WEISS’S QUESTION

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Abstract. There is a function \( f : \mathbb{R}^+ \to \{0, 1\} \) such that if \( F(x, y) = f(d(x, y)) \) \((d(x, y)\) is the distance between \( x \) and \( y \)), then there is no uncountable homogeneous set for \( F \). If CH holds, we show that there is a similar coloring of \( \mathbb{R}^n \) with \( \aleph_1 \) colors so that uncountable sets contain all colors.

An important result in partition theory, due to Sierpiński, states the existence of a coloring \( F : [\mathbb{R}]^2 \to \{0, 1\} \) with no uncountable homogenous set [3]. Under CH, Erdős proved the stronger statement that there is a coloring of \([\mathbb{R}]^2\) with \( \aleph_1 \) colors such that each color occurs in each uncountable set [2]. In [4], Todorcevic proved this without any extra assumption.

William Weiss (Toronto) asked if there is a Sierpiński-type function \( F : [\mathbb{R}]^2 \to \{0, 1\} \) so that the color \( F(x, y) \) only depends on the value of \( d(x, y) \), the distance between \( x \) and \( y \). In Theorem 1, we give an affirmative answer. In Theorem 2, utilizing an idea of Erdős’s above theorem, we prove that, under CH, there is a like coloring of \( \mathbb{R}^n \) with \( \aleph_1 \) colors, so that in each uncountable set every color occurs.

Notation. Definitions. We use the notions and definitions of axiomatic set theory. In particular, each ordinal is a von Neumann ordinal, each cardinal is identified with the least ordinal of that cardinality. \( c \) is cardinal continuum, that is, \( |\mathbb{R}| \).

If \( S \) is a set, \( \kappa \) a cardinal, then \( [S]^\kappa = \{ x \subseteq S : |x| = \kappa \} \).

\( \mathbb{R} \) is the set of reals, \( \mathbb{R}^+ = \{ x \in \mathbb{R} : x > 0 \} \). \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space, \( d(x, y) \) is the distance between \( x \) and \( y \). If \( x, y \in \mathbb{R}^n \), then \((x, y)\) denotes the scalar product of \( x \) and \( y \), \( \|x\| = \sqrt{(x, x)} \). CH is the Continuum Hypothesis: \( c = \aleph_1 \).

Theorem 1. There is a function \( f : \mathbb{R}^+ \to \{0, 1\} \) such that if \( F(x, y) = f(d(x, y)) \), then there is no uncountable homogeneous set for \( F \).

Proof. Instead of defining a function \( f \) with \( \text{Dom}(f) = \mathbb{R}^+ \), we shall define a function \( f \) with \( \text{Dom}(f) = \mathbb{R} - \{0\} \) that satisfies \( f(z) = f(-z) \).

Let \( B = \{ b_\alpha : \alpha < \gamma \} \) be a Hamel basis.

If \( z \in \mathbb{R} - \{0\} \), then it can uniquely be written as

\[
    z = \sum_{i=1}^{n} \mu_i(z) b_{\alpha_i},
\]

where \( \mu_i(z) \in \mathbb{Q} - \{0\} \), \( \alpha_1 < \cdots < \alpha_n \). We call \( \langle \mu_1(z), \ldots, \mu_n(z) \rangle \) the type of \( z \), \( \text{tp}(z) \), and \( \{\alpha_1, \ldots, \alpha_n\} \) the support of \( z \), \( \text{supp}(z) \).

Claim 1. There are countably many types. If \( \text{tp}(x) = \langle \mu_1, \ldots, \mu_n \rangle \), then \( \text{tp}(-x) = \langle -\mu_1, \ldots, -\mu_n \rangle \).
**Proof.** Immediate.

Set \( \mu(z) = \mu_1(z) \) and define
\[
f(z) = \begin{cases} 
1, & \mu(z)z > 0 \\
0, & \mu(z)z < 0.
\end{cases}
\]

**Claim 2.** \( f(z) = f(-z) \).

**Proof.** As by Claim 1 \( \mu(-z)(-z) = \mu(z)z \).

We define the coloring \( F : [\mathbb{R}]^2 \rightarrow \{0, 1\} \) by
\[
F(x, y) = f(|x - y|) = f(x - y).
\]

Assume that \( \{x_\xi : \xi < \omega_1\} \) is an uncountable homogeneous set for \( F \). We repeatedly shrink the system to more and more regular uncountable subsystems as follows. Choose \( Z_0 \in [\omega_1]^{\aleph_1} \) such that all elements \( \{x_\xi : \xi \in Z_0\} \) have the same type: \( \text{tp}(x_\xi) = (\lambda_1, \ldots, \lambda_n) \) (possible as there are countably many types). Then, applying the \( \Delta \)-system lemma, choose \( Z_1 \in [Z_0]^{\aleph_1} \) such that \( \{\text{supp}(x_\xi) : \xi \in Z_1\} \) forms a \( \Delta \)-system, \( \text{supp}(x_\xi) = s \cup s_\xi \) \( (\xi \in Z_1) \) with \( s \cap s_\xi = s_\xi \cap s_\eta = \emptyset \) for \( \xi \neq \eta \in Z_1 \). Next pick \( Z_2 \in [Z_1]^{\aleph_1} \) such that \( s \) occupies the same positions in the supports, possible as there are finitely many possibilities for the relative positions of \( s \) inside the supports. Let \( j \) be the least position of some element of \( s_\xi \) (by the above, it is the same for all \( \xi \in Z_2 \)).

The elements \( \{\alpha_\xi^j : \xi \in Z_2\} \) are different, by the Dushnik–Miller theorem \([1] \) there is \( Z_3 \in [Z_2]^{\aleph_1} \) such that \( \{\alpha_\xi^j : \xi < \omega_1\} \) is increasing.

If \( \xi < \eta \) are in \( Z_3 \), then \( \mu(x_\xi - x_\eta) = \lambda_j \), so they have the same sign. As there is no uncountable increasing or decreasing sequence of reals, there are \( \xi_0 < \eta_0 \) and \( \xi_1 < \eta_1 \) in \( Z_3 \) such that \( x_{\xi_0} < x_{\eta_0} \) and \( x_{\xi_1} > x_{\eta_1} \), consequently \( F(x_{\xi_0}, x_{\eta_0}) \neq F(x_{\xi_1}, x_{\eta_1}) \) and so \( \{x_\xi : \xi < \omega_1\} \) is not homogeneous.

For the rest of the paper, fix a positive integer \( n \).

In order to prepare for Theorem 2, we introduce some notions.

**Definition.** If \( a_1, \ldots, a_k \in \mathbb{R}^n \), set
\[
W(a_1, \ldots, a_k) = \{\lambda_1 a_1 + \cdots + \lambda_k a_k : \lambda_1 + \cdots + \lambda_k = 1\}.
\]

The elements of \( A \subseteq \mathbb{R}^n \) are in \( k \)-general position if \( a_{k+1} \notin W(a_1, \ldots, a_k) \) for \( a_1, \ldots, a_{k+1} \in A \) distinct. Note that if the elements of \( A \) are in \( k \)-general position and \( 2 \leq k' \leq k \), then they are in \( k' \)-general position. If \( a_1, \ldots, a_{k+1} \) are in \( k \)-general position, then \( W(a_1, \ldots, a_{k+1}) \) is a translate of a \( k \)-dimensional subspace.

**Lemma 1.** If \( a_1, \ldots, a_{k+1} \) are in \( k \)-general position \( (1 \leq k \leq n) \), then \( \{a_i - a_1 : 2 \leq i \leq k + 1\} \) is linearly independent in the subspace \( W(a_1, \ldots, a_{k+1}) - a_1 \).

**Proof.** Assume indirectly that \( \sum_{i=2}^{k+1} \lambda_i(a_i - a_1) = 0, \lambda_j \neq 0. \) Then
\[
a_j = \sum_{2 \leq i \leq k+1} \frac{-\lambda_i}{\lambda_j} a_i + \left( \sum_{2 \leq i \leq k+1} \frac{\lambda_i}{\lambda_j} \right) a_1
\]
and here the sum of the coefficients is \( \lambda_j/\lambda_j = 1. \)
Lemma 2. If $H = W(a_1, \ldots, a_{k+1}), y, y' \in H$, $z = y - y'$, $(z, a_i - a_1) = 0$ $(2 \leq i \leq k + 1)$, then $z = 0$.

Proof. By the definition of $H$, $z = \sum_{i=1}^{k+1} \lambda_i a_i$ for some coefficients with $\sum_{i=1}^{k+1} \lambda_i = 0$. Then

$$(z, z) = \sum_{i=1}^{k+1} \lambda_i (z, a_i) = \sum_{i=1}^{k+1} \lambda_i (z, a_i - a_1) + (\sum_{i=1}^{k+1} \lambda_i) (z, a_1) = 0$$

and so $z = 0$.

Lemma 3. Assume that $H = W(a_1, \ldots, a_{k+1})$, $r_1, \ldots, r_{k+1}$ are reals. Then there is at most one $y \in H$ such that $d(a_i, y) = r_i$ $(1 \leq i \leq k + 1)$.

Proof. The case $k = 1$ is immediate.

Assume that $2 \leq k \leq n$ and $\|a_i - y\|^2 = \|a_i - y'\|^2 = r_i^2$ $(1 \leq i \leq k + 1)$. Multiplying out, one obtains

$$\|(a_i - y) + (y' - y)\|^2 = \|a_i - y\|^2 - 2(a_i - y, y' - y) + \|y' - y\|^2 = \|a_i - y\|^2,$$

that is,

$$2(a_i - y, y' - y) - \|y' - y\|^2 = 0.$$

If we subtract the $i = 1$ case, we get

$$2(a_i - a_1, y' - y) = 0 \ (2 \leq i \leq k + 1),$$

which implies $y = y'$ by Lemma 2.

Lemma 4. Let $H = W(a_1, \ldots, a_{k+1})$. Then there is at most one point $y \in H$ such that $d(a_i, y) = d(a_j, y)$ $(i \neq j)$.

Proof. The case $k = 1$ is obvious.

Assume $k \geq 2$. Assume that $y, y' \in H$ both satisfy the condition. Then

$$\|a_i - y\|^2 = \|a_j - y\|^2 = \|(a_i - y) + (a_j - a_i)\|^2 = \|a_i - y\|^2 + 2(a_i - y, a_j - a_i) + \|a_j - a_i\|^2,$$

consequently

$$2(a_i - y, a_j - a_i) + \|a_j - a_i\|^2 = 0.$$

Calculating this for $y'$ and subtracting the result, we have

$$(y' - y, a_j - a_i) = 0 \ (i \neq j)$$

and we conclude with Lemma 2.

Lemma 5. If $A \subseteq \mathbb{R}^n$, $|A| = \aleph_1$, then there are $1 \leq k \leq n$, $H \subseteq \mathbb{R}^n$, a translate of a $k$-dimensional subspace, $B \subseteq A \cap H$ such that $|B| = \aleph_1$ and if $k \geq 2$, then $B$ is in $k$-general position.

Proof. Let $k$ be minimal such that there exists a translate $H$ of a $k$-dimensional subspace with $|H \cap A| = \aleph_1$. 


If \( k = 1 \), set \( B = A \cap H \).

Assume \( k \geq 2 \). We select the points \( \{x_\alpha : \alpha < \omega_1\} \subseteq A \cap H \) which are in \( k \)-general position, by transfinite recursion on \( \alpha \). Assume that we have reached step \( \alpha \). Select

\[
x_\alpha \in A \cap H - \bigcup \{W(x_{\beta_1}, \ldots, x_{\beta_k}) : \beta_1 < \cdots < \beta_k < \alpha\}.
\]

The choice is possible, as countably many translates of \((k-1)\)-dimensional subspaces do not cover a translate of a \(k\)-dimensional subspace.

Finally, set \( B = \{x_\alpha : \alpha < \omega_1\} \).

**Theorem 2 (CH).** If \( 1 \leq n < \omega \), then there is a coloring \( f : \mathbb{R}^+ \to \omega_1 \), such that if \( F(x, y) = f(d(x, y)) \) \((\{x, y\} \in [\mathbb{R}^n]^2) \), then \( \{F(x, y) : \{x, y\} \in [B]^2\} = \omega_1 \) for every \( B \in [\mathbb{R}^n]^{\aleph_1} \).

**Proof.** Enumerate as \( \{\langle A_\alpha, k_\alpha, H_\alpha \rangle : \alpha < \omega_1\} \) all triples \( \langle A, k, H \rangle \) where \( A \subseteq \mathbb{R}^n \), \( |A| = \aleph_0 \), \( 1 \leq k \leq n \), \( H \) is a translate of a \( k \)-dimensional subspace, \( A \subseteq H \), if \( 2 \leq k \leq n \), then \( A \) is in \( k \)-general position, \( H = W(a_1, \ldots, a_{k+1}) \) \( \{a_1, \ldots, a_{k+1}\} \in [A]^{k+1} \).

Enumerate \( \mathbb{R}^n \) as \( \mathbb{R}^n = \{q_\alpha : \alpha < \omega_1\} \).

For \( \alpha < \omega_1 \), let \( M_\alpha \subseteq \mathbb{R}^n \), \( N_\alpha \subseteq \mathbb{R} \) be the smallest sets such that

(a) \( \{q_\beta : \beta < \alpha\} \subseteq M_\alpha \);
(b) if \( a, b \in M_\alpha \), then \( d(a, b) \in N_\alpha \);
(c) if \( a_1, \ldots, a_{k+1} \in M_\alpha \) are in \( k \)-general position, \( r_1, \ldots, r_{k+1} \in N_\alpha \), \( y \in W(a_1, \ldots, a_{k+1}) \) is such that \( d(a_i, y) = r_i \) \((1 \leq i \leq k+1) \), then \( y \in M_\alpha \);
(d) if \( a_1, \ldots, a_{k+1} \in M_\alpha \) are in \( k \)-general position, \( y \in W(a_1, \ldots, a_{k+1}) \) is such that \( d(a_1, y) = d(a_2, y) = \cdots = d(a_{k+1}, y) \), then \( y \in M_\alpha \);
(e) if \( \beta < \alpha \), then \( H_\beta \cap (M_\alpha + M_\alpha) = \aleph_0 \).

The existence of \( M_\alpha \) and \( N_\alpha \) can easily be shown by a Skolem-type construction: by closing off \( \{q_\beta : \beta < \alpha\} \) under the finitary operations described by (b), (c), (d), and (e).

Note that

(A) \( M_0 = N_0 = \emptyset \);
(B) \( |M_\alpha|, |N_\alpha| \leq \aleph_0 \) \((\alpha < \omega_1)\);
(C) \( M_\beta \subseteq M_\alpha, N_\beta \subseteq N_\alpha \) \((\beta < \alpha)\);
(D) \( \bigcup \{M_\alpha : \alpha < \gamma\}, \bigcup \{N_\alpha : \alpha < \gamma\} \) \((\gamma < \omega_1)\).

We are going to define \( f|N_\alpha \) by transfinite recursion on \( \alpha \). Assume we already constructed \( f|N_\alpha \) and we would like to extend it to \( N_{\alpha+1} \).

Enumerate the countable set

\[
\{\langle \beta, H_\beta \cap (M_{\alpha+1} - M_\alpha) \rangle : \beta < \alpha, A_\beta \subseteq M_\alpha \}\times \alpha
\]

as \( \{\langle \beta_i, p_i, \xi_i \rangle : i < \omega \} \).

By recursion on \( i < \omega \), we choose an element \( x_i \in A_{\beta_i} \) such that

(1) \( d(x_i, p_i) \in N_{\alpha+1} - N_\alpha \);
(2) \( d(x_i, p_i) \neq d(x_j, p_j) \) \((j < i)\).

We have to show that at step \( i \) we can choose \( x_i \).

**Claim 1.** There are at most \( n \) elements \( x \in A_{\beta_i} \) which do not satisfy (1).

**Proof.** Indeed, assume that for the distinct \( x(1), \ldots, x(n+1) \in A_{\beta_i} \), we have that \( d(x(m), p_i) \in N_\alpha \) \((1 \leq m \leq n+1)\). That is, there are reals \( r(1), \ldots, r(n+1) \in N_\alpha \)
such that \( d(x(m), p_i) = r(m) \) (\( 1 \leq m \leq n + 1 \)). For each sequence \( \langle r(1), \ldots, r(n + 1) \rangle \), \( r(1), \ldots, r(n + 1) \in \mathbb{N} \) there is at most one \( p \) such that \( d(x(m), p) = r(m) \) (here we use that \( x(1), \ldots, x(n + 1) \) are in \( k_\alpha \)-general position and Lemma 3) so, by (c) of the definition of \( M_\alpha \) and \( \mathbb{N}_\alpha \), it contains this \( p \), that is, \( p_i \in \mathbb{N}_\alpha \), a contradiction. \( \square \)

**Claim 2.** There are at most \( n_i \) elements \( x \in A_{\beta_i} \) which do not satisfy (2).

**Proof.** Assume indirectly that there are more than \( n_i \) elements that do not satisfy (2). Then, for some \( j < i \), there are \( x(1), \ldots, x(n + 1) \in A_{\beta_i} \) such that \( d(x(m), p_i) = d(x(j), p_j) \), that is, there is \( p \) such that

\[
d(x(1), p) = d(x(2), p) = \cdots = d(x(n + 1), p).
\]

But by (d) of the above definition, then each such \( p \) must be in \( M_\alpha \), which contradicts the assumption on \( p_i \). \( \square \)

By Claims 1 and 2, we can choose \( x_i \) and proceed. With all \( x_i \) selected, we extend \( f \) to \( \mathbb{R}^+ \cap \mathbb{N}_{\alpha + 1} \) so that \( f(d(x_i, p_i)) = \xi_i \) (\( i < \omega \)). This is possible by (1) and (2).

We claim that the function \( f \) is as required.

Let \( B \in [\mathbb{R}^\alpha]^{\omega_1} \) and \( \xi < \omega_1 \) be so that \( F(x, y) \neq \xi \) for \( x, y \in B \). By Lemma 5, there is a \( \beta < \omega_1 \) such that \( A_\beta \subseteq B \) and \( |H_\beta \cap B| = \aleph_1 \). If \( \alpha < \omega_1 \) is so large that \( \beta, \xi < \alpha \) and \( A_\beta \subseteq M_\alpha \) hold, then for every \( p \in (H_\beta \cap B) - M_\alpha \) there is \( x \in A_\alpha \) with \( F(x, p) = \xi \), consequently \( H_\beta \cap B \subseteq M_\alpha \), that is, \( H_\beta \cap B \) is countable, a contradiction.

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