it is also possible to define the notion of Haar null sets in a similar fashion, using witness functions instead of witness measures.

Haar nullness and Haar meagerness both mean that a set is negligibly small in a certain senses, but they formalize very different kinds of smallness. (Intuitively a set is not Haar null if it has a large amount of elements and a set is not Haar meager if its elements are widely distributed.) For example Theorem 6.4.14 shows that under certain conditions the whole Polish group can be written as the union of a Haar null and a Haar meager set.

On the other hand, the analogy between these notions means that many of the results for Haar null sets are also true for Haar meager sets and it is often possible to prove them using similar methods. For example the classical Erdős-Sierpiński duality theorem states the following:

Theorem 1.4 (Erdős-Sierpiński). Assuming the Continuum Hypothesis, there is a bijection \( f : \mathbb{R} \to \mathbb{R} \) such that for every set \( A \subseteq \mathbb{R} \), \( f(A) \) is meager if and only if \( A \) has Lebesgue measure zero and \( f(A) \) has Lebesgue measure zero if and only if \( A \) is meager.

(In this “small” group the Haar null sets are the sets with Lebesgue measure zero and the Haar meager sets are the meager sets.)
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The proof of this result can be generalized to larger groups, but it strongly relies on CH and does not guarantee that $f$ preserves topological properties. Despite this, we can use this duality as an informal guiding principle, which can provide valuable conjectures and justifies the parallel study of these two notions.

Most of the material presented in thesis is based on the survey paper [46], which is joint work with M. Elekes. Following this survey, we include a unified collection of the basic facts about this area and we prove many of these results in a more general setting than that of the papers where they have appeared originally. This overview is extended with a more detailed examination of two topics that appear in chapter 3 and chapter 4.

This first chapter and chapter 2 collects the notations, assumptions and elementary facts that will be used through this thesis. (Among these, section 2.5 presents a lemma that was published in [79].)

After this, chapter 3 examines the details of our core notions using tools from descriptive set theory. This inquiry is motivated by the fact that a set $A$ is said to be Haar null (or Haar meager) if there is a Borel set $B \supseteq A$ which satisfies certain conditions, and here the choice of Borelness (instead of other similar conditions) is somewhat arbitrary. With the exception of section 3.1, this chapter presents the results of [79].

Then chapter 4 examines the notion of strongly Haar meager sets, which is closely related to Haar meager sets. (A set is strongly Haar meager if it has a witness function which is just the identity function of the group restricted to a compact subset.) There were many signs which have suggested that this notion is equivalent to ordinary Haar meagerness, but the results in this section show that (in the Polish group $\mathbb{Z}^\omega$) there exists a Haar meager set that is not strongly Haar meager. This chapter presents the results of [44], which is joint work with M. Elekes, M. Poór and Z. Vidnyánszky.

Then chapter 5 collects and examines several additional notions of smallness that are all closely related to Haar nullness.

This is followed by chapter 6, which examines four basic properties which are satisfied by the sets of Haar measure zero and meager sets in the locally compact groups. Although these results cannot be directly generalized to the case when the group is not necessarily locally compact, we will obtain several limited variants, which are still useful in many situations.

Finally, in chapter 7 we present a collection of ideas and tricks, which are useful to determine whether a certain set is Haar null or Haar meager, and then in chapter 8 we give a brief list of papers that apply Haar null and Haar meager sets in various branches of mathematics.
1.1 Topological notions and assumptions

The symbols \( \mathbb{N} \) and \( \omega \) both refer to the set of nonnegative integers. We write \( \mathbb{N} \) if we consider this set as a topological space (with the discrete topology) and \( \omega \) if we use it only as a cardinal, ordinal or index set. (For example we write the Polish space of the countably infinite sequences of natural numbers as \( \mathbb{N}^\omega \).) We consider the nonnegative integers as von Neumann ordinals, i.e. we identify the nonnegative integer \( n \) with the set \( \{0, 1, 2, \ldots, n\} \).

We use the notation \( [m, n] = [m, n] \cap \mathbb{Z} \) for sets consisting of consecutive integers, and as usual, \( \mathbb{Z}_+ \) denotes the set of positive integers.

If \( S \) is a subset of a topological space, \( \text{int}(S) \) is the interior of \( S \) and \( \overline{S} \) is the closure of \( S \).

We consider \( \mathbb{N}, \mathbb{Z} \) and all finite sets to be topological spaces with the discrete topology. (Note that this convention allows us to simply write the Cantor set as \( 2^\omega = \{0, 1\}^\omega \).)

\( \mathcal{P}(S) \) denotes the power set of a set \( S \). For a set \( S \subseteq X \times Y \), \( x \in X \) and \( y \in Y \), \( S_x \) is the \( x \)-section \( S_x = \{ y : (x, y) \in S \} \) and \( S^y \) is the \( y \)-section \( S^y = \{ x : (x, y) \in S \} \).

We will use \( \mathcal{K}(X) \) to denote the nonempty compact sets of a topological space \( X \), equipped with the Vietoris topology. The well-known result \([69, \text{Theorem 4.25}]\) states that if \( X \) is Polish, then \( \mathcal{K}(X) \) is also Polish.

**Remark 1.1.1.** In \([69]\), \( \mathcal{K}(X) \) is defined as the space of all compact subsets of \( X \), but we defined \( \mathcal{K}(X) \) as the space of all nonempty compact subsets of \( X \). This modified definition is frequently used in the literature, as otherwise there are many situations where the trivial case of \( \emptyset \) is exceptional. It is easy to see that \([69, \text{Theorem 4.25}]\) remains valid with our definition.

We will also use the following fact about this space:

**Fact 1.1.2** (Michael \([77, 4.1.3]\)). If \( X \) is a zero-dimensional Polish space (that is, \( X \) is Polish and has a basis consisting of clopen sets), then \( \mathcal{K}(X) \) is also zero-dimensional.

If \( X \) is a topological space, then

\( \mathcal{B}(X) \) denotes its Borel subsets (\( \mathcal{B}(X) \) is the \( \sigma \)-algebra generated by the open sets, see \([69, \text{Chapter II}]\)),

\( \mathcal{M}(X) \) denotes its meager subsets (a set is meager if it is the union of countably many nowhere dense sets and a set is nowhere dense if the interior of its closure is empty, see \([69, \text{§8.A}]\)).

If the space \( X \) is Polish (that is, separable and completely metrizable), then

\( \Sigma^1_1(X) \) denotes its analytic subsets (a set is analytic if it is the continuous image of a Borel set, see \([69, \text{Chapter III}]\)),
$\Pi_1^1(X)$ denotes its coanalytic subsets (a set is coanalytic if its complement is analytic, see [63, Chapter IV]).

If the topological space $X$ is clear from the context, we simply write $\mathcal{B}$, $\mathcal{M}$, $\Sigma_1^1$ and $\Pi_1^1$.

In a metric space $(X,d)$, $\text{diam}(S) = \sup\{d(x,y) : x, y \in S\}$ denotes the diameter of the subset $S$. If $x \in X$ and $r > 0$, then $B(x,r) = \{x' \in X : d(x,x') < r\}$ and $\overline{B}(x,r) = \{x' \in X : d(x,x') \leq r\}$ denotes respectively the open and the closed ball with center $x$ and radius $r$ in $X$.

**Definition 1.1.3.** In this thesis we identify an outer measure $\mu$ with its restriction to the $\mu$-measurable sets and “measure” means either an outer measure or the corresponding complete measure. (We will not encounter situations where it would be important to consider a measure as a mapping with a concrete, restricted domain.)

Recall that if $\mu$ is an (outer) measure on a set $X$, we say that $A \subseteq X$ is $\mu$-measurable when $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$ for every $B \subseteq X$. A measure $\mu$ is said to be Borel if all Borel sets are $\mu$-measurable. The support of the measure $\mu$ is denoted by $\text{supp}\, \mu$.

Almost all of our results will be about topological groups. A set $G$ is called a topological group if is equipped with both a group structure and a Hausdorff topology and these structures are compatible, that is, the multiplication map $G \times G \to G$, $(g,h) \mapsto gh$ and the inversion map $G \to G$, $g \mapsto g^{-1}$ are continuous functions. We use the convention that whenever we require a group to have some topological property (for example a “compact group”, a “Polish group”, . . . ), then it means that the group must be a topological group and have that property (as a topological space). The identity element of a group $G$ will be denoted by $1_G$.

Most of the results in this thesis are about certain subsets of Polish groups. Unless otherwise noted, $(G, \cdot)$ denotes an arbitrary Polish group. We denote the group operation by multiplication even when we assume that the (abstract) group under consideration is abelian, but we write the group operation of well-known concrete abelian groups like $(\mathbb{R}, +)$ or $(\mathbb{Z}^\omega, +)$ as addition.

Some results are limited to Polish groups that admit a two-sided invariant metric. (A metric $d$ on $G$ is called two-sided invariant (or simply invariant) if $d(g_1h,g_2k) = d(h,k)$ for any $g_1, g_2, h, k \in G$.) Groups with this property are also called TSI groups. This class of groups properly contains all Polish, abelian groups, since each metric group $G$ admits a left-invariant metric which, obviously, is invariant when $G$ is abelian. Any invariant metric on a Polish group is automatically complete. For proofs of these facts and more results about TSI groups see for example [59, §8].

Some basic results can be generalized for non-separable groups, but we will only deal with the separable case. On the other hand, many papers about this topic only consider
abelian groups or some class of vector spaces and this thesis presents many of these results in a more general setting (following \[46\]).

In the special case when $G$ is locally compact, our notions will coincide with simpler notions (see section 2.2) and the majority of the results in this thesis become either easier to prove or false, so the interesting case is when $G$ is not locally compact.

### 1.2 Generalized Haar null sets

Recall that according to \[\text{Definition 1.1}\] a set $A \subseteq G$ is Haar null if there exists a Borel set $B \supseteq A$ which satisfies a certain condition (there exists a Borel probability measure $\mu$ that assigns measure zero to all translates of $B$). Here the choice of Borelness is somewhat arbitrary and there are some papers (including the original paper \[23\] of Christensen) which require universal measurability instead of Borelness. (Recall that a subset $A$ of a Polish space $X$ is called \textit{universally measurable} if it is $\mu$-measurable for any $\sigma$-finite Borel measure $\mu$ on $X$.)

This technical difference is usually not important in practical applications (most important results can be proved for both notions in the same way), but e.g. \[\text{Corollary 3.1.6}\] shows that there are generalized Haar null sets that are not Haar null. When a paper uses both notions, sets satisfying this alternative definition are called “generalized Haar null sets”:

\[\text{Definition 1.2.1.}\] A set $A \subseteq G$ is said to be a \textit{generalized Haar null} if there are a universally measurable set $B \supseteq A$ and a Borel probability measure $\mu$ on $G$ such that $\mu(g Bh) = 0$ for every $g, h \in G$. A measure $\mu$ satisfying this is called a \textit{witness measure} for $A$. The system of generalized Haar null subsets of $G$ is denoted by $\mathcal{GHN} = \mathcal{GHN}(G)$.

\[\text{Remark 1.2.2.}\] As every Borel set is universally measurable, every Haar null set is generalized Haar null.

For more information about the role of this Borel hull $B$ and results about replacing Borelness with other similar conditions see section 2.3 and chapter 3.

### 1.3 Notations for sequences

In some parts of this thesis we use some notation related to sequences (i.e. functions $s$ whose domain is either $\omega$ or $\{0, 1, \ldots, n - 1\}$ for some natural number $n$). This section summarizes the (fairly standard) notations that are related to sequences and their manipulation.
As usual, $s_k$ denotes the element of the sequence $s$ with index $k$ (i.e. $s_k = s(k)$ for an index $k$ that is in the domain of $s$). If $S$ is an arbitrary set, then $S^{<\omega} = \bigcup_{n \in \omega} S^n$ denotes the set of finite sequences of elements of $S$ and $\emptyset$ denotes the empty sequence $\emptyset \in S^0$. For $s \in S^{<\omega}$, $|s|$ denotes the length of $s$.

If $s$ and $s'$ are two sequences and $s$ is finite ($s'$ may be infinite), then $s \sqcup s'$ denotes concatenation of $s$ and $s'$. In particular if $s \in S^{<\omega}$ and $\ell \in S$ is an additional element, then $s \sqcup \ell$ denotes the sequence of length $|s| + 1$ which consists of the elements of $s$ followed by $\ell$ as the last element. If $x$ is a (finite or infinite) sequence of length at least $n$, then $x \mid n$ denotes the sequence formed by the first $n$ elements of $x$.

If $s, s'$ are two sequences then we say that $s \subseteq s'$ if there is a sequence $t$ such that $s' = s \sqcup t$ (if we consider functions and sequences as sets of pairs, this is just the usual inclusion relation). For a sequence $s \in S^{<\omega}$, let $[s] \subseteq S^\omega$ be the set of sequences which have $s$ as an initial segment, i.e.

$$[s] = \{ x \in S^\omega : s \subseteq x \}.$$

If $S$ is endowed with the discrete topology, then it is well-known that $\{[s] : s \in S^{<\omega} \}$ is a basis of the topology of $S^\omega$.

We say that a set $A \subseteq S^{<\omega}$ is cofinal if for every $s \in S^{<\omega}$ there exists an $a \in A$ such that $s \subseteq a$ (i.e. $s$ is an initial segment of $a$).
Chapter 2

Basic properties

2.1 Notions of smallness

Both “Haar null” and “Haar meager” are notions of smallness (i.e. we usually think of Haar null and Haar meager sets as small or negligible). This point of view is justified by the fact that both the system of Haar null sets and the system of Haar meager sets are $\sigma$-ideals.

Definition 2.1.1. A system $\mathcal{I}$ (of subsets of some set) is called a $\sigma$-ideal if

(I) $\emptyset \in \mathcal{I}$,
(II) $A \in \mathcal{I}, B \subseteq A \Rightarrow B \in \mathcal{I}$ and
(III) if $A_n \in \mathcal{I}$ for all $n \in \omega$, then $\bigcup_n A_n \in \mathcal{I}$.

To prove that these systems are indeed $\sigma$-ideals we will need some technical lemmas.

Lemma 2.1.2. If $\mu$ is a Borel probability measure on $G$ and $U$ is a neighborhood of $1_G$, then there are a compact set $C \subseteq G$ and $t \in G$ with $\mu(C) > 0$ and $C \subseteq tU$.

Proof. Applying [33, Theorem 17.11], there exists a compact set $\hat{C} \subseteq G$ with $\mu(\hat{C}) \geq \frac{1}{2}$. Fix an open set $V$ with $1_G \in V \subseteq \overline{V} \subset U$. The collection of open sets $\{tV : t \in \hat{C}\}$ covers $\hat{C}$ and $\hat{C}$ is compact, so $\hat{C} = \bigcup_{t \in F}(tV \cap \hat{C})$ for some finite set $F \subseteq \hat{C}$. It is clear that $\mu(tV \cap \hat{C})$ must be positive for at least one $t \in F$. Choosing $C = t\overline{V} \cap \hat{C}$ clearly satisfies our requirements.

Corollary 2.1.3. If $\mu$ is a Borel probability measure on $G$, $V \subseteq G$ is an open set, $B \subseteq G$ is an arbitrary set which satisfies that $\mu(gBh) = 0$ for every $g, h \in G$, then there exists a Borel probability measure $\mu'$ that satisfies $\mu'(gBh) = 0$ for every $g, h \in G$ and has a compact support that is contained in $V$. 

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Proof. For an arbitrary $v \in V$, the set $U = v^{-1}V$ is a neighborhood of $1_G$. Applying the previous lemma, it is easy to check that $\mu'(X) = \frac{\mu(X \cap C)}{\mu(C)}$ satisfies our requirements. (Recall that a “measure” means either an outer measure or the corresponding complete measure, therefore we may write $\mu(gBh)$ for any set $B \subseteq G$.)

**Lemma 2.1.4.** Let $d$ be a metric on $G$ that is compatible with the topology of $G$. If $L \subseteq G$ is compact and $\varepsilon > 0$ is arbitrary, then there exists a neighborhood $U$ of $1_G$ such that $d(x \cdot u, x) < \varepsilon$ for every $x \in L$ and $u \in U$.

**Proof.** (Reproduced from [38, Lemma 2].) By the continuity of the function $(x, u) \mapsto d(x \cdot u, x)$, for every $x \in L$ there are neighborhoods $V_x$ of $x$ and $U_x$ of $1_G$ such that the image of $V_x \times U_x$ is a subset of $[0, \varepsilon)$. Let $F \subseteq L$ be a finite set such that $L \subseteq \bigcup_{x \in F} V_x$. It is easy to check that $U = \bigcap_{x \in F} U_x$ satisfies our conditions.

**Theorem 2.1.5** (Christensen, Mycielski).

1. The system $\mathcal{HN}$ of Haar null sets is a $\sigma$-ideal.
2. The system $\mathcal{GHN}$ of generalized Haar null sets is a $\sigma$-ideal.

**Proof.** It is trivial that both $\mathcal{HN}$ and $\mathcal{GHN}$ satisfy (I) and (II) in Definition 2.1.1. The proof of (III) that is reproduced here is from the appendix of [26], where a corrected version of the proof in [78] is given. Proving this fact is easier in abelian Polish groups (see [28, Theorem 1]) and when the group is metrizable with a complete left invariant metric (this would allow the proof of [78, Theorem 3] to work without modifications). The appendix of [26] mentions the other approaches and discusses the differences between them.

The proof of (III) for Haar null sets and for generalized Haar null sets is very similar. The following proof will be for Haar null sets, but if “Borel set” is replaced with “universally measurable set” and “Haar null” is replaced with “generalized Haar null”, it becomes the proof for generalized Haar null sets.

Let $A_n$ be Haar null for all $n \in \omega$. By definition there are Borel sets $B_n \subseteq G$ and Borel probability measures $\mu_n$ on $G$ such that $A_n \subseteq B_n$ and $\mu_n(gB_nh) = 0$ for every $g, h \in G$. Let $d$ be a complete metric on $G$ that is compatible with the topology of $G$ (as $G$ is Polish, it is completely metrizable).

We construct for all $n \in \omega$ a compact set $C_n \subseteq G$ and a Borel probability measure $\tilde{\mu}_n$ such that the support of $\tilde{\mu}_n$ is $C_n$, $\tilde{\mu}_n(gB_nh) = 0$ for every $g, h \in G$ (i.e. $\tilde{\mu}_n$ is a witness measure) and the “size” of the sets $C_n$ decreases “quickly”.

The construction will be recursive. For the initial step use Corollary 2.1.3 to find a Borel probability measure $\tilde{\mu}_0$ that satisfies $\tilde{\mu}_0(gB_0h) = 0$ for every $g, h \in G$ and that has compact support $C_0 \subseteq G$. Assume that $\tilde{\mu}_{n'}$ and $C_{n'}$ are already defined for all $n' < n$. By Lemma 2.1.3 there exists a neighborhood $U_n$ of $1_G$ such that if $u \in U_n$, then
\( d(k \cdot u, k) < 2^{-n} \) for every \( k \) in the compact set \( C_0C_1C_2 \cdots C_{n-1} \). Applying Corollary 2.1.3 again we can find a Borel probability measure \( \tilde{\mu}_n \) that satisfies \( \tilde{\mu}_n(gB_nh) = 0 \) for every \( g, h \in G \) and that has a compact support \( C_n \subseteq U_n \).

If \( c_n \in C_n \) for all \( n \in \omega \), then it is clear that the sequence \((c_0c_1c_2 \cdots c_n)_{n \in \omega}\) is a Cauchy sequence. As \((G, d)\) is complete, this Cauchy sequence is convergent; we write its limit as the infinite product \( c_0c_1c_2 \cdots \). The map \( \varphi : \prod_{n \in \omega} C_n \rightarrow G, \varphi((c_0, c_1, c_2, \ldots)) = c_0c_1c_2 \cdots \) is the uniform limit of continuous functions, hence it is continuous.

Let \( \mu^\Pi \) be the product of the measures \( \tilde{\mu}_n \) on the product space \( C^\Pi \) defined as \( C^\Pi = \prod_{n \in \omega} C_n \). Let \( \mu = \varphi_* (\mu^\Pi) \) be the push-forward of \( \mu^\Pi \) along \( \varphi \) onto \( G \), i.e.
\[
\mu(X) = \mu^\Pi(\varphi^{-1}(X)) = \mu^\Pi\left( \{(c_0, c_1, c_2, \ldots) \in C^\Pi : c_0c_1c_2 \cdots \in X \} \right).
\]

We claim that \( \mu \) witnesses that \( A = \bigcup_{n \in \omega} A_n \) is Haar null. Note that \( A \) is contained in the Borel set \( B = \bigcup_{n \in \omega} B_n \), so it is enough to show that \( \mu(gB_h) = 0 \) for every \( g, h \in G \).

As \( \mu \) is \( \sigma \)-additive, it is enough to show that \( \mu(gB_nh) = 0 \) for every \( g, h \in G \) and \( n \in \omega \).

Fix \( g, h \in G \) and \( n \in \omega \). Notice that if \( c_j \in C_j \) for every \( j \neq n, j \in \omega \), then
\[
\tilde{\mu}_n\left( \{ c_n \in C_n : c_0c_1c_2 \cdots c_n \cdots \in gB_nh \} \right) = \tilde{\mu}_n\left( (c_0c_1 \cdots c_{n-1})^{-1} \cdot gB_nh \cdot (c_{n+1}c_{n+2} \cdots)^{-1} \right) = 0
\]
because \( \tilde{\mu}_n(g'B_nh') = 0 \) for all \( g', h' \in G \). Applying Fubini’s theorem in the product space \( \left( \prod_{j \neq n} C_j \right) \times C_n \) to the product measure \( \left( \prod_{j \neq n} \tilde{\mu}_j \right) \times \tilde{\mu}_n \) yields that
\[
0 = \mu^\Pi\left( \{(c_0, c_1, \ldots, c_n, \ldots) \in C^\Pi : c_0c_1 \cdots c_n \cdots \in gB_nh \} \right).
\]

By the definition of \( \mu \) this means that \( \mu(gB_nh) = 0 \).

The analogous statement for Haar meager sets was proved as [23, Theorem 2.9] in the abelian case, and as [38, Theorem 3] in the general case:

**Theorem 2.1.6** (Darji, Doležal-Rmoutil-Vejnar-Vlasák). *The system \( \mathcal{HM} \) of Haar meager sets is a \( \sigma \)-ideal.*

**Proof.** Again, the proof of (I) and (II) in Definition 2.1.1 is obvious. We reproduce the proof of (III) from [38, Theorem 3]. This proof will be very similar to the proof of Theorem 2.1.5, but restricting the witnesses to a smaller “part” of \( G \) is simpler in this case (we do not need an analogue of Corollary 2.1.3).

Let \( A_n \) be Haar meager for all \( n \in \omega \). By definition there are Borel sets \( B_n \subseteq G \), compact metric spaces \( K_n \neq \emptyset \) and continuous functions \( f_n : K_n \rightarrow G \) such that \( f_n^{-1}(gB_nh) \) is
meager in \( K_n \) for every \( g, h \in G \). Let \( d \) be a complete metric on \( G \) that is compatible with the topology of \( G \).

We construct for all \( n \in \omega \) a compact metric space \( \tilde{K}_n \) and a continuous function \( \tilde{f}_n : \tilde{K}_n \to G \) satisfying that \( \tilde{f}_n^{-1}(gB_nh) \) is meager in \( \tilde{K}_n \) for every \( g, h \in G \) (i.e. \( \tilde{f}_n \) is a witness function) and the “size” of the images \( \tilde{f}_n(K_n) \subseteq G \) decreases “quickly”.

Unlike the Haar null case, we do not have to apply recursion in this construction. By Lemma 2.1.4 there exists a neighborhood \( U_n \) of \( 1_G \) such that if \( u \in U_n \), then \( d(k \cdot u, k) < 2^{-n} \) for every \( k \) in the compact set \( f_0(K_n)f_1(K_1) \cdots f_{n-1}(K_{n-1}) \). Let \( x_n \in f_n(K_n) \) be an arbitrary element and \( \tilde{K}_n = f_n^{-1}(x_nU_n) \). The set \( \tilde{K}_n \) is compact (because it is a closed subset of a compact set) and nonempty. Let \( \tilde{f}_n : \tilde{K}_n \to G \), \( \tilde{f}_n(k) = x_n^{-1}f_n(k) \), this is clearly continuous.

**Claim 2.1.7.** For every \( n \in \omega \) and \( g, h \in G \), \( \tilde{f}_n^{-1}(gB_nh) \) is meager in \( \tilde{K}_n \).

**Proof.** Fix \( n \in \omega \) and \( g, h \in G \). The set \( \tilde{f}_n^{-1}(U_n) \) is open in \( K_n \) and because \( f \) is a witness function, the set \( \tilde{f}_n^{-1}(gB_nh) = f_n^{-1}(x_n gB_nh) \) is meager in \( K_n \). This means that \( \tilde{f}_n^{-1}(U_n) \cap \tilde{f}_n^{-1}(gB_nh) \) is meager in \( \tilde{f}_n^{-1}(U_n) \). Since each open subset of \( K_n \) is comeager in its closure and the closure of \( \tilde{f}_n^{-1}(U_n) = f_n^{-1}(x_n U_n) \) is \( f_n^{-1}(x_nU_n) = \tilde{K}_n \), simple formal calculations yield that \( \tilde{f}_n^{-1}(gB_nh) \cap \tilde{K}_n \) is meager in \( \tilde{K}_n \).

Let \( K \) be the compact set \( \prod_{n \in \omega} \tilde{K}_n \) and for \( n \in \omega \) let \( \psi_n \) be the continuous function \( \psi_n : K \to G \),

\[
\psi_n(k) = \tilde{f}_0(k_0) \cdot \tilde{f}_1(k_1) \cdot \cdots \cdot \tilde{f}_{n-1}(k_{n-1}).
\]

By the choice of \( U_n \) we obtain \( d(\psi_{n-1}(k), \psi_n(k)) \leq 2^{-n} \) for every \( k \in K \). Using the completeness of \( d \) this means that the sequence of functions \( (\psi_n)_{n \in \omega} \) is uniformly convergent. Let \( f : K \to G \) be the limit of this sequence. \( f \) is continuous, because it is the uniform limit of continuous functions.

We claim that \( f \) witnesses that \( A = \bigcup_{n \in \omega} A_n \) is Haar meager. Note that \( A \) is contained in the Borel set \( B = \bigcup_{n \in \omega} B_n \), so it is enough to show that \( f^{-1}(gBh) \) is meager in \( K \) for every \( g, h \in G \). As meager subsets of \( K \) form a \( \sigma \)-ideal, it is enough to show that \( f^{-1}(gB_nh) \) is meager in \( K \) for every \( g, h \in G \) and \( n \in \omega \).

Fix \( g, h \in G \) and \( n \in \omega \). Notice that if \( k_j \in \tilde{K}_j \) for every \( j \neq n, j \in \omega \), then Claim 2.1.7 means that

\[
\{k_n \in \tilde{K}_n : f(k_0, k_1, \ldots, k_n, \ldots) \in gB_nh\} =
\{k_n \in \tilde{K}_n : \tilde{f}_0(k_0) \cdot \tilde{f}_1(k_1) \cdot \cdots \cdot \tilde{f}_n(k_n) \cdots \in gB_nh\}
= \tilde{f}_n^{-1}\left(\left(\tilde{f}_0(k_0) \cdot \tilde{f}_1(k_1) \cdots \tilde{f}_{n-1}(k_{n-1})\right)^{-1} \cdot gB_nh \cdot \left(\tilde{f}_{n+1}(k_{n+1}) \cdot \tilde{f}_{n+2}(k_{n+2}) \cdots \right)^{-1}\right)
\]
is meager in $\hat{K}_n$. Applying the Kuratowski-Ulam theorem (see e.g. [69, Theorem 8.41]) in the product space $\left(\prod_{j \neq n} \hat{K}_j\right) \times \hat{K}_n$, the Borel set $f^{-1}(gB_n)h$ is meager.

As the group $G$ acts on itself via multiplication, it is useful if this action does not convert “small” sets into “large” ones. This means that a “nice” notion of smallness must be a translation invariant system.

**Definition 2.1.8.** A system $\mathcal{I} \subseteq \mathcal{P}(G)$ is called translation invariant if $A \in \mathcal{I} \iff gAh \in \mathcal{I}$ for every $A \subseteq G$ and $g, h \in G$.

**Proposition 2.1.9.** The $\sigma$-ideals $\mathcal{H}_N$, $G\mathcal{H}_N$ and $\mathcal{H}_M$ are all translation invariant.

**Proof.** This is clear from Definition 1.1, Definition 1.2.1 and Definition 1.3.

If a nontrivial notion of smallness has these “nice” properties, then the following lemma states that countable sets are small and nonempty open sets are not small. Applying this simple fact for our $\sigma$-ideals is often useful in simple cases.

**Lemma 2.1.10.** Let $\mathcal{I}$ be a translation invariant $\sigma$-ideal that contains a nonempty set but does not contain all subsets of $G$. If $A \subseteq G$ is countable, then $A \in \mathcal{I}$, and if $U \subseteq G$ is nonempty open, then $U \notin \mathcal{I}$.

**Proof.** If $x \in G$, then any nonempty set in $\mathcal{I}$ has a translate that contains $\{x\}$ as a subset, hence $\{x\} \in \mathcal{I}$. Using that $\mathcal{I}$ is closed under countable unions, this yields that if $A \subseteq G$ is countable, then $A \in \mathcal{I}$. To prove the other claim, suppose for a contradiction that $U \subseteq G$ is a nonempty open set that is in $\mathcal{I}$. It is clear that $G = \bigcup_{g \in G} gU$, and as $G$ is Lindelöf, $G = \bigcup_{n \in \omega} g_nU$ for some countable subset $\{g_n : n \in \omega\} \subseteq G$. But here $g_nU \in \mathcal{I}$ (because $\mathcal{I}$ is translation invariant) and thus $G \in \mathcal{I}$ (because $\mathcal{I}$ is closed under countable unions), and this means that $\mathcal{I}$ contains all subsets of $G$, and this is a contradiction.

**Remark 2.1.11.** Let $\mathcal{I}$ be one of the $\sigma$-ideals $\mathcal{H}_N$, $G\mathcal{H}_N$ and $\mathcal{H}_M$. If $G$ is countable, then $\mathcal{I} = \{\emptyset\}$, otherwise $\mathcal{I}$ contains a nonempty set and does not contain all subsets of $G$.

### 2.2 Connections to Haar measure and meagerness

This section discusses the connection between sets with Haar measure zero and Haar null sets and the connection between meager sets and Haar meager sets. In the simple case when $G$ is locally compact we will find that equivalence holds for both pairs, justifying the names “Haar null” and “Haar meager”. When $G$ is non-locally-compact, we will see
that the first connection is broken by the fact that there is no Haar measure on the group. For the other pair we will see that Haar meager sets are always meager, but there are Polish groups where the converse is not true.

First we recall some well-known facts about Haar measures. For proofs and more detailed discussion see for example [51, §15].

**Definition 2.2.1.** If \((X, \Sigma)\) is a measurable space with \(\mathcal{B}(X) \subseteq \Sigma\), then a measure \(\mu : \Sigma \to [0, 1]\) is called regular if \(\mu(U) = \sup \{\mu(K) : K \subseteq U, K \text{ is compact}\}\) for every \(U\) open set and \(\mu(A) = \inf \{\mu(U) : A \subseteq U, U \text{ is open}\}\) for every set \(A\) in the domain of \(\mu\).

**Definition 2.2.2.** If \(G\) is a topological group (not necessarily Polish), a measure \(\lambda : \mathcal{B}(G) \to [0, 1]\) is called a left Haar measure if it satisfies the following properties:

(I) \(\lambda(F) < \infty\) if \(F\) is compact,
(II) \(\lambda(U) > 0\) if \(U\) is a nonempty open set,
(III) \(\lambda(gB) = \lambda(B)\) for all \(B \in \mathcal{B}(G)\) and \(g \in G\) (left invariance),
(IV) \(\lambda\) is regular.

If left invariance is replaced by the property \(\lambda(Bg) = \lambda(B)\) for all \(B \in \mathcal{B}(G)\) and \(g \in G\) (right invariance), the measure is called a right Haar measure.

**Theorem 2.2.3** (existence of the Haar measure). If \(G\) is a locally compact group (not necessarily Polish), then there exists a left (right) Haar measure on \(G\) and if \(\lambda_1, \lambda_2\) are two left (right) Haar measures, then \(\lambda_1 = c \cdot \lambda_2\) for a positive real constant \(c\).

If \(G\) is compact, (I) means that the left and right Haar measures are finite measures, and this fact can be used to prove the following result:

**Theorem 2.2.4.** If \(G\) is a compact group, then all left Haar measures are right Haar measures and vice versa.

This result is also trivially true in abelian locally compact groups, but not true in all locally compact groups. However, the following result remains true:

**Theorem 2.2.5.** If \(G\) is a locally compact group, then the left Haar measures and the right Haar measures are absolutely continuous relatively to each other, that is, for every Borel set \(B \subseteq G\), either every left Haar measure and every right Haar measure assigns measure zero to \(B\) or no left Haar measure and no right Haar measure assigns measure zero to \(B\).

This allows us to define the following notion:

**Definition 2.2.6.** Suppose that \(G\) is a locally compact group and fix an arbitrary left (or right) Haar measure \(\lambda\). We say that a set \(N \subseteq G\) has Haar measure zero if \(N \subseteq B\) for some Borel set \(B\) with \(\lambda(B) = 0\). The collection of these sets is denoted by \(\mathcal{N} = \mathcal{N}(G)\).
Definition 2.2.2 defines the Haar measures only on the Borel sets. If \( \lambda \) is an arbitrary left (or right) Haar measure, we can complete it using the standard techniques. The domain of the completion will be \( \sigma(B(G) \cup \mathcal{N}) \) (the \( \sigma \)-algebra generated by \( \mathcal{N} \) and the Borel sets). For every set \( A \) in this \( \sigma \)-algebra, let

\[
\lambda(A) = \inf \left\{ \sum_{j \in \omega} \lambda(B_j) : B_j \in B(G), A \subseteq \bigcup_{j \in \omega} B_j \right\}.
\]

This completion will be a complete measure that agrees with the original \( \lambda \) on Borel sets and satisfies properties (I) – (IV) from Definition 2.2.2 (or right invariance instead of left invariance if \( \lambda \) was a right Haar measure). We will identify a left (or right) Haar measure with its completion and we will also call this extension (slightly imprecisely) a left (or right) Haar measure.

The following theorem justifies the choice of the name of “Haar null” sets. The foundational paper [23] shows this in the abelian case; [78, Theorem 1] gives a complete proof (using a slightly different definition of Haar measures).

**Theorem 2.2.7** (Christensen, Mycielski). If \( G \) is a locally compact Polish group, then the system of sets with Haar measure zero is the same as the system of Haar null sets and is the same as the system of generalized Haar null sets, that is, \( \mathcal{N}(G) = \mathcal{H}\mathcal{N}(G) = \mathcal{G}\mathcal{H}\mathcal{N}(G) \).

**Proof.** \( \mathcal{N}(G) \subseteq \mathcal{H}\mathcal{N}(G) \):
Let \( \lambda \) be a left Haar measure and \( \lambda' \) be a right Haar measure. If \( N \in \mathcal{N}(G) \) is arbitrary, then by definition there is a Borel set \( B \) satisfying \( N \subseteq B \) and \( \lambda(B) = 0 \). The left invariance of \( G \) means that \( \lambda(gB) = 0 \) for every \( g \in G \). Applying Theorem 2.2.5 this means that \( \lambda'(gB) = 0 \) for every \( g \in G \), and applying the right invariance of \( \lambda' \) we get that \( \lambda'(gBh) = 0 \) for every \( g, h \in G \). Using the regularity of \( \lambda' \), it is easy to see that there is a compact set \( K \) with \( 0 < \lambda'(K) < \infty \). The measure \( \mu(X) = \frac{\lambda'(K \cap X)}{\lambda'(K)} \) is clearly a Borel probability measure. \( \mu \ll \lambda' \) means that \( \mu(gBh) = 0 \) for every \( g, h \in G \), so \( B \) and \( \mu \) satisfy the requirements of Definition 1.1.

\( \mathcal{H}\mathcal{N}(G) \subseteq \mathcal{G}\mathcal{H}\mathcal{N}(G) \):
This is obviously true in all Polish groups, see Remark 1.2.2.

\( \mathcal{G}\mathcal{H}\mathcal{N}(G) \subseteq \mathcal{N}(G) \):
Suppose that \( A \in \mathcal{G}\mathcal{H}\mathcal{N}(G) \). By definition there exists a universally measurable \( B \subseteq G \) and a Borel probability measure \( \mu \) such that \( \mu(gBh) = 0 \) for every \( g, h \in G \). Notice that we will only use that \( \mu(Bh) = 0 \) for every \( h \in G \), so we will also prove that (using the terminology of section 5.1) all generalized right Haar null sets have Haar measure zero. Let \( \lambda \) be a left Haar measure on \( G \). Let \( m \) be the multiplication map \( m : G \times G \to G \), \( (x, y) \mapsto x \cdot y \).
There exists an open neighborhood $\mathcal{O}_{x}\subseteq G$ such that $x \cdot \mathcal{O}_{x} \subseteq G$ for every $x \in G$. This follows from the preimage of a universally measurable set under the continuous map $m$. (This follows from that the preimage of a Borel set under $m$ is Borel, and for every $\sigma$-finite measure $\nu$ on $G \times G$, the preimage of a set of $m_{*}(\nu)$-measure zero under $m$ must be of $\nu$-measure zero. Here $m_{*}(\nu)$ is the push-forward measure: $m_{*}(\nu)(X) = \nu(\{ (x, y) : x \cdot y \in X \})$.)

Applying Fubini’s theorem in the product space $G \times G$ to the product measure $\mu \times \lambda$ (which is a $\sigma$-finite Borel measure) we get that

$$(\mu \times \lambda)(m^{-1}(B)) = \int_{G} \lambda(\{ y : x \cdot y \in B \}) \, d\mu(x) = \int_{G} \mu(\{ x : x \cdot y \in B \}) \, d\lambda(y),$$

hence

$$\int_{G} \lambda(x^{-1}B) \, d\mu(x) = \int_{G} \mu(By^{-1}) \, d\lambda(y).$$

As $\mu$ is a witness measure, the right hand side is the integral of the constant $0$ function. On the left hand side $\lambda(x^{-1}B) = \lambda(B)$, as $\lambda$ is left invariant (note that $B$ is $\lambda$-measurable, because $B$ is universally measurable and $\lambda$ is $\sigma$-finite). Thus $0 = \int_{G} \lambda(B) \, d\mu(x) = \lambda(B)$.

As $A \subseteq B$, this means that $\lambda(A) = 0$, $A \in \mathcal{N}(G)$. \hfill $\Box$

We reproduce the proof of the classical theorem which shows that (left and right) Haar measures do not exist on topological groups that are not locally compact. We will apply the following generalized version of the Steinhaus theorem. (The original version by Steinhaus considered the Lebesgue measure on the group $(\mathbb{R}, +)$, Weil generalized this for the case of Haar measures [72, page 50].)

Theorem 2.2.8 (Steinhaus, Weil). If $G$ is a topological group, $\lambda$ is a left Haar measure on $G$ and $C \subseteq G$ is compact with $\lambda(C) > 0$, then $1_{G} \in \text{int}(C \cdot C^{-1})$.

Proof. As $\lambda$ is a Haar measure and $C$ is compact, $\lambda(C) < \infty$. Using the regularity of $\lambda$, there is an open set $U \supseteq C$ that satisfies $\lambda(U) < 2\lambda(C)$.

Claim 2.2.9. There exists an open neighborhood $V$ of $1_{G}$ such that $V \cdot C \subseteq U$.

Proof. For every $c \in C$ the multiplication map $m : G \times G \to G$ is continuous at $(1_{G}, c)$, so $c \in V_{c} \cdot W_{c} \subseteq U$ for some open neighborhood $V_{c}$ of $1_{G}$ and some open neighborhood $W_{c}$ of $c$. As $C$ is compact and $\bigcup_{c \in C} W_{c} \supseteq C$, there is a finite set $F$ with $\bigcup_{c \in F} W_{c} \supseteq C$. Then $V = \bigcap_{c \in F} V_{c}$ satisfies $V \cdot W_{c} \subseteq U$ for every $c \in F$, so $V \cdot C \subseteq U$. \hfill $\Box$

Now it is enough to prove that $V \subseteq C \cdot C^{-1}$. Choose an arbitrary $v \in V$. Then $v \cdot C$ and $C$ are subsets of $U$ and $\lambda(v \cdot C) = \lambda(C) > \frac{\lambda(U)}{2}$ (we used the left invariance of $\lambda$). This means that $v \cdot C \cap C \neq \emptyset$, so there exists $c_{1}, c_{2} \in C$ with $vc_{1} = c_{2}$, but this means that $v = c_{2}c_{1}^{-1} \in C \cdot C^{-1}$. \hfill $\Box$
We note that in Polish groups it is possible to find a compact subset with positive Haar measure in every set with positive Haar measure. Hence the following version of the previous theorem is also true:

**Corollary 2.2.10** (Steinhaus, Weil). If \( G \) is a locally compact Polish group, \( \lambda \) is a left Haar measure on \( G \) and \( A \subseteq G \) is \( \lambda \)-measurable with \( \lambda(A) > 0 \), then \( 1_G \in \text{int}(A \cdot A^{-1}) \).

Other, more general variants of the Steinhaus theorem are examined in Section 6.2.

Now we are ready to prove that Haar measures only exist in the locally compact case:

**Theorem 2.2.11.** If \( G \) is a topological group and \( \lambda \) is a left Haar measure on \( G \), then \( G \) is locally compact.

**Proof.** \( \lambda(G) > 0 \) as \( G \) is open. Using the regularity of \( G \), there exists a compact set \( C \) with \( \lambda(C) > 0 \). The set \( C \cdot C^{-1} \) is compact (it is the image of the compact set \( C \times C \) under the continuous map \((x, y) \mapsto xy^{-1}\)). Applying Theorem 2.2.8 yields that \( C \cdot C^{-1} \) is a neighborhood of \( 1_G \), but then for every \( g \in G \) the set \( g \cdot C \cdot C^{-1} \) is a compact neighborhood of \( g \), and this shows that \( G \) is locally compact. \( \square \)

The connection between meager sets and Haar meager sets is simpler. The following results are from \([29]\) (the first paper about Haar meager sets, which only considers abelian Polish groups) and \([38]\) (where the concept of Haar meager sets is extended to all Polish groups).

**Theorem 2.2.12** (Darji, Doležal-Rmoutil-Vejnar-Vlasák). Every Haar meager set is meager, \( \mathcal{HM}(G) \subseteq \mathcal{M}(G) \).

**Proof.** Let \( A \) be a Haar meager subset of \( G \). By definition there exists a Borel set \( B \supseteq A \), a (nonempty) compact metric space \( K \) and a continuous function \( f : K \to G \) such that \( f^{-1}(gBh) \) is meager in \( K \) for every \( g, h \in G \).

Consider the set
\[
S = \{(g, k) : f(k) \in gB\} \subseteq G \times K,
\]
which is Borel because it is the preimage of \( B \) under the continuous map \((g, k) \mapsto g^{-1} \cdot f(k)\). For every \( g \in G \), the \( g \)-section of this set is \( S_g = \{k \in K : f(k) \in gB\} = f^{-1}(gB) \), and this is a meager set in \( K \). Hence, by the Kuratowski-Ulam theorem, \( S \) is meager in \( G \times K \). Using the Kuratowski-Ulam theorem again, for comeager many \( k \in K \), the section \( S^k = \{g \in G : f(k) \in gB\} = f(k) \cdot B^{-1} \) is meager in \( G \). Since \( K \) is compact, there is at least one such \( k \). Then the inverse of the homeomorphism \( b \mapsto f(k) \cdot b^{-1} \) maps the meager set \( S^k \) to \( B \), and this shows that \( B \) is meager. \( \square \)

**Theorem 2.2.13** (Darji, Doležal-Rnooutil-Vejnar-Vlasák). In a locally compact Polish group \( G \) meagerness is equivalent to Haar meagerness, that is, \( \mathcal{HM}(G) = \mathcal{M}(G) \).
Proof. We only need to prove the inclusion $\mathcal{M}(G) \subseteq \mathcal{H}\mathcal{M}(G)$. As $G$ is locally compact, there is a nonempty open set $U \subseteq G$ such that $\overline{U}$ is compact. Let $f : \overline{U} \to G$ be the identity map restricted to $\overline{U}$. If $M$ is meager in $G$, then there exists a meager Borel set $B \supseteq M$. The set $gBh$ is meager in $G$ for every $g, h \in G$ (as $x \mapsto gxh$ is a homeomorphism), so $f^{-1}(gBh) = gBh \cap \overline{U}$ is meager in $\overline{U}$ for every $g, h \in G$.

**Theorem 2.2.14** (Darji, Doležal-Rmoutil-Vejnar-Vlasák). In a non-locally-compact Polish group $G$ that admits a two-sided invariant metric meagerness is a strictly stronger notion than Haar meagerness, that is, $\mathcal{H}\mathcal{M}(G) \subsetneq \mathcal{M}(G)$.

Proof. We know that $\mathcal{H}\mathcal{M}(G) \subseteq \mathcal{M}(G)$. To construct a meager but not Haar meager set, we will use a theorem from [87]. As the proof of this purely topological theorem is relatively long, we do not reproduce it here.

**Theorem 2.2.15** (Solecki). Assume that $G$ is a non-locally-compact Polish group that admits a two-sided invariant metric. Then there exists a closed set $F \subseteq G$ and a continuous function $\varphi : F \to 2^\omega$ such that for any $x \in 2^\omega$ and any compact set $C \subseteq G$ there is a $g \in G$ with $gC \subseteq \varphi^{-1}\{x\}$.

Using this we construct a closed nowhere dense set $M$ that is not Haar meager. The system $\{f^{-1}\{x\} : x \in 2^\omega\}$ contains continuum many pairwise disjoint closed sets. If we fix a countable basis in $G$, only countably many of these sets contain an open set from that basis. If for $x_0 \in 2^\omega$ the set $M = f^{-1}\{x_0\}$ does not contain a basic open set, then it is nowhere dense (as it is closed with empty interior). On the other hand, it is clear that $M$ is not Haar meager, as for every compact metric space $K$ and continuous function $f : K \to G$ there exists a $g \in G$ such that $gf(K) \subseteq M$, thus $f^{-1}(g^{-1}M) = K$.

### 2.3 Naive versions and their limitations

It is a tempting idea to replace the notions of Haar null and Haar meager sets with the following simpler notions:

**Definition 2.3.1.** A set $A \subseteq G$ is called *naively Haar null* if there is a Borel probability measure $\mu$ on $G$ such that $\mu(gAh) = 0$ for every $g, h \in G$.

**Definition 2.3.2.** A set $A \subseteq G$ is called *naively Haar meager* if there is a (nonempty) compact metric space $K$ and a continuous function $f : K \to G$ such that $f^{-1}(gAh)$ is meager in $K$ for every $g, h \in G$.

These are closely related to our “regular” notions; indeed, comparing them with the “regular” definitions one can immediately see that the following fact holds:
Fact 2.3.3.

1. A set $A \subseteq G$ is Haar null when it has a Borel hull $B \supseteq A$ which is naively Haar null.
2. A set $A \subseteq G$ is generalized Haar null when it has a universally measurable hull $B \supseteq A$ which is naively Haar null.
3. A set $A \subseteq G$ is Haar meager when it has a Borel hull $B \supseteq A$ which is naively Haar meager.

It is clear that the “regular” notions imply the corresponding naive notions; however, the following theorem shows that these implications cannot be reversed in a relatively general class of groups:

Theorem 2.3.4 (Elekes-Vidnyánszky / Doležal-Rmoutil-Vejnar-Vlasák). Let $G$ be an uncountable abelian Polish group.

1. There exists a subset of $G$ that is naively Haar null but not Haar null.
2. There exists a subset of $G$ that is naively Haar meager but not Haar meager.

We do not reproduce the relatively long proof of these results. The proof of (1) can be found in [48, Theorem 1.3], the proof of (2) can be found in [38, Theorem 16]. Also note that if $G$ is non-locally-compact, then this result is the corollary of Theorem 3.1.5.

While this would not be a problem by itself, there are also several examples which show that these naive notions do not form $\sigma$-ideals. As these examples cover a large class of Polish groups, these notions cannot be used as notions of smallness, and therefore they are relatively rare in the literature compared to the “regular” variants. (However, they can be interesting when they are combined by a notion that is similar to Borelness, e.g. as in section 3.1.)

The rest of this section shows three of these examples:

Example 2.3.5 (Dougherty [40]). Let $G$ be an uncountable Polish group. Assuming the Continuum Hypothesis, there exists a subset $W$ in the product group $G \times G$ such that both $W$ and $(G \times G) \setminus W$ are naively Haar null and naively Haar meager.

Proof. Let $<_W$ be a well-ordering of $G$ in order type $\omega_1$, and let $W = \{(g, h) \in G \times G : g <_W h\}$ be this relation considered as a subset of $G \times G$.

Let $\mu_1$ be an non-atomic measure on $G$ and $\mu_2$ be a measure on $G$ that is concentrated on a single point. It is clear that $(\mu_1 \times \mu_2)(gW) = 0$ for every $g, h \in G$ (as $gW \cap \text{supp}(\mu_1 \times \mu_2)$ is countable). Similarly $(\mu_2 \times \mu_1)((G \times G) \setminus W)h = 0$ for every $g, h \in G$, but these mean that $W$ and $(G \times G) \setminus W$ are both naively Haar null.

Let $K \subseteq G$ be a nonempty perfect compact set (it is well known that such set exists) and let $f_1, f_2 : K \to G$ be the continuous functions $f_1(k) = (k, 1_G)$, $f_2(k) = (1_G, k)$.
(here \(1_G\) could be replaced by any fixed element of \(G\)). Then \(f_i^{-1}(gWh)\) is countable (hence meager) for every \(g, h \in G\), so \(W\) is naively Haar meager, similarly \(f_2\) shows that \((G \times G) \setminus W\) is also naively Haar meager.

**Example 2.3.6.** The Polish group \((\mathbb{R}^2, +)\) is the union of countably many sets that are both naively Haar null and naively Haar meager.

**Proof.** The paper [34] constructs a decomposition of the plane with the following properties:

**Theorem 2.3.7** (Davies). Suppose that \((\theta_i)_{i \in \omega}\) is a countably infinite system of directions, such that \(\theta_i\) and \(\theta_j\) are not parallel if \(i \neq j\). Then the plane can be decomposed as a disjoint union \(\mathbb{R}^2 = \bigcup_{i \in \omega} S_i\) such that each line in the direction \(\theta_i\) intersects the set \(S_i\) in at most one point.

We do not include the proof of this theorem.

Later [38, Example 5.4] notices that the sets \(S_i\) \((i \in \omega)\) in this decomposition are naively Haar null: Let \(\mu_i\) be the 1-dimensional Lebesgue measure on an arbitrary line with direction \(\theta_i\). Then \(\mu_i(g + S_i + h) = 0\) for every \(g, h \in \mathbb{R}^2\), because at most one point of \(g + S_i + h\) is contained in the support of \(\mu_i\). This means that \(S_i\) is indeed naively Haar null.

A similar argument shows that these sets are also naively Haar meager: Let \(K_i\) be a nonempty perfect compact subset of an arbitrary line with direction \(\theta_i\), and let \(f_i : K_i \rightarrow \mathbb{R}^2\) be the restriction of the identity function. Then \(f_i^{-1}(g + S_i + h)\) contains at most one point for every \(g, h \in \mathbb{R}^2\), hence it is meager. This proves that \(S_i\) is naively Haar meager.

The following example from [38, Proposition 17] yields the results of **Example 2.3.5** in a different class of groups. The cited paper only proves this for the naively Haar meager case, but states that it can be proved analogously in the naively Haar null case. We do not reproduce this proof, as it is significantly longer than the proof of **Example 2.3.5**.

**Example 2.3.8** (Doležal-Rmoutil-Vejnar-Vlasák). Let \(G\) be an uncountable abelian Polish group. Assuming the Continuum Hypothesis, there exists a subset \(X \subseteq G\) such that both \(X\) and \(G \setminus X\) are naively Haar null and naively Haar meager.

### 2.4 Alternative formulations

In the literature there are several results which prove that some condition is equivalent to one of our core definitions (**Definition 1.1**, **Definition 1.2.1** or **Definition 1.3**). In fact,
some papers use one of these alternative formulations to define Haar null, strongly Haar null or Haar meager sets.

In the witness measure is required to be a Borel probability measure, but some alternative conditions yield equivalent definitions.

Recall that Definition 1.1 (or, respectively, Definition 1.2.1) states that a set \( A \subseteq G \) is Haar null (or generalized Haar null) if and only if there is a Borel (or universally measurable) set \( B \supseteq A \) that satisfies the condition (1) in the following theorem. This formulation demands that the witness measure must be a Borel probability measure; however there are some alternative conditions that yield equivalent definitions:

**Theorem 2.4.1.** For a set \( B \subseteq G \) the following are equivalent:

1. there exists a Borel probability measure \( \mu \) on \( G \) such that \( \mu(gBh) = 0 \) for every \( g, h \in G \) (i.e. \( B \) is a naively Haar null set; see section 2.3 for more information about this notion)
2. there exists a Borel probability measure \( \mu \) on \( G \) such that \( \mu \) has compact support and \( \mu(gBh) = 0 \) for every \( g, h \in G \),
3. there exists a Borel measure \( \mu \) on \( G \) such that \( 0 < \mu(X) < \infty \) for some \( \mu \)-measurable set \( X \subseteq G \) and \( \mu(gBh) = 0 \) for every \( g, h \in G \),
4. there exists a Borel measure \( \mu \) on \( G \) such that \( 0 < \mu(C) < \infty \) for some compact set \( C \subseteq G \) and \( \mu(gBh) = 0 \) for every \( g, h \in G \) (the paper [64] calls a Borel set \( B \) shy if it has this property).

**Proof.** The implications (2) \( \Rightarrow \) (4) \( \Rightarrow \) (3) are trivial. (3) \( \Rightarrow \) (1) is true, because if \( \mu \) and \( X \) satisfies the requirements of (3), then \( \tilde{\mu}(Y) = \frac{\mu(Y \cap X)}{\mu(X)} \) is a Borel probability measure and \( \tilde{\mu} \leq \mu \) means that \( \tilde{\mu}(gBh) = 0 \) for every \( g, h \in G \). Finally, the implication (1) \( \Rightarrow \) (2) follows from Corollary 2.1.3.

For universally measurable sets we may generalize [64, Fact 4] to extend this equivalence:

**Theorem 2.4.2** (Hunt-Sauer-Yorke). For a universally measurable set \( B \subseteq G \) the following are equivalent:

1. there exists a Borel probability measure \( \mu \) on \( G \) such that \( \mu(gBh) = 0 \) for every \( g, h \in G \) (i.e. \( B \) is a generalized Haar null set),
5. there exists a Borel probability measure \( \mu \) on \( G \) and a generalized Haar null set \( N \subseteq G \) such that \( \mu(gBh) = 0 \) for every \( g, h \in G \setminus N \).

**Proof.** The direction (1) \( \Rightarrow \) (5) is trivial. To prove the other direction, assume that (5) holds. We may assume without loss of generality that \( N \) is universally measurable. Fix a Borel probability measure \( \nu \) on \( G \) such that \( \nu(gNh) = 0 \) for all \( g, h \in G \) and let \( \overline{\nu} \) denote the Borel probability measure \( \overline{\nu}(X) = \nu(\{x^{-1} : x \in X\}) \).
Consider the measure $\hat{\mu}$ defined by the convolution

$$\hat{\mu}(X) = \nu * \mu * \nu(X) = (\nu \times \mu \times \nu)(\{(p, q, r) \in G^3 : pqr \in X\}),$$

which is clearly a Borel probability measure on $G$. To prove that $\hat{\mu}$ satisfies condition (1) we need to check that if we fix arbitrary $g, h \in G$, then

$$\hat{\mu}(gBh) = (\nu \times \mu \times \nu)(\{(p, q, r) \in G^3 : pqr \in gBh\}) = 0.$$

We may apply Fubini’s theorem to see that

$$\hat{\mu}(gBh) = \int_G \int_G \int_G \chi_{p^{-1}gBhr^{-1}}(q) \ d\nu(p) \ d\mu(q) \ d\nu(r) =$$

$$= \int_G \int_G \mu(p^{-1}gBhr^{-1}) \ d\nu(p) \ d\nu(r) =$$

$$= \int_G \int_G \mu(pgBhr) \ d\nu(p) \ d\nu(r).$$

We know that $\nu(h^{-1}N) = \nu(Ng^{-1}) = 0$ and so the set $(G \setminus h^{-1}N) \times (G \setminus Ng^{-1})$ has full $(\nu \times \nu)$-measure in $G \times G$. This implies that

$$\hat{\mu}(gBh) = \int_{G \setminus h^{-1}N} \int_{G \setminus Ng^{-1}} \mu(pgBhr) \ d\nu(p) \ d\nu(r).$$

But here $p \in G \setminus Ng^{-1}$ and $r \in G \setminus h^{-1}N$ means that $pg, hr \in G \setminus N$ and therefore $\mu(pgBhr) = 0$, concluding our proof.

We remark that the proof only used the fact that $N$ is a generalized left-and-right Haar null set. (This is a weaker notion than generalized Haar null sets, we discuss it in section 5.1)

The following result characterizes Haar null sets with witness functions, analogously to Haar meager sets:

**Theorem 2.4.3** (Banakh-Głąb-Jabłońska-Swaczyna). For a Borel set $B \subseteq G$ the following are equivalent:

1. $G$ is Haar null,
2. there exists an injective continuous map $f : 2^{\omega} \rightarrow G$ such that $f^{-1}(gBh) \in \mathcal{N}(2^{\omega})$ for all $g, h \in G$,
3. there exists a continuous map $f : 2^{\omega} \rightarrow G$ such that $f^{-1}(gBh) \in \mathcal{N}(2^{\omega})$ for all $g, h \in G$.

Here $\mathcal{N}(2^{\omega})$ is the $\sigma$-ideal of sets of Haar measure zero on the Cantor cube $2^{\omega}$. 
The relatively long proof of this result can be found as [9, Theorem 4.3]. The paper [9] only considers the case of abelian Polish groups, but the proof of this result remains valid in the case when \( G \) is not necessarily abelian. For related results stated in this thesis, see also Theorem 2.4.4 and Theorem 5.3.3.

The following result gives an equivalent characterization of Haar null sets which allows proving that a Borel set is Haar null by constructing measures that assign small, but not necessarily zero measures to the translates of that set. In [72, Theorem 1.1] Matoušková proves this theorem for separable Banach spaces, but her proof can be generalized to work in arbitrary Polish groups.

**Theorem 2.4.4** (Matoušková). A Borel set \( B \) is Haar null if and only if for every \( \varepsilon > 0 \) and neighborhood \( U \) of \( 1_G \), there exists a Borel probability measure \( \mu \) on \( G \) such that the support of \( \mu \) is contained in \( U \) and \( \mu(g Bh) < \varepsilon \) for every \( g, h \in G \).

**Proof.** Let \( P(G) \) be the set of Borel probability measures on \( G \). As \( G \) is Polish, [34, Theorem 17.23] states that \( P(G) \) (endowed with the weak topology) is also a Polish space. In particular this means that it is possible to fix a compatible metric \( d \) such that \( (P(G), d) \) is a complete metric space. If \( \mu, \nu \in P(G) \), let \( (\mu \ast \nu)(X) = (\mu \times \nu)(\{(x, y) : xy \in X\}) \) be their convolution. It is straightforward to see that \( \ast \) is associative (but not commutative in general, as we did not assume that \( G \) is commutative). The map \( \ast : P(G) \times P(G) \to P(G) \) is continuous, for a proof of this see e.g. [34, Proposition 2.3]. Let \( \delta(X) = 1 \) if \( 1_G \in X \), and \( \delta(X) = 0 \) if \( 1_G \notin X \), then it is clear that \( \delta \in P(G) \) is the identity element for \( \ast \).

First we prove the “only if” part. Let \( (U_n)_{n \in \omega} \) be open sets with \( \bigcap_n U_n = \{1_G\} \). For every \( n \in \omega \) fix a Borel probability measure \( \mu_n \) such that

1. \( \text{supp} \mu_n \subseteq U_n \) and
2. \( \mu_n(g Bh) < \frac{1}{n+1} \) for every \( g, h \in G \).

It is easy to see from property (I) that the sequence \( (\mu_n)_{n \in \omega} \) (weakly) converges to \( \delta \), and this and the continuity of \( \ast \) means that for any \( \nu \in P(G) \) the sequence \( (d(\nu, \nu \ast \mu_n))_{n \in \omega} \) converges to zero. This allows us to replace \( (\mu_n)_{n \in \omega} \) with a subsequence which also satisfies that \( d(\nu, \nu \ast \mu_n) < 2^{-n} \) for every measure \( \nu \) from the finite set

\[ \{\mu_{j_0} \ast \mu_{j_1} \ast \ldots \ast \mu_{j_r} : r < n \text{ and } 0 \leq j_0 < j_1 < \ldots < j_r < n\}. \]

(Notice that property (II) clearly remains true for any subsequence.) Using this assumption and the completeness of \( (P(G), d) \) we can define (for every \( n \in \omega \)) the “infinite convolution” \( \mu_n \ast \mu_{n+1} \ast \ldots \) as the limit of the Cauchy sequence \( (\mu_n \ast \mu_{n+1} \ast \mu_{n+2} \ast \ldots \ast \mu_{n+j})_{j \in \omega} \).

We will show that the choice \( \mu = \mu_0 \ast \mu_1 \ast \ldots \) witnesses that \( B \) is Haar null.

We have to prove that \( \mu(g Bh) = 0 \) for every \( g, h \in G \). To show this fix arbitrary \( g, h \in G \) and \( n \in \omega \); we will show that \( \mu(g Bh) \leq \frac{1}{n+1} \). Let \( \alpha_n = \mu_0 \ast \mu_1 \ast \ldots \ast \mu_{n-1} \) and
Lemma 2.1.2. If yields that there are a compact set section 1.3.

We will use a modified version of the usual construction. Note that if for every $B$

Applying then $\mu$

continuous function $\phi$

S

We will choose a set of finite 0-1 sequences rely on the notation introduced in

Proof. Lemma 2.4.5.

the well-known result that for every (nonempty) compact metric space $K$

To prove the “if” part of the theorem, suppose that there exists a $\varepsilon > 0$ and a neighborhood $U$ of $1_G$ such that for every Borel probability measure $\mu$ on $G$ if supp $\mu \subseteq U$, then $\mu(g Bh) \geq \varepsilon$ for some $g, h \in G$. Let $\mu$ be an arbitrary Borel probability measure. Applying Lemma 2.1.2 yields that there are a compact set $C \subseteq G$ and $c \in G$ with $\mu(C) > 0$ and $C \subseteq cU$. Define $\mu'(X) = \frac{\mu(cX \cap C)}{\mu(C)}$, then $\mu'$ is a Borel probability measure with supp $\mu' \subseteq U$, hence $\mu'(g Bh) \geq \varepsilon$ for some $g, h \in G$. This means that $\mu(cg Bh) \neq 0$, so $\mu$ is not a witness measure for $B$, and because $\mu$ was arbitrary, $B$ is not Haar null. \qed

To prove our next result we will need a technical lemma. This is a modified version of the well-known result that for every (nonempty) compact metric space $K$, there exists a continuous surjective map $\varphi : 2^\omega \to K$.

Lemma 2.4.5. If $(K, d)$ is a (nonempty) compact metric space, then there exists a continuous function $\varphi : 2^\omega \to K$ such that if $M$ is meager in $K$, then $\varphi^{-1}(M)$ is meager in $2^\omega$.

Proof. We will use a modified version of the usual construction. Note that if diam($K$) < 1, then this function $\varphi$ will be surjective (we will not need this fact). In this proof we rely on the notation introduced in section 1.3.

We will choose a set of finite 0-1 sequences $S \subseteq 2^{<\omega}$, and for every $s \in S$ we will choose a point $k_s \in K$. The construction of $S$ will be recursive: for every $n \in \omega$ we define a set $S_n$ and let $S = \bigcup_{n \in \omega} S_n$. Our choices will satisfy the following properties:

(I) for every $n \in \omega$ and $x \in 2^\omega$ there exists a unique $s = s(x, n) \in S_n$ with $s \preceq x$, and $s(x, n) < s(x, n')$ if $n < n'$.

(II) for every $n \in \omega$ and $x \in 2^\omega$ the set $C_{x,n} = \bigcap_{0 \leq j < n} \overline{B}(k_{s(x,j)}, \frac{1}{j+1})$ is nonempty.
First we let \( S_0 = \{0\} \) and choose an arbitrary \( k_0 \in K \), then these trivially satisfy (I) and (II).

Suppose that we already defined \( S_0, S_1, \ldots, S_{n-1} \) and let \( s \in S_{n-1} \) be arbitrary. Notice that for \( x, x' \in [s] \), \( C_{x,n-1} = C_{x',n-1} \) and denote this common set with \( C_s \). The set \( C_s \) is (nonempty) compact and it is covered by the open sets \( \{B(c, \frac{1}{n}) : c \in C_s\} \), hence we can select a collection \( (c_j^{(s)})_{j \in I_s} \) where \( I_s \) is a finite index set such that \( \bigcup_{j \in I_s} B(c_j^{(s)}, \frac{1}{n}) \supseteq C_s \). We may increase the cardinality of this collection by repeating one element several times if necessary, and thus we can assume that the index set \( I_s \) is of the form \( 2^\ell_s \) for some integer \( \ell_s \geq 1 \) (i.e. it consists of the 0-1 sequences with length \( \ell_s \)). Now we can define \( S_n = \{s \triangle t : s \in S_{n-1}, t \in 2^{\ell_s}\} \). If \( s' \in S_n \), then there is a unique \( s \in S_{n-1} \) such that \( s \preceq s' \), if \( t \) satisfies that \( s' = s \triangle t \) (i.e. \( t \) is the final segment of \( s' \)), then let \( k_{s'} = c_t^{(s)} \).

It is straightforward to check that these choices satisfy (I). Property (II) is satisfied because if \( x \in 2^\omega \), and \( s, s' \) and \( t \) are the sequences that satisfy \( s = s(x, n-1) \) and \( s \triangle t = s' = s(x, n) \), then \( k_{s'} = c_t^{(s)} \in C_{x,n} \), because it is contained in both \( C_{x,n-1} = C_s \) and \( B(k_s, \frac{1}{n}) \), and this shows that \( C_{x,n} \) is nonempty.

We use this construction to define the function \( \varphi \): let \( \{\varphi(x)\} = \bigcap_{n \in \omega} C_{x,n} \) (as the system of nonempty compact sets \( (C_{x,n})_{n \in \omega} \) is descending and the intersection of this system is indeed a singleton). This function is continuous, because if \( \varepsilon > 0 \) and \( \varphi(x) = k \in K \), then \( 2 \cdot \frac{1}{n+1} < \varepsilon \) for some \( n_0 \), and then for every \( x' \in \text{the clopen set } [s(x,n_0)] \) the set \( B(k_{s(x,n_0)}, \frac{1}{n_0+1}) = B(k_{s(x',n_0)}, \frac{1}{n_0+1}) \) has diameter \( \varepsilon \) and contains both \( \varphi(x) \) and \( \varphi(x') \).

Now we prove that if \( U \subseteq 2^\omega \) is open, then \( \varphi(U) \) contains an open set \( V \). It is clear from (I) that \( \{[s] : s \in S\} \) is a base of the topology of \( 2^\omega \). This means that there exist a \( n \in \omega \setminus \{0\} \) and \( s \in S_{n-1} \) satisfying \( [s] \subseteq U \). Let \( V = \bigcap_{0 \leq j < n} B(k_{t_j}, \frac{1}{j+1}) \) where \( t_j = s(x, j) \) for an arbitrary \( x \in [s] \) (this is well-defined as \( t_j \in S_j \) is the only element of \( S_j \) with \( s(x, j) \preceq s(x, n-1) = s \)). It is clear that \( V \) is open and \( V \subseteq C_{x,n-1} = C_s \) (where \( x \in [s] \) is arbitrary, this is again well-defined), to show that \( V \subseteq \varphi(U) \) let \( v \in V \) be arbitrary. Define \( u_0 \in 2^{\ell_s} \) such that \( v \in B(k_s, \frac{1}{n}) \) (this is possible as \( \{B(k_s, \frac{1}{n}) : u_0 \in 2^{\ell_s}\} \) was a cover of \( C_{x,n-1} = C_s \)). Repeat this to define \( u_1 \in 2^{\ell_{u_0}} \) such that \( v \in B(k_{s-u_0}, \frac{1}{n+1}) \), then \( u_2 \in 2^{\ell_{u_0-u_1}} \) such that \( v \in B(k_{s-u_0-u_1}, \frac{1}{n+2}) \) etc. and let \( y \) be the infinite sequence \( y = s \triangle u_0 \triangle u_1 \triangle u_2 \ldots \). It is easy to see that \( \varphi(y) = v \), as \( d(\varphi(y), v) \leq 2 \cdot \frac{1}{n+1} \) for every \( n \in \omega \).

Finally we show that if \( M \in M(K) \) is arbitrary, then \( \varphi^{-1}(M) \in M(2^\omega) \). As \( M \) is meager \( M \subseteq \bigcup_{n \in \omega} F_n \) for a system \( (F_n)_{n \in \omega} \) of nowhere dense closed sets. If \( \varphi^{-1}(M) \) is
not meager, then \( \varphi^{-1}(F_n) \) contains an open set for some \( n \in \omega \). But then \( F_n \) contains an open set, which is a contradiction.

We use this lemma to show that in Definition 1.3 we can also restrict the choice of the compact metric space \( K \). In the following theorem the equivalence \( (1) \Leftrightarrow (2) \) is [33, Proposition 3] and the equivalence \( (1) \Leftrightarrow (3) \) is [29, Theorem 2.11].

**Theorem 2.4.6** (Doležal-Vlasák / Darji). For a Borel set \( B \subseteq G \) the following are equivalent:

1. there exists a (nonempty) compact metric space \( K \) and a continuous function \( f : K \to G \) such that \( f^{-1}(gBh) \) is meager in \( K \) for every \( g, h \in G \) (i.e. \( B \) is Haar meager),
2. there exists a continuous function \( f : 2^\omega \to G \) such that \( f^{-1}(gBh) \) is meager in \( 2^\omega \) for every \( g, h \in G \),
3. there exists a (nonempty) compact set \( C \subseteq G \), a continuous function \( f : C \to G \) such that \( f^{-1}(gBh) \) is meager in \( C \) for every \( g, h \in G \).

**Proof.** (1) \( \Rightarrow \) (2):

This implication is an easy consequence of Lemma 2.4.3. If \( K \) and \( f \) satisfies the requirements of (1) and \( \varphi \) is the function granted by Lemma 2.4.3, then \( \tilde{f} = f \circ \varphi : 2^\omega \to G \) will satisfy the requirements of (2), because it is continuous and for every \( g, h \in G \) the set \( f^{-1}(gBh) \) is meager in \( K \), hence \( \varphi^{-1}(f^{-1}(gBh)) = \tilde{f}^{-1}(gBh) \) is meager in \( 2^\omega \).

(2) \( \Rightarrow \) (3):

If \( G \) is countable, the only Haar meager subset of \( G \) is the empty set. In this case, any nonempty \( C \subseteq G \) and continuous function \( f : C \to G \) is sufficient. If \( G \) is not countable, then it is well known that there is a (compact) set \( C \subseteq G \) that is homeomorphic to \( 2^\omega \). Composing the witness function \( f : 2^\omega \to G \) granted by (2) with this homeomorphism yields a function that satisfies our requirements (together with \( C \)).

(3) \( \Rightarrow \) (1):

This implication is trivial.

---

### 2.5 Haar null sets in \( \mathbb{Z}^\omega \)

In this section we describe another characterization of Haar null sets, which is limited to the frequently studied Polish group \( (\mathbb{Z}^\omega, +) \) (countably infinite sequences of integers with the elementwise addition as the group operation) and uses *witness sequences* instead of witness measures. This characterization will be an important ingredient in the proof of Theorem 3.2.4.
This characterization was published in the paper \[79\]; we will apply it in \textit{chapter 3}.

Let \(\varrho_k\) be the uniform probability measure on \([0, k]\), that is, the (Borel probability) measure on \(\mathbb{Z}\) defined by
\[
\varrho_k(X) = \frac{|X \cap [0, k]|}{k+1}.
\]

If \((a_n)_{n \in \omega}\) is a sequence of positive integers, then let \(\mu_a\) be the Borel probability measure on \(\mathbb{Z}^\omega\) defined as the product \(\bigotimes_{n \in \omega} \varrho_{a_n}\). Clearly
\[
\text{supp} \mu_a = \prod_{n \in \omega} [0, a_n].
\]

The following result characterizes the Borel Haar null subsets of \(\mathbb{Z}^\omega\):

**Theorem 2.5.1.** A Borel subset \(B \subseteq \mathbb{Z}^\omega\) is Haar null if and only if there exists a sequence of positive integers \((a_n)_{n \in \omega}\) such that \(\mu_a(B + x) = 0\) for every \(x \in \mathbb{Z}^\omega\).

**Definition 2.5.2.** We call a sequence \((a_n)_{n \in \omega}\) satisfying this a \textit{witness sequence} for \(B\).

This theorem is motivated by \[88, \text{Theorem 4.1}\], but that result works in a more general setting and shows that one can always choose a witness measure from another class of “simple measures”. It would be possible to modify the proof of \[88, \text{Theorem 4.1}\] to prove our result, but due to technical difficulties we give a different, self-contained proof.

**Proof of Theorem 2.5.1.** The “if” part of the statement is trivial. To prove the “only if” part, assume that \(B\) is a Borel Haar null subset of \(\mathbb{Z}^\omega\).

It is not very hard to prove that every Haar null set has a witness measure with compact support, for a proof of this, see e.g. \[39, \text{Theorem 4.1.4}\]. Using this, let \(\mu\) be a witness measure for \(B\) such that \(\text{supp} \mu\) is compact and therefore its projection \(\{x_n : x \in \text{supp} \mu\} \subseteq \mathbb{Z}\) is compact (i.e. finite) for every \(n \in \omega\).

To simplify the calculations, also suppose that \(\text{supp} \mu\) only contains sequences with nonpositive elements. This is always possible, as we may replace \(\mu\) by \(\mu'(X) = \mu(X + \ell)\) where \(\ell \in \mathbb{Z}^\omega\) is the sequence \(\ell_n = \max\{x_n : x \in \text{supp} \mu\}\).

Let \(M_n = -\min\{x_n : x \in \text{supp} \mu\}\). It is clear that
\[
\text{supp} \mu \subseteq \prod_{n \in \omega} [-M_n, 0].
\]

Choose a sequence \(N_n\) of (large) positive integers such that \(N_n > 2M_n\) (for every \(n \in \omega\)) and moreover
\[
\prod_{n \in \omega} \left(1 - \frac{M_n}{N_n + 1}\right) > 0.
\]
Let $\nu$ be the measure $\nu = \mu * \mu_N$. This is the convolution of Borel probability measures, hence itself a Borel probability measure on $\mathbb{Z}^\omega$. Applying the definition of convolution, then using that $\mu$ is a witness measure we can see that for every $x \in \mathbb{Z}^\omega$

$$\nu(B + x) = \int_{\mathbb{Z}^\omega} \mu(B + x - y) \, d\mu_N(y) = 0,$$

i.e. $\nu$ is also a witness measure for $B$. It is easy to see that

$$\text{supp } \nu \subseteq \prod_{n \in \omega} [-M_n, N_n].$$

The measure $\nu$ is a “uniformized” variant of $\mu$, in fact if we ignore some “border zones” then the measure will be “uniform” on the “central zone” and this central zone will have positive measure. This will allow us to restrict $\nu$ to this central zone, normalize it and get a witness measure that is of the form $\mu_a$ for a witness sequence $a \in \mathbb{Z}_+^\omega$.

This heuristic statement can be formalized as the following claim:

**Claim 2.5.3.** Define $a_n = N_n - M_n$. The set $\text{supp } \mu_a = \prod_{n \in \omega} [0, a_n]$ has positive $\nu$-measure. Moreover,

$$\mu_a(X) = \frac{\nu(X \cap \text{supp } \mu_a)}{\nu(\text{supp } \mu_a)}$$

holds for every $X \subseteq \mathbb{Z}^\omega$ (and these are defined for the same sets).

**Proof.** For every Borel set $X \subseteq \text{supp } \mu_a$ one can apply Fubini’s theorem to get

$$\nu(X) = \int_{\mathbb{Z}^\omega} \mu_N(X - y) \, d\mu(y) = \int_{\text{supp } \mu} \mu_N(X - y) \, d\mu(y).$$

If $y \in \text{supp } \mu$ is arbitrary, then $-M_n \leq y(n) \leq 0$ for every $n \in \omega$, hence

$$\varrho_{N_n}(W - y(n)) = \varrho_{N_n}(W)$$

for every $W \subseteq [0, a_n]$ (using that $a_n = N_n - M_n$ and $\varrho_k$ is a uniform distribution). Moreover, notice that $\mu_N$ is the product of the measures $(\varrho_{N_n})_{n \in \omega}$ and the measure which assigns $\mu_N(P - y)$ to the Borel set $P \subseteq \mathbb{Z}^\omega$ is the product of the measures which assign $\varrho_{N_n}(W - y(n))$ to the set $W \subseteq \mathbb{Z}$. Thus the equality $\varrho_{N_n}(W - y(n)) = \varrho_{N_n}(W)$, which holds for every $W \subseteq [0, a_n]$, yields that these two product measures coincide when they are restricted to Borel subsets of $\prod_{n \in \omega} [0, a_n] = \text{supp } \mu_a$. If we apply this to a Borel subset $X \subseteq \text{supp } \mu_a$, then we get $\mu_N(X - y) = \mu_N(X)$ for every $y \in \text{supp } \mu$. This implies that

$$\nu(X) = \int_{\text{supp } \mu} \mu_N(X) \, d\mu(y) = \mu_N(X). \quad (2.5.1)$$
For \( n \in \omega \) and \( W \subseteq \mathbb{Z} \) it is straightforward from the definitions that
\[
\varrho_{n}(W) = \int_{W} f_{n}(w) \, d\varrho_{N_{n}}(w)
\]
where \( f_{n} : \mathbb{Z} \to \mathbb{R} \) is the density function defined as
\[
f_{n}(w) = \begin{cases} \frac{N_{n}+1}{a_{n}+1} & \text{if } w \in [0, a_{n}], \\ 0 & \text{if } w \in \mathbb{Z} \setminus [0, a_{n}]. \end{cases}
\]

If \( X \subseteq \mathbb{Z}^{\omega} \) is a Borel set, then taking the product of these yields that
\[
\mu_{a}(X) = \int_{X} f(x) \, d\mu_{N}(x)
\]
where \( \lambda = \prod_{n \in \omega} \frac{N_{n}+1}{a_{n}+1} \) and \( f : \mathbb{Z}^{\omega} \to \mathbb{R} \) is the density function
\[
f(x) = \begin{cases} \lambda & \text{if } x(n) \in [0, a_{n}] \text{ for all } n \in \omega, \\ 0 & \text{otherwise.} \end{cases}
\]

Notice that \( \lambda \) is trivially positive, but \( \lambda < \infty \) because we assumed that
\[
\prod_{n \in \omega} \left( 1 - \frac{M_{n}}{N_{n}+1} \right) > 0.
\]

In particular, for a Borel set \( X \subseteq \text{supp} \mu_{a} \) (\( = \prod_{n \in \omega} [0, a_{n}] \)) this means that
\[
\mu_{a}(X) = \lambda \cdot \mu_{N}(X) \quad (2.5.2)
\]

Considering the special case when \( X = \text{supp} \mu_{a} \), we get
\[
\lambda \cdot \mu_{N}(\text{supp} \mu_{a}) = \mu_{a}(\text{supp} \mu_{a}) = 1,
\]
\[
\mu_{N}(\text{supp} \mu_{a}) = \frac{1}{\lambda} > 0. \quad (2.5.3)
\]

Using (2.5.1), (2.5.2) and (2.5.3) we can see that \( \nu(X) = \mu_{a}(X) \cdot \nu(\text{supp} \mu_{a}) \) for every \( X \subseteq \text{supp} \mu_{a} \). Here \( \nu(\text{supp} \mu_{a}) > 0 \) is implied by (2.5.1) and (2.5.3). Therefore for an arbitrary \( X \subseteq \mathbb{Z}^{\omega} \), \( \mu_{a}(X) = \frac{\nu(X \cap \text{supp} \mu_{a})}{\nu(\text{supp} \mu_{a})} \), and this concludes the proof of the claim. \( \square \)

As we return to proving Theorem 2.5.1, we already know that for every \( x \in \mathbb{Z}^{\omega} \), \( \nu(B + x) = 0 \). This implies that for every \( x \in \mathbb{Z}^{\omega} \), \( \nu((B + x) \cap \text{supp} \mu_{a}) = 0 \) and hence (using the recently proved Claim 2.5.3) \( \mu_{a}(B + x) = 0 \). This means that \( a \) is indeed a witness sequence. \( \square \)
Chapter 3

Complexity of hulls

3.1 Analytic and coanalytic hulls

Recall that Definition 1.1 and Definition 1.3 say that a set $A$ is Haar null and, respectively, Haar meager when it has a Borel hull $B \supseteq A$ which satisfies certain conditions (i.e. it is naively Haar null or naively Haar meager). The examples in section 2.3 show that this condition about a “simple” hull is necessary, but the choice of Borel sets as “simple” sets is somewhat arbitrary. Indeed, Definition 1.2.1 introduces the generalized Haar null sets, where universally measurable hulls are used instead of Borel hulls, and that choice also yields a useful notion of smallness.

It is a natural idea to replace Borel hulls with analytic hulls (i.e. continuous images of Borel sets) or coanalytic hulls (i.e. complements of analytic sets). Note that Lusin’s theorem (see [69, 29.7]) implies that all analytic or coanalytic sets are universally measurable and therefore there are no technical difficulties with the witness measures.

It turns out that an analytic hull always implies the existence of a Borel hull:

**Theorem 3.1.1** (Solecki, Doležal-Rmoutil-Vejnar-Vlasák).

1. Every analytic naively Haar null set is Haar null.
2. Every analytic naively Haar meager set is Haar meager.

**Proof.** The claim (1) appears in [87]; while the claim (2) is [88, Proposition 8]. The ideas behind these proofs are almost identical, but due to the limitations of the traditional notions we present them separately.

First, for (1) assume that $A$ is an analytic naively Haar null set and $\mu$ is a Borel probability measure on $G$ such that $\mu(gAh) = 0$ for every $g, h \in G$. We will prove that there exists a Borel set $B \supseteq A$ such that $\mu(gBh) = 0$ is satisfied for every $g, h \in G$. 
Claim 3.1.2. The family of sets

\[ \Phi = \{ X \subseteq G : X \text{ is analytic and } \mu(gXh) = 0 \text{ for every } g, h \in G \} \]

is coanalytic on analytic, that is, for every Polish space \( Y \) and \( P \in \Sigma_1^1(Y \times G) \), the set \( \{ y \in Y : P_y \in \Phi \} \) is \( \Pi_1^1 \).

Proof. Let \( Y \) be Polish space and \( P \in \Sigma_1^1(Y \times G) \) and let

\[ \hat{P} = \{ (g, h, y, \gamma) \in G \times G \times Y \times G : \gamma \in gP_yh \} \]

Then \( \hat{P} \) is analytic, as it is the preimage of \( P \) under \((g, h, y, \gamma) \mapsto (y, g^{-1}h^{-1})\). We will use the following result, which can be found as [69, Theorem 29.26]:

Theorem 3.1.3 (Kondô-Tugué). Let \( X, Y \) be standard Borel spaces and \( A \subseteq X \times Y \) an analytic set. Then the set

\[ \{ (\nu, x, r) \in P(Y) \times X \times \mathbb{R} : \nu(A_{x}) > r \} \]

is analytic. (Standard Borel spaces are introduced in [69, Section 12], we only use the fact that every Polish space is a standard Borel space.)

Using this result yields that

\[ \{ (\nu, g, h, y, r) \in P(G) \times G \times G \times Y \times \mathbb{R} : \nu(\hat{P}_{(g, h, y)}) > r \} \]

is analytic, therefore its section at \((\nu = \mu, r = 0)\), which is the set

\[ \{ (g, h, y) \in G \times G \times Y : \mu(\hat{P}_{(g, h, y)}) > 0 \} \]

is also analytic. Projecting this on \( Y \) yields that

\[ \{ y \in Y : \mu(\hat{P}_{(g, h, y)}) > 0 \text{ for some } g, h \in G \} \]

is analytic, but then

\[ \{ y \in Y : \mu(\hat{P}_{(g, h, y)}) = 0 \text{ for all } g, h \in G \} = \{ y \in Y : P_y \in \Phi \} \]

is coanalytic. \( \square \)

Now, since \( A \in \Phi \), by the dual form of the First Reflection Theorem (see [69, Theorem 35.10] and the remarks following it) there exists a Borel set \( B \) with \( B \supseteq A \) and \( B \in \Phi \), and this \( B \) (together with \( \mu \)) satisfies our requirements.
We note that in section 5.3, we will use a modified variant of this method to prove the stronger result Theorem 5.3.11.

Then, for (2) let us assume that $A$ is an analytic naively Haar meager set and $f$ is a continuous function $f : K \to G$ for some nonempty compact set $K$ such that $f^{-1}(gAh)$ is meager in $K$ for every $g, h \in G$. We will prove that (for this $K$ and $f$) there exists a Borel set $B \supseteq A$ such that $f^{-1}(gBh)$ is meager in $K$ for every $g, h \in G$.

Claim 3.1.4. The family of sets

$$\Phi = \{X \subseteq G : X \text{ is analytic and } f^{-1}(gXh) \text{ is meager in } K \text{ for every } g, h \in G\}$$

is coanalytic on analytic, that is, for every Polish space $Y$ and $P \in \Sigma_1^1(Y \times G)$, the set $\{y \in Y : P_y \in \Phi\}$ is $\Pi_1^1$.

Proof. Let $Y$ be a Polish space and $P \in \Sigma_1^1(Y \times G)$ and let

$$\tilde{P} = \{(g, h, y, k) \in G \times G \times Y \times K : f(k) \in gP_yh\}.$$

Then $\tilde{P}$ is analytic, as it is the preimage of $P$ under $(g, h, y, k) \mapsto (y, g^{-1}f(k)h^{-1})$. Novikov’s theorem (see e.g. [69, Theorem 29.22]) states that if $U$ and $V$ are Polish spaces and $A \subseteq U \times V$ is analytic, then $\{u \in U : A_u \text{ is not meager in } V\}$ is analytic. This yields that $\{(g, h, y) : \tilde{P}_{(g,h,y)} \text{ is meager in } K\}$ is coanalytic, but then

$$\{y \in Y : \tilde{P}_{(g,h,y)} \text{ is meager in } K \text{ for every } g, h \in G\} = \{y \in Y : P_y \in \Phi\}$$

is also coanalytic.

Now, as in the proof of (1), since $A \in \Phi$, we may apply the dual form of the First Reflection Theorem (see [69, Theorem 35.10 and the remarks following it]) and conclude that there exists a Borel set $B$ with $B \supseteq A$ and $B \in \Phi$. This $B$ clearly satisfies our requirements.

On the other hand, the existence of a coanalytic hull does not imply the existence of a Borel hull:

Theorem 3.1.5 (Elekes-Vidnyánszky, Doležal-Vlasák). Assume that $G$ is a non-locally-compact abelian Polish group.

(1) There exists a coanalytic naively Haar null set $A \subseteq G$ which is not Haar null.

(2) There exists a coanalytic naively Haar meager set $A \subseteq G$ which is not Haar meager.
The proof of claim (1) can be found in [17], while claim (2) is [39, Theorem 13]. We do not include these relatively long proofs, but we remark that they were proved with methods that are analogous to the calculations in section 3.3 and section 3.2. Also note that these theorems were proved in the abelian case, but an analogous proof would work in the case when $G$ is TSI.

Notice that a set $A \subseteq G$ provided by claim (1) of Theorem 3.1.5 is generalized Haar null, but not Haar null:

**Corollary 3.1.6.** If $G$ is non-locally-compact and abelian, then $\mathcal{GHN}(G) \supseteq \mathcal{HN}(G)$.

This (and its generalization to TSI groups) is the best known result for [17, Question 5.4], which asks the following:

**Question 3.1.7 (Elekes-Vidnyánszky).** Is $\mathcal{GHN}(G) \supseteq \mathcal{HN}(G)$ in all non-locally-compact Polish groups?

### 3.2 Ranks of hulls in the Borel hierarchy

In [Definition 2.2.2] we required that every Haar measures must be regular, i.e. if $G$ is a locally compact Polish group, $\mu$ is a left or right Haar measure on $G$ and $A \subseteq G$ is $\mu$-measurable, then

$$\mu(A) = \inf\{\mu(U) : A \subseteq U, U \text{ is open}\}.$$  

This immediately implies that if $A \subseteq G$ is $\mu$-measurable, then there exists a $G_\delta$ set $A' \supseteq A$ such that $\mu(A') = \mu(A)$, in particular if $A$ is a set of Haar measure zero, then it is contained in a $G_\delta$ set of Haar measure zero.

This naturally inspires the question [78, $P_1$]:

**Question 3.2.1 (Mycielski).** Suppose that $G$ is a Polish group and $Y \subseteq G$ is Haar null. Does there exist a $G_\delta$ Haar null set including $Y$?

In the case of abelian Polish groups this was answered by [17, Theorem 1.3]:

**Theorem 3.2.2 (Elekes-Vidnyánszky).** If $G$ is a non-locally-compact abelian Polish group then there exists a Borel Haar null set $B \subseteq G$ that cannot be covered by a $G_\delta$ Haar null set. In fact, for each $1 \leq \xi < \omega_1$ there exists a Haar null Borel set that cannot be covered by a $\Pi^0_\xi$ Haar null set.

The paper [39] also proved the analog of this result for Haar meager sets:

**Theorem 3.2.3 (Doležal-Vlasák).** Let $G$ be a non-locally compact abelian Polish group and $1 \leq \xi < \omega_1$. Then there is a strongly Haar meager Borel set that cannot be covered by a $\Sigma^0_\xi$ Haar meager set.
Note that the measure $\leftrightarrow$ category duality usually involves a $\Sigma \leftrightarrow \Pi$ swap; therefore it is not surprising that this theorem considers $\Sigma_0^\xi$ hulls instead of $\Pi_0^\xi$ hulls. The notion of strongly Haar meager sets is described in Chapter 4; every strongly Haar meager set is Haar meager.

We do not include proofs of these two theorems, but we prove the following result, which is a refined version of Theorem 3.2.2.

**Theorem 3.2.4.** In the (non-locally-compact abelian Polish) group $\mathbb{Z}^\omega$ for every $2 \leq \xi < \omega_1$ there exists a Haar null set $E$ that is the difference of two $\Pi_0^\xi$ sets but is not contained in any $\Pi_0^\xi$ Haar null set.

Compared to Theorem 3.2.2, this theorem uses more detailed calculations to derive a sharper upper bound for the complexity of the example set $E$. These calculations rely on the characterization introduced in Section 2.3, which is limited to the well-known and important non-locally-compact abelian Polish group $\mathbb{Z}^\omega$. Following e.g. the ideas in [88, Theorem 4.1] it would be possible to prove a similar result in a more general setting, but that would introduce more technical difficulties.

**Proof.** First we will construct some simple functions which will act as “building blocks” in the construction of $E$.

Let us define the function

$$\theta : \mathbb{Z}_+ \times \{0,1\} \times \mathbb{Z} \to \mathbb{Z}, \quad \theta(n,b,z) = (n-1)(n+4) + b(n+2) + z.$$ 

Elementary calculations show that $\theta(n,1,0) = \theta(n,0,0) + (n+2)$, $\theta(n+1,0,0) = \theta(n,1,0) + (n+2)$ and hence if we restrict $\theta$ to the set

$$T = \{(n,b,z) : n \in \mathbb{Z}_+, b \in \{0,1\}, z \in [0,n+1]\},$$

then $\theta(T) = \mathbb{N}$ is the set of nonnegative integers, and the restricted function $\theta|T$ is an order preserving bijection (when $T$ is ordered lexicographically and $\mathbb{N}$ has its usual ordering). Let $\iota : \mathbb{N} \to T$ be the inverse of this restriction.

We can let $\theta$ act elementwise on sequences of length $\omega$, that is, we can define

$$t : \mathbb{Z}_+^\omega \times 2^\omega \times \mathbb{Z}^\omega \to \mathbb{Z}^\omega, \quad (t(a,x,g))(k) = \theta(a(k),x(k),g(k)) \text{ for all } k \in \omega.$$ 

Later we will use the fact that if $a \in \mathbb{Z}_+^\omega$ and $x \in 2^\omega$ are fixed, then $t(a,x,g) = t(a,x,0) + g$, i.e. $g \mapsto t(a,x,g)$ is a translation.

Analogously, we may also let $\iota$ act elementwise on sequences of length $\omega$ to get a function $i : \mathbb{N}^\omega \to T^\omega$. It is clear that both $t$ and $i$ are continuous (in fact, Lipschitz).
With a slight abuse of notation let us identify $T^\omega$ and the set
\[ T = \{(a,x,g) \in \mathbb{Z}_+^\omega \times 2^\omega \times \mathbb{Z}^\omega : (\forall k)(g(k) \in [0,a(k) + 1])\} \]
($T^\omega$ contains sequences of triples, $T$ contains triples of sequences, the natural map between them is a homeomorphism). As $i$ is the inverse of a restriction of $\theta$, the same holds for $i$ and $t$: every $(a,x,g) \in T$ satisfies $i(t(a,x,g)) = (a,x,g)$ and every $s \in \mathbb{N}^\omega$ satisfies $t(i(s)) = s$.

Let us fix a partial function $f : \mathbb{Z}_+^\omega \times 2^\omega \mapsto \mathbb{Z}^\omega$ which satisfies the conditions of the following theorem:

**Theorem 3.2.5.** If $2 \leq \xi < \omega_1$, then there exists a partial function $f : \mathbb{Z}_+^\omega \times 2^\omega \mapsto \mathbb{Z}^\omega$ which satisfies the following properties:

(i) $f$ is $\Sigma_0^0$-measurable and $\text{graph}(f)$ is the difference of two $\Pi_0^0$ subsets of $\mathbb{Z}_+^\omega \times 2^\omega \times \mathbb{Z}^\omega$;

(ii) if $(a,x) \in \text{dom}(f)$, then $f(a,x) \in \text{supp} \mu_a$;

(iii) if $a \in \mathbb{Z}_+^\omega$ and $S \in \Pi_0^0(2^\omega \times \mathbb{Z}^\omega)$ satisfy that $\text{graph}(f_a) \subseteq S$, then there exists $x \in 2^\omega$ such that $\mu_a(S_x) > 0$.

(Here for $a \in \mathbb{Z}_+^\omega$, $f_a$ denotes the partial function $f_a : 2^\omega \mapsto \mathbb{Z}^\omega$, $f_a(x) = f(a,x)$.)

We will prove this theorem later in section 3.3. With the help of this function $f$ we can define $E$ as

\[ E = t(\text{graph}(f)) = \{t(a,x,g) : (a,x,g) \in \text{graph}(f)\} = \{t(a,x,f(a,x)) : (a,x) \in \text{dom}(f)\}. \]

**Claim 3.2.6.** $E$ is the difference of two $\Pi_0^0$ subsets of $\mathbb{Z}^\omega$.

**Proof.** If we apply first the definition of $\mathcal{T}$ and then Property (ii) of Theorem 3.2.5, then we get

\[ \mathcal{T} \ni \{(a,x,g) \in \mathbb{Z}_+^\omega \times 2^\omega \times \mathbb{Z}^\omega : g \in \text{supp} \mu_a\} \ni \text{graph}(f). \]

This implies that $\text{graph}(f) = i(t(\text{graph}(f))) = i(E)$ and hence

\[ E = i^{-1}(\text{graph}(f)). \]

As Property (i) states that $\text{graph}(f) \subseteq \mathcal{T}$ is the difference of two $\Pi_0^0$ subset of $\mathbb{Z}_+^\omega \times 2^\omega \times \mathbb{Z}^\omega$, it is also the difference of two $\Pi_0^0$ subsets of $\mathcal{T}$ (using that $\mathcal{T}$ is closed). This means that its preimage under the continuous function $i : \mathbb{N}^\omega \mapsto \mathcal{T}$ is the difference of two $\Pi_0^0$ subsets of $\mathbb{N}^\omega$. As $\mathbb{N}^\omega$ is a closed subset of $\mathbb{Z}^\omega$, this yields that $E = i^{-1}(\text{graph}(f))$ is indeed the difference of two $\Pi_0^0$ subsets of $\mathbb{Z}^\omega$. \qed
Claim 3.2.7. E is Haar null.

Proof. We will show that \( a_0 = (1,1,\ldots) \) is a witness sequence for this fact. (By the way, the corresponding witness measure, \( \mu_{a_0} \) is just the usual coin-flip measure with \( \text{supp}(\mu_{a_0}) = \{0,1\}^\omega \subset \mathbb{Z}^\omega \).) It is clearly sufficient to show that \(|(E+r) \cap \{0,1\}^\omega| \leq 1\) for all \( r \in \mathbb{Z}^\omega \). This is equivalent to saying that if \( e, e' \in E \) and \( e \neq e' \), then \(|e(k) - e'(k)| \geq 2\) for at least one \( k \in \omega \).

Fix arbitrary \( e, e' \in E \) with \( e \neq e' \). By the definition of \( E \) there are \( a, a' \in \mathbb{Z}^\omega \) and \( x, x' \in \mathbb{Z}^\omega \) such that \( e = t(a, x, f(a, x)) \) and \( e' = t(a', x', f(a', x')) \). As we assumed that \( e \neq e' \), we can find a \( k \in \omega \) where \( (a(k), x(k)) \neq (a'(k), x'(k)) \). Without loss of generality, we may assume that \( (a(k), x(k)) < (a'(k), x'(k)) \) lexicographically. By Property (ii) of Theorem 3.2.5 we know that \( f(a, x) \in \text{supp}(\mu_a) = \prod_{k \in \omega} [0, a(k)] \), hence \( 0 \leq (f(a, x))(k) \leq a(k) \) and analogously \( 0 \leq (f(a', x'))(k) \leq a'(k) \). Straightforward and elementary calculations (using these bounds and the definition of \( t \) and \( \theta \)) show that \( e(k) + 2 \leq e'(k) \) both in the case when \( a(k) < a'(k) \) and in the case when \( a(k) = a'(k) \) (and hence \( x(k) < x'(k) \), i.e. \( x(k) = 0 \) and \( x'(k) = 1 \)). These allow us to conclude that \( E \) is indeed Haar null. \( \square \)

Claim 3.2.8. There is no Haar null set \( H \in \Pi^0_\xi(\mathbb{Z}^\omega) \) containing \( E \).

Proof. Suppose that \( H \in \Pi^0_\xi \) is such a set. By Theorem 2.5.1 there exists a witness sequence \( a \in \mathbb{Z}^\omega \) such that \( \mu_a(H + r) = 0 \) for every \( r \in \mathbb{Z}^\omega \). As \( t \) is continuous, the section map \( t_a : 2^\omega \times \mathbb{Z}^\omega \to \mathbb{Z}^\omega \), \( (x, g) \mapsto t(a, x, g) \) is also continuous, hence \( S = t_a^{-1}(H) \subseteq 2^\omega \times \mathbb{Z}^\omega \) is a \( \Pi^0_\xi \) set.

It is easy to check that \( \text{graph}(f_a) \subseteq S \), and therefore by Property (iii) of Theorem 3.2.5 there exists an \( x^* \in 2^\omega \) such that \( \mu_a(S_{x^*}) > 0 \). By the definition of \( S \), \( t(a, x^*, S_{x^*}) \subseteq t_a(S) \subseteq H \). But \( g \mapsto t(a, x^*, g) \) is a translation, so a translate of \( H \) contains \( S_{x^*} \), but \( S_{x^*} \) has positive \( \mu_a \)-measure and hence this contradicts that \( a \) is a witness sequence for \( H \). \( \square \)

This concludes the proof of Theorem 3.2.4. \( \square \)

3.3 A function with a surprisingly thick graph

In this section we will prove Theorem 3.2.5. This result was inspired by the result [17, Theorem 3.1], which constructs a similar partial function \( f \), but uses methods that do not provide information about the Borel rank of \( \text{graph}(f) \).
Theorem 3.2.5. If $2 \leq \xi < \omega_1$, then there exists a partial function $f : \omega_2 \times 2^\omega \to \omega$ which satisfies the following properties:

(i) $f$ is $\Sigma^0_\xi$-measurable and $\text{graph}(f)$ is the difference of two $\Pi^0_\xi$ subsets of $\omega_2 \times 2^\omega \times \omega$;

(ii) if $(a,x) \in \text{dom}(f)$, then $f(a,x) \in \text{supp} \mu_a$;

(iii) if $a \in \omega_2$ and $S \in \Pi^0_\xi(2^\omega \times \omega)$ satisfy that $\text{graph}(f_a) \subseteq S$, then there exists $x \in 2^\omega$ such that $\mu_a(S_x) > 0$.

(Here for $a \in \omega_2$, $f_a$ denotes the partial function $f_a : 2^\omega \to \omega$, $f_a(x) = f(a,x)$.)

We remark that a partial function $f : X \to Y$ is $\Sigma^0_\xi$-measurable if the preimages of open subsets of $Y$ are $\Sigma^0_\xi$ subsets of $X$.

The proof will use a large section uniformization result by Holický. We will apply it in the form of the following statement, which is an immediate corollary of the results in [61].

Corollary 3.3.1. Assume that $X$ and $Y$ are Polish spaces, $2 \leq \alpha \leq \omega_1$, and $\mu : X \times B(Y) \to [0,1]$ satisfies

(a) $\mu(x, \cdot)$ is a Borel probability measure on $Y$ for every $x \in X$, and

(b) $\{x \in X : \mu(x,H) > r\}$ is open for every open $H \subset Y$ and $r \in \mathbb{R}$.

Assume that $A \in \Sigma^0_\alpha(X \times Y)$ and define $P = \{p \in X : \mu(p,A_p) > 0\}$.

Then there exists a partial function $f : X \to Y$ such that $f$ is $\Sigma^0_\alpha$-measurable, $\text{dom}(f) = P$ and $\text{graph}(f) \subseteq A$. Moreover, this partial function also satisfies that $\text{graph}(f) \in \Pi^0_\alpha(P \times Y)$.

Proof of Corollary 3.3.1. To prove that this holds for some $X$, $Y$, $\alpha$, $\mu$, and $A$, first apply [61, Lemma 2.1] (and the remarks preceding it) for $X$, $Y$, $\alpha$, $\mu$, $B = A$, and $\alpha_0 = 1$. The lemma yields that $P \in \Sigma^0_\alpha(X)$, because $\alpha_0 = 1$ implies $\alpha^* = \alpha$ (here “$B$”, “$\alpha_0$”, and “$\alpha^*$” are introduced in the statement of the lemma).

This means we can apply [61, Theorem 3.5] for $X$, $Y$, $\alpha$, $\mu$, $B = A \cap (P \times Y)$, and $\alpha_0 = 1$ and choose $f = \xi$ (here “$B$”, “$\alpha_0$”, “$\alpha^*$”, and a function “$\xi$” are introduced in the statement of the theorem, the theorem states that $\xi$ has all the necessary properties; we use that $\alpha^* = \alpha$ holds again).}

Proof of Theorem 3.2.5. Let $U \in \Sigma^0_\xi(2^\omega \times 2^\omega \times \omega)$ be universal for the $\Sigma^0_\xi$ subsets of $2^\omega \times \omega$, that is, for every $B \in \Sigma^0_\xi(2^\omega \times \omega)$ there exists $x \in 2^\omega$ such that $U_x = B$ (for the existence of such a set see [61, Theorem 22.3]). The preimage of this set under the continuous map $(x,g) \mapsto (x,x,g)$ is the set

$$U' = \{(x,g) \in 2^\omega \times \omega : (x,x,g) \in U\},$$
which is hence a $\Sigma^0_\xi$ set. Later we will use that $U'_x = U_{x,x}$ for every $x \in 2^\omega$.

Notice that the set
\[
S = \{(a, x, g) \in \mathbb{Z}_+^\omega \times 2^\omega \times \mathbb{Z}^\omega : g \in \text{supp}\mu_a\}
\]
is equal to
\[
\{(a, x, g) \in \mathbb{Z}_+^\omega \times 2^\omega \times \mathbb{Z}^\omega : (\forall n \in \omega)(0 \leq g(n) \leq a(n))\}
\]
and hence is trivially closed.

We can combine these to define the $\Sigma^0_\xi$ set
\[
U'' = (\mathbb{Z}_+^\omega \times U') \cap S \quad (\subseteq \mathbb{Z}_+^\omega \times 2^\omega \times \mathbb{Z}^\omega).
\]

We will apply Corollary 3.3.1 for the Polish spaces $X = \mathbb{Z}_+^\omega \times 2^\omega$ and $Y = \mathbb{Z}^\omega$, the map $\mu : \mathbb{Z}_+^\omega \times 2^\omega \times B(\mathbb{Z}^\omega) \to [0, 1]$ defined by $\mu((a, x), S) = \mu_a(S)$, the set $A = U''$ and $\alpha = \xi$.

As in Corollary 3.3.1, define $P = \{p \in \mathbb{Z}_+^\omega \times 2^\omega : \mu(p, A_p) > 0\}$. It is clear that condition (a) is satisfied.

**Claim 3.3.2.** Condition (b) of Corollary 3.3.1 is also satisfied, that is, $\{(a, x) \in \mathbb{Z}_+^\omega \times 2^\omega : \mu((a, x), H) > r\}$ is open for every open $H \subset \mathbb{Z}^\omega$ and $r \in \mathbb{R}$.

**Proof.** $H$ can be written as $H = \bigcup_{i \in \omega}[s_i]$ for some sequences $s_i \in \mathbb{Z}^{<\omega}$ and we may also assume that this union is disjoint. Then $\mu((a, x), H) = \mu_a(H) = \sum_{i \in \omega} \mu_a([s_i])$.

Notice that for $K \in \omega$, the set $\{(a, x) \in \mathbb{Z}_+^\omega \times 2^\omega : \sum_{i \in K} \mu_a([s_i]) > r\}$ is open, as $\sum_{i \in K} \mu_a([s_i])$ depends only on the first $\max_{i \in K} |s_i|$ elements of the sequence $a$. This means that
\[
\{(a, x) \in \mathbb{Z}_+^\omega \times 2^\omega : \mu((a, x), H) > r\}
= \left\{(a, x) \in \mathbb{Z}_+^\omega \times 2^\omega : \sum_{i \in \omega} \mu_a([s_i]) > r\right\}
= \bigcup_{K \in \omega} \left\{(a, x) \in \mathbb{Z}_+^\omega \times 2^\omega : \sum_{i \in K} \mu_a([s_i]) > r\right\}
\]

is a union of open sets and this proves our claim. \qed

The application of Corollary 3.3.1 proves the existence of a partial function $f : \mathbb{Z}_+^\omega \times 2^\omega \to \mathbb{Z}^\omega$ such that $f$ is $\Sigma^0_\xi$-measurable, $\text{dom}(f) = P$, $\text{graph}(f) \subseteq A$, and $\text{graph}(f) \in \Pi^0_\xi(P \times \mathbb{Z}^\omega)$. To finish the proof of Theorem 3.2.5 we show that this $f$ satisfies Properties (i)–(iii).
As $P = \text{dom}(f) = f^{-1}(\mathbb{Z}_\omega)$ is a $\Sigma^0_\xi$ set, $\text{graph}(f) \in \Pi^0_\xi(P \times \mathbb{Z}_\omega)$ is clearly the difference of two $\Pi^0_\xi$ subsets of $\mathbb{Z}^\omega_\xi \times 2^\omega \times \mathbb{Z}_\omega$, concluding the proof of Property (i).

Property (ii) is clear because for all $(a, x) \in \text{dom}(f)$,

$$f(a, x) \in U''_{a, x} = U'_x \cap \text{supp}(\mu_a) \subseteq \text{supp} \mu_a.$$ 

To prove Property (iii), suppose to the contrary that there exists $a \in \mathbb{Z}^\omega_\xi$ and $S \in \Pi^0_\xi(2^\omega \times \mathbb{Z}_\omega)$ such that $\text{graph}(f_a) \subseteq S$ but for all $x \in 2^\omega$, $\mu_a(S_x) = 0$. The complement of $S$ is the $\Sigma^0_\xi$ set $B = (2^\omega \times \mathbb{Z}_\omega) \setminus S$. By the universality of $U$, there is an $x^* \in 2^\omega$ such that $U_{x^*} = B$. We know that for every $x \in 2^\omega$, $\mu_a(B_x) = 1 - \mu_a(S_x) = 1 > 0$, in particular $\mu_a(B_x) = \mu_a(U_{x^*, x^*}) = \mu_a(U'_{x^*}) > 0$. It is clear from the definitions that $U''_{a, x^*} = U'_{x^*} \cap \text{supp}(\mu_a)$, and so $\mu_a(U''_{a, x^*}) > 0$.

Therefore $(a, x^*) \in P = \text{dom}(f)$ and then $\text{graph}(f) \subseteq U''$ yields that

$$f(a, x^*) \in U''_{a, x^*} \subseteq U'_{x^*}, = U_{x^*, x^*} = B_{x^*}.$$ 

But we also supposed that $\text{graph}(f_a) \subseteq S$, and this yields $f(a, x^*) \in S_{x^*} = \mathbb{Z}_\omega \setminus B_{x^*}$, a contradiction.

This proves that $f$ indeed satisfies the requirements of Theorem 3.2.5.

### 3.4 Related results and questions

Theorem 3.2.4 was motivated by the following question:

**Question 3.4.1** (Elekes-Vidnyánszky). What is the least complexity of a Haar null set that cannot be covered by a $G_\delta$ Haar null set? And in general, what is the least complexity of a Haar null set that cannot be covered by a $\Pi^0_\xi$ Haar null set?

Our result implies that if $2 \leq \xi < \omega_1$, then there is a $\Delta^0_{\xi+1}$ Haar null subset of $\mathbb{Z}_\omega$ that is not contained in any $\Pi^0_\xi$ Haar null set. This does not give a complete answer for the question, but narrows down the possibilities to two classes of the Borel hierarchy: the least possible complexity class of a Haar null set that is not contained in a $\Pi^0_\xi$ Haar null set must be either $\Sigma^0_\xi$ or $\Delta^0_{\xi+1}$.

Moreover, our result shows that even if we subdivide $\Delta^0_{\xi+1}$ into the classes of the so-called difference hierarchy, our example is in one of the lowest classes.

Therefore in $\mathbb{Z}_\omega$ essentially only the following question is left open:

**Question 3.4.2.** For a given $2 \leq \xi < \omega_1$, is there a $\Sigma^0_\xi$ Haar null set in $\mathbb{Z}_\omega$ that cannot be covered by a $\Pi^0_\xi$ Haar null set?
The cases of other non-locally-compact abelian Polish groups are also left open. As $\mathbb{Z}^\omega$ is among the “nicest” non-locally-compact Polish groups, it is plausible that the answer will be similar in those other groups.

While the questions of Elekes and Vidnyánszky only consider the abelian case, it is natural to generalize them for arbitrary Polish groups:

**Question 3.4.3.** For a given Polish group $G$ and $2 \leq \xi < \omega_1$, what is the least complexity of a Haar null set in $G$ that cannot be covered by a $\Pi^0_\xi$ Haar null set?

The paper [5] examines this generalized question and proves the following result, answering Question 3.4.3 for a particular (non-abelian, Polish) group $H$ and $\xi = 2$:

**Theorem 3.4.4 (Banakh).** There exists a Polish meta-abelian group $H$ containing a subgroup $F \subset H$ such that $F$ is a $F_\sigma$ Haar null set in $H$ but every $G_\delta$ set $G \subset H$ containing $B$ is thick and hence is not Haar null in $H$.

(A topological group $H$ is called *meta-abelian* if it contains a closed normal abelian subgroup $A \subset H$ such that the quotient group $H/A$ is abelian.)

This paper also mentions the following variant of this question:

**Question 3.4.5 (Elekes-Vidnyánszky, Banakh).** Is each countable subset of an uncountable Polish group $G$ contained in a $G_\delta$ Haar null subset of $G$?

However, in Theorem 6.4.12 we will prove that this happens to be equivalent to the well-known question Question 6.4.1, which we will discuss in section 6.4.

It is natural to ask whether Theorem 3.2.4 has an analog for Haar meager sets:

**Question 3.4.6.** For a given Polish group $G$ and $1 \leq \xi < \omega_1$, what is the least complexity of a Haar meager set in $G$ that cannot be covered by a $\Sigma^0_\xi$ Haar meager set?
Chapter 4

Strongly Haar meager sets

Strongly Haar meager sets are the variant of Haar meager sets where we require the witness function to be the identity function of $G$ restricted to a compact subset $C \subseteq G$. This is motivated by the fact that when we prove that some set is Haar meager, we frequently use witness functions of this kind.

4.1 Overview

Definition 4.1.1. A set $A \subseteq G$ is said to be strongly Haar meager if there are a Borel set $B \supseteq A$ and a (nonempty) compact set $C \subseteq G$ such that $gBh \cap C$ is meager in $C$ for every $g, h \in G$.

The following basic question is [29, Problem 2]:

Question 4.1.2 (Darji). Is every Haar meager set strongly Haar meager?

(Note that the paper [29] only considered the case of abelian groups.)

The result [6, Theorem 5.13] shows that in a certain class of abelian Polish groups the Haar meager sets and the strongly Haar meager sets coincide:

Definition 4.1.3. A topological group $G$ is called hull-compact if each compact subset of $G$ is contained in a compact subgroup of $G$.

Theorem 4.1.4 (Banakh-Głąb-Jabłońska-Swaczyna). If the abelian Polish group $G$ is hull-compact, then every Haar meager subset of $G$ is strongly Haar meager.

Example 4.1.5. It is not very hard to verify that the abelian Polish group $(\mathbb{Q}/\mathbb{Z})^\omega$ is hull-compact (where we endow $\mathbb{Q}$ with the discrete topology).
However, we answer the question of Darji negatively by constructing a counterexample in the group $\mathbb{Z}^\alpha$:

**Theorem 4.1.6.** In the abelian Polish group $\mathbb{Z}^\omega$, there exists a $G_\delta$ set $R$ that is Haar meager but not strongly Haar meager.

In **Theorem 4.4.7** we also prove that this counterexample is as simple as possible: every $F_\sigma$ Haar meager set is strongly Haar meager. We also note that in **Claim 4.4.5** we prove that our example is in fact a so-called Haar nowhere dense set.

As an additional motivation, we mention that according to part (2) of $\mathbb{[6, Theorem 13.8]}$ the generic variants of these notions (when we require not only one witness, but comeager many witnesses) do coincide:

**Theorem 4.1.7** (Banakh-Głąb-Jabłońska-Swaczyna). Assume that $G$ is an abelian Polish group and $B \subseteq G$ is a Borel set. Then the following are equivalent:

1. in the Polish space $C(\{0,1\}^\omega, G)$ of continuous functions from $\{0,1\}^\omega$ to $G$ (endowed with the compact-open topology) the set
   \[ \{ f \in C(\{0,1\}^\omega, G) : f \text{ witnesses that } B \text{ is Haar meager} \} \text{ is comeager}, \]

2. in the Polish space $\mathcal{K}(G)$ of nonempty, compact subsets of $G$
   \[ \{ C \in \mathcal{K}(G) : C \text{ witnesses that } B \text{ is strongly Haar meager} \} \text{ is comeager}. \]

In (1) we only considered the potential witness functions whose domain is the Cantor set $\{0,1\}^\omega$. This is a natural restriction, as according to **Theorem 2.4.6** every Haar meager set has a witness function whose domain is $\{0,1\}^\omega$.

### 4.2 Translating the compact sets apart

This section is motivated by the results and ideas in $\mathbb{[47, 87, 88]}$. These papers prove slightly weaker claims than our **Theorem 4.2.1**, but they work in a more general setting. (Our proof relies on the structure of $\mathbb{Z}^\omega$ to make the calculations shorter and simpler.)

Let
\[ H = \{ (K, x) \in \mathcal{K}(\mathbb{Z}^\omega) \times \mathbb{Z}^\omega : x \in K \}. \]

Note that $H$ is a closed set in the product space $\mathcal{K}(\mathbb{Z}^\omega) \times \mathbb{Z}^\omega$.

**Theorem 4.2.1.** There exists a map $t : \mathcal{K}(\mathbb{Z}^\omega) \to \mathbb{Z}^\omega$ so that the map $T : H \to \mathbb{Z}^\omega$ defined by
\[ T(K, x) = x + t(K) \]
is a homeomorphism between $H$ and the set

$$F = T(H) = \bigcup_{K \in K(\mathbb{Z}^\omega)} (K + t(K)),$$

where the union is disjoint. Moreover, $F$ is a closed subset of $\mathbb{Z}^\omega$ and satisfies that

$$(K + t(K) + \{-1, 0, 1\}^\omega) \cap F = K + t(K)$$

for each $K \in K(\mathbb{Z}^\omega)$.

Proof. As Fact 1.1.2 states that $K(\mathbb{Z}^\omega)$ is zero-dimensional, we may apply [63, Theorem 7.2] to get an embedding

$$c : K(\mathbb{Z}^\omega) \to \{-1, 1\}^\omega.$$ 

Define the (clearly continuous) function

$$b : K(\mathbb{Z}^\omega) \to \mathbb{N}^\omega, \quad b(K)_n = \max\{|x_n| : x \in K\} + 1.$$ 

For each $n \in \omega$ let

$$t(K)_n = 3 \cdot b(K)_n \cdot c(K)_n.$$ 

It is clear that $t$ is continuous and if we introduce the map

$$T : H \to \mathbb{Z}^\omega, \quad T(K, x) = x + t(K)$$

then it is also continuous and satisfies that

$$T(H) = \bigcup_{K \in K(\mathbb{Z}^\omega)} (K + t(K)).$$

Fact 4.2.2. Assume that $(K, x) \in H$, $y = T(K, x)$ and $n \in \omega$. Then it is easy to verify that

1. $|y_n| > b(K)_n$ and
2. $c(K)_n = \begin{cases} +1 & \text{if } y_n > 0, \\ -1 & \text{if } y_n < 0. \end{cases}$

Claim 4.2.3. $T$ is injective.

Proof. Assume that $(K, x), (K', x') \in H$ and $T(K, x) = T(K', x')$. Using part (2) of Fact 4.2.2, $c(K) = c(K')$, but then $K = K'$, as $c$ is injective. This also implies that

$$x = T(K, x) - t(K) = T(K', x') - t(K') = x'.$$
Claim 4.2.4. $F = T(H)$ is closed and the map $T^{-1} : F \to H$ is continuous.

Proof. Assume that $y^{(m)} \in F$ for each $m \in \omega$ and this sequence converges to some $y^* \in \mathbb{Z}^\omega$. As $F = T(H)$, there are compact sets $K^{(m)} \in \mathcal{K}(\mathbb{Z}^\omega)$ and sequences $x^{(m)} \in K^{(m)}$ such that $y^{(m)} = T(K^{(m)}, x^{(m)})$ for each $m \in \omega$. It is sufficient to prove that $(K^{(m)}, x^{(m)})_{m \in \omega}$ converges to some $(K^*, x^*) \in H$. (If this holds, then $y^* = T(K^*, x^*) \in F$ also demonstrates that $F$ is closed.)

Using part (2) of Fact 4.2.2 and the convergence of $y^{(m)}$ yields that $(c(K^{(m)}))_{m \in \omega}$ converges to some element $\gamma \in \{-1, 1\}^\omega$. Our next step is to prove that $\gamma$ is contained in the image of $c$.

Notice that

$$f : \omega \to \mathbb{N}, \quad f(n) = \sup_{m \in \omega} |y_n^{(m)}|$$

is a well-defined function, because for each $n \in \omega$ the sequence $(|y_n^{(m)}|)_{m \in \omega}$ is convergent and therefore bounded. Using part (1) of Fact 4.2.2,

$$f(n) \geq |y_n^{(m)}| > b(K^{(m)})_n \quad \text{for each } n, m \in \omega.$$

Applying the definition of $b$, this implies that for each $m \in \omega$,

$$K^{(m)} \subseteq \mathcal{K}\left\{ z \in \mathbb{Z}^\omega : |z_n| < f(n) \text{ for each } n \in \omega \right\}.$$

As the (nonempty) compact subsets of a compact space form a compact space themselves, $(K^{(m)})_{m \in \omega}$ has a subsequence that converges to some compact set $K^*$. Applying this, the continuity of $c$ and the fact that $\lim_{m \in \omega} c(K^{(m)}) = \gamma$ exists, we obtain that

$$\gamma = \lim_{m \in \omega} c(K^{(m)}) = c(K^*).$$

As $c$ was an embedding, this implies that $K^{(m)}$ converges to $K^*$ (when $m \to \infty$).

As $t$ is continuous and $y^{(m)}$ is convergent, this implies that $x^{(m)} = y^{(m)} - t(K^{(m)})$ is also convergent to some $x^* \in \mathbb{Z}^\omega$. Finally $(K^*, x^*) \in H$ follows from the fact that $(K^{(m)}, x^{(m)}) \to (K^*, x^*)$ and $H$ is a closed set. $\square$

Now we know that $F$ is closed and $T$ is a homeomorphism between $H$ and $F$. To conclude the proof of Theorem 4.2.1, we will fix an arbitrary $K \in \mathcal{K}(\mathbb{Z}^\omega)$ and show that

$$(K + t(K) + \{-1, 0, 1\}^\omega) \cap F = (K + t(K)).$$

Fix an arbitrary $z \in (K + t(K) + \{-1, 0, 1\}^\omega) \cap F$. There are $x \in K$, $\varepsilon \in \{-1, 0, 1\}^\omega$, and
$K' \in \mathcal{K}(\mathbb{Z}^\omega)$ and $x' \in K'$ such that
\[ z = x + t(K) + \varepsilon = x' + t(K'). \]

For each $n \in \omega$ we may use both parts of Fact 4.2.2 and the fact that $b(K)_n \geq 1$ to verify that
\[ c(K)_n = 1 \iff z_n > 0 \iff c(K')_n = 1. \]

As $c$ is injective, this means that $K' = K$, so $z \in K + t(K)$, as claimed.

**Remark 4.2.5.** An analogous proof would work in the case when $\{-1, 0, 1\}^\omega$ is replaced in the statement by any other compact subset $C \subseteq \mathbb{Z}^\omega$ that contains the all zero sequence.

### 4.3 Construction of the witness function

The goal of this section is to construct a compact metric space $K_0$ and a function $f_0 : K_0 \to \mathbb{Z}^\omega$ that will witness the Haar meagerness of our example. In order to do this, we will introduce some structures and prove elementary claims about their properties.

We say that a sequence $s$ is $m$-segmented if it is the concatenation of constant sequences of length $2^m$. More formally, this means the following:

**Definition 4.3.1.** Let $S$ be an arbitrary set and $m \in \mathbb{N}$ be a nonnegative integer. An infinite sequence $s \in S^{\omega^\omega}$ is $m$-segmented if $s_{q \cdot 2^m + r_1} = s_{q \cdot 2^m + r_2}$ for all integers $q \in \mathbb{N}$ and $0 \leq r_1, r_2 < 2^m$. A finite sequence $s \in S^{<\omega^\omega}$ is $m$-segmented if $|s| = Q \cdot 2^m$ for some $Q \in \mathbb{N}$ and $s_{q \cdot 2^m + r_1} = s_{q \cdot 2^m + r_2}$ for all integers $0 \leq q < Q$ and $0 \leq r_1, r_2 < 2^m$.

Notice that an $m$-segmented sequence is also $m'$-segmented for all $0 \leq m' \leq m$.

**Definition 4.3.2.** Let us fix a sequence $b_s \in \{0, 1\}^{<\omega^\omega}$ for each $s \in \omega^{<\omega}$ in a way that it satisfies the following properties:

1. $b_\emptyset = \emptyset$,
2. $|b_s|$ is divisible by $2^{|s|}$,
3. if $s \in \omega^{<\omega}$ and $\beta$ is a nonempty finite $|s|$-segmented sequence such that $|b_s| + |\beta|$ is divisible by $2^{|s| + 1}$, then there is exactly one $\ell \in \omega$ such that $b_{s \cdot \ell} = b_s \cdot \beta$,
4. conversely, if $s \in \omega^{<\omega}$ and $\ell \in \omega$, then $b_{s \cdot \ell}$ can be written as $b_{s \cdot \ell} = b_s \cdot \beta$ where $\beta$ is a nonempty finite $|s|$-segmented sequence satisfying that $|b_s| + |\beta|$ is divisible by $2^{|s| + 1}$.

It is clear that using recursion we can choose a system $\{b_s\}_{s \in \omega^{<\omega}}$ that satisfies these properties.
**Definition 4.3.3.** Define the set $C_s \subseteq \{0, 1\}^\omega \subseteq \mathbb{Z}^\omega$ by

$$C_s = \{b_s \cap x : x \in \{0, 1\}^\omega \text{ and } x \text{ is } |s|\text{-segmented}\}.$$ 

**Fact 4.3.4.** The sets $C_s$ have the following properties:

1. $C_0 = \{0, 1\}^\omega$,
2. $C_s$ is compact for each $s \in \omega^{<\omega}$,
3. if $s \subseteq s'$, then $C_s \supseteq C_{s'}$.

**Proof.** Properties (1) and (2) are trivial, property (3) can be proved by using induction on the value of $(|s'| - |s|)$ and applying property (4) of Definition 4.3.2. \qed

**Claim 4.3.5.** Assume that $s \in \omega^{<\omega}$ and $U$ is a nonempty relatively open subset of $C_s$. Then there are infinitely many indices $\ell \in \omega$ such that $C_s \cap \ell \subseteq U$.

**Proof.** Fix an arbitrary element $u \in U$. As $U$ is relatively open, there exists an $n_0 \in \omega$ such that $C_s \cap [u \mid n] \subseteq U$ for each $n \geq n_0$. We may assume that $n_0 > |b_s|$.

Let us define the infinite set

$$N = \{n \in \omega : n \geq n_0 \text{ and } 2^{|s|+1} \text{ divides } n\}$$

and consider an arbitrary $n \in N$.

As $U \subseteq C_s$, the sequence $u$ can be written as $b_s \cap x$ where $x \in \{0, 1\}^\omega$ is an $|s|$-segmented sequence. Using property (2) of Definition 4.3.2, $2^{|s|}$ divides $n - |b_s|$, therefore it is clear that $\beta = x \mid (n - |b_s|)$ is $|s|$-segmented. Also notice that $\beta$ is nonempty because $n \geq n_0 > |b_s|$ and $|b_s| + |\beta| = n$ is divisible by $2^{|s|+1}$, and thus by property (3) of Definition 4.3.2, there exists an index $\ell_n \in \omega$ such that

$$u \mid n = b_s \cap (x \mid (n - |b_s|)) = b_s \cap \beta = b_s \cap \ell_n.$$ 

Clearly $C_s \cap \ell_n \subseteq [b_s \cap \ell_n]$ and according to property (3) of Fact 4.3.4 $C_s \cap \ell_n \subseteq C_s$, therefore $C_s \cap \ell_n \subseteq U$. This is sufficient, because property (3) of Definition 4.3.2 implies that $\ell_n \neq \ell_{n'}$ if $n \neq n'$. \qed

We will also use the following encoding of $\omega^{<\omega}$ in $\{0, 1\}^{<\omega}$:

**Definition 4.3.6.** For $n \in \omega$, let $h(n) = (0, 0, \ldots, 0, 1) \in \{0, 1\}^{n+1}$ be the sequence consisting of $n$ zeroes and then “1” as the last element. Generalizing this, for a finite sequence $s = (s_0, s_1, \ldots, s_{k-1}) \in \omega^{<\omega}$, let $h(s) = h(s_0) \cap h(s_1) \cap \ldots \cap h(s_{k-1})$.

Let $h_0(s) \in \{0, 1\}^\omega$ denote the infinite sequence $h(s) \cap (0, 0, \ldots)$ (that is, $h(s)$ extended to infinite length by appending zeroes).
Fact 4.3.7. The functions $h : \omega^{<\omega} \to \{0,1\}^{<\omega}$ and $h_0 : \omega^{<\omega} \to \{0,1\}^{\omega}$ have the following properties:

1. $h$ and $h_0$ are both injective,
2. if $s, s' \in \omega^{<\omega}$, then $s \subseteq s' \iff [h(s)] \supseteq [h(s')] \iff [h(s)] \ni h_0(s')$,
3. if $s \in \omega^{<\omega}$, $k \in \omega$ and $\ell \in \omega$ is large enough, then $[h(s \upharpoonright \ell)] \subseteq [h_0(s) \upharpoonright k]$.

Intuitively we will use $h_0(s)$ as a “height” assigned to the sequence $s$: when we construct the compact metric space $K_0$, we will “lift” a copy of $C_s$ to height $h_0(s)$ for each $s \in \omega$. More precisely, this means the following:

Definition 4.3.8. Let $K_0$ be the closure of the set
\[ \bigcup_{s \in \omega^{<\omega}} C_s \times \{h_0(s)\} \]
in the space $\mathbb{Z}^\omega \times \{0,1\}^\omega$ and let $f_0 : K_0 \to \mathbb{Z}^\omega$ be the restriction of the projection $\mathbb{Z}^\omega \times \{0,1\}^\omega \to \mathbb{Z}^\omega$ to the set $K_0$.

Claim 4.3.9. $K_0$ is compact.

Proof. According to Definition 4.3.3, $C_s \subseteq \{0,1\}^{\omega} \subseteq \mathbb{Z}^\omega$ for each $s \in \omega^{<\omega}$. This clearly implies that the closed set $K_0$ is a subset of $\{0,1\}^\omega \times \{0,1\}^\omega$, a compact set. \(\square\)

To study the structure of the set $K_0$, we introduce
\[ D_s = \bigcup_{\sigma \in \omega^{<\omega}, \sigma \supseteq s} C_\sigma \times \{h_0(\sigma)\} \subseteq \mathbb{Z}^\omega \times \{0,1\}^{\omega}. \]

Using this notation, the definition of $K_0$ can be written as $K_0 = \overline{D_0}$. It is clear that if $s \supseteq s'$ then $D_s \subseteq D_{s'}$, and in particular $D_s \subseteq D_0 \subseteq K_0$ holds for each $s \in \omega^{<\omega}$.

Claim 4.3.10. For each $s \in \omega^{<\omega}$,
\[ \overline{D}_s = K_0 \cap (\mathbb{Z}^\omega \times [h(s)]) = K_0 \cap (C_s \times [h(s)]) \]
and therefore $\overline{D}_s$ is relatively clopen in $K_0$.

Proof. By property (2) of Fact 4.3.7, $s \subseteq \sigma$ if and only if $h_0(\sigma) \in [h(s)]$, so
\[ D_0 \cap (\mathbb{Z}^\omega \times [h(s)]) = D_s. \]
Elementary calculations show that if $A, B$ are two subsets of a topological space and $B$ is clopen, then $\overline{A \cap B} = \overline{A} \cap \overline{B}$. Using this for the clopen set $\mathbb{Z}^\omega \times [h(s)]$ yields that

$$K_0 \cap (\mathbb{Z}^\omega \times [h(s)]) = \overline{D_s}.$$

This clearly shows that $\overline{D_s}$ is relatively open in $K_0$.

To prove the second equality, notice that $C_\sigma \subseteq C_s$ for any sequence $\sigma \supseteq s$ and thus $D_\sigma \subseteq C_s \times \{0,1\}^\omega$. Taking closure and then intersecting with $\overline{D_s} = K_0 \cap (\mathbb{Z}^\omega \times [h(s)])$ yields that $\overline{D_s} = K_0 \cap (C_s \times [h(s)])$, as stated. \hfill $\Box$

We will use the following lemma to prove that certain subsets of $K_0$ are meager (and in fact, nowhere dense):

**Lemma 4.3.11.** If $X \subseteq \mathbb{Z}^\omega$ satisfies that

$$\{s' \in \omega^{<\omega} : X \cap C_{s'} = \emptyset\}$$

is cofinal in $\omega^{<\omega}$,

then $f_0^{-1}(X)$ is nowhere dense in $K_0$.

**Proof.** Let $U$ be an arbitrary nonempty, relatively open subset of $K_0$. Then $U$ can be written as $U = K_0 \cap V$ where $V$ is open (in $\mathbb{Z}^\omega \times \{0,1\}^\omega$).

$K_0 = \overline{D_0}$ and $K_0$ intersects the open set $V$, therefore $D_0$ also intersects $V$. This implies that there is a sequence $s \in \omega^{<\omega}$ and a point $x \in C_s$ such that $(x, h_0(s)) \in V$.

As $V$ is an open set in the product space $\mathbb{Z}^\omega \times \{0,1\}^\omega$, there are $n, k \in \omega$ such that $[x \upharpoonright n] \times [h_0(s) \upharpoonright k] \subseteq V$.

As $C_s \cap [x \upharpoonright n]$ is a nonempty relatively open subset of $C_s$, we may apply **Claim 4.3.5** to get an infinite set $L \subseteq \omega$ of indices such that $C_{s \upharpoonright \ell} \subseteq C_s \cap [x \upharpoonright n]$ for each $\ell \in L$.

Property (3) of **Fact 4.3.10** implies we may choose an index $\ell \in L$ which also satisfies that $[h(s \upharpoonright \ell)] \subseteq [h_0(s) \upharpoonright k]$.

Using the condition of the lemma we can find an $s' \in \omega^{<\omega}$ such that $s' \supseteq s \upharpoonright \ell$ and $X \cap C_{s'} = \emptyset$. **Claim 4.3.11** states that $\overline{D_{s'}} = K_0 \cap (C_{s'} \times [h(s')])$ is a relatively clopen subset of $K_0$. This implies that $f_0(\overline{D_{s'}}) \subseteq C_{s'}$, but then applying $X \cap C_{s'} = \emptyset$ yields $f_0^{-1}(X) \cap \overline{D_{s'}} = \emptyset$. Also notice that

$$\overline{D_{s'}} = K_0 \cap (C_{s'} \times [h(s')]) \subseteq K_0 \cap (C_{s \upharpoonright \ell} \times [h(s \upharpoonright \ell)]) \subseteq K_0 \cap ([x \upharpoonright n] \times [h_0(s) \upharpoonright k]) \subseteq K_0 \cap V = U.$$

This means that in an arbitrary nonempty, relatively open subset $U$ of $K_0$ we found a (clearly nonempty) subset $\overline{D_{s'}}$ that is relatively open in $K_0$ and disjoint from $f_0^{-1}(X)$. This implies that $f_0^{-1}(X)$ is nowhere dense in $K_0$. \hfill $\Box$
4.4 Counterexample for the question of Darji

Now we are ready to prove the main result of this chapter:

**Theorem 4.1.6.** In the abelian Polish group $\mathbb{Z}^\omega$, there exists a $G_\delta$ set $R$ that is Haar meager but not strongly Haar meager.

**Proof.** Fix a map $t : K(\mathbb{Z}^\omega) \to \mathbb{Z}^\omega$ which satisfies the conditions of [Theorem 4.2.1]. Recall that

$$H = \{(K, x) \in K(\mathbb{Z}^\omega) \times \mathbb{Z}^\omega : x \in K\}$$

is a closed set and according to [Theorem 4.2.1] the map

$$T : H \to \mathbb{Z}^\omega, \quad T(K, x) = x + t(K)$$

is a homeomorphism between $H$ and the closed set $F = T(H) \subseteq \mathbb{Z}^\omega$, and this set $F$ satisfies that

$$(K + t(K) + \{-1, 0, 1\}^\omega) \cap F = K + t(K)$$

for each $K \in K(\mathbb{Z}^\omega)$. As

$$\{-1, 0, 1\}^\omega = \{0, 1\}^\omega - \{0, 1\}^\omega = C_\emptyset - C_\emptyset,$$

we can reformulate this fact:

**Fact 4.4.1.** The set $F$ satisfies that

$$(K + t(K) + C_\emptyset - C_\emptyset) \cap F = K + t(K) \quad \text{for each } K \in K(\mathbb{Z}^\omega).$$

We will use the following definition:

**Definition 4.4.2.** If $s \in \omega^{<\omega}$, then let $T_s = \{t \in \mathbb{Z}^\omega : t + C_s \subseteq F\}$ denote the set of translations which move $C_s$ into $F$.

**Claim 4.4.3.** $T_s$ is closed for each $s \in \omega^{<\omega}$.

**Proof.** We prove that for an arbitrary $s \in \omega^{<\omega}$, the set $T_s$ contains all of its limit points. Assume that $t_i \in T_s$ for each $i \in \omega$ and $t^* = \lim_{i \to \infty} t_i$ exists. We know that if $c \in C_s$ and $i \in \omega$, then $t_i + c \in F$. As $F$ is closed, this implies that if $c \in C_s$, then $\lim_{i \to \infty} (t_i + c) = t^* + c \in F$. This shows that $t^* \in T_s$, concluding the proof. \[ \square \]

Using these sets of translations, we can define the set $R$:

**Definition 4.4.4.** Let

$$R = F \setminus \bigcup_{s \in \omega^{<\omega}} \bigcup_{t \in \omega} (T_s + C_s - t).$$
It is clear from this definition that $R$ is $G_δ$ and $R \subseteq F$.

The rest of the proof consists of two parts: We will first prove Claim 4.4.5, which will imply that $R$ is Haar meager, then we will prove Claim 4.4.6, which will imply that $R$ is not strongly Haar meager.

Claim 4.4.5. For the compact metric space $K_0$ and function $f_0$ defined in Definition 4.3.8, if $g \in \mathbb{Z}^ω$, then $f_0^{-1}(R + g)$ is a nowhere dense subset of $K_0$. (Using the terminology of [9], this states that $R$ is Haar nowhere dense.)

Proof. Fix an arbitrary $g \in \mathbb{Z}^ω$. According to Lemma 4.3.11, it is sufficient to prove that $f_0' = 0 : (R + g) \setminus C_0 = \emptyset$. This states that $g$ is cofinal in $\omega^ω$. We fix an arbitrary $s \in \omega^ω$ and prove that there exists a $s' \in \omega^ω$ such that $s' \subseteq s$ and $(R + g) \cap C_{s'} = \emptyset$. Clearly $(R + g) \cap C_{s'} = \emptyset$ if and only if $R \cap (C_{s'} - g) = \emptyset$.

We distinguish two cases:

Case 1: $C_s - g \subseteq F$.
In this case Definition 4.4.4 implies that $-g \in T_s$. Pick an arbitrary $\ell \in \omega$ (for example, let $\ell = 0$) and let $s' = s^\ell$. Then

$$C_{s'} - g \subseteq T_s + C_s = T_s + C_s^\ell \subseteq \bigcup_{\sigma \in \omega^ω} \bigcup_{\ell \in \omega} (T_0 + C_{\sigma^\ell})$$

and therefore Definition 4.4.4 implies that $(C_{s'} - g) \cap R = \emptyset$.

Case 2: $C_s - g \nsubseteq F$.
In this case the set $U = C_s \setminus (F + g)$ is nonempty and relatively open in $C_s$ (because $F$ is a closed subset of $\mathbb{Z}^ω$). Applying Claim 4.3.5 we can select an index $\ell \in \omega$ such that $C_{s^\ell} \subseteq U$. This means that $s' = s^\ell$ is a good choice:

$$C_{s'} - g \subseteq U - g = (C_s - g) \setminus F \subseteq (C_s - g) \setminus R$$

using the fact that $R \subseteq F$.

Claim 4.4.6. If $K \subseteq \mathbb{Z}^ω$ is a nonempty compact set, then there is an element $g \in \mathbb{Z}^ω$ such that $(R + g) \cap K$ is a comeager subset of $K$.

Proof. Fix a nonempty compact set $K \in K(\mathbb{Z}^ω)$. We will prove that $g = -t(K)$ satisfies that $(R - t(K)) \cap K$ is a comeager subset of $K$. (Recall that we fixed a map $t$ that satisfies the conditions of Definition 4.4.2.) As $x \mapsto x + t(K)$ is a homeomorphism from $K$ to $K + t(K)$, it is enough to prove that $R \cap (K + t(K))$ is a comeager subset of $K + t(K)$. 

Recall that according to Definition 4.4.4,

\[ R = F \setminus \bigcup_{s \in \omega^2} \bigcup_{\ell \in \omega} (T_s + C_s \cdot \ell). \]

It is clear that \( K + t(K) \subseteq F \), therefore

\[ R \setminus (K + t(K)) = (K + t(K)) \setminus \bigcup_{s \in \omega^2} \bigcup_{\ell \in \omega} (T_s + C_s \cdot \ell) = \bigcap_{s \in \omega^2} \bigcap_{\ell \in \omega} ((K + t(K)) \setminus (T_s + C_s \cdot \ell)). \]

To prove that this countable intersection is comeager in \( K + t(K) \), it is enough to prove that if we fix \( s \in \omega^2 \) and \( \ell \in \omega \), then the set

\[ (K + t(K)) \setminus (T_s + C_s \cdot \ell) \]

is comeager in \( K + t(K) \). This set is clearly relatively open, because \( C_s \cdot \ell \) is compact and Claim 4.4.3 states that \( T_s \) is closed. Therefore it is enough to prove that this set is dense in \( K + t(K) \). If we fix an arbitrary nonempty relatively open subset \( U \) of \( K + t(K) \), then we need to check that

\[ U \cap ((K + t(K)) \setminus (T_s + C_s \cdot \ell)) = U \setminus (T_s + C_s \cdot \ell) \neq \emptyset. \]

Fix an arbitrary \( u \in U \). As \( U \) is relatively open, we may find an index \( n \in \omega \) such that \([u \upharpoonright n] \cap (K + t(K)) \subseteq U\). We will use recursion to define a sequence \( x \in K + t(K) \subseteq \mathbb{Z}^\omega \).

First let

1. \( x \upharpoonright n = u \upharpoonright n \).

After this we will select the elements of the sequence \( x \) one by one. Assume that we already defined \( x \upharpoonright j \) for an index \( j \geq n \) and

2. if \( j \equiv 0 \pmod{2^{|s|+1}} \), then let

\[ x_j = \min\{y_j : y \in (K + t(K)) \cap [x \upharpoonright j]\}, \]

3. if \( j \equiv 2^{|s|} \pmod{2^{|s|+1}} \), then let

\[ x_j = \max\{y_j : y \in (K + t(K)) \cap [x \upharpoonright j]\}, \]

4. otherwise, choose an arbitrary element \( x_j \) which satisfies that

\[ x_j \in \{y_j : y \in (K + t(K)) \cap [x \upharpoonright j]\}. \]

It is easy to check that \( (K + t(K)) \cap [x \upharpoonright j] \) remains nonempty during this procedure and therefore \( x \in K + t(K) \). Note that in conditions (2) and (3) the minimum and maximum are well-defined because \((K + t(K)) \cap [x \upharpoonright j]\) is compact.

Property (1) implies that \( x \in U \), we wish to prove that \( x \notin T_s + C_s \cdot \ell \). Assume for the
contrary that $x = t^* + c^*$ for some elements $t^* \in T_s$ and $c^* \in C_s \setminus \ell$.

Recall that
$$T_s = \{ t \in \mathbb{Z}^\omega : t + C_s \subseteq F \} \quad \text{(Definition 4.3.2)}$$

and
$$C_s = \{ b_s \cap x : x \in \{0, 1\}^\omega \text{ and } x \text{ is } |s|\text{-segmented} \} \quad \text{(Definition 4.3.3)}.$$

$t^* \in T_s$ means that $t^* + C_s = x - c^* + C_s \subseteq F$. As $x \in K + t(K)$, $c^* \in C_s \setminus \ell \subseteq C_0$ and $C_s \subseteq C_0$, we know that $x - c^* + C_s \subseteq K + t(K) + C_0 - C_0$. According to Fact 4.4.1,
$$(K + t(K) + C_0 - C_0) \cap F = K + t(K),$$
therefore $t^* + C_s = x - c^* + C_s \subseteq K + t(K)$.

For each index $j \in \omega$, consider the sequences
$$r(j, 0) = (c^* \upharpoonright j)^{\cap}(0, 0, \ldots) \quad \text{and} \quad r(j, 1) = (c^* \upharpoonright j)^{\cap}(1, 1, \ldots)$$
consisting of the first $j$ elements of $c^*$, followed by zeroes and ones respectively. If $j \geq |b_s|$ and $j$ is divisible by $2^{|s|}$ then it is straightforward to check that $r(j, 0), r(j, 1) \in C_s$ (using Definition 4.3.3 and the fact that $c^* \in C_s \setminus \ell \subseteq C_s$).

Fix a $j \in \omega$ such that $j \equiv 0 \pmod{2^{|s|+1}}$ and $j > \max\{n, |b_s - \ell|\}$. According to property (2) of $x$,
$$x_j = \min\{y_j : y \in (K + t(K)) \cap [x \upharpoonright j]\}.$$

Notice that $y = x - c^* + r(j, 0)$ satisfies that $y \in x - c^* + C_s \subseteq K + t(K)$ (because $r(j, 0) \in C_s$) and $y \in [x \upharpoonright j]$ (because $c^* \upharpoonright j = r(j, 0) \upharpoonright j$). This implies that
$$x_j \leq y_j = (x - c^* + r(j, 0))_j = x_j - c^*_j + 0 \quad \Rightarrow \quad c^*_j \leq 0.$$

Now apply an analogous argument for the index $j' = j + 2^{|s|}$: According to property (3) of $x$,
$$x_{j'} = \max\{y_{j'} : y \in (K + t(K)) \cap [x \upharpoonright j']\}.$$

Notice that $y = x - c^* + r(j', 1)$ satisfies that $y \in x - c^* + C_s \subseteq K + t(K)$ (because $r(j', 1) \in C_s$) and $y \in [x \upharpoonright j']$ (because $c^* \upharpoonright j' = r(j', 1) \upharpoonright j'$). This implies that
$$x_{j'} \geq y_{j'} = (x - c^* + r(j', 1))_{j'} = x_{j'} - c^*_{j'} + 1 \quad \Rightarrow \quad c^*_{j'} \geq 1.$$

As $c^* = b_{s - \ell} \cap z$ for some $(|s| + 1)$-segmented sequence $z \in \{0, 1\}^\omega$ and the length of $b_{s - \ell}$ is divisible by $2^{|s|+1}$, we know that $c^*_j = c^*_{j+2^{|s|}} = c^*_j$. This contradicts that $c^*_j \leq 0$ and $c^*_j \geq 1$, proving that our indirect assumption was incorrect.
We proved that \( x \in U \) and \( x \notin (T_s + C_{s, \ell}) \), and this implies that \( (K + t(K)) \setminus (T_s + C_{s, \ell}) \) is indeed a dense subset of \( K + t(K) \).

This concludes the proof of Theorem 4.1.6.

The following theorem shows that our \( G_\delta \) counterexample is as simple as possible:

**Theorem 4.4.7.** If \( G \) is a Polish group and \( A \subseteq G \) is an \( F_\sigma \) Haar meager subset, then \( A \) is strongly Haar meager.

**Proof.** Assume that the continuous map \( f : K \to G \) witnesses that \( A \) is Haar meager (where \( K \) is a nonempty compact metric space). We show that the compact set \( C = f(K) \subseteq G \) witnesses that \( A \) is strongly Haar meager. Assume that the set \( B \) is a translate of \( A \) (that is, \( B = gAh \) for some \( g, h \in G \)). It is clearly enough to prove that \( B \cap C \) is meager in \( C \).

We will use the following facts which are all well-known and easy to prove:

1. An \( F_\sigma \) set is meager if and only if it has empty interior,
2. The preimage of an \( F_\sigma \) set under a continuous function is also \( F_\sigma \),
3. If the preimage of a set \( X \) under a continuous function has empty interior, then the set \( X \) itself has empty interior relative to the image of the function.

As \( B \) is an \( F_\sigma \) set, (2) yields that \( f^{-1}(B) \) is also \( F_\sigma \). As \( f \) was a witness function, \( f^{-1}(B) \) is meager, but then (1) yields that \( f^{-1}(B) \) has empty interior. But \( f^{-1}(B) = f^{-1}(B \cap C) \) because \( C \) is the image of \( f \), therefore (3) yields that \( B \cap C \) has empty interior relative to \( C \). Using (1) again (\( B \cap C \) is an \( F_\sigma \) subset of \( C \)), this yields that \( B \cap C \) is indeed a meager subset of \( C \).

\[ \square \]

### 4.5 Related questions

As we answered Question 4.1.2 negatively, constructing a Haar meager but not strongly Haar meager set in \( \mathbb{Z}^\omega \), only the following open-ended question remains of Question 4.1.2:

**Question 4.5.1.** What can we say about the (abelian) Polish groups where every Haar meager set is strongly Haar meager?

Notice that both Example 4.1.3 and Theorem 4.1.6 studied groups that can be written as countable products of countable discrete groups. In fact, our ideas allow us to describe the situation in this frequently studied, simple class of Polish groups:

**Claim 4.5.2.** Consider a group \( G = \prod_{i \in \omega} G_i \) where each \( (G_i, +) \) is a countable abelian group endowed with the discrete topology. Then the following are equivalent:
(1) every Haar meager subset of $G$ is strongly Haar meager,
(2) every compact subset of $G$ is contained in a locally compact subgroup,
(3) for all but finitely many $i \in \omega$ the group $G_i$ is a torsion group.

We do not include a proof of this claim, as our proof involves proving generalizations of Theorem 4.2.1 and Claim 4.4.6 that are more complicated and harder to understand, but do not require additional interesting ideas.

**Remark 4.5.3.** In the previous claim the implication (2) $\Rightarrow$ (1) remains true for any abelian Polish group $G$. This can be proved by slightly modifying the proof of [6, Theorem 5.13].

In addition to Question 4.1.2 the paper [29] contains another problem about strongly Haar meager sets, [29, Problem 3]:

**Question 4.5.4** (Darji). Does the collection of strongly Haar meager sets form a $\sigma$-ideal?

In fact, even the following variant of this question seems to be open:

**Question 4.5.5.** Does the collection of strongly Haar meager sets form an ideal? (Or equivalently, is the union of two strongly Haar meager sets strongly Haar meager?)

As the Haar meager sets form a $\sigma$-ideal, the interesting case for these questions is studying groups that contain Haar meager, but not strongly Haar meager sets.
Chapter 5

Other related notions

In this chapter we introduce more variants of Haar null sets which are studied in the literature.

5.1 Left and right Haar null sets

When we defined Haar null sets in Definition 1.1, we used multiplication from both sides by arbitrary elements of $G$. If we replace this by multiplication from one side, we get the following notions:

**Definition 5.1.1.** A set $A \subseteq G$ is said to be left Haar null (or right Haar null) if there are a Borel set $B \supseteq A$ and a Borel probability measure $\mu$ on $G$ such that $\mu(gB) = 0$ for every $g \in G$ ($\mu(Bg) = 0$ for every $g \in G$).

**Definition 5.1.2.** A set $A \subseteq G$ is said to be left-and-right Haar null if there are a Borel set $B \supseteq A$ and a Borel probability measure $\mu$ on $G$ such that $\mu(gB) = \mu(Bg) = 0$ for every $g \in G$.

If “Borel set” is replaced by “universally measurable set”, we can naturally obtain the generalized versions of these notions. As the papers about this topic happen to follow Christensen in defining “Haar null set” to mean generalized Haar null set in the terminology of this thesis, most results in this section were originally stated for these generalized versions.

There are situations where these “one-sided” notions are more useful than the usual, symmetric one. (For example [90] uses them to state results related to automatic continuity and the generalizations of the Steinhaus theorem. We mention some of these results in section 6.2.) However in some groups these are not “good” notions of smallness in the sense of section 2.1.
Translation invariance is not problematic even for the “one-sided” notions: Suppose that e.g. $B$ is Borel left Haar null and a Borel probability measure $\mu$ satisfies $\mu(gB) = 0$ for every $g \in G$. If we consider a right translate $Bh$ (this is the interesting case, invariance under left translation is trivial), then $\mu'(X) = \mu(Xh^{-1})$ is a Borel probability measure which satisfies $\mu'(g \cdot Bh) = 0$ for every $g \in G$. An analogous argument shows that the system of right Haar null sets is translation invariant; using Proposition 5.1.3 it follows from these that the system of left-and-right Haar null sets is also translation invariant. It is clear that this reasoning works for the generalized versions, too.

Unfortunately, there are Polish groups where these notions fail to form $\sigma$-ideals. In [00] Solecki gives a sufficient condition which guarantees that the generalized left Haar null sets form a $\sigma$-ideal and gives another sufficient condition which guarantees that the left Haar null sets do not form a $\sigma$-ideal. We state these results and show examples of groups satisfying these conditions without proofs:

**Definition 5.1.3** (Solecki). A Polish group $G$ is called amenable at 1 if for any sequence $(\mu_n)_{n \in \omega}$ of Borel probability measures on $G$ with $1_G \in \text{supp} \mu_n$, there are Borel probability measures $\nu_n$ and $\nu$ such that

1. $\nu_n \ll \mu_n$,
2. if $K \subseteq G$ is compact, then $\lim_n (\nu \ast \mu_n)(K) = \nu(K)$.

This class is closed under taking closed subgroups and continuous homomorphic images. A relatively short proof also shows that (II) is equivalent to

(II') if $K \subseteq G$ is compact, then $\lim_n (\nu_n \ast \nu)(K) = \nu(K)$.

This means that if $G$ is amenable at 1, then so is the opposite group $G^{\text{opp}}$. ($G^{\text{opp}}$ is the set $G$ considered as a group with $(x, y) \mapsto y \cdot x$ as the multiplication.)

Examples of groups which are amenable at 1 include:

1. abelian Polish groups,
2. locally compact Polish groups,
3. countable direct products of locally compact Polish groups such that all but finitely many factors are amenable,
4. inverse limits of sequences of amenable, locally compact Polish groups with continuous homomorphisms as bonding maps.

**Theorem 5.1.4** (Solecki). If $G$ is amenable at 1, then the generalized left Haar null sets form a $\sigma$-ideal.

Our note about the opposite group (and the fact that the intersection of two $\sigma$-ideals is also a $\sigma$-ideal) means that generalized right Haar null sets and generalized left-and-right Haar null sets also form $\sigma$-ideals.
The following definition is the sufficient condition for the “bad” case. Note that this condition is also symmetric (in the sense that if \( G \) satisfies it, then \( G^{\text{opp}} \) also satisfies it).

**Definition 5.1.5** (Solecki). A Polish group \( G \) is said to **have a free subgroup at 1** if it has a non-discrete free subgroup whose all finitely generated subgroups are discrete.

The paper [90] lists several groups which all have a free subgroup at 1, we mention some of these:

1. countably infinite products of Polish groups containing discrete free non-Abelian subgroups,
2. \( S_{\infty} \), the group of permutations of \( \mathbb{N} \) with the topology of pointwise convergence,
3. \( \text{Aut}(\mathbb{Q}, \leq) \), the group of order-preserving self-bijections of the rationals with the topology of pointwise convergence on \( \mathbb{Q} \) viewed as discrete (that is, a sequence \( (f_n)_{n\in\omega} \in G^\omega \) is said to be convergent if for every \( q \in \mathbb{Q} \) there is a \( n_0 \in \omega \) such that the sequence \( (f_n(q))_{n\geq n_0} \) is constant),

**Theorem 5.1.6** (Solecki). If \( G \) has a free subgroup at 1, then there are a Borel left Haar null set \( B \subseteq G \) and \( g \in G \) such that \( B \cup Bg = G \). As the left Haar null sets are translation invariant and \( G \) is not left Haar null, this means that they do not form an ideal.

As the notion of having a free subgroup at 1 is symmetric, the same is true for right Haar null sets.

In abelian groups it is trivial that the notions introduced by **Definition 5.1.1** and **Definition 5.1.2** are all equivalent to the notion of Haar null sets (and the generalized versions are equivalent to generalized Haar null sets).

The following remark is also trivial, but sometimes useful:

**Remark 5.1.7**. If \( A \subseteq G \) is a conjugacy invariant set (that is, \( gAg^{-1} = A \) for every \( g \in G \)), then \( gAh = gh \cdot h^{-1}Ah = ghA \) (and similarly \( gAh = Agh \)) for every \( g, h \in G \), hence if \( A \) is left (or right) Haar null, then \( A \) is Haar null.

For locally compact groups **Theorem 2.2.7** and the remark in its proof shows that all generalized right Haar null sets are Haar null. This implies that in locally compact groups the eight variants of Haar null (generalized or not, left or right or left-and-right or “plain”) are all equivalent to having Haar measure zero.

The following diagram summarizes the situation in the general case:
left Haar null \iff (1) \iff left Haar null and right Haar null \iff (2) \iff left-and-right \iff Haar null \iff (3)

(Here \( \not\Rightarrow \) means that there is a counterexample in a suitable Polish group and \( \Rightarrow \) means that the implication is true in all Polish groups.)

The not-implications marked by (1) are demonstrated by the already stated Theorem 5.1.6. We will prove the implication (2) as Proposition 5.1.8 and demonstrate the not-implication (3) by Example 5.1.9.

**Proposition 5.1.8.** A set \( A \subseteq G \) is left-and-right Haar null if and only if it is both left Haar null and right Haar null.

This simple result was mentioned in [91]; it can be proved by modifying the proof of [78, Theorem 2].

**Proof.** We only have to show that if \( A \) is both left Haar null and right Haar null, then it is left-and-right Haar null, as the other direction is trivial. By definition there exist a Borel set \( B \) and Borel probability measures \( \mu_1, \mu_2 \) on \( G \) such that \( \mu_1(gB) = \mu_2(Bg) = 0 \) for every \( g \in G \).

Define 
\[
\mu(X) = (\mu_1 \times \mu_2) \left( \{(x, y) \in G^2 : xy \in X\} \right),
\]
then \( \mu \) is clearly a Borel probability measure and if the characteristic function of a set \( S \) is denoted by \( \chi_S \), then using Fubini’s theorem we have
\[
\mu(gB) = \int_G \int_G \chi_{gB}(yx) \ d\mu_1(x) \ d\mu_2(y) = \int_G \mu_1(y^{-1}gB) \ d\mu_2(y) = 0
\]
and
\[
\mu(Bg) = \int_G \int_G \chi_{Bg}(yx) \ d\mu_2(y) \ d\mu_1(x) = \int_G \mu_2(Bg^{-1}x) \ d\mu_1(x) = 0
\]
and these show that \( \mu \) satisfies the requirements of Definition 5.1.2. \( \square \)

The following two counterexamples appear in [91]. This paper uses relatively elementary techniques and examines the structure of the group

\[
\mathcal{H}[0,1] = \{ f : f \text{ is continuous, strictly increasing, } f(0) = 0, f(1) = 1 \}.
\]

(This is the group of order-preserving self-homeomorphisms of \([0,1]\), the group operation is composition, the topology is the compact-open topology.) We state these results without proofs:
Example 5.1.9 (Shi-Thomson). The group $\mathcal{H}[0,1]$ has a Borel subset that is left-and-right Haar null but not Haar null.

Example 5.1.10 (Shi-Thomson). There exists a Borel set $B \subseteq \mathcal{H}[0,1]$ and a Borel probability measure $\mu$ such that $\mu(Bg) = 0$ for every $g \in \mathcal{H}[0,1]$ (this implies that $B$ is right Haar null), but $\mu(gB) \neq 0$ for some $g \in \mathcal{H}[0,1]$ (i.e. $\mu$ does not witness that $B$ is left Haar null).

When the Polish group $G$ has a free subgroup at 1, then Theorem 5.1.6 implies that the left Haar null sets are distinct from the right Haar null sets; [00, Question 5.1] asks if the opposite of this is true for groups that are amenable at 1:

Question 5.1.11 (Solecki). Let $G$ be an amenable at 1 Polish group. Do left Haar null subsets of $G$ coincide with right Haar null subsets of $G$?

We conclude this part by stating [89, Theorem 6.1], which provides a necessary and sufficient condition for the equivalence of the notions generalized left Haar null and generalized Haar null in a special class of groups.

Theorem 5.1.12 (Solecki). Let $H_n (n \in \omega)$ be countable groups and consider the group $G = \prod_n H_n$. The following conditions are equivalent:

1. In $G$ the system of generalized left Haar null sets is the same as the system of generalized Haar null sets.
2. For each universally measurable set $A \subseteq G$ that is not generalized Haar null, $1_G \in \text{int}(AA^{-1})$.
3. For each closed set $F \subseteq G$ that is not (generalized) Haar null, $FF^{-1}$ is dense in some non-empty open set.
4. For all but finitely many $n \in \omega$ all elements of $H_n$ have finite conjugacy classes in $H_n$, that is, for all but finitely many $n \in \omega$ and for all $x \in H_n$ the set $\{yxy^{-1} : y \in H_n\}$ is finite.

5.2 Openly Haar null sets

Definition 5.2.1. A set $A \subseteq G$ is said to be open Haar null if there is a Borel probability measure $\mu$ on $G$ such that for every $\varepsilon > 0$ there is an open set $U \supseteq A$ such that $\mu(gUh) < \varepsilon$ for every $g, h \in G$. If $\mu$ has these properties, we say that $\mu$ witnesses that $A$ is open Haar null.

The notion of open Haar null sets was introduced in [88], and more thoroughly examined in [20].

The following simple proposition shows that this is a stronger property than Haar nullness:
Proposition 5.2.2. Every openly Haar null set is contained in a $G_δ$ Haar null set.

Proof. For every $n \in \omega$ there is an open set $U_n \supseteq A$ such that $\mu(gU_nh) < \frac{1}{n+1}$ for every $g, h \in G$. Then $B = \bigcap_{n \in \omega} U_n$ is a $G_δ$ set that is Haar null because it satisfies $\mu(gBh) \leq \mu(gU_nh) < \frac{1}{n+1}$ for every $g, h \in G$, hence $\mu(gBh) = 0$ for every $g, h \in G$. □

Remark 5.2.3. In non-locally-compact abelian Polish groups “openly Haar null” is a strictly stronger property than “Haar null”, because Theorem 3.2.2 states that in these groups there is a Haar null set that is not contained in a $G_δ$ Haar null set.

To prove that the system of openly Haar null sets forms a $\sigma$-ideal, we will need the following technical lemma:

Lemma 5.2.4. If $\mu$ witnesses that $A \subseteq G$ is openly Haar null and $V$ is a neighborhood of $1_G$, then there exists a measure $\mu'$ which also witnesses that $A$ is openly Haar null and has a compact support that is contained in $V$.

Proof. Lemma 2.1.2 states that there are a compact set $C \subseteq G$ and $c \in G$ such that $\mu(C) > 0$ and $C \subseteq cV$. Let us define $\mu'(X) = \frac{\mu(cX \cap C)}{\mu(C)}$, this is clearly a Borel probability measure and has a compact support that is contained in $V$. Fix an arbitrary $\varepsilon > 0$. We will find an open set $U \supseteq A$ such that $\mu'(gUh) < \varepsilon$ for every $g, h \in G$. Notice that

$$\mu'(gUh) < \varepsilon \iff \mu(ugUh \cap C) < \mu(C) \cdot \varepsilon \iff \mu(ugUh) < \mu(C) \cdot \varepsilon.$$ 

There exists an open set $U \supseteq A$ with $\mu(gUh) < \mu(C) \cdot \varepsilon$ for every $g, h \in G$, and this satisfies $\mu(ugUh) < \mu(C) \cdot \varepsilon$ for every $g, h \in G$, hence $\mu'$ has the required properties. □

Theorem 5.2.5 (Cohen-Kallman). The system of openly Haar null sets is a translation invariant $\sigma$-ideal.

Proof. It is trivial that the system of openly Haar null sets satisfy (I) and (II) in Definition 2.1.1. The following proof of (III) is described in the appendix of [26] and is similar to the proof of Theorem 2.1.6.

Let $A_n$ be openly Haar null for all $n \in \omega$, we prove that $A = \bigcup_{n \in \omega} A_n$ is also openly Haar null. For every set $A_n$ fix a measure $\mu_n$ which witnesses that $A_n$ is openly Haar null. Let $d$ be a complete metric on $G$ that is compatible with the topology of $G$.

We construct for all $n \in \omega$ a compact set $C_n \subseteq G$ and a Borel probability measure $\tilde{\mu}_n$ such that the support of $\tilde{\mu}_n$ is $C_n$, $\tilde{\mu}_n(gU_nh) < \varepsilon$ for every $g, h \in G$ and the “size” of the sets $C_n$ decreases “quickly”.

The construction will be recursive. For the initial step use Lemma 5.2.4 to find a measure $\tilde{\mu}_0$ witnessing that $A_0$ is openly Haar null and has compact support $C_0 \subseteq G$. Assume
Lemma 2.1.4 \[ \text{there exists a neighborhood } V_n \text{ of } 1_G \text{ such that if } v \in V_n, \text{ then } d(k \cdot v, k) < 2^{-n} \text{ for every } k \text{ in the compact set } C_0C_1C_2 \cdots C_{n-1}. \]

Applying Lemma 5.2.6 again we can find a measure \( \hat{\mu}_n \) with compact support \( C_n \subseteq V_n \) which is witnessing that \( A_n \) is openly Haar null.

If \( c_n \in C_n \) for all \( n \in \omega \), then it is clear that the sequence \( (c_0c_1c_2 \cdots c_n)_{n \in \omega} \) is a Cauchy sequence. As \((G, d)\) is complete, this Cauchy sequence is convergent; we write its limit as the infinite product \( c_0c_1c_2 \cdots \). The map \( \varphi : \prod_{n \in \omega} C_n \to G, \varphi((c_0, c_1, c_2, \ldots)) = c_0c_1c_2 \cdots \) is the pointwise limit of continuous functions, hence it is Borel.

Let \( \mu^\Pi \) be the product of the measures \( \hat{\mu}_n \) on the product space \( C^\Pi \) defined as \( C^\Pi = \prod_{n \in \omega} C_n \). Let \( \mu = \varphi_* (\mu^\Pi) \) be the push-forward of \( \mu^\Pi \) along \( \varphi \) onto \( G \), i.e.

\[
\mu(X) = \mu^\Pi(\varphi^{-1}(X)) = \mu^\Pi(\{(c_0, c_1, c_2, \ldots) \in C^\Pi : c_0c_1c_2 \cdots \in X\}).
\]

We claim that \( \mu \) witnesses that \( A = \bigcup_{n \in \omega} A_n \) is openly Haar null. Fix an arbitrary \( \varepsilon > 0 \), we will show that there is an open set \( U \supseteq A \) such that \( \mu(gUh) < \varepsilon \) for every \( g, h \in G \). It is enough to find open sets \( U_n \supseteq A_n \) such that \( \mu(gU_nh) < \varepsilon \cdot 2^{-(n+2)} \) for every \( g, h \in G \) and \( n \in \omega \), because then \( U = \bigcup_{n \in \omega} U_n \) satisfies that for every \( g, h \in G \)

\[
\mu(gUh) \leq \sum_{n \in \omega} \mu(gU_nh) \leq \frac{\varepsilon}{2} < \varepsilon.
\]

Fix \( g, h \in G \) and \( n \in \omega \). Choose an \( U_n \supseteq A_n \) open set satisfying that \( \hat{\mu}_n(gU_nh) < \varepsilon \cdot 2^{-(n+2)} \) for every \( g, h \in G \). Notice that if \( c_j \in C_j \) for every \( j \neq n, j \in \omega \), then

\[
\hat{\mu}_n(\{c_n \in C_n : c_0c_1c_2 \cdots c_n \cdots \in gU_nh\}) = \hat{\mu}_n((c_0c_1 \cdots c_{n-1})^{-1} \cdot gU_nh \cdot (c_{n+1}c_{n+2} \cdots)^{-1}) < \varepsilon \cdot 2^{-(n+2)}
\]

because \( \hat{\mu}_n(g'U_nh') < \varepsilon \cdot 2^{-(n+2)} \) for all \( g', h' \in G \). Applying Fubini’s theorem in the product space \( \prod_{j \neq n} C_j \times C_n \) to the product measure \( \prod_{j \neq n} \hat{\mu}_j \times \hat{\mu}_n \) yields that

\[
\varepsilon \cdot 2^{-(n+2)} > \mu^\Pi(\{(c_0, c_1, \ldots, c_n, \ldots) \in C^\Pi : c_0c_1 \cdots c_n \cdots \in gU_nh\}).
\]

By the definition of \( \mu \) this means that \( \mu(gU_nh) < \varepsilon \cdot 2^{-(n+2)} \). \( \square \)

Unfortunately, there are groups where this ideal contains only the empty set. The following results about this “collapse” are proved as \([26, \text{Propositions 5 and 3}]\):

**Proposition 5.2.6** (Cohen-Kallman). Assume that \( G \) is a Polish group. If for every compact subset \( C \subseteq G \) and every nonempty open subset \( U \subseteq G \) there are \( g, h \in G \) with \( gCh \subseteq U \), then the empty set is the only openly Haar null subset of \( G \).
Proposition 5.2.7 (Cohen-Kallman). In particular \( G = H[0,1] \), the group of order-preserving self-homeomorphisms of \([0,1]\) (endowed with the compact-open topology) has this property, hence in \( H[0,1] \) only the empty set is openly Haar null.

On the other hand, \([26, \text{Proposition 2}]\) shows several groups and classes of groups where the ideal of openly Haar null sets is nontrivial:

Proposition 5.2.8 (Cohen-Kallman). In the Polish group \( G \) there is a nonempty openly Haar null subset if at least one of the following conditions holds:

1. \( G \) is uncountable and admits a two-sided invariant metric,
2. \( G = S_\infty \) is the group of permutations of \( \mathbb{N} \) with the topology of pointwise convergence,
3. \( G = \text{Aut}(\mathbb{Q}, \leq) \) is the group of order-preserving self-bijections of the rationals with the topology of pointwise convergence on \( \mathbb{Q} \) viewed as discrete (i.e. a sequence \( (f_n)_{n \in \omega} \in G^\omega \) is said to be convergent if for every \( q \in \mathbb{Q} \) there is a \( n_0 \in \omega \) such that the sequence \( (f_n(q))_{n \geq n_0} \) is constant),
4. \( G = U(l^2) \) is the unitary group on the separable infinite-dimensional complex Hilbert space with the strong operator topology,
5. \( G \) admits a continuous surjective homomorphism onto a group which has a nonempty openly Haar null subset.

In section 6.4 we will mention Theorem 6.4.3, which is an interesting application of this notion.

5.3 Generically Haar null sets

Definition 5.3.1. Assume that \( A \subseteq G \) is universally measurable. Let

\[ T(A) = \{ \mu \in P(G) : \mu(gAh) = 0 \quad \text{for all} \quad g, h \in G \} \]

be the set of its witness measures. We say that \( A \) is generically Haar null if \( T(A) \) is comeager in \( P(G) \) (the Polish space of Borel probability measures).

This notion was introduced by Dodos in the paper \([35]\), which only considers abelian groups. In this first paper these sets were called “strongly Haar null sets”, despite the fact that this notion is unrelated to the strongly Haar meager sets introduced in chapter 4. Later Dodos generalized this notion for arbitrary Polish groups in \([37]\), which uses the name “generically Haar null sets”.

The “trick” of this definition is that the system of generically Haar null sets is obviously closed under countable unions because \( T(\bigcup_{n \in \omega} A_n) \supseteq \bigcap_{n \in \omega} T(A_n) \) and the countable
intersection of comeager sets is still comeager. The following proposition states this fact and the other useful properties which are evident from the definition:

**Proposition 5.3.2.** The system of generically Haar null sets is translation-invariant and forms a $\sigma$-ideal in the $\sigma$-algebra of universally measurable sets. Every generically Haar null set is a generalized Haar null set.

Notice that we only defined generically Haar null sets among the universally measurable sets.

In the special case when $G$ is abelian, this notion coincides with the “generic” version of [Theorem 2.4.3](#) part (3):

**Theorem 5.3.3** (Banakh-Głąb-Jabłońska-Swaczyna). Assume that $G$ is an abelian Polish group and $B \subseteq G$ is a Borel set. Then the following are equivalent:

1. in the Polish space $C(2^\omega, G)$ of continuous functions from $2^\omega$ to $G$ (endowed with the compact-open topology) the set
   $$\{ f \in C(2^\omega, G) : f^{-1}(gB) \in \mathcal{N}(2^\omega) \}$$
   is comeager,

2. in the Polish space $\mathcal{P}(G)$ of Borel probability measures on $G$ the set
   $$\{ \mu \in \mathcal{P}(G) : \mu \text{ witnesses that } B \text{ is Haar null} \}$$
   is comeager.

As in [Theorem 2.4.3](#), $\mathcal{N}(2^\omega)$ denotes the $\sigma$-ideal of sets of Haar measure zero on the Cantor cube $2^\omega$. The proof of this result can be found as part (2) of [6, Theorem 13.8](#).

It is also possible to define an analogous one-sided notion:

**Definition 5.3.4.** Assume that $A \subseteq G$ is universally measurable. Let

$$T_l(A) = \{ \mu \in \mathcal{P}(G) : \mu(gA) = 0 \text{ for all } g \in G \}$$

be the set of its left witness measures. We say that $A$ is generically left Haar null if $T_l(A)$ is comeager in $\mathcal{P}(G)$.

It is clear that generically Haar null sets are always generically left Haar null and the two notions coincide in abelian groups.

**Proposition 5.3.5.** The system of generically left Haar null sets is translation-invariant and forms a $\sigma$-ideal in the $\sigma$-algebra of universally measurable sets. Every generically left Haar null set is a generalized left Haar null set.
Notice that generically left Haar null sets are closed under countable unions even in those groups where left Haar null sets (or generalized left Haar null sets) do not have this property (see Theorem 5.1.6).

The following trichotomy gives additional motivation for this notion:

**Theorem 5.3.6** (Dodos). Each analytic set \( A \subseteq G \) satisfies one of the following:

1. \( T(A) = \emptyset \) (i.e. \( A \) is not generalized Haar null),
2. \( T(A) \) is meager and dense in \( P(G) \),
3. \( T(A) \) is comeager in \( P(G) \) (i.e. \( A \) is generically Haar null).

**Proof.** First we prove that if \( A \) is generalized Haar null, then \( T(A) \) is dense in \( P(G) \) (this part of the proof is from \([36]\), Proposition 5 (i)). Let \( \delta_x \) denote the Dirac measure concentrated at \( x \) (i.e. \( \delta_x(X) = 1 \) if \( x \in X \) and \( \delta_x(X) = 0 \) otherwise). The system

\[
F = \{ \sum_{i=1}^n \alpha_i \delta_{x_i} : n \in \omega, 0 \leq \alpha_i \leq 1, \sum_{i=1}^n \alpha_i = 1, x_i \in G \}
\]

is a dense subset of \( P(G) \) (see \([69]\), Theorem 17.19). It is enough to prove that if \( \varphi = \sum_{i=1}^n \alpha_i \delta_{x_i} \in F \) is arbitrary, then any of its neighborhoods contains an element of \( T(A) \). If \( U \subset P(G) \) is a neighborhood of \( \varphi \), then there are open sets \( U_i \subset G \) (\( 1 \leq i \leq n \)) such that \( U_i \) is a neighborhood of \( x_i \) and \( U \) contains all measures of the form \( \sum_{i=1}^n \alpha_i \mu_i \) where \( \mu_i \in P(G) \) and \( \text{supp} \mu_i \subset U_i \). Corollary 2.1.3 states that there are witness measures \( \mu_i \in T(A) \) with \( \text{supp} \mu_i \subset U_i \); the convex combination \( \sum_{i=1}^n \alpha_i \mu_i \) is also a witness measure, concluding this proof.

Now it is enough to prove that \( T(A) \) is always either meager in \( P(G) \) or comeager in \( P(G) \) (this part of the proof is from \([35]\), Theorem A). This proof relies on the following fact about the geometrical structure of the space of probability measures:

**Definition 5.3.7.** Let \( S \) be a set equipped with a function \( c : [0,1] \times S \times S \to S \) that defines the convex combinations of elements of \( S \) (for example \( S = P(X) \) with \( c(t,\mu,\nu) = t\mu + (1-t)\nu \)). A subset \( F \subseteq S \) is called a face of \( S \) if it is convex (that is, \( 0 \leq t \leq 1, p, q \in F \Rightarrow c(t,p,q) \in F \)) and extremal (that is, \( 0 < t < 1, c(t,p,q) \in F \Rightarrow p \in F \& q \in F \)).

**Theorem 5.3.8** (Dodos). Let \( X \) be a Polish space and \( F \) be a face of \( P(X) \) with the Baire property. Then \( F \) is either a meager or a comeager subset of \( P(X) \).

The proof of this result can be found as \([35]\), Theorem B].

It is easy to see that if \( A \) is a universally measurable generalized Haar null set, then \( T(A) \) is a face of \( P(G) \).

To finish the proof of the theorem it is enough to prove the following claim:
Claim 5.3.9. For an analytic set $A \subseteq G$, $T(A)$ has the Baire property.

Proof. It is clear that the set

$$S = \{((g, h), gah) : g, h \in G, a \in A\} \subseteq (G \times G) \times G$$

is analytic. If we apply Theorem 3.1.3 for the set $S$, $X = G \times G$ and $Y = G$, then we can see that

$$\{((\mu, (g, h), r) \in P(G) \times (G \times G) \times \mathbb{R} : \mu(gAh) > r\}$$

is analytic. If we take the section of this set at $r = 0$ and project this section on $P(G)$, then we can see that

$$\{\mu \in P(G) : \exists g, h \in G : \mu(gAh) > 0\} = P(G) \setminus T(A)$$

is analytic, $T(A)$ is coanalytic. It follows from Corollary 29.14 that analytic and coanalytic sets have the Baire property, concluding the proof.

It is clear that the following one-sided variant of this trichotomy can be proved analogously:

Theorem 5.3.10 (Dodos). Each analytic set $A \subseteq G$ satisfies one of the following:

1. $T_{l}(A) = \emptyset$ (i.e. $A$ is not generalized left Haar null),
2. $T_{l}(A)$ is meager and dense in $P(G)$,
3. $T_{l}(A)$ is comeager in $P(G)$ (i.e. $A$ is generically left Haar null).

Following and , we stated the trichotomy Theorem 5.3.6 for analytic sets. This is formally stronger than stating it for Borel sets, but it turns out that the analytic generically Haar null sets are just the analytic subsets of the Borel generically Haar null sets:

Theorem 5.3.11 (Dodos). Let $A \subseteq G$ be an analytic generalized Haar null set. Then there exists a Borel Haar null set $B \supseteq A$ such that $T(A) \setminus T(B)$ is meager in $P(G)$.

Proof. This result originally appears as Corollary 12, in a paper which only considers the question in the abelian case. We use a different method to prove this result (in all Polish groups $G$); this method is from where it was used to prove part (1) of Theorem 3.1.1.

In the proof of we proved Claim 5.3.9 which states that $T(A)$ has the Baire property. This implies that $T(A)$ can be written as $T(A) = R \cup M$ where $R$ is a nonempty Borel subset of $P(G)$ and $M$ is meager in $P(G)$. 
Claim 5.3.12. The family of sets

$$\Phi = \{ X \subseteq G : X \text{ is analytic and } \mu(gXh) = 0 \text{ for every } g, h \in G \text{ and } \mu \in R \}$$

is coanalytic on analytic, that is, for every Polish space $Y$ and $P \in \Sigma_1^1(Y \times G)$, the set

$$\{ y \in Y : P_y \in \Phi \}$$

is $\Pi_1^1$.

Proof. Let $Y$ be Polish space and $P \in \Sigma_1^1(Y \times G)$ and let

$$\tilde{P} = \{ (g, h, y, \gamma) \in G \times G \times Y \times G : \gamma \in gP_yh \}.$$ 

Then $\tilde{P}$ is analytic, as it is the preimage of $P$ under $(g, h, y, \gamma) \mapsto (y, g^{-1} \gamma h^{-1})$. Applying Theorem 3.1.3 yields that

$$\{ (\mu, g, h, y, r) \in P(G) \times G \times G \times Y \times \mathbb{R} : \mu(\tilde{P}(g, h, y)) > r \}$$

is analytic, therefore its section

$$\{ (\mu, g, h, y) \in P(G) \times G \times G \times Y : \mu(\tilde{P}(g, h, y)) > 0 \}$$

is also analytic. If we intersect this with the Borel set $R \times G \times G \times Y$, we get the analytic set

$$\{ (\mu, g, h, y) \in R \times G \times G \times Y : \mu(\tilde{P}(g, h, y)) > 0 \}.$$ 

Projecting this on $Y$ yields that

$$\{ y : \mu(\tilde{P}(g, h, y)) > 0 \text{ for some } g, h \in G \text{ and } \mu \in R \}$$

is analytic. Using this and the fact that $\tilde{P}(g, h, y) = gP_yh$ (by definition) yields that

$$\{ y \in Y : \mu(\tilde{P}(g, h, y)) = 0 \text{ for all } g, h \in G \text{ and } \mu \in R \} = \{ y \in Y : P_y \in \Phi \}$$

is coanalytic.

Now, since $A \in \Phi$, by the dual form of the First Reflection Theorem (see [33, Theorem 35.10] and the remarks following it) there exists a Borel set $B$ with $B \supseteq A$ and $B \in \Phi$. $B \in \Phi$ means that $R \subseteq T(B)$, in particular, $B$ is Haar null because $R$ is nonempty. It is clear from the definition that $T(B) \subseteq T(A)$, therefore

$$T(A) \setminus T(B) \subseteq T(A) \setminus R \subseteq M,$$

which shows that $T(A) \setminus T(B)$ is indeed meager.

The following result is an immediately corollary of this theorem:
Corollary 5.3.13. An analytic set $A \subseteq G$ is generically Haar null if and only if there is a Borel generically Haar null set $B \subseteq G$ such that $A \subseteq B$.

It is relatively easy to prove that a Borel generically Haar null set is not only Haar null, but also meager. (This idea appears as part (ii) of [36, Proposition 5].) In fact, this is also true for generically left Haar null sets:

Proposition 5.3.14 (Dodos). If $B$ is a Borel generically left Haar null set, then $B$ is meager.

Proof. Assume for the contrary that $B$ is not meager. This implies that $B$ is comeager in some open subset of $G$ (see e.g. [69, 8.26]). If $D$ is a countable dense subset of $G$, then the product $DB = \{db : d \in D, b \in B\}$ is comeager in $G$. It follows from [21, Theorem 10] that $\mu(DB) = 1$ for the generic measure $\mu \in P(G)$. On the other hand, the generic measure $\mu \in P(G)$ satisfies that $\mu(gB) = 0$ for all $g \in G$ (because $B$ is generically left Haar null), thus $\mu(DB) \leq \sum_{g \in D} \mu(gB) = 0$, a contradiction.

Notice that this proof works even if we replace “Borel” with “universally measurable and with the Baire property” in the proposition.

The following proposition implies that this $\sigma$-ideal is nontrivial (contains nonempty sets) in abelian groups:

Proposition 5.3.15 (Dodos). Assume that $G$ is an abelian Polish group. If $A \subset G$ is a $\sigma$-compact Haar null set, then $A$ is generically Haar null.

The proof of this result can be found as part (iii) of [36, Proposition 5].

Notice that this proposition implies that if (the abelian Polish group) $G$ is locally compact, then the closed Haar null sets are generically Haar null. The converse of this is also true:

Theorem 5.3.16 (Dodos). Assume that $G$ is an abelian Polish group. Then $G$ is locally compact if and only if every closed Haar null set is generically Haar null.

The proof of this result can be found as [35, Corollary 9].

In section 6.2 we will state a variant of the Steinhaus theorem (Theorem 6.2.9) which uses the generically left Haar null sets as small sets.

For more information about related notions in the special case of abelian Polish groups, see [3].
5.4 Aronszajn, cube, and Gauss null sets

In separable Banach spaces there are various other notions of nullsets. Let us describe some of these very briefly.

A Borel measure $\mu$ on a separable (real) Banach space $X$ is called a cube measure if it is the push-forward of the standard measure (the product of the Lebesgue measures) on $[0,1]^\mathbb{N}$ along the map $r \mapsto x + \sum_i r_i x_i$, where $x, x_i \in X$ are fixed and $\sum_i |x_i| < \infty$. A Borel measure $\mu$ on $X$ is called Gaussian if its push-forward $\mu \circ (x^*)^{-1}$ is a Gaussian measure (the distribution of a normal random variable) on $\mathbb{R}$ for every $x^* \in X^*$. A Borel measure is called non-degenerate if all closed hyperplanes are of measure zero.

Definition 5.4.1. Let $X$ be a (real) separable Banach space and $B \subseteq X$ a Borel set. Then $B$ is called

- **Aronszajn null** if for every sequence $(x_i)_{i \in \mathbb{N}} \subset X$ with dense span there are Borel sets $(B_i)_{i \in \mathbb{N}}$ with $B = \bigcup_i B_i$ such that each $B_i$ has one-dimensional Lebesgue measure zero on every line parallel to $x_i$,

- **cube null** if it is null with respect to every non-degenerate cube measure on $X$,

- **Gauss null** if it is null with respect to every non-degenerate Gaussian measure on $X$.

The fundamental result in this area is Csörnyei’s theorem stating that these three notions actually coincide [27]. As for the connection with Haar null sets, it is clear from the definition that all cube null sets are Haar null. However the converse implication does not hold in general, as Theorem 7.4.2 implies that the compact sets are Haar null in a separable Banach space, but one can show that there are compact sets that are not cube null. For more information on these notions, especially for the connection between these notions and differentiability of Lipschitz functions, see e.g. [83].

It is also worth mentioning that there is a further important related notion, that of the so called $\Gamma$-null sets, but we do not give the somewhat technical definition here. This can also be found in [83].
Chapter 6

Analogs of the results from the locally compact case

In this chapter we discuss generalizations and analogs of a few theorems that are well-known for locally compact groups. Unfortunately, although these are true in locally compact groups for the sets of Haar measure zero and the meager sets, neither of them remains completely valid for Haar null sets and Haar meager sets. However, in some cases weakened versions remain true, and these often prove to be useful.

6.1 Fubini’s theorem and the Kuratowski-Ulam theorem

Fubini’s theorem, and its topological analog, the Kuratowski-Ulam theorem (see e.g. [??, 8.41]) describes small sets in product spaces. They basically state that a set (which is measurable in the appropriate sense) in the product of two spaces is small if and only if co-small many sections of it are small. Notice that because the product of left (or right) Haar measures of two locally compact groups of is trivially a left (or right) Haar measure of the product group, (a special case of) Fubini’s theorem connects the sets of Haar measure zero in the two groups and the sets of Haar measure zero in the product group.

Unfortunately, analogs of these theorems are proved only in very special cases, and there are counterexamples known in otherwise “nice” groups. We provide a simple counterexample (which can be found as [??, Example 20]) that works in both the Haar null and Haar meager case.

Example 6.1.1 (folklore). There exists a closed set \( A \subseteq \mathbb{Z}^\omega \times \mathbb{Z}^\omega \) that is neither Haar null nor Haar meager, but in one direction all its sections are Haar null and Haar...
meager. (In the other direction, non-Haar-null and non-Haar-meager many sections are non-Haar-null and non-Haar-meager.)

Proof. The group operation of $\mathbb{Z}^\omega \times \mathbb{Z}^\omega$ is denoted by $+$. The set with these properties will be

$$A = \{(s, t) \in \mathbb{Z}^\omega \times \mathbb{Z}^\omega : t_n \geq s_n \geq 0 \text{ for every } n \in \omega\}.$$ 

It is clear from the definition that $A$ is closed.

Note that for $t \in \mathbb{Z}^\omega$, the section $A_t = \{s \in \mathbb{Z}^\omega : t_n \geq s_n \geq 0 \text{ for every } n \in \omega\}$ is compact (as it is the product of finite sets with the discrete topology), and it follows from Theorem 7.4.2 and Corollary 7.4.7 that all compact sets are Haar null and Haar meager in $\mathbb{Z}^\omega$.

To show that $A$ is not Haar null and not Haar meager, we will use the technique described in section 7.6 and show that for every compact set $C$ the section $A_C$ is not Haar null. As $C = \emptyset$ satisfies this, we may assume that $C \neq \emptyset$. Let $\pi_n^1(s, t) = s_n$ and $\pi_n^2(s, t) = t_n$, then $\pi_n^1, \pi_n^2 : \mathbb{Z}^\omega \times \mathbb{Z}^\omega \to \mathbb{Z}$ are continuous functions. Let $a$ and $b$ be the sequences satisfying $a_n = -\min \pi_n^1(C)$ and $b_n = -\min \pi_n^2(C) + \max \pi_n^1(C) - \min \pi_n^1(C)$. It is straightforward to check that this choice guarantees that if $(s, t) \in C + (a, b)$, then $t_n \geq s_n \geq 0$ for every $n \in \omega$.

Finally, if $s \in \mathbb{Z}^\omega$ satisfies $s_n \geq 0$ for every $n \in \omega$, then similar, but simpler arguments show that the section $A_s = \{t \in \mathbb{Z}^\omega : t_n \geq s_n \text{ for every } n \in \omega\}$ contains a translate of every compact set $C \subseteq \mathbb{Z}^\omega$, hence it is neither Haar null nor Haar meager. Similarly, the set $\{s \in \mathbb{Z}^\omega : s_n \geq 0 \text{ for every } n \in \omega\}$ is also neither Haar null nor Haar meager, so we proved the statement about the sections in the other direction.

The following counterexample appears as [23, Theorem 6]:

**Example 6.1.2** (Christensen). Let $H$ be a separable infinite dimensional Hilbert space (with addition as the group operation) and let $S^1$ be the unit circle in the complex plane (with complex multiplication as the group operation). There exists in the product group $H \times S^1$ a Borel set $A$ such that

(I) For every $h \in H$, the section $A_h$ has Haar measure one in $S^1$.

(II) For every $s \in S^1$, the section $A_s$ is Haar null in $H$.

(III) The complement of $A$ is Haar null in the product group $H \times S^1$.

Note that here $S^1$ is a compact group.

The connection between (I) and (III) in this example is not accidental:
Theorem 6.1.3 (Christensen, Borwein-Moors). If \((G_1, \cdot)\) is an abelian Polish group and \((G_2, \cdot)\) is a locally compact abelian Polish group and \(A \subset G_1 \times G_2\) is universally measurable, then the following are equivalent:

1. \(A\) is generalized Haar null in \(G_1 \times G_2\).
2. there is a generalized Haar null set \(N \subset G_1\) such that for all \(g_1 \in G_1 \setminus N\), the section \(A_{g_1}\) is Haar null in \(G_2\) (that is, the Haar measure on \(G_2\) assigns measure zero to it).

This connection is mentioned in [23] and proved in [18, Theorem 2.3].

We prove the following generalized version of this result:

Theorem 6.1.4. Suppose that \(G\) and \(H\) are Polish groups and \(B \subseteq G \times H\) is a Borel subset. Then if \(G\) is locally compact, then the following conditions are all equivalent.

1. there exists a Haar null set \(E \subseteq H\) and a Borel probability measure \(\mu\) on \(G\) such that every \(h \in H \setminus E\) and \(g_1, g_2 \in G\) satisfies \(\mu(g_1B^h g_2) = 0\) (i.e. these sections are Haar null and \(\mu\) is witness measure for each of them),
2. \(B\) is Haar null in \(G \times H\),
3. there exists a Haar null set \(E \subseteq H\) such that for every \(h \in H \setminus E\) the section \(B^h\) has Haar measure zero in \(G\).

Proof. (1) \(\Rightarrow\) (2):
As the set \(E\) is Haar null, there exists a Borel set \(E' \supseteq E\) and a Borel probability measure \(\nu\) on \(H\) such that \(\nu(h_1E'h_2) = 0\) for every \(h_1, h_2 \in H\). We show that the measure \(\mu \times \nu\) witnesses that \(B\) is Haar null in \(G \times H\). Fix arbitrary \(g_1, g_2 \in G\) and \(h_1, h_2 \in H\), we have to prove that \((\mu \times \nu)((g_1, h_1) \cdot B \cdot (g_2, h_2)) = 0\).

Notice that

\[(g_1, h_1) \cdot B \cdot (g_2, h_2) \subseteq (G \times (h_1 \cdot E' \cdot h_2)) \cup ((g_1, h_1) \cdot (B \setminus (G \times E')) \cdot (g_2, h_2))\]

and in this union the \((\mu \times \nu)\)-measure of the first term is zero (by the choice of \(\mu\) and \(E')\). For every \(h \in H\), the \(h\)-section of the Borel set \((g_1, h_1) \cdot (B \setminus (G \times E')) \cdot (g_2, h_2)\) is \(g_1 \cdot (B \setminus (G \times E'))^{h_1^{-1}h_2^{-1}} \cdot g_2\), and this is either the empty set (if \(h_1^{-1}h_2^{-1} \in E'\)) or a set of \(\mu\)-measure zero (if \(h_1^{-1}h_2^{-1} \in H \setminus E'\)). Applying Fubini’s theorem in the product space \(G \times H\) to the product measure \(\mu \times \nu\) yields that \((\mu \times \nu)((g_1, h_1) \cdot B \cdot (g_2, h_2)) = 0\), and this is what we had to prove.

(2) \(\Rightarrow\) (3) (when \(G\) is locally compact):
Let \(B \subseteq G \times H\) be Haar null and suppose that \(\mu\) is a witness measure, i.e. \(\mu((g_1, h_1) \cdot B \cdot (g_2, h_2)) = 0\) for every \(g_1, g_2 \in G\) and \(h_1, h_2 \in H\). Fix a left Haar measure \(\lambda\) on \(G\), let
\( \delta \) denote the Dirac measure at 1
\( H \) (i.e. for \( X \subseteq H, \delta(X) = 1 \) if \( 1 \in X \) and 0 otherwise) and let \( \lambda = \lambda \times \delta \). Let \( E = \{ h \in H : \lambda(B^h) \neq 0 \} \), it is clearly enough to prove that \( E \subseteq H \) is Haar null. First we use a standard argument to show that \( E \) is a Borel set.

If \((X, \mathcal{S})\) is a measurable space, \( Y \) is a separable metrizable space, \( P(Y) \) is the space of Borel probability measures on \( Y \) and let \( \lambda \) measurable (for \( S \subseteq B \)), then [63, Theorem 17.25] states that the map \( X \times P(Y) \to \mathbb{R}_+, (x, \varrho) \mapsto \varrho(A_x) \) is measurable (for \( S \subseteq B(P(Y)) \)). Applying this for \((X, \mathcal{S}) = (H, \mathcal{B}(H)), Y = G, A = B \) yields that \( \tilde{E} = \{ (h, \varrho) \in H \times P(G) : \varrho(B^h) \neq 0 \} \) is Borel (the Borel preimage of an open set in \( \mathbb{R}_+ \)). If \( \varrho \) is a Borel probability measure on \( G \) that is equivalent to \( \lambda \) in the sense that they have the same zero sets, then \( E = \tilde{E}^\varrho \) is Borel, because it is the section of a Borel set.

We will show that the measure \( \nu(X) = \mu(G \times X) \) witnesses that \( E \) is Haar null. Fix arbitrary \( h_1, h_2 \in H \), we have to prove that \( 0 = \nu(h_1 Eh_2) = \mu(G \times (h_1 Eh_2)) \).

Consider the set
\[
S = \{ ((u_G, u_H), (v_G, v_H)) \in ((G \times H) \times (G \times H)) : (u_G, u_H) \cdot (v_G, v_H) \in (1_G, h_1) \cdot B \cdot (1_G, h_2) \},
\]
it is easy to see that this is a Borel set. Applying Fubini’s theorem in the product space \((G \times H) \times (G \times H)\) to the product measure \( \mu \times \lambda \) yields that
\[
(\mu \times \lambda)(S) = \int_{G \times H} \mu(1_G, h_1) \cdot B \cdot (v_G^{-1}, h_2v_H^{-1}) \ d\lambda((v_G, v_H)) = 0
\]
because \( \mu \) witnesses that \( B \) is Haar null. Applying Fubini’s theorem again for the other direction yields that
\[
0 = (\mu \times \lambda)(S) = \int_{G \times H} \lambda(u_G^{-1}, u_H^{-1}h_1) \cdot B \cdot (1_G, h_2) \ d\mu((u_G, u_H)) = \\
= \int_{G \times H} \lambda(g) \ (g, 1_H) \in (u_G^{-1}, u_H^{-1}h_1) \cdot B \cdot (1_G, h_2) \ d\mu((u_G, u_H)) = \\
= \int_{G \times H} \lambda(g) \ (u_{Gh}, h_1^{-1}u_{Gh}h_2^{-1}) \ d\mu((u_G, u_H)) = \\
= \int_{G \times H} \lambda(B^{h_1^{-1}u_{Gh}h_2^{-1}}) \ d\mu((u_G, u_H)) = \\
= \int_{G \times H} \lambda(B^{h_1^{-1}u_{Gh}h_2^{-1}}) \ d\mu((u_G, u_H)).
\]

It is clear from the definition of \( E \) that the function
\[
\varphi : G \times H \to \mathbb{R}_+, \quad (u_G, u_H) \mapsto \lambda(B^{h_1^{-1}u_{Gh}h_2^{-1}})
\]
takes strictly positive values on the set $G \times (h_1 Eh_2)$. However, our calculations showed that $\int \varphi \, d\mu$ is zero, hence the $\mu$-measure of the set $G \times (h_1 Eh_2)$ must be zero.

(3) $\Rightarrow$ (1) (when $G$ is locally compact):

Fix a left Haar measure $\lambda$ on $G$. (3) states that $\lambda(B^h) = 0$ for every $h \in H \setminus E$. Using Theorem 2.5.2, it is easy to see that $\lambda(g_1B^h g_2) = 0$ for every $g_1, g_2 \in G$. Let $C \subseteq G$ be a Borel set such that $0 < \lambda(C) < \infty$ (the regularity of the Haar measure guarantees a compact $C$ satisfying this) and let $\mu$ be the Borel probability measure $\mu(X) = \frac{\lambda(X \cap C)}{\lambda(C)}$. Then $\mu \ll \lambda$ guarantees that $\mu$ satisfies condition (1).

We also prove the analog of this for Haar meager sets:

**Theorem 6.1.5.** Suppose that $G$ and $H$ are Polish groups and $B \subseteq G \times H$ is a Borel subset. Then if $G$ is locally compact, then the following conditions are all equivalent. Moreover, the implication (1) $\Rightarrow$ (2) remains valid even if $G$ is not necessarily locally compact.

1. there exists a Haar meager set $E \subseteq H$, a (nonempty) compact metric space $K$ and a continuous function $f : K \to G$ such that every $h \in H \setminus E$ and $g_1, g_2 \in G$ satisfies that $f^{-1}(g_1B^h g_2)$ is meager in $K$ (i.e. these sections are Haar meager and $f$ is a witness function for each of them),

2. $B$ is Haar meager in $G \times H$,

3. there exists a Haar meager set $E \subseteq H$ such that for every $h \in H \setminus E$ the section $B^h$ is meager in the locally compact group $G$.

**Proof.** (1) $\Rightarrow$ (2):

As the set $E$ is Haar meager, there exist a Borel set $E' \supseteq E$, a (nonempty) compact metric space $K'$ and a continuous function $\varphi : K' \to H$ such that $\varphi^{-1}(h_1 E' h_2)$ is meager in $K'$ for every $h_1, h_2 \in H$.

Let $f \times \varphi : K \times K' \to G \times H$ be the function $(f \times \varphi)(k, k') = (f(k), \varphi(k'))$. We show that the function $f \times \varphi$ witnesses that $B$ is Haar meager in $G \times H$. Fix arbitrary $g_1, g_2 \in G$ and $h_1, h_2 \in H$, we have to prove that $(f \times \varphi)^{-1}((g_1, h_1) \cdot B \cdot (g_2, h_2))$ is meager in $K \times K'$.

Notice that

$$(g_1, h_1) \cdot B \cdot (g_2, h_2) \subseteq (G \times (h_1 \cdot E' \cdot h_2)) \cup ((g_1, h_1) \cdot (B \setminus (G \times E')) \cdot (g_2, h_2))$$

and in this union

$$(f \times \varphi)^{-1}(G \times (h_1 \cdot E' \cdot h_2)) = K \times \varphi^{-1}(h_1 \cdot E' \cdot h_2)$$

and using the choice of $\varphi$ and the Kuratowski-Ulam theorem it is clear that this is a meager subset of $K \times K'$. For every $h \in H$, the $h$-section of the Borel set $(g_1, h_1) \cdot (B \setminus
Let \( f \) be the function (where \( k \) is the case of measure). Let \( B \) be the function \( \phi \) yields that the set \( K \) is meager in \( k \). This means that for every \( h \), then \( G \) is indeed meager in \( K \) or a set whose preimage under \( f \) is meager in \( K \) (if \( h^{-1}h_2^{-1} \in H \setminus E' \)).

\[
(f \times \varphi)^{-1} \left( (g_1, h_1) \cdot (B \setminus (G \times E')) \cdot (g_2, h_2) \right) \subseteq K \times K'
\]
is meager in \( K \). Applying the Kuratowski-Ulam theorem in the product space \( K \times K' \) yields that the set \( (f \times \varphi)^{-1}((g_1, h_1) \cdot B \cdot (g_2, h_2)) \) is indeed meager in \( K \times K' \), and this is what we had to prove.

(2) \( \Rightarrow \) (3) (when \( G \) is locally compact):
Let \( B \subseteq G \times H \) be Haar meager and suppose that \( f : K \to G \times H \) is a witness function (where \( K \) is a nonempty compact metric space), i.e. \( f^{-1}((g_1, h_1) \cdot B \cdot (g_2, h_2)) \) is meager in \( K \) for every \( g_1, g_2 \in G \) and \( h_1, h_2 \in H \). Let \( f(k) = (f_G(k), f_H(k)) \) for every \( k \in K \), then \( f_G : K \to G \) and \( f_H : K \to H \) are continuous functions. Let \( E = \{ h \in H : B^h \) is not meager in \( G \} \), then [92, Theorem 16.1] states that \( E \) is Borel. It is enough to prove that \( E \subseteq H \) is Haar meager. We show that this is witnessed by the function \( f_H : K \to H \). Fix arbitrary \( h_1, h_2 \in H \), we have to prove that \( f_H^{-1}(h_1 Eh_2) \) is meager in \( K \).

As in the case of measure, define the Borel set

\[
S = \{ ((u_G, u_H), (v_G, v_H)) \in ((G \times H) \times (G \times H)) : (u_G, u_H) \cdot (v_G, v_H) \in (1_G, h_1) \cdot B \cdot (1_G, h_2) \},
\]
and the function \( \psi : G \to G \times H, \psi(g) = (g, 1_H) \) (\( \psi \) is the analog of \( \tilde{\lambda} \) from the case of measure). Let \( f \times \psi : K \times G \to (G \times H) \times (G \times H) \) be the function \( (f \times \psi)(k, g) = (f(k), \psi(g)) \). First notice that for every \( g \in G \), the \( g \)-section of the set \( (f \times \psi)^{-1}(S) \subseteq K \times G \) is

\[
((f \times \psi)^{-1}(S))^g = \{ k \in K : (f \times \psi)((k, g)) \in S \} =
\]

\[
= \{ k \in K : (f(k), \psi(g)) \in S \} = \{ k \in K : (f(k), (g, 1_H)) \in S \} =
\]

\[
= \{ k \in K : f(k) \in (1_G, h_1) \cdot B \cdot (1_G, h_2) \cdot (g, 1_H)^{-1} \} =
\]

\[
f^{-1}((1_G, h_1) \cdot B \cdot (g^{-1}, h_2))
\]
and this is meager in \( K \) (as \( f \) witnesses that \( B \) is Haar meager), hence the Kuratowski-Ulam theorem yields that \( (f \times \psi)^{-1}(S) \) is meager in \( K \times G \).

Applying the Kuratowski-Ulam theorem in the other direction yields that the set

\[
\{ k \in K : ((f \times \psi)^{-1}(S))^k \text{ is not meager in } G \}
\]
is meager in \( K \). Here if we let \( u_G = f_G(k) \) and \( u_H = f_H(k) \), then
\[
((f \times \psi)^{-1}(S))_k = \{ g \in G : (f \times \psi)((k, g)) \in S \} = \{ g \in G : (u_G, u_H, (g, 1_H)) \in S \} = \{ g \in G : (g, 1_H) \in (u_G, u_H)^{-1} \cdot (1_G, h_1) \cdot B \cdot (1_G, h_2) \} = \{ g \in G : (u_G^{-1}, u_H^{-1}h_1) \cdot B \cdot (1_G, h_2) \} = \{ g \in G : (u_Gg, h_1^{-1}u_Hh_2^{-1}) \in B \} = u_G^{-1}B^{-1}_{h_1^{-1}f_H(k)h_2^{-1} = (f_G(k))^{-1}B^{-1}_{h_1^{-1}f_H(k)h_2^{-1}}.
\]

Thus we know that
\[
\{ k \in K : (f_G(k))^{-1}B^{-1}_{h_1^{-1}f_H(k)h_2^{-1}} \text{ is not meager in } G \} \text{ is meager in } K,
\]
and because meagerness is translation invariant
\[
\{ k \in K : B_{h_1^{-1}f_H(k)h_2^{-1}} \text{ is not meager in } G \} \text{ is meager in } K.
\]

But notice that
\[
f_H^{-1}(h_1 Eh_2) = \{ k \in K : f_H(k) \in h_1 Eh_2 \} = \{ k \in K : h_1^{-1}f_H(k)h_2^{-1} \in E \} = \{ k \in K : B_{h_1^{-1}f_H(k)h_2^{-1}} \text{ is not meager in } G \}
\]
so we proved that \( f_H \) is indeed a witness measure for \( E \).

(3) \( \Rightarrow \) (1) (when \( G \) is locally compact):

Notice that the proof of Theorem 2.2.13 shows that in a locally compact Polish group every meager set is Haar meager and there is a function which is a witness function for each of them. The implication that we have to prove is clearly a special case of this observation.

Finally, we state another special case when the analog of the Kuratowski-Ulam theorem is valid. The proof of this result can be found in [33, Theorem 18].

**Theorem 6.1.6** (Doležal-Rmoutil-Vejnar-Vlasák). Suppose that \( G \) and \( H \) are Polish groups and \( A \subseteq G \) and \( B \subseteq H \) are analytic sets. Then \( A \times B \) is Haar meager in \( G \times H \) if and only if at least one of \( A \) and \( B \) is Haar meager in the respective group.

### 6.2 The Steinhaus theorem

The Steinhaus theorem and its generalizations state that if a set \( A \subseteq G \) is not small (and satisfies some measurability condition, e.g. it is in the \( \sigma \)-ideal generated by the
Borel sets and the small sets), then $AA^{-1} = \{ xy^{-1} : x, y \in A \}$ contains a neighborhood of $1_G$.

The original result of Steinhaus stated this in the group $(\mathbb{R}, +)$ and used the sets of Lebesgue measure zero as the small sets. Weil extended this for an arbitrary locally compact group, using the sets of Haar measure zero as the small sets.

**Corollary 2.2.10** (Steinhaus, Weil). If $G$ is a locally compact Polish group, $\lambda$ is a left Haar measure on $G$ and $A \subseteq G$ is $\lambda$-measurable with $\lambda(A) > 0$, then $1_G \in \text{int}(AA^{-1})$.

We already proved this result and used it to prove that there are no (left or right) Haar measures in non-locally-compact groups.

**Remark 6.2.1.** The proof of this result also works for non-locally-compact groups, but in those groups the result is vacuously true.

It is natural to ask the following question (a question of this type appears as e.g. \[85, Problem 2.7\]):

**Question 6.2.2.** Let $A$ be a Borel (or universally measurable) subset in the Polish group $G$ that is not Haar null (or not generalized Haar null, or not Haar meager). What can we say about the groups $G$ where $1_G \in \text{int}(AA^{-1})$ is necessarily satisfied?

Unfortunately, there are groups where the answer is negative. This is demonstrated by \[89, Theorem 6.1\], which we already stated as **Theorem 5.1.12** in the section about left and right Haar null sets. However, there are weakened versions of the Steinhaus theorem which turn out to be true and useful.

One of these is stated as \[85, Theorem 2.8\] and is a slight variation of a result in \[22\]. This result does not claim that $AA^{-1}$ will be a neighborhood of $1_G$, only that finitely many conjugates of it will cover a neighborhood of $1_G$. Also, it uses generalized right Haar null sets as small sets (see section 5.1 for the definition), which is a weaker notion than generalized Haar null sets (hence this result does not imply the variant where “right” is omitted from the text).

**Theorem 6.2.3** (Christensen, Rosendal). Suppose that $A \subseteq G$ is a universally measurable subset which is not generalized right Haar null. Then for any neighborhood $W$ of $1_G$ there are $n \in \omega$ and $h_0, h_1, h_2, \ldots, h_{n-1} \in W$ such that

$$h_0AA^{-1}h_0^{-1} \cup h_1AA^{-1}h_1^{-1} \cup \ldots \cup h_{n-1}AA^{-1}h_{n-1}^{-1}$$

is a neighborhood of $1_G$. 
Suppose that the conclusion fails for $A$ and $W$, that is, for every $n \in \omega$ and $h_0, h_1, \ldots h_{n-1} \in W$ and any neighborhood $V \ni 1_G$, there is some

$$g \in V \setminus \left( h_0 AA^{-1} h_0^{-1} \cup h_1 AA^{-1} h_1^{-1} \cup \ldots \cup h_{n-1} AA^{-1} h_{n-1}^{-1} \right).$$

Then we can inductively choose a sequence $(g_j)_{j \in \omega}$ such that $g_j \to 1_G$ and for every $j_0 < j_1 < j_2 < \ldots$ index sequence and $r \in \omega$

(I) the infinite product $g_{j_0} g_{j_1} g_{j_2} \cdots$ converges (this can be achieved e.g. by requiring $d(g_{j_0} \cdots g_{j_{r-1}}, g_{j_0} \cdots g_{j_r}) < 2^{-j_r}$ where $d$ is a fixed complete metric on $G$).

(II) $g_j \not\in (g_{j_0} \cdots g_{j_{r-1}})^{-1} AA^{-1} (g_{j_0} \cdots g_{j_{r-1}})$

Using (I) we can define a continuous map $\varphi : 2^\omega \to G$ by

$$\varphi(\alpha) = g_0^{\alpha(0)} g_1^{\alpha(1)} g_2^{\alpha(2)} \cdots,$$

where $g^0 = 1_G$ and $g^1 = g$. Let $\lambda$ be the Haar measure on the Cantor group $(\mathbb{Z}_2)^\omega = 2^\omega$ and notice that as $A$ is not generalized right Haar null, there is some $g \in G$ such that

$$\lambda(\varphi^{-1}(Ag)) = \varphi_*(\lambda)(Ag) > 0$$

So by Corollary 2.2.10,

$$\varphi^{-1}(Ag)(\varphi^{-1}(Ag))^{-1}$$

contains a neighborhood of the identity $(0, 0, \ldots)$ in $2^\omega$. This neighborhood must contain an element with exactly one “1” coordinate, so there are $\alpha, \beta \in \varphi^{-1}(Ag)$ and $m \in \omega$ such that $\alpha(m) = 1$, $\beta(m) = 0$ and $\alpha(j) = \beta(j)$ for every $j \in \omega, j \neq m$. This means that there are $h = g_{j_0} g_{j_1} \cdots g_{j_{r-1}}$, $j_0 < j_1 < \ldots < j_{r-1} < m$ and $k \in G$ such that $\varphi(\alpha) = h g_{j_0} g_{j_1} \cdots g_{j_{r-1}} k$ and $\varphi(\beta) = h k$. It follows that

$$h g_{j_0} g_{j_1} \cdots g_{j_{r-1}} k^{-1} h^{-1} = h g_{j_0} g_{j_1} \cdots g_{j_{r-1}} A = AA^{-1}$$

and so $g_m \in h^{-1} AA^{-1} h = (g_{j_0} \cdots g_{j_{r-1}})^{-1} AA^{-1} (g_{j_0} \cdots g_{j_{r-1}})$, contradicting the choice of $g_m$.

The following special case is often useful:

**Corollary 6.2.4.** Suppose that $A \subseteq G$ is a universally measurable subset which is conjugacy invariant (that is, $gAg^{-1} = A$ for every $g \in G$; in abelian groups every subset has this property) and not generalized Haar null. Then $AA^{-1}$ is a neighborhood of the identity.

**Proof.** If $A$ is conjugacy invariant, then as we noted, $gAh = gAg^{-1} \cdot gh = Ag h$ and thus $A$ is generalized right Haar null if and only if $A$ is generalized Haar null. Moreover, if
A is conjugacy invariant, then \( gAA^{-1}g^{-1} = (gAg^{-1}) \cdot (gAg^{-1})^{-1} = AA^{-1} \), hence \( AA^{-1} \) is also conjugacy invariant and thus in this case \( \text{Theorem 6.2.3} \) states that \( AA^{-1} \) is a neighborhood of identity.

Variants of the Steinhaus theorem can be used to prove results about automatic continuity (results stating that all homomorphisms \( \pi : G \rightarrow H \) which satisfy certain properties are continuous). For example the classical result \( \text{Corollary 2.2.10} \) yields that any universally measurable homomorphism from a locally compact Polish group into another Polish group is continuous (see e.g. \[85\], Corollary 2.4; a map is said to be \textit{universally measurable} if the preimages of open sets are universally measurable). We prove the following automatic continuity result as a corollary of \( \text{Theorem 6.2.3} \).

**Corollary 6.2.5** (Christensen). If \( G \) and \( H \) are Polish groups, \( H \) admits a two-sided invariant metric and \( \pi : G \rightarrow H \) is a universally measurable homomorphism, then \( \pi \) is continuous.

\textit{Proof.} It is enough to prove that \( \pi \) is continuous at \( 1_G \) (a homomorphism is continuous if and only if it is continuous at the identity element). Let \( V \subseteq H \) be an arbitrary neighborhood of \( 1_H \).

Using the continuity of the map \( (x, y) \mapsto xy^{-1} \) at \((1_H, 1_H) \in H \times H\), there is an open set \( U \) with \( 1_H \in U \) satisfying that \( UU^{-1} \subseteq V \). If \( d \) is a two-sided invariant metric on \( H \), then the open balls \( B(1_H, r) \) are conjugacy invariant sets and form a neighborhood base of \( 1_H \), hence we may also assume that \( U \) is conjugacy invariant.

\( G \) can be covered by countably many right translates of \( \pi^{-1}(U) \), because there is a countable set \( S \subseteq \pi(G) \) that is dense in \( \pi(G) \) and thus the system \( \{U \cdot s : s \in S\} \) covers \( \pi(G) \). This means that the universally measurable set \( \pi^{-1}(U) \) is not right Haar null, hence we may apply \( \text{Theorem 6.2.3} \) to see that for some \( n \in \omega \) and \( h_0, h_1, \ldots, h_{n-1} \in G \) the set

\[
W = \bigcup_{0 \leq j < n} h_j \pi^{-1}(U)(\pi^{-1}(U))^{-1}h_j^{-1}
\]

is a neighborhood of \( 1_G \). For every \( g \in W \) there is a \( 0 \leq j < n \) such that \( g \in h_j \pi^{-1}(U)(\pi^{-1}(U))^{-1}h_j^{-1} \), but then

\[
\pi(g) \in \pi(h_j)UU^{-1}\pi(h_j)^{-1} = \pi(h_j)U\pi(h_j)^{-1} \cdot (\pi(h_j)U\pi(h_j)^{-1})^{-1} = UU^{-1},
\]

thus \( \pi(W) \subseteq UU^{-1} \subseteq V \) and this shows that \( \pi \) is continuous at \( 1_G \).

\text{Proof.} □

In \[90\] Solecki proved that if \( G \) is amenable at \( 1 \) (see \( \text{Definition 5.1.3} \)), then a simpler variant of the Steinhaus theorem is true, but if \( G \) has a free subgroup at \( 1 \) (see \( \text{Definition 5.1.5} \)) and satisfies some technical condition, then this variant is false. The following theorems state these results.
Theorem 6.2.6 (Solecki). If $G$ is amenable at 1 and $A \subseteq G$ is universally measurable and not generalized left Haar null, then $A^{-1}A$ contains a neighborhood of $1_G$.

Definition 6.2.7. A Polish group $G$ is called strongly non-locally-compact if for any neighborhood $U$ of $1_G$ there exists a neighborhood $V$ of $1_G$ such that $U$ cannot be covered by finitely many sets of the form $gVh$ with $g, h \in G$.

Note that the examples we mentioned after Definition 5.1.5 are all strongly non-locally-compact.

Theorem 6.2.8 (Solecki). If $G$ has a free subgroup at 1 and is strongly non-locally-compact, then there is a Borel set $A \subseteq G$ which is not left Haar null and satisfies $1_G \notin \text{int}(A^{-1}A)$.

Using the notion of generically left Haar null sets (which we describe in section 5.3), it is possible to prove another weakened version of Question 6.2.2.

Theorem 6.2.9 (Dodos). Suppose that $A \subseteq G$ is an analytic set which is not generically left Haar null. Then $A^{-1}A$ is not meager in $G$.

The relatively long proof of this result can be found as [37, Theorem A].

Combining this and Pettis’ theorem yields the following corollary:

Corollary 6.2.10 (Dodos). If $A \subseteq G$ is analytic and not generically left Haar null, then $1_G \in \text{int}(A^{-1}AA^{-1}A)$.

The above mentioned Pettis’ theorem (which is also known as Piccard’s theorem and appears as e.g. [10, Theorem 2.9.6] and [62, Theorem 9.9]) is the analog of the Steinhaus theorem which uses the meager sets as small sets.

As “Haar meager” is a stronger notion than “meager”, the following result strengthens this classical result in the abelian case:

Theorem 6.2.11 (Jabłońska). Let $G$ be an abelian Polish group. If $A \subseteq G$ is a Borel set that is not Haar meager, then $AA^{-1}$ is a neighborhood of $1_G$.

The proof of this theorem can be found in [15]; the method of the proof is similar to that of Theorem 6.2.3.

For more information about this question in the special case of abelian Polish groups, see the paper [6].
6.3 The countable chain condition

The countable chain condition (often abbreviated as ccc) is a well-known property of partially ordered sets or structures with associated partially ordered sets. A partially ordered set \((P, \leq)\) is said to satisfy the countable chain condition if every strong antichain in \(P\) is countable. (A set \(A \subseteq P\) is a strong antichain if \(\forall x, y \in A : (x \neq y \Rightarrow \nexists z \in P : (z \leq x \text{ and } z \leq y))\).) This is called countable “chain” condition because in some particular cases this condition happens to be equivalent to a condition about lengths of certain chains.

We will apply the countable chain condition to notions of smallness in the following sense:

**Definition 6.3.1.** Suppose that \(X\) is a set and \(S \subseteq \mathcal{A} \subseteq \mathcal{P}(X)\). We say that \(S\) has the countable chain condition in \(\mathcal{A}\) if there is no uncountable system \(U \subseteq \mathcal{A} \setminus S\) such that \(U \cap V \in S\) for any two distinct \(U, V \in U\).

In our cases \(\mathcal{A}\) will be a \(\sigma\)-algebra and \(S\) will be the \(\sigma\)-ideal of “small” sets. Notice that \(S\) has the countable chain condition in \(\mathcal{A}\) if and only if the partially ordered set \((\mathcal{A} \setminus S, \subseteq)\) satisfies the countable chain condition. Also notice that if \(\tilde{S} \subseteq S\) and \(\tilde{\mathcal{A}} \supseteq \mathcal{A}\), then “\(\tilde{S}\) has the countable chain condition in \(\tilde{\mathcal{A}}\)” is a stronger statement than “\(S\) has the countable chain condition in \(\mathcal{A}\)”.

We will generalize the following two classical results (these are stated as [69, Exercise 17.2] and [69, Exercise 8.31]).

**Proposition 6.3.2.** If \(\mu\) is a \(\sigma\)-finite measure, the \(\sigma\)-ideal of sets with \(\mu\)-measure zero has the countable chain condition in the \(\sigma\)-algebra of \(\mu\)-measurable sets.

**Proposition 6.3.3.** In a second countable Baire space, the \(\sigma\)-ideal of meager sets has the countable chain condition in the \(\sigma\)-algebra of sets with the Baire property.

The theorems in section 2.2 state that if \(G\) is locally compact, then \(\mathcal{N} = \mathcal{H}\mathcal{N} = \mathcal{G}\mathcal{H}\mathcal{N}\) (i.e. of Haar measure zero \(\Leftrightarrow\) Haar null \(\Leftrightarrow\) generalized Haar null) and \(\mathcal{M} = \mathcal{H}\mathcal{M}\) (i.e. meager \(\Leftrightarrow\) Haar meager). The special case of Proposition 6.3.2 where \(\mu = \lambda\) for a left Haar measure \(\lambda\) on \(G\) means that the \(\sigma\)-ideal \(\mathcal{H}\mathcal{N} = \mathcal{G}\mathcal{H}\mathcal{N}\) has the countable chain condition in the \(\sigma\)-algebra of \(\lambda\)-measurable sets. As every universally measurable set is \(\lambda\)-measurable, this clearly implies that \(\mathcal{H}\mathcal{N} = \mathcal{G}\mathcal{H}\mathcal{N}\) has the countable chain condition in the \(\sigma\)-algebra of universally measurable sets. Analogously, Proposition 6.3.3 means that in a locally compact Polish group \(G\) the \(\sigma\)-ideal \(\mathcal{H}\mathcal{M}\) has the countable chain condition in the \(\sigma\)-algebra of sets with the Baire property.

For the case of measure Christensen asked in [23, Problem 2] whether is this true in the non-locally-compact case (the paper [23] considers only abelian groups, but the problem is interesting in general).
The following simple example shows that the answer for this is negative in the group $\mathbb{Z}^\omega$ (a variant of this is stated in [10, Proposition 1]). It also answers the analogous question in the case of category.

**Example 6.3.4.** For $A \subseteq \omega$ let

$$S(A) = \{ s \in \mathbb{Z}^\omega : s_n \geq 0 \text{ if } n \in A \text{ and } s_n < 0 \text{ if } n \notin A \}.$$  

Then the system $\{ S(A) : A \in \mathcal{P}(\omega) \}$ consists of continuum many pairwise disjoint Borel (in fact, closed) subsets of $\mathbb{Z}^\omega$ which are neither generalized Haar null nor Haar meager.

**Proof.** It is clear that $S(A)$ is closed for every $A \subseteq \omega$. If $A$ and $B$ are two different subsets of $\omega$, then some $n \in \omega$ satisfies for example $n \in A \setminus B$ and thus $\forall s \in S(A) : s_n \geq 0$, but $\forall s \in S(B) : s_n < 0$.

Finally, for every $A \subseteq \omega$ the set $S(A)$ contains a translate of every compact subset $C \subseteq \mathbb{Z}^\omega$, because if we define $t(C) \in \mathbb{Z}^\omega$ by

$$t_n^{(C)} = \begin{cases} \min\{c_n : c \in C\} & \text{if } n \in A, \\ -1 - \max\{c_n : c \in C\} & \text{if } n \notin A, \end{cases}$$

then clearly $C + t(C) \subseteq A$. Applying Lemma 7.6.2 concludes our proof. □

In [87] Solecki showed that the situation is the same in all non-locally-compact groups that admit a two-sided invariant metric. This is a corollary of Theorem 2.2.15, which we already stated without proof. As we will use Lemma 7.6.2 again, this also answers the question in the case of category.

**Example 6.3.5** (Solecki). Suppose that $G$ is non-locally-compact and admits a two-sided invariant metric. Then none of $\mathcal{H}\mathcal{N}$, $\mathcal{G}\mathcal{H}N$ and $\mathcal{H}\mathcal{M}$ has the countable chain condition in $\mathcal{B}(G)$.

**Proof.** By Theorem 2.2.15 there exists a closed set $F \subseteq G$ and a continuous function $\varphi : F \to 2^\omega$ such that for any $x \in 2^\omega$ and any compact set $C \subseteq G$ there is a $g \in G$ with $gC \subseteq \varphi^{-1}(\{x\})$. Then the system $\{\varphi^{-1}(\{x\}) : x \in 2^\omega\}$ consists of continuum many pairwise disjoint closed sets and they all satisfy the requirements of Lemma 7.6.2, hence they are neither generalized Haar null nor Haar meager. □
6.4 Decomposition into a Haar null and a meager set

In a locally compact group $G$ the regularity of the Haar measures implies that the group can be written as $G = N \cup M$ where $N$ is of Haar measure zero and $M$ is meager.

In [28] Darji asks if this holds for non-locally-compact groups:

**Question 6.4.1** (Darji). Can every uncountable Polish group be written as the union of two sets, one meager and the other Haar null?

**Remark 6.4.2.** More precisely, [28] asks whether every uncountable Polish group can be written as the union of a meager and a generalized Haar null set (in the terminology of this thesis). However, it is easy to see that this distinction is inconsequential, as every meager set is contained in an $F_\sigma$ (in particular, Borel) meager set.

Although this question is open, there are known results which answer it affirmatively in various groups or classes of groups.

For example, the following surprisingly short calculation from [26] shows that the existence of a nonempty openly Haar null set in the group implies the existence of a decomposition. (Openly Haar null sets are introduced in section 5.2.)

**Theorem 6.4.3** (Cohen-Kallman). If there is a nonempty openly Haar null set in $G$, then every countable subset $C \subseteq G$ is contained in a comeager Haar null set. In particular, $G$ may be written as the union $G = A \cup B$ where $A$ is a Haar null set and $B$ is meager in $G$.

**Proof.** If there is a nonempty openly Haar null set in $G$, then applying Lemma 2.1.11 yields that every countable set in $G$ is openly Haar null. In particular a dense countable set $C' \supseteq C$ is openly Haar null. Then Proposition 5.2.2 yields that $C' \subseteq A$ for a $G_\delta$ Haar null set $A$. The dense $G_\delta$ set $A$ is a countable intersection of dense open sets, and the complement of a dense open set is nowhere dense, thus we may choose $B = G \setminus A$. \qed

Although there are Polish groups where only the empty set is openly Haar null, the combination of this fact and [26, Proposition 2] (which we already stated without proof as Proposition 5.2.8) answers the question of Darji affirmatively in several well-known groups:

**Theorem 6.4.4** (Cohen-Kallman). The following uncountable Polish groups can be written as the union of a meager and a Haar null set:

1. Uncountable groups that admit a two-sided invariant metric,
2. $S_\infty$, the group of permutations of $\mathbb{N}$,
Chapter 6. Analogs of the results from the locally compact case

(3) $\text{Aut}(\mathbb{Q}, \leq)$, the group of order-preserving self-bijections of the rationals,
(4) $\mathcal{U}(\ell^2)$, the unitary group on the separable infinite-dimensional complex Hilbert space.

Before these, (2) had been already proved in [11] (using a different approach which is briefly described in section 7.3).

The result [30, Corollary 5.2] also gives an affirmative answer in two well-known automorphism groups in addition to the already mentioned $\text{Aut}(\mathbb{Q}, \leq)$:

**Theorem 6.4.5** (Darji-Elekes-Kalina-Kiss-Vidnyánszky). The following uncountable Polish groups can be written as the union of a meager and a Haar null set:

1. $\text{Aut}(\mathcal{R})$, where $\mathcal{R}$ is the countably infinite random graph (also called the Radó graph and the Erdős-Rényi graph),
2. $\text{Aut}(\mathcal{B}_\infty)$, where $\mathcal{B}_\infty$ is the countable atomless Boolean algebra.

These concrete results are special cases of [30, Corollary 5.1], which states the following:

**Theorem 6.4.6** (Darji-Elekes-Kalina-Kiss-Vidnyánszky). Let $G$ be a closed subgroup of $S_\infty$. Assume that $G$ satisfies the FACP (finite algebraic closure property), that is, for every finite $S \subset \mathbb{N}$ the set \{b : |G_S(b)| < \infty\} is finite, where $G_S$ is the pointwise stabilizer of $S$ under the action of $G$. Moreover, suppose that the set $F = \{g \in G : \text{Fix}(g) \text{ is infinite}\}$ is dense in $G$. Then $G$ can be written as the union of a meager and a Haar null set.

These results from [30] are corollaries of the examination of the “size” of the conjugacy classes in these groups; we omit their relatively long proofs.

It is well known that $\text{Aut}(\mathcal{B}_\infty)$ (the group considered in part (2) of Theorem 6.4.5) is isomorphic to the homeomorphism group of the Cantor set. The paper [33] answers Question 6.4.1 in two other important homeomorphism groups:

**Theorem 6.4.7** (Darji-Elekes-Kalina-Kiss-Vidnyánszky). The following uncountable Polish groups can be written as the union of a meager and a Haar null set:

1. $\mathcal{H}([0,1])$, the group of order-preserving self-homeomorphisms of the unit interval,
2. $\mathcal{H}(S^1)$, the group of order-preserving self-homeomorphisms of the circle.

Both groups are endowed with the compact-open topology, which coincides with the topology of uniform convergence in these cases.

These results are also corollaries of the examination of the size of the conjugacy classes, for their proofs see [33, Corollary 6.1] and [33, Corollary 6.2]. Notice that the example of $\mathcal{H}([0,1])$ shows that decomposition into a meager and a Haar null set may be possible even if there are no nonempty openly Haar null sets in the group (Proposition 5.2.7).

The result [33, Proposition 6.3] allows us to pull back decompositions from factor groups:
Proposition 6.4.8 (Darji-Elekes-Kalina-Kiss-Vidnyánszky). Let $G$ and $H$ be Polish groups and suppose that there exists a continuous, surjective homomorphism $\varphi : G \to H$. If $H$ can be written as the union of a meager and a Haar null set then $G$ can be decomposed in such a way as well.

Proof. If $\varphi$ satisfies these conditions, then it follows from [57, Theorem 2.3.3] that $\varphi$ is necessarily an open mapping.

Let $H = N \cup M$ where $N$ is Haar null and $M$ is meager. We will show that $G = \varphi^{-1}(N) \cup \varphi^{-1}(M)$ is a good decomposition of $G$.

By [10, Proposition 8], the inverse image of a Haar null set under a continuous, surjective homomorphism is also Haar null, hence $\varphi^{-1}(N)$ is Haar null in $G$.

As $M$ is meager, there are closed, nowhere dense sets $(S_n)_{n \in \omega}$ such that $M \subseteq \bigcup_{n \in \omega} S_n$. For each $n$, the set $\varphi^{-1}(S_n)$ is closed (because $\varphi$ is continuous) and nowhere dense (because $\varphi$ is open); these imply that $\varphi^{-1}(M)$ is meager.

Applying this, it is easy to prove the following:

Corollary 6.4.9. The following uncountable Polish groups can be written as the union of a meager and a Haar null set:

1. $(\text{Diff}_+^k[0,1], \circ)$, the group of $k$-times differentiable order-preserving self-homeomorphisms of $[0,1]$ (where $k \geq 1$ is an integer),
2. $\mathcal{H}(\mathbb{D}^2)$, the group of orientation-preserving self-homeomorphisms of the closed disc $\mathbb{D}^2$.

Proof. (1) The map $\varphi : (\text{Diff}_+^k[0,1], \circ) \to (\mathbb{R}^+, \cdot)$, $f \mapsto f'(0)$ is a continuous, surjective homomorphism. As $\mathbb{R}^+$ is locally compact, the existence of a decomposition is well-known.

(2) The map $\psi : \mathcal{H}(\mathbb{D}^2) \to \mathcal{H}(\mathbb{S}^1)$, $h \mapsto h|_{\mathbb{S}^1}$ is a continuous homomorphism. It is easy to see that if we restrict $\psi$ to the set

$$\{h : h \text{ preserves the center and is linear on the radiuses} \} \subset \mathcal{H}(\mathbb{D}^2),$$

then we get a bijective map. This shows that $\psi$ is surjective. The other condition of Proposition 6.4.8 follows from [Theorem 6.4.7] (2), which states that $\mathcal{H}(\mathbb{S}^1)$ can be written as the union of a meager and a Haar null set.

We note that part (1) was originally proved in [26] using part (e) of [26, Proposition 2] (which we already stated without proof as part (5) of Proposition 5.2.8); while part (2) is [33, Corollary 6.5].
The questions [33, Question 6.6] and [33, Question 6.7] highlight the following special cases of [Question 6.4.1]:

**Question 6.4.10** (Darji-Elekes-Kalina-Kiss-Vidnyánszky). Is it possible to write the following uncountable Polish groups as the union of a meager and a Haar null set:

1. \( H(D^n) \), the group of orientation-preserving self-homeomorphisms of the closed \( n \)-ball \( D^n \) (where \( n \geq 3 \) is an integer),
2. \( H(S^n) \), the group of orientation-preserving self-homeomorphisms of the \( n \)-sphere \( S^n \) (where \( n \geq 2 \) is an integer),
3. the homeomorphism group of the Hilbert cube \([0,1]^\omega\).

After these particular cases, we would like to mention an equivalent form of Question 6.4.1, which connects it to a different family of problems. The following, seemingly unrelated question appears as [47, Question 5.5] and later in [17]:

**Question 6.4.11** (Elekes-Vidnyánszky, Banakh). Is each countable subset of an uncountable Polish group \( G \) contained in a \( G_\delta \) Haar null subset of \( G \)?

However, it turns out that this is equivalent to Question 6.4.11:

**Theorem 6.4.12.** In an uncountable Polish group \( G \) the following are equivalent:

1. Each countable subset of \( G \) is contained in a \( G_\delta \) Haar null subset of \( G \).
2. There exists a countable subset of \( G \) that is dense and contained in a \( G_\delta \) Haar null subset of \( G \).
3. \( G \) can be written as the union of a meager and a Haar null set.

**Proof.** (1) \( \Rightarrow \) (2): As \( G \) is Polish, there exists a countable dense subset \( D \) in \( G \).

(2) \( \Rightarrow \) (3) Let \( D \subset G \) be countable dense set that is contained in a \( G_\delta \) Haar null set \( N \subset G \). \( N \) is dense, because it contains the dense set \( D \) as a subset. It is well-known that the complement of a dense \( G_\delta \) set is meager; therefore \( G = (G \setminus N) \cup N \) is a suitable decomposition.

(3) \( \Rightarrow \) (1): Assume that \( G = M_0 \cup N_0 \) where \( M_0 \) is meager and \( N_0 \) is Haar null. As \( M_0 \) is meager, there exists an \( F_\sigma \) meager set \( M \) such that \( M_0 \subseteq M \subset G \) (this follows from the fact that the closure of a nowhere dense set is still nowhere dense). Then the \( G_\delta \) set \( N = G \setminus M \) is Haar null, because it is a subset of \( N_0 \).

Let \( C \subset G \) be an arbitrary countable set. We will show that there exists a \( g \in G \) such that \( C \subset gN = \{ g \cdot x : x \in N \} \) (this is sufficient, because the translates of a \( G_\delta \) Haar null set are also \( G_\delta \) Haar null sets). The set of “bad” translations is

\[
\{ g \in G : C \not\subseteq gN \} = \bigcup_{c \in C} \{ g \in G : c \not\in gN \} = \bigcup_{c \in C} \{ g \in G : c \cdot g^{-1} \in M \} = \bigcup_{c \in C} Mg
\]
which is a meager set (because it is a countable union of meager sets); this clearly shows that there exists a \( g \in G \) such that \( C \subset gN \).

As every Haar meager set is meager (but the converse is not true in general – see section 2.2), it is natural to ask the following stronger version of Question 6.4.1, which appeared first as [63, Question 4]:

**Question 6.4.13** (Jabłońska). *Can every uncountable abelian Polish group be written as the union of two sets, one Haar meager and the other Haar null?*

We state the following partial answer, its proof can be found as [38, Theorem 25]:

**Theorem 6.4.14** (Doležal-Rmoutil-Vejnar-Vlasák). *Let \( G \) be an abelian Polish group such that its identity element has a local basis consisting of open subgroups. Then \( G \) is the union of a Haar null set and a Haar meager set.*

Recall that the condition of this result has several equivalent formulations:

**Theorem 6.4.15.** *Let \( G \) be a Polish group. Then the following are equivalent:*

1. \( G \) is isomorphic to a closed subgroup of \( S_\infty \);
2. \( G \) admits a countable local basis at the identity consisting of open subgroups;
3. \( G \) admits a countable basis closed under left multiplication;
4. \( G \) admits a compatible left-invariant ultrametric.

The proof of this classical result can be found as [14, Theorem 1.5.1].

The following result shows that this kind of decomposition can be lifted from a certain kind of subgroup onto the whole group:

**Theorem 6.4.16** (Doležal-Rmoutil-Vejnar-Vlasák). *Let \( G \) be a Polish group and \( H \leq G \) be an uncountable closed subgroup of the center of \( G \). Assume that \( H \) can be written as the union of a Haar null set in \( H \) and a Haar meager set in \( H \). Then \( G \) is also the union of a Haar null set in \( G \) and a Haar meager set in \( G \).

In particular, this holds for \( G = \mathbb{R}^\omega \) or \( G = X \) where \( X \) is a Banach space.*

The proof of this result can be found as [38, Theorem 22].
Chapter 7

Common techniques

In this chapter we introduce six techniques, which are frequently useful in practice. The first five of these can be used to show that a set is small, and the last one can be used to show that a set is not small. Note that some of the results from the earlier chapters (for example, the basic properties in section 2.1 or the equivalent definitions in section 2.4) are also very useful in practice.

7.1 Probes

Probes are a very basic technique for constructing witness measures. The core of this idea is fairly straightforward and the only surprising thing about probes is the fact that despite their simplicity they are often useful.

Probes were introduced with the following definition in [13] (a paper which examines the Haar null sets in completely metrizable linear spaces).

Definition 7.1.1. Suppose that $V$ is an (infinite-dimensional) completely metrizable linear space. A finite dimensional subspace $P \subseteq V$ is called a probe for a set $A \subseteq V$ if the Lebesgue measure on $P$ witnesses that $A$ is Haar null.

Remark 7.1.2. Strictly speaking, these Lebesgue measures are not probability measures, therefore they cannot be witness measures in the sense of Definition 1.1. However, Theorem 2.4.1 implies that if $B$ is Borel and the Lebesgue measure $\lambda_P$ on a subspace $P$ satisfies that $\lambda_P(B + v) = 0$ for every $v \in V$ then, then $B$ is Haar null.

There is nothing “magical” about this definition, but it is easy to handle these simple witness measures in the calculations and if there is a probe for a set $A \supseteq V$, then by definition $A$ is Haar null. In arbitrary Polish groups it is easy to generalize this idea and consider a witness measure which is the “natural” measure supported on a small
and well-understood subgroup or subset. If the considered set and the candidate for the probe are not too contrived, then it is often easy to see that it is indeed a probe.

For Haar meager sets the analogue of this is basically proving that the set is strongly Haar meager (see Chapter 4) and this is witnessed by a “naturally chosen” set.

The proof of the following example illustrates the usage of probes.

**Example 7.1.3.** In the Polish group \((C[0,1], +)\) of continuous real-valued functions on \([0,1]\), the set \(M = \{ f \in C[0,1] : f \) is monotone on some interval\} is Haar null.

**Proof.** For a proper interval \(I \subseteq [0,1]\) let

\[ M(I) = \{ f \in C[0,1] : f \) is monotone on \(I\}. \]

As the Haar null sets form a \(\sigma\)-ideal and

\[ M = \bigcup \{ M([q,r]) : 0 \leq q < r \leq 1 \text{ and } q, r \in \mathbb{Q}\}, \]

it is enough to show that \(M(I)\) is Haar null for every proper interval \(I \subseteq [0,1]\). It is straightforward to check that \(M(I)\) is Borel (in fact, closed).

Fix a function \(\varphi \in C[0,1]\) such that its restriction to \(I\) is not of bounded variation. We show that the one-dimensional subspace \(\mathbb{R}\varphi = \{ c \cdot \varphi : c \in \mathbb{R}\}\) is a probe for \(M(I)\), that is, the measure \(\mu\) on \(C[0,1]\) that is defined by

\[ \mu(X) = \lambda(\{ c \in \mathbb{R} : c \cdot \varphi \in X\}) \]

(where \(\lambda\) is the Lebesgue measure on \(\mathbb{R}\)) is a witness measure for \(M(I)\).

We have to prove that \(\mu(M(I) + f) = 0\) for every \(f \in C[0,1]\). By definition

\[ \mu(M(I) + f) = \lambda(\{ c \in \mathbb{R} : c \cdot \varphi \in M(I) + f\}) \]

and here the set \(S_f = \{ c \in \mathbb{R} : c \cdot \varphi \in M(I) + f\}\) has at most one element, because if \(c_1, c_2 \in S_f\), then \(c_1 \cdot \varphi = m_1 + f\) and \(c_2 \cdot \varphi = m_2 + f\) for some functions \(m_1, m_2 \in M(I)\) that are monotone on \(I\) and hence \((c_1 - c_2) \cdot \varphi = m_1 - m_2\) is of bounded variation when restricted to \(I\), but this is only possible if \(c_1 = c_2\). Thus \(\lambda(S_f) = \mu(M(I) + f) = 0\) and this shows that \(M(I)\) is Haar null.

For additional examples which demonstrate the usage of probes, see the paper [64] (which introduces probes and gives several simple examples) or the paper [63] which proves Example 7.3.1 using a two-dimensional probe. The latter paper is of particular interest because it also proves (at the beginning of section 2) that Example 7.3.1 cannot be proved using a one-dimensional probe.
7.2 Application of the Steinhaus theorem

Sometimes the application of one of the results in Section 6.2 can yield very short proofs for the Haar nullness and Haar meagerness of certain sets. Unfortunately, this technique is restricted in the sense that “good” analogs of the Steinhaus theorem are known only in special classes of groups.

We illustrate this technique by proving Example 7.1.3 again.

Example 7.2.1. In the Polish group \( (C[0,1], +) \) of continuous real-valued functions on \([0, 1]\), the set \( M = \{ f \in C[0,1] : f \text{ is monotone on some interval} \} \) is Haar null and Haar meager.

Proof. As we noted in the proof using probes, if \( I \subseteq [0, 1] \) is a proper interval, then the set

\[
M(I) = \{ f \in C[0,1] : f \text{ is monotone on } I \}
\]

is Borel and it is enough to see that this set is Haar null and Haar meager for every proper interval \( I \subseteq [0, 1] \) (we use the fact that the Haar meager sets also form a \( \sigma \)-ideal).

Assume for contradiction that there exists a proper interval \( I \subseteq [0, 1] \) such that \( M(I) \) is either not Haar null or not Haar meager. If \( M(I) \) is not Haar null, then Corollary 6.2.3 implies that \( M(I) - M(I) \) is a neighborhood of the constant 0 function. Similarly, if \( M(I) \) is not Haar meager, then Theorem 6.2.11 implies that \( M(I) - M(I) \) is a neighborhood of the constant 0 function.

It is well-known and easy to prove that the difference of two monotone functions is a function of bounded variation; this implies that every \( f \in M(I) - M(I) \) satisfies that the restriction \( f|_I \) is of bounded variation. However, it is easy to construct a continuous function \( f \) such that \( \|f\|_\infty \) is small and \( f|_I \) is not of bounded variation; this clearly contradicts the fact that \( M(I) - M(I) \) is a neighborhood of the constant 0 function. \( \Box \)

The proof of Proposition 7.4.3 is another example of this technique.

7.3 The Wiener measure as witness

The Wiener measure (which we will denote by \( \mu \) in this section) is a well-studied Borel probability measure on the Polish group \( (C[0,1], +) \). There are lots of results which show that something is true for “most” continuous functions by proving that \( \mu(E) = 0 \) where \( E \) is the set of “exceptional” functions.

Although \( \mu(E) = 0 \) does not necessarily imply that \( E \) is Haar null, it is often possible to use the same methods to show that there is a Borel set \( B \) such that \( E \subseteq B \subseteq C[0,1] \) and \( \mu(B + g) = 0 \) for every \( g \in C[0,1] \).
We illustrate this technique on the main result of [63]. This original proof by Hunt relied on constructing a two-dimensional probe. Later, Holický and Zajíček gave an alternative proof in [62] using the Wiener measure as a witness measure; here we reproduce this second proof.

**Example 7.3.1** (Hunt). In the Polish group \((C[0,1],+)\) of continuous real-valued functions on \([0,1]\), the set

\[
E = \{ f \in C[0,1] : f \text{ has a derivative } f'(x) \in \mathbb{R} \text{ at some point } 0 \leq x \leq 1 \}
\]

is Haar null.

**Proof (Holický-Zajíček).** We will prove that the set

\[
E_R = \{ f \in C[0,1] : f \text{ has a finite right derivative at some point } 0 \leq x < 1 \}
\]

is Haar null; then by symmetry the set

\[
E_L = \{ f \in C[0,1] : f \text{ has a finite left derivative at some point } 0 < x \leq 1 \}
\]

is also Haar null and this is enough because clearly \(E \subseteq E_R \cup E_L\).

For a function \(f \in C[0,1]\), we say that \(f\) is Lipschitz from the right at \(x\) if

\[
\limsup_{y \to x^+} \left| \frac{f(y) - f(x)}{y - x} \right| < \infty.
\]

It is clearly sufficient to prove the following claim:

**Claim 7.3.2.** The set

\[
B = \{ f \in C[0,1] : f \text{ is Lipschitz from the right at some point } 0 \leq x < 1 \}
\]

has the following properties:

1. \(E_R \subseteq B\),
2. \(B\) is Borel (in fact, \(F_\sigma\)),
3. \(\mu(B + g) = 0\) for every \(g \in C[0,1]\) (where \(\mu\) is the Wiener measure).

Here (1) is true because if \(f\) has a finite right derivative \(d\) at some point \(0 \leq x < 1\) then

\[
\limsup_{y \to x^+} \left| \frac{f(x) - f(y)}{x - y} \right| = |d| < \infty.
\]
(2) follows from the fact that \( B = \bigcup_n E_n \) where
\[
E_n = \{ f \in C[0, 1] : \text{there is a } 0 \leq x < 1 - \frac{1}{n} \text{ such that for all } 0 < h < 1 - x, |f(x + h) - f(x)| \leq nh \}.
\]
Elementary calculations (which can be found in [52, Chapter 11]) show that \( E_n \) is closed.

(3) is the nontrivial part of this claim. To prove it, first notice that if \( f \) is Lipschitz from the right at \( x \), then there exists \( k \in \mathbb{N} \) and \( \delta > 0 \) such that if \( x < u < v < x + \delta \), then
\[
|f(v) - f(u)| < k \cdot (v - x).
\]
If \( n \in \mathbb{N} \) is large enough \( (n > \frac{1}{5\delta}) \), then there exists an \( i \in \{0, 1, \ldots, n - 1\} \) such that \( \frac{i - 1}{n} < x \leq \frac{i}{n} < \frac{i + 1}{n} < \frac{i + 2}{n} < x + \delta \), and hence for all \( j \in \{0, 1, 2\} \),
\[
\left| f \left( \frac{i + j + 1}{n} \right) - f \left( \frac{i + j}{n} \right) \right| < \frac{4k}{n}.
\]
If we formalize this observation, then we get
\[
B \subseteq \bigcup_{k \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcap_{n \geq m} \bigcap_{i=0}^{n-1} \bigcap_{j=0}^{2} \left\{ f \in C[0, 1] : \left| f \left( \frac{i + j + 1}{n} \right) - f \left( \frac{i + j}{n} \right) \right| < \frac{4k}{n} \right\}.
\]
Now we fix an arbitrary \( g \in C[0, 1] \) and conclude the proof of the claim by showing that \( \mu(B + g) = 0 \). As \( B + g = \{ h \in C[0, 1] : h - g \in B \} \),
\[
B + g \subseteq \bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \bigcap_{i=0}^{n-1} M_{n,k,i},
\]
where
\[
M_{n,k,i} = \bigcap_{j=0}^{2} \left\{ h \in C[0, 1] : \left| h \left( \frac{i + j + 1}{n} \right) - h \left( \frac{i + j}{n} \right) + \Delta_{i,j,n} \right| < \frac{4k}{n} \right\}
\]
and \( \Delta_{i,j,n} \) is the difference of two values of \( g \) (we will not use its value).

If \( h \) is a random function with the Wiener measure as its distribution, then it is well known that any difference of the form \( h(t_2) - h(t_1) \) (where \( t_2 > t_1 \)) has normal distributions with variance \( t_2 - t_1 \) and moreover, a collection of these differences is independent if they correspond to non-overlapping intervals.

If \( X \) is a random variable with normal distribution and variance \( \sigma^2 \) in an arbitrary
probability space \((\Omega, \mathcal{F}, P)\) and \(\Delta \in \mathbb{R}\) and \(r > 0\) are arbitrary values, then clearly

\[
P(|X + \Delta| < r) \leq \frac{1}{\sqrt{2 \pi \sigma^2}} \cdot 2r
\]

because \(\frac{1}{\sqrt{2 \pi \sigma^2}}\) is the maximal value of the density function of \(X\).

Using this for the differences of the values of \(h\) yields

\[
\mu(M_{n,k,i}) \leq \left(\frac{1}{\sqrt{2 \pi \frac{1}{n}}} \cdot 2 \cdot \frac{4k}{n}\right)^3 = n^{-\frac{3}{2}} \cdot \frac{128 \sqrt{2}}{\pi^\frac{3}{2}} \cdot k^\frac{3}{2},
\]

\[
\mu\left(\bigcup_{i=0}^{n-1} M_{n,k,i}\right) \leq n^{-\frac{3}{2}} \cdot \frac{128 \sqrt{2}}{\pi^\frac{3}{2}} \cdot k^3 \xrightarrow{n \to \infty} 0
\]

and therefore

\[
\mu(B + g) \leq \mu\left(\bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \bigcup_{i=0}^{n-1} M_{n,k,i}\right) = 0
\]

as we claimed. \(\Box\)

Note that \([12]\) shows that the set

\[
\{f \in C[0,1] : f \text{ has a derivative } f'(x) \in \mathbb{R} \cup \{-\infty, \infty\} \text{ at some point } 0 \leq x \leq 1\}
\]

is not Haar null, but its complement is also not Haar null.

## 7.4 Compact sets are small

This technique is based on the idea that in non-locally-compact groups the compact sets are “small” in the sense that they have empty interior. This naturally inspires the following question:

**Question 7.4.1.** Is it true that the compact subsets are Haar null (or Haar meager) in every non-locally-compact Polish group?

While this question is still open, there are several partial results which give positive answers when some additional assumptions are satisfied. It is frequently possible to use these partial results as lemmas. As both the system of Haar null sets and the system of Haar meager sets are \(\sigma\)-ideals, they also imply that \(K_\sigma\) sets (countable unions of compact sets) are Haar null and/or Haar meager in these groups.

The following theorem is \([10, \text{Proposition 12}]\), one of the earliest results in this topic.
Theorem 7.4.2. Let $G$ be a non-locally-compact Polish group admitting a two-sided invariant metric. Then every compact subset of $G$ is Haar null.

Proof. We will use Theorem 2.4.3 to prove this result; this proof is not essentially different from the proof in [10], but separates the ideas specific to compact sets (this proof) and the construction of a limit measure (the proof of Theorem 2.4.4).

Fix a two-sided invariant metric $d$ on $G$ and let $C \subseteq G$ be an arbitrary compact subset. We need to prove that for every $\delta > 0$ and neighborhood $U$ of $1_G$ there exists a Borel probability measure $\mu$ on $G$ such that the support of $\mu$ is contained in $U$ and $\mu(gCh) < \delta$ for every $g, h \in G$.

Fix $\delta > 0$ and a neighborhood $U$ of $1_G$. We may assume that $U$ is open. As $G$ is non-locally-compact, the open set $U$ is not totally bounded, hence there exists an $\varepsilon > 0$ such that $U$ cannot be covered by finitely many open balls of radius $2\varepsilon$.

As $C$ is compact, hence totally bounded, there exists a $N \in \omega$ such that $C$ can be covered by $N$ open balls of radius $\varepsilon$. This means that if $X \subseteq C$ and every $x, x' \in X$ satisfies $x \neq x' \Rightarrow d(x, x') \geq 2\varepsilon$, then $|X| \leq N$ (because each of the $N$ open balls of radius $\varepsilon$ covering $C$ may contain at most one element of $X$). Using the invariance of $d$ this yields that for every $g, h \in G$ if $X \subseteq gCh$ and every $x, x' \in X$ satisfies $x \neq x' \Rightarrow d(x, x') \geq 2\varepsilon$, then $|X| \leq N$.

As $U$ cannot be covered by finitely many open balls of radius $2\varepsilon$, it is possible to choose a sequence $(u_n)_{n \in \omega}$ such that $u_n \in U$ and $u_n \notin \bigcup_{i=0}^{N-1} B(u_i, 2\varepsilon)$ for every $n \in \omega$. Choose an integer $M$ that is larger than $\frac{N}{\delta}$ and let $Y = \{u_n : 0 \leq n < M\}$. Let $\mu$ be the measure on $Y$ which assigns measure $\frac{1}{M}$ to every point in $Y$. If $g, h \in G$ are arbitrary, then $\mu(gCh) = \frac{|gCh \cap Y|}{M}$ and here every $y, y' \in gCh \cap Y$ satisfies $y \neq y' \Rightarrow d(y, y') \geq 2\varepsilon$, and hence $|gCh \cap Y| \leq N$, and thus $\mu(gCh) \leq \frac{N}{M} < \delta$. \hfill $\square$

The following result works in all non-locally-compact Polish groups, but only proves that the compact sets are right Haar null (this is a weaker notion than Haar nullness, see section 5.1 for the definition and properties).

Proposition 7.4.3. Let $G$ be a non-locally-compact Polish group. Then every compact subset of $G$ is right Haar null.

Proof. Suppose that $C \subseteq G$ is compact but not right Haar null. Applying Theorem 6.2.3 yields that there exist a $n \in \omega$ and $h_0, h_1, \ldots, h_{n-1} \in G$ such that

$$h_0CC^{-1}h_0^{-1} \cup h_1CC^{-1}h_1^{-1} \cup \ldots \cup h_{n-1}CC^{-1}h_{n-1}$$

is a neighborhood of $1_G$. But $CC^{-1}$ is compact (as it is the image of $C \times C$ under the continuous map $(x, y) \mapsto xy^{-1}$), thus its conjugates are also compact, and the union
of finitely many compact sets is also compact, and this is a contradiction, because a neighborhood cannot be compact in $G$. \hfill \Box

The paper \cite{39} investigates the question in the case of Haar meager sets, we state the main results without proofs. This article introduces the \emph{finite translation property} with the following definition:

**Definition 7.4.4.** A set $A \subseteq G$ is said to have the \emph{finite translation property} if for every open set $\emptyset \neq U \subseteq G$ there exists a finite set $M \subseteq U$ such that for every $g, h \in G$ we have $gMh \not\subseteq A$.

The first part of the proof is the following result which allows using this property to prove that a set is strongly Haar meager (this is a stronger notion than Haar meagerness, see Chapter 4 for the definition and properties). The role of this result is roughly similar to the role of Theorem 2.4.4 in the case of measure; its proof involves a relatively complex recursive construction.

**Theorem 7.4.5.** If an $F_\sigma$ set $A \subseteq G$ has the finite translation property, then $A$ is strongly Haar meager.

The second part is showing that the compact sets have the finite translation property when there is a two-sided invariant metric; the proof is relatively simple and very similar to the one used in Theorem 7.4.2.

**Theorem 7.4.6.** Let $G$ be a non-locally-compact Polish group admitting a two-sided invariant metric. Then every compact subset of $G$ has the finite translation property.

These results yield the analogue of Theorem 7.4.2. Note that in the case when $G$ is abelian, it is also possible to prove this as a corollary of Theorem 6.2.11 using the method of the proof of Proposition 7.4.3.

**Corollary 7.4.7.** Let $G$ be a non-locally-compact Polish group admitting a two-sided invariant metric. Then every compact subset of $G$ is (strongly) Haar meager.

In addition to these, \cite{39} also shows that compact sets have the finite translation property in $S_\infty$ (the group of all permutations of a countably infinite set).

We illustrate the usage of this technique with a simple example.

**Example 7.4.8.** In the non-locally-compact Polish group $(\mathbb{Z}^\omega, +)$ there are subsets $A, B \subseteq \mathbb{Z}^\omega$ such that they are neither Haar null nor Haar meager, but for every $x \in \mathbb{Z}^\omega$ the intersection $(A + x) \cap B$ is both Haar null and Haar meager.
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Proof. It is well-known that in $\mathbb{Z}^\omega$ a closed set $C$ is compact if and only if
\[ C \subseteq \prod_{n \in \omega} \{u_n, u_n + 1, u_n + 2, \ldots, v_n - 1, v_n\} \text{ for some } u, v \in \mathbb{Z}^\omega, \]
as sets of this kind are closed subsets in a product of compact sets and in the other direction if $C \subseteq \mathbb{Z}^\omega$ is compact, then the projections map it into compact subsets of $\mathbb{Z}$.

Let $A = \{ a \in \mathbb{Z}^\omega : a_n \leq 0 \text{ for every } n \in \omega \}$ and $B = \{ b \in \mathbb{Z}^\omega : b_n \geq 0 \text{ for every } n \in \omega \}$. It is easy to check that these sets satisfy the condition of Lemma 7.6.2 (the result used in the last technique) and this implies that $A$ and $B$ are neither Haar null nor Haar meager.

On the other hand, the set $(A + x) \cap B = \{ z \in \mathbb{Z}^\omega : 0 \leq z_n \leq x_n \}$ is compact, hence Theorem 7.4.2 and Corollary 7.4.7 shows that it is Haar null and Haar meager.

Note that this phenomenon is impossible in the locally compact case, where non-small sets have density points and if we translate a density point of one set onto a density point of the other, then the intersection will be non-small. On the other hand, [75, Theorem 4] proves that if $(G, +)$ is abelian and non-locally-compact, then there are sets $A, B \subset G$ which are not Haar null, but satisfy that $(A + x) \cap B$ is Haar null for every $x \in G$.

7.5 Random construction

This is a technique that is useful when one wants to prove that a not very small set is still small enough to be Haar null. (For example this technique may work for sets that are not Haar meager, because it does not prove Haar meagerness.) The main idea of this technique is that a witness measure for a Haar null set $A \subseteq G$ is a Borel probability measure and one can use the language of probability theory (e.g. random variables, conditional probabilities, stochastic processes) to construct it and prove that it is indeed a witness measure.

For example, the paper [11] applies this to prove a result in $S_\infty$, the group of all permutations of the natural numbers (endowed with the topology of pointwise convergence). We illustrate this technique by reproducing the core ideas of this proof. (The proof also contains relatively long calculations which we omit.)

Theorem 7.5.1 (Dougherty-Mycielski). In the Polish group $S_\infty$ let $X$ be the set of permutations that have infinitely many infinite cycles and finitely many finite cycles. Then the complement of $X$ is Haar null.

Proof (sketch). We will find a Borel probability measure $\mu$ on $S_\infty$ such that $\mu(gXh) = 1$ for every $g, h \in S_\infty$. Using that $X$ is conjugacy invariant $\mu(gXh) = \mu(gh(h^{-1}Xh)) =$
\(\mu(ghX)\) and here \(\{gh : g, h \in S_\infty\} = S_\infty = \{g^{-1} : g \in S_\infty\}\), thus it is enough to show that \(\mu(g^{-1}X) = 1\) for every \(g \in S_\infty\).

We define the probability measure \(\mu\) by describing a procedure which chooses a random permutation \(p\) with distribution \(\mu\). (This way we can describe a relatively complicated measure in a way that keeps the calculations manageable.)

Fix a sequence \(k_0 < k_1 < k_2 < \ldots\) of natural numbers which are large enough (the actual growth rate is used by the omitted parts of the proof). The procedure will choose values for \(p(0), p^{-1}(0), p(1), p^{-1}(1), p(2), p^{-1}(2), \ldots\) in this order, skipping those which are already defined (e.g. if we choose \(p(0) = 1\) in the first step, then the step for \(p^{-1}(1)\) is omitted, as we already know that \(p^{-1}(1) = 0\)). When we have to choose a value for \(p(n)\), we choose randomly a natural number \(n' < k_n\) which is still available as an image (that is, \(n' \geq n\) and \(n'\) is not among the already determined values \(p(0), p(1), \ldots, p(n-1)\)); we assign equal probabilities to each of these choices. Similarly, when we have to choose a value for \(p^{-1}(n)\), we choose randomly a natural number \(n' \leq k_n\) which is still available as a preimage (that is, \(n' > n\) and \(n'\) is not among the already determined values \(p^{-1}(0), p^{-1}(1), \ldots, p^{-1}(n-1)\)); we assign equal probabilities to each of these choices again. When we are finished with these steps, the resulting object \(p\) is clearly a well-defined permutation, as every \(n \in \omega\) has exactly one image and exactly one preimage assigned to it. We can assume that \(k_n\) is large enough to satisfy \(k_n > 2n + 1\) and this guarantees that we never “run out” of choices.

We say that \(p_0\) is a possible partial result, if it can arise after finitely many steps of this process. We omit the relatively long combinatorial arguments which show that the following claim is true:

**Claim 7.5.2.** Assume that \(p_0\) is a possible partial result, \(g \in S_\infty\) is an arbitrary element and \(M\) is a natural number. Then there is a natural number \(N\) such that the conditional probability with respect to \(\mu\), under the condition of extending \(p_0\), of the event that the permutation \(p\) chosen by our process will be such that \(gp\) has no finite cycles including a number greater than \(N\) and no two of the numbers \(N + 1, \ldots, N + M\) are in the same cycle of \(gp\) is at least \(\frac{1}{2}\).

Using this claim, it is possible to show the following claim by induction on \(i\) (we also omit this part of the proof):

**Claim 7.5.3.** Assume that (as in the previous claim) \(p_0\) is a possible partial result, \(g \in S_\infty\) is an arbitrary element and \(M\) is a natural number. Then for every \(i \in \omega\) the conditional probability (with respect to \(\mu\), under the condition of extending \(p_0\)) of the event that the permutation \(p\) chosen by our process satisfies that \(gp\) has only finitely many finite cycles and at least \(M\) infinite cycles is at least \(1 - 2^{-i}\).
Applying this second claim for every $i \in \omega$ in the special case when $p_0$ is the empty partial permutation yields that the (unconditional) probability (with respect to $\mu$) of the event that the permutation $p$ chosen by our process satisfies that $gp$ has only finitely many finite cycles and at least $M$ infinite cycles is 1. Since this is true for every $M \in \omega$, the permutation $gp$ has infinitely many infinite cycles with $\mu$-probability 1. This shows that $\mu(g^{-1}X) = 1$ for the arbitrary permutation $g$, so we are done.

This set $X$ is the union of countably many conjugacy classes of permutations, one for each finite list of sizes for the finite cycles in the permutation; the paper [41] also shows that none of these conjugacy classes are Haar null.

### 7.6 Sets containing translates of all compact sets

Proving that a set is not Haar null from the definitions requires showing that all Borel probability measures fail to witness that it is Haar null, which is frequently harder than just showing one measure witnessing that the set is Haar null. The situation is similar for Haar meager sets, where even the choice of the domain of the witness function is not straightforward, although the equivalence $(1) \Leftrightarrow (2)$ in Theorem 2.4.6 can be used to eliminate this additional choice.

However, there are many cases where it is sufficient to apply the following notion and lemma:

**Definition 7.6.1.** We say that a set $A \subseteq G$ is compact catcher if every compact set can be translated into it (i.e. for every compact set $C \subseteq G$ there are $g, h \in G$ such that $gCh \subseteq A$).

Note that these sets are called “thick sets” in [6, Definition 7.1].

**Lemma 7.6.2.** If $A \subseteq G$ is compact catcher, then it is neither generalized Haar null nor Haar meager.

This lemma appears e.g. as [31, Lemma 2.1], but there are many papers where this simple idea is integrated into the proof of some result.

**Proof.** If $A$ were generalized Haar null, then by Theorem 2.4.1 there would be a universally measurable set $B \supseteq A$ and a Borel probability measure $\mu$ with compact support $C \subseteq G$ such that $\mu(g'Bh') = 0$ for every $g', h' \in G$, but there are $g, h \in G$ such that $gCh \subseteq A \subseteq B$ and thus $\mu(g^{-1}Bh^{-1}) \geq \mu(C) = 1$, a contradiction.

Similarly, if $A$ would be Haar meager, then there would be a Borel set $B \supseteq A$, a (nonempty) compact metric space $K$ and a continuous function $f : K \to G$ such that
$f^{-1}(g'Bh')$ is meager in $K$ for every $g', h' \in G$, but there are $g, h \in G$ such that $gf(K)h \subseteq A \subseteq B$ and thus $f^{-1}(g^{-1}Bh^{-1}) \supseteq f^{-1}(f(K)) = K$ is not meager in $K$, a contradiction. \qed
Chapter 8

A brief outlook

It is beyond the scope of this thesis to collect the countless number of results and applications of Haar null sets in various fields of mathematics. However, we now give a highly incomplete list of works using Haar null sets, and encourage the reader to use these as starting points for further reading.

First we mention the paper \[81\] which also surveys applications of Haar null sets in addition to presenting their core properties. However, this paper was published in 2005 and therefore it does not contain several recent ideas.

The recent paper \[6\] introduces and systematically investigates (in the abelian case) the so called Haar-I sets, a common generalization of the notions of Haar null and Haar meager sets.

One of the original motivations of Christensen was to consider versions of automatic continuity as discussed e.g. in Corollary 6.2.5, in this topic see the excellent survey paper \[85\], and also \[86\] and \[53\].

The Rademacher theorem states that a Lipschitz function between Euclidean spaces is differentiable almost everywhere. The second motivation of Christensen was to extend this result to Banach spaces, see the paper \[24\] by Christensen, and also \[7, 16, 18, 95\].

A closely related result is the Alexandrov theorem stating that convex functions on Euclidean spaces are twice differentiable almost everywhere. For extensions of this result to Banach spaces see e.g. \[73, 74\].

There are various other interesting directions of research in functional analysis involving Haar null sets, see e.g. \[42, 50, 71, 72\].

When Haar null sets were rediscovered (under the name “shy sets”) in \[64\], they were the first ones to apply Haar null sets in the theory of dynamical systems. Since then numerous other such applications have been found, see e.g. \[2, 13, 19, 68\].
Multifractal analysis is a large area within geometric measure theory. There are numerous interesting results concerning the multifractal analysis of co-Haar null functions and measures, see e.g. [11, 12, 80].

There are countless results in geometric measure theory calculating various fractal dimensions of images, graphs, and level sets of certain functions, e.g. the generic ones, co-Haar null ones, Brownian motion, etc. A recurring theme is that the generic set is as small as it can be, while the co-Haar null one is a large as it can be. See e.g. [1, 9, 10, 13, 31, 45, 57, 64, 84, 106, 110, 116, 121, 133, 138, 149, 154, 166, 171, 182, 187, 199, 203, 215, 220, 232, 236, 248, 355, 360–377, 389, 405, 422, 26, 30, 35, 41, 91].

Let us also mention a few results that are measure theoretic duals of classical results of real analysis about generic continuous functions: [62, 63, 94] concern nowhere differentiable functions, and [8] is the Bruckner-Garg theorem describing the topological structure of the level sets. Interestingly, in the case of [94] and [8] the generic behaviour happens with “probability strictly between 0 and 1”, that is, the set of functions exhibiting the behaviour in question is neither Haar null nor co-Haar null.

A large part of modern descriptive set theory deals with homeomorphism groups of compact metric spaces, and automorphism groups of countable first-order structures, such as e.g. $\mathcal{H}[0, 1]$, the group of increasing homeomorphisms of the unit interval, $S_\infty$, the permutation group of the natural numbers or $\text{Aut}(\mathbb{Q}, <)$, the group of increasing bijections of the rational numbers. There have been numerous papers describing the structure of the random element of these groups (that is, the properties which are true for all elements except for a Haar null set), see e.g. [26, 33, 36, 41, 71].

There is also quite some literature dealing with the set theoretic aspects of Haar null sets. The so called cardinal invariants of (various versions of) Haar null sets are calculated in [1] and [13]; other related results can be found in e.g. [88] and [145].

Finally, there are also interesting but somewhat sporadic results involving Haar null sets in the theory of differential equations: [39, 60] and in the theory of functional equations: [20, 23, 50, 63, 81].
Bibliography


Summary

The thesis studies notions that allow us to formally state that a subset of a Polish group (a separable, completely metrizable topological group) is negligibly small. The two most important notions are Haar null sets (which generalize the notion of having Haar measure zero to groups that are not locally compact and therefore do not admit a Haar measure) and Haar meager sets (which coincide with the meager sets in locally compact groups, and provide a better analog of Haar null sets in the Polish groups that are not locally compact).

Chapter 3 examines the details of our core notions using tools from descriptive set theory. This inquiry is motivated by the fact that a set \( A \) is said to be Haar null if there is a Borel set \( B \supseteq A \) which satisfies certain conditions, and here the choice of Borelness (instead of other similar conditions) is somewhat arbitrary.

In locally compact groups it is easy to see that a set of Haar measure zero is always contained in a \( G_\delta \) set of Haar measure zero. On the other hand, our main result in this area proves that in the (frequently studied and “nice”) abelian Polish group \((\mathbb{Z}^\omega, +)\) for every \( 2 \leq \xi < \omega_1 \) there exists a Haar null set \( E \) that is the difference of two \( \Pi^0\xi \) sets but is not contained in any \( \Pi^0\xi \) Haar null set.

Chapter 4 examines the relation between the notion of Haar meager sets and the analogous notion of strongly Haar meager sets, which was already introduced in the paper \([29]\) where Darji defined the Haar meager sets. There were many signs which have suggested that these notions are equivalent, but we proved that in the abelian Polish group \( \mathbb{Z}^\omega \) there exists a Haar meager set that is not strongly Haar meager. The results in this chapter are joint work with M. Elekes, M. Poór and Z. Vidnyánszky.

In addition to these particular results, this thesis also contains an overview of the fundamental results related to Haar null and Haar meager sets. This material is based on the survey paper \([46]\), which is joint work with M. Elekes. The main goal of this work was to provide a practical introduction to this area of study. We included the basic facts about Haar null and Haar meager sets, examined the connections of these notions to other related notions, and provided a toolbox of proof techniques that might be useful in this area. In addition to collecting the earlier results, this part also proves some of them in a more general setting than that of the paper where they were originally published.
Magyar nyelv összefoglaló

Az alapfogalmak bemutatása után a 3. részben a leíró halmazelmélet eszközeivel vizsgáljuk ezt a témát. Ebből a szempontból az vet fel kérdéseket, hogy egy halmazt akkor nevezünk kicsinek, ha van egy Borel halmaz, amire bizonyos feltételek teljesülnek, és itt valamennyire önkényes volt az a döntés, hogy Borelséget írjunk el más hasonló jellemző helyett.

Könnyen belátható, hogy ha egy lokálisan kompakt csoportokban egy halmaz a Haar-nullhalmaz, akkor befoglalható egy Haar-nullhalmaz szerint nullhalmaz $G_δ$ halmazba. Ezzel szemben ebben a témában megmutatjuk azt, hogy a $(\mathbb{Z}^ω, +)$ csoportban (ami kommutatív, lengyel, és nem lokálisan kompakt) minden $2 ≤ ξ < ω_1$ rendszámra létezik olyan halmaz, ami Haar null és eláll két $Π_ξ^0$ halmaz különbségeként, de nem fedhet le egy $Π_ξ^0$ Haar null halmazzal.

A 4. rész megvizsgálja a Haar els kategóriájú halmazok kapcsolatát a rokon ersen Haar els kategóriájú (angolul “strongly Haar meager”) fogalommal, amelyet Darji ugyanabban a [24] cikkben vezetett be, mint a Haar els kategóriájú halmazok fogalmát. Sok jel utalt arra, hogy ez a két fogalom valójában ekvivalens, de mi megmutattuk, hogy a $(\mathbb{Z}^ω, +)$ csoportban létezik olyan Haar els kategóriájú halmaz, ami nem ersen Haar els kategóriájú. Ez a rész Elekes Mártonnal, Poór Márkkal és Vidnyánszky Zoltánnal közös eredményeket mutat be.

Ezek a konkrétabb részkérdéseken túl ez a disszertáció áttekinti a Haar null és Haar els kategóriájú halmazokhoz kötőd alapvet eredményeket. Ezek a részek a [30] összefoglaló cikken alapulnak, ami Elekes Mártonnal közös munka. Ennek az anyagnak az a f célja, hogy jól alkalmazható formában bemutassa ezt a kutatási területet. Itt összegyjtöttük a Haar null és Haar els kategóriájú halmazokkal kapcsolatos alapvet tényeket, megvizsgáltuk a rokon fogalommakkal való kapcsolatokat és összeállítottuk olyan bizonyítási módszereknél egy gyártényét, amelyek hasznosak lehetnek ezen a területen. A korábbi eredmények összegyjtésén túl ezek a részek általánosítanak is bizonyos eredményeket.