The Mathematical Analysis of Voting and Districting Rules

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Chapter 1

Introduction

Considering electoral systems, we identify three major components (voting rules, districting procedures, and apportioning methods), which can be fruitfully analyzed from a normative (axiomatic) viewpoint and are related to computational problems of various difficulty. In this thesis we focus on the mathematical analysis of voting rules and the political districting problem, where Parts I and II contain our results on voting and districting, respectively.

The axiomatic analysis goes back to Arrow [6] who established for at least three alternatives the non-existence of a non-dictatorial social choice rule (SCR) defined on the universal domain, satisfying the Pareto property and the independence of irrelevant alternatives. In a similar vein Gibbard [38] and Satterthwaite [74] demonstrated for at least three alternatives the nonexistence of a non-dictatorial surjective social choice function (SCF) satisfying strategy-proofness (i.e. non-manipulability).

In Chapter 2 we characterize the preference domains on which proper scoring methods are non-dictatorial, satisfy the Pareto-property and Arrow’s “independence of irrelevant alternatives”. Under a weak richness condition, these domains are obtained by fixing one preference ordering and including all its cyclic permutations (“Condorcet cycles”). We then ask on which domains the Borda count is non-manipulable. It turns out that it is non-manipulable on a broader class of domains when combined with appropriately chosen tie-breaking rules. On the other hand, we also prove that the rich domains on which the Borda count is non-manipulable for all possible tie-breaking rules are again the cyclic permutation domains. Chapter 2 is based on Barbie, Puppe, and Tasnádi [11].

Going one step further, in Chapter 3 based on Puppe and Tasnádi [65] we also determine the domains on which the Borda count is Nash-implementable. Therefore, we char-
acterize the preference domains on which the Borda count satisfies Maskin monotonicity. The basic concept is again the notion of a cyclic permutation domain which arises by fixing one particular ordering of alternatives and including all its cyclic permutations. However, it turns out that compared with strategy-proofness the Borda count is monotonic on a larger class of domains. We show that the maximal domains on which the Borda count satisfies Maskin monotonicity are the “cyclically nested permutation domains” which are obtained from the cyclic permutation domains in an appropriately specified recursive way.

To escape from the impossibility results in [6], [38], and [74] we investigate domain restrictions in Chapters 2 and 3. We follow another possible route in Chapter 4 by replacing the axiomatic setting with an optimization problem. In particular, distance rationalization of voting rules is based on the minimization of the distance to some plausible criterion, such as unanimity or the Condorcet criterion. We propose a new alternative: the optimization of the distance to undesirable voting rules, namely, the dictatorial voting rules. Applying a plausible metric between social choice functions, we obtain two results: (i) the plurality rule minimizes the sum of the distances to the dictatorial rules and can be regarded in some sense as a compromise lying between all dictatorial rules; (ii) the reverse-plurality rule maximizes the distance to the closest dictator. Chapter 4 is based on Bednay, Moskalenko, and Tasnádi [12].

In Part II we turn to the analysis of the political districting problem. Districting plays a crucial role in all electoral systems endowed with single member districts. The underlying problem is that an actual redistricting, which has to be carried out in order to prevent malapportionment (i.e. unequal representation of citizens), may favor a specific party. The resulting bias could be caused by an ex ante unbiased districting procedure (where by ex ante unbiasedness we mean that the bias and its extent towards a party is not known prior to the employment of the given districting procedure) or consciously by the body in charge of the redistricting process. The latter goes back to 1812 when Elbrige Gerry Governor of Massachusetts reluctantly signed a redistricting plan favoring Jeffersonian democrats into law. The word gerrymandering, standing for the activity of carrying out a redistricting in favor of a certain party, was coined from Gerry’s name and the salamander like looking districts of Massachusetts in 1812. Since then gerrymandering is a hot issue in the United States and is still happening because in most US states the legislature has primary responsibility for creating a redistricting plan, often subject to approval by the state governor. Recently, it also became an important question in Hungary.

In Chapter 5 we study the districting problem from an axiomatic point of view in a framework with two parties, deterministic voter preferences and geographical constraints.
The axioms are normatively motivated and reflect a notion of fairness to voters. Our main result is an “impossibility” theorem demonstrating that all anonymous districting rules are necessarily complex in the sense that they either use information beyond the mere number of districts won by the parties, or they violate an appealing consistency requirement according to which an acceptable districting rule should induce an acceptable districting of appropriate subregions. The chapter is based on our work in Puppe and Tasnádi [68].

In Section 6.1 we demonstrate for discrete districting problems with geographical constraints that determining an (ex post) unbiased districting, which requires that the number of representatives of a party should be proportional to its share of votes, turns out to be a computationally intractable (NP-complete) problem. This raises doubts as to whether an independent jury will be able to come up with a “fair” redistricting plan in case of a large population, that is, there is no guarantee for finding an unbiased districting (even if such exists). This section contains our results published in Puppe and Tasnádi [66].

In Sections 6.2 and 6.3 we show that optimal partisan redistricting with geographical constraints is a computationally intractable (NP-complete) problem too. In particular, even when voter’s preferences are deterministic, a solution is generally not obtained by concentrating opponent’s supporters in “unwinnable” districts (“packing”) and spreading one’s own supporters evenly among the other districts in order to produce many slight marginal wins (“cracking”). These two sections build on Puppe and Tasnádi [67] and Fleiner, Nagy and Tasnádi [32].

A hierarchical extension of the districting problem was investigated by Kobayashi and Tasnádi [48].
Part I

Voting
Chapter 2

Non-Manipulable Domains for the Borda Count

Ever since their publication, the two most important results of social choice theory, the impossibility theorems of Arrow and Gibbard-Satterthwaite, have led to a steady search for possibility results on restricted domains (see Gaertner [34] for an overview). The usual approach is to fix an appropriate set of admissible preferences, and to investigate which social choice rules satisfy Arrow’s conditions, respectively which social choice functions are non-manipulable, on that preference domain. Classic examples of this approach are Black [18] and Moulin [55] who consider the domain of single-peaked preferences.

A somewhat different view on the question has been developed by Dasgupta and Maskin [27]. These authors consider specific preference aggregation rules such as majority rule, plurality rule or the Borda count, and ask on what domains these rules satisfy desirable conditions in the spirit of Arrow’s conditions. This chapter follows their approach. Specifically, we restrict our attention here to the Borda count and, slightly more generally, to scoring methods (cf. Moulin [56]). We characterize the preference domains on which scoring methods satisfy Arrow’s conditions (“Arrovian domains”). In contrast to Dasgupta and Maskin [27], we impose the original independence of irrelevant alternatives condition, not their stronger neutrality condition. By consequence, the Arrovian domains for the Borda count determined here encompass the domains that satisfy Dasgupta and Maskin’s characterizing condition of “quasi-agreement.” Our analysis also shows that all Arrovian

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1See also Barberá, Sonnenschein and Zhou [10] and Barberá, Gul and Stacchetti [9] for multi-dimensional extensions, and Nehring and Puppe [60, 61] for a unifying approach to impossibility and possibility results based on “generalized single-peaked preferences” in the context of strategy-proofness.
domains for the Borda count that are minimally “rich” in the sense that any social alternative is on top of at least one preference ordering, are obtained by fixing one preference ordering over the alternatives and including all its cyclic permutations. Remarkably, these are precisely the configurations of preferences that give rise to the Condorcet paradox. The rich domains on which the Borda count “works well” thus turn out to be exactly the problematic domains for majority voting.

We then consider the question on which domains the Borda count is strategy-proof. Since the Borda count does in general not select one single social alternative, we have to consider tie-breaking rules here. It turns out that the Borda count is strategy-proof with any given tie-breaking rule on all Arrovian domains. The converse is not true, however. We show by example that there exist rich domains on which the Borda count violates the independence of irrelevant alternatives condition but is nevertheless strategy-proof when combined with *some* suitable tie-breaking rule. On the other hand, under the richness condition, strategy-proofness of the Borda count with all tie-breaking rules yields again exactly the Arrovian domains (one fixed preference ordering together with all its cyclic permutations).

Our analysis confirms the general view of the literature that the Borda count is highly vulnerable to strategic manipulation. This intuition is made precise here in two ways. First, for any preference ordering, there is only one rich domain that contains the given preference ordering and that renders the Borda count non-manipulable. By contrast, for other choice rules there are frequently many different rich and non-manipulable domains that contain a given preference ordering; for instance, there are many rich single-peaked domains that contain a given single-peaked preference ordering. Secondly, any fixed rich domain on which the Borda count is non-manipulable is as small as it could possibly be, since it contains just as many orderings as there are social states. Again, this strong restriction does not apply to single-peaked domains, for instance. Thus, the overall conclusion

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2The apparent conflict of this conclusion with Dasgupta and Maskin’s robustness result for majority voting is due to the fact that we do not impose neutrality here.

3Recently, there have been different approaches to measuring the degree of “vulnerability” of voting procedures to strategic manipulation, see Aleskerov and Kurbanov and Smith, among others. The most relevant study in our context is Favardin, Leppelley and Serais who characterize (for the case of three alternatives) the preference profiles at which the Borda count is manipulable. Their conclusion is that the Borda count is significantly more vulnerable than, say, the Copeland method. Note that, in contrast to this literature, our aim is not to determine the relative frequency of possible manipulation on an unrestricted domain, but to characterize the restricted domains on which manipulation can never occur.
from our analysis is that the Borda count fares poorly in terms of strategic manipulation, in the sense that there are very few non-manipulable domains all of which are, moreover, very small.

Our work has inspired a similar analysis of the plurality rule by Sanver [73]. Without fixing a specific rule Hatsumi, Berga and Serizawa [40] found that the separable domain[^1] is a maximal domain for strategy-proofness and no-vetoer (that is, no voter has veto power).

### 2.1 Basic Notation and Definitions

Let $X$ be a finite universe of social states or social alternatives. By $\mathcal{P}_X$, we denote the set of all linear orderings (irreflexive, transitive and total binary relations) on $X$, and by $\mathcal{P} \subseteq \mathcal{P}_X$ a generic subdomain of the unrestricted domain $\mathcal{P}_X$. Moreover, denote by $\mathcal{R}$ the set of all weak orderings (reflexive, transitive and complete binary relations).

**Definition 2.1 (Social choice rule).** A mapping $F : \bigcup_{n=1}^{\infty} \mathcal{P}^n \rightarrow \mathcal{R}$ that assigns a social preference ordering $F(\succ_1, \ldots, \succ_n) \in \mathcal{R}$ to each $n$-tuple of linear orderings and all $n$ is called a social choice rule (SCR).

Thus, we do allow for non-trivial indifferences on the social level but not on the individual level. Note also that we require a SCR to be defined for societies with any finite number of agents. For some of our results this will be important. Alternatively, we could have assumed a continuum of agents as e.g. in Dasgupta and Maskin [27].

A SCR $F$ satisfies the *Pareto rule* on $\mathcal{P}$ if, for all $x, y \in X$, all $\succ_i \in \mathcal{P}$ and all $n$,

$$[x \succ_i y \text{ for all } i = 1, \ldots, n] \Rightarrow x \succ y,$$

where $\succ$ is the strict part of the social preference relation $\succeq = F(\succ_1, \ldots, \succ_n)$.

A SCR $F$ is called *non-dictatorial* on $\mathcal{P}$ if, either $\#\mathcal{P} = 1$ or, for all $n \geq 2$ and all $i = 1, \ldots, n$, there exist $x, y \in X$ and $\succ_i \in \mathcal{P}$ such that $x \succ_i y$ and $y \succeq x$, where $\succeq = F(\succ_1, \ldots, \succ_n)$.

A SCR $F$ satisfies *independence of irrelevant alternatives (IIA)* on $\mathcal{P}$ if, for all $x, y \in X$, all $n$ and all $\succ_i, \succ_i' \in \mathcal{P}$,

$$[\succ_i | \{x, y\} = \succ_i' | \{x, y\} \text{ for all } i = 1, \ldots, n] \Rightarrow \succeq | \{x, y\} = \succeq' | \{x, y\}.$$

[^1]: A preference relation is separable if for any set $A \subseteq X$ and any object $x \notin A$ we have that $\{x\} \cup A$ is preferred to $A$ if and only if $x$ is preferred to the empty set. If each voter can choose its preference relation from the set of all separable preferences, then the respective domain of preference profiles is called separable.
where $\succeq = F(\succ_1, \ldots, \succ_n)$, $\succeq' = F(\succ'_1, \ldots, \succ'_n)$, and $\succeq |_{\{x,y\}}$ denotes the restriction of the binary relation $\succ$ to the pair $\{x,y\}$.

**Definition 2.2 (Scoring method).** Let $q$ be the cardinality of $X$, and let $s : \{1, \ldots, q\} \to \mathbb{R}$ satisfy $s(1) \geq s(2) \geq \ldots \geq s(q)$ and $s(1) > s(q)$. Moreover, let $rk[x, \succ]$ denote the rank of alternative $x$ in the ordering $\succ$ (i.e. $rk[x, \succ] = 1$ if $x$ is the top alternative in the ranking $\succ$, $rk[x, \succ] = 2$ if $x$ is second-best, and so on). A SCR is a scoring method if for some function $s$, all $x, y \in X$, all $n$ and all $\succ_i$, $i = 1, \ldots, n$,

$$x \succeq y \iff \sum_{i=1}^{n} s(rk[x, \succ_i]) \geq \sum_{i=1}^{n} s(rk[y, \succ_i]),$$

where $\succeq$ is the social preference corresponding to $(\succ_1, \ldots, \succ_n)$. The scoring method corresponding to the function $s : \{1, \ldots, q\} \to \mathbb{R}$ will be denoted by $F^s$. A scoring method is called proper if $s$ is strictly decreasing.

**Definition 2.3 (Borda count).** The Borda count, denoted by $F^B$, is the proper scoring method corresponding to the function $s(k) = q + 1 - k$ for $k = 1, \ldots, q$.

Clearly, all scoring methods are non-dictatorial; moreover, any proper scoring method satisfies the Pareto rule. On the other hand, scoring methods do not generally satisfy the IIA condition. A characterization of the domains on which scoring methods satisfy this condition will be provided in Section 2.2.

**Definition 2.4 (Social choice function).** A mapping $f : \bigcup_{n=1}^{\infty} \mathcal{P}^n \to X$ that assigns a social alternative to each $n$-tuple of linear orderings and all $n$ is called a social choice function (SCF).

A SCF $f$ satisfies unanimity on $\mathcal{P}$ if, for all $x \in X$, all $\succ_i \in \mathcal{P}$ and all $n$,

$$[rk[x, \succ_i] = 1 \text{ for all } i = 1, \ldots, n] \Rightarrow x = f(\succ_1, \ldots, \succ_n).$$

A SCF $f$ is called non-manipulable, or strategy-proof on $\mathcal{P}$ if for all $n$, all $\succ_i, \succ'_i \in \mathcal{P}$ and all $\succ_{-i} \in \mathcal{P}^{n-1}$,

$$f(\succ_i, \succ_{-i}) \succeq_i f(\succ'_i, \succ_{-i}).$$

**Example 2.1 (Borda count with tie-breaking rule).** For our purposes, a tie-breaking rule is simply a linear ordering $\tau$ on $X$. For given a tie-breaking rule $\tau$, any SCR $F$ uniquely defines a SCF $f$ by associating to each preference profile $(\succ_1, \ldots, \succ_n)$ the $\tau$-best element of $F(\succ_1, \ldots, \succ_n) \subseteq X$. Below we will specifically consider the Borda count $F^B$ together with a tie-breaking rule $\tau$; the resulting SCF will be denoted by $f^B_\tau$. Note that $f^B_\tau$ satisfies unanimity.
Obviously, sufficiently “small” domains can give rise to strategy-proofness in a trivial way. For instance, any SCF is vacuously strategy-proof on any domain consisting of one single preference ordering. We will therefore be often interested in domains that are “rich” in the sense that any alternative is on top of some preference ordering.

Definition 2.5 (Rich domain). A domain $\mathcal{P}$ is called rich if for any $x \in X$ there exists $\succ \in \mathcal{P}$ such that $rk[x, \succ] = 1$.

### 2.2 Arrovian Domains for Scoring Methods

It is well-known that scoring methods violate the IIA condition on the unrestricted domain $\mathcal{P}_X$. However, scoring methods may well satisfy this condition on restricted domains. We will say that $\mathcal{P}$ is an Arrovian domain for the SCR $F$ if $F$ is non-dictatorial and satisfies the Pareto rule as well as IIA on $\mathcal{P}$.

Definition 2.6 (Equal score difference). A domain $\mathcal{P}$ satisfies the equal score difference (ESD) condition with respect to $s$ if, for all $x, y \in X$, either all orderings in $\mathcal{P}$ agree on $\{x, y\}$, or if not, then

$$s(rk[x, \succ]) - s(rk[y, \succ]) = s(rk[x, \succ']) - s(rk[y, \succ'])$$

for all $\succ, \succ' \in \mathcal{P}$ such that $\succ|_{\{x,y\}} = \succ'|_{\{x,y\}}$.

Note that for the Borda count the latter condition reduces to

$$rk[x, \succ] - rk[y, \succ] = rk[x, \succ'] - rk[y, \succ']$$

for all $\succ, \succ' \in \mathcal{P}$ such that $\succ|_{\{x,y\}} = \succ'|_{\{x,y\}}$, which we will also refer to as the equal rank difference (ERD) condition.

Theorem 2.1. A domain is Arrovian for the proper scoring method $F^s$ if and only if it satisfies the equal score difference condition with respect to $s$.

Proof. Clearly, any scoring method is non-dictatorial and satisfies the Pareto rule on any domain. Let $\mathcal{P}$ satisfy the equal score difference condition. In order to verify IIA consider

---

5This condition is often imposed in the literature. It is much weaker than the richness condition used in Nehring and Puppe [60].
any \(x, y \in X\) and \(\succ_i \in \mathcal{P}\) for \(i = 1, \ldots, n\). Suppose that \(x \succeq y\), where \(\succeq = F^s(\succ_1, \ldots, \succ_n)\), i.e. suppose that
\[
\sum_{i=1}^{n} [s(\text{rk}[x, \succ_i]) - s(\text{rk}[y, \succ_i])] \geq 0. \tag{2.1}
\]
If all orderings in \(\mathcal{P}\) agree on \(\{x, y\}\), we must have \(x \succ y\), and this relative ranking of \(x\) and \(y\) holds for the social preference corresponding to \(\text{any}\) profile. Thus, assume that not all orderings in \(\mathcal{P}\) agree on the pair \(\{x, y\}\). Then, by the equal score difference condition, the inequality (3.1) is preserved when any voter \(i\)'s ordering \(\succ_i \in \mathcal{P}\) is replaced by an ordering \(\succ'_i \in \mathcal{P}\) that agrees with \(\succ_i\) on \(\{x, y\}\). This shows that \(F^s\) satisfies IIA on \(\mathcal{P}\).

Conversely, suppose that the domain \(\mathcal{P}\) does not satisfy the equal score difference condition. Then, there exist \(x, y \in X\) and three orderings \(\succ, \succ', \succ'' \in \mathcal{P}\) such that
\[
l := s(\text{rk}[y, \succ'']) - s(\text{rk}[x, \succ'']) > 0
\]
and
\[
s(\text{rk}[x, \succ]) - s(\text{rk}[y, \succ]) =: m > m' := s(\text{rk}[x, \succ']) - s(\text{rk}[y, \succ']) > 0.
\]
Choose \(n_1\) and \(n_2\) such that
\[
\frac{l}{m'} > \frac{n_1}{n_2} > \frac{l}{m},
\]
and consider the following two profiles of \(n_1 + n_2\) individuals. In the first profile, denoted by \(\Pi = (\succ, \ldots, \succ, \succ'', \ldots, \succ'')\), the first \(n_1\) voters have the preference \(\succ\) and the remaining \(n_2\) voters have the preference \(\succ''\); in the second profile, denoted by \(\Pi' = (\succ', \ldots, \succ', \succ'', \ldots, \succ'')\), the first \(n_1\) voters have the preference \(\succ'\), and the remaining \(n_2\) voters have the preference \(\succ''\). By construction, \(x\) is ranked strictly above \(y\) in the social ranking \(F^s(\Pi)\) corresponding to the first profile, while \(y\) is strictly above \(x\) in the social ranking \(F^s(\Pi')\) corresponding to the second profile. This yields the desired violation of IIA.

\[\square\]

### 2.3 A Special Case: The Borda Count

The restrictiveness of the equal score difference condition depends on the scoring rule. For instance, suppose that \(X = \{x, y, z\}\) and consider any scoring method \(s\) that does not coincide with the Borda count, i.e. \(s(2) - s(1) \neq s(3) - s(2)\). It is easily seen that any domain that satisfies the equal score difference condition with respect to such \(s\) can consist of at most two preference orderings on \(X\). More generally, one can show that, for arbitrary \(X\), no scoring method different from the Borda count can satisfy the equal score
difference condition on any rich domain. On the other hand, for the Borda count there
are rich domains satisfying the corresponding (equal rank difference) condition. In the
following, we will provide a “global” characterization of all such domains. Before we do so,
we briefly want to compare our equal rank difference condition to Dasgupta and Maskin’s
condition of “quasi-agreement.” That condition requires that any triple \( \{x, y, z\} \) admit
one member, say \( x \), such that all orderings in the domain agree on either (i) \( x \) being the
best element among the three, or (ii) \( x \) being the middle element, or (iii) \( x \) being the
worst element among the triple. Dasgupta and Maskin [27] show that the property of
quasi-agreement characterizes the domains on which the Borda count satisfies an appro-
priate neutrality condition stronger than Arrow’s independence of irrelevant alternatives
considered here. By consequence, quasi-agreement is more restrictive than the equal rank
difference condition. This can be directly verified by contraposition, as follows. Suppose
that a domain violates the equal rank difference condition, i.e. there exist three orderings
\( \succ^1, \succ^2 \) and \( \succ^3 \) such that
\[
 rk[y, \succ^1] - rk[x, \succ^1] > rk[y, \succ^2] - rk[x, \succ^2] > 0 \tag{2.2}
\]
and \( rk[y, \succ^3] - rk[x, \succ^3] < 0 \). By [2.2], there exists a third alternative \( z \) such that
\( x \succ^1 z \succ^1 y \) but not \( (x \succ^2 z \succ^2 y) \), in which case the three orderings violate quasi-
agreement on the triple \( \{x, y, z\} \).

Equal rank difference as well as quasi-agreement are “local” conditions; the former
imposes restrictions on any pair, the latter on any triple. It is therefore not evident how
these conditions are reflected in the “global” structure of the corresponding domains. We
now provide an alternative characterization of equal rank difference domains that makes
this global structure explicit. An ordering \( \succ' \) is called a cyclic permutation of \( \succ \) if \( \succ' \) can be
obtained from \( \succ \) by sequentially shifting the bottom element to the top while leaving the
order between all other alternatives unchanged. Thus, for instance, the cyclic permutations
of the ordering \( abcd \) are \( dabc, cdab \) and \( bdca \). The set of all cyclic permutations of a fixed
ordering \( \succ \) is denoted by \( Z(\succ) \). Say that a domain \( \mathcal{P} \) is hierarchically cyclic if there exists
a partition \( \{X_1, ..., X_r\} \) of \( X \) such that for all \( \succ \in \mathcal{P} \) and all \( i \in \{1, ..., r\} \);

(i) \( x \succ y \) whenever \( x \in X_i, y \in X_j \) and \( j > i \), and

(ii) \( \{\succ' |_{X_i} : \succ' \in \mathcal{P}\} \subseteq Z(\succ |_{X_i}) \) or \( \#\{\succ' |_{X_i} : \succ' \in \mathcal{P}\} \leq 2 \)

Thus, a domain is hierarchically cyclic if the universe of alternatives can be partitioned
in such a way that (i) the partition elements themselves are ordered unambiguously and
identically by all orderings, and (ii) within each partition element $X_i$, the restrictions to $X_i$ give rise to at most two different orderings on $X_i$, or they are cyclic permutations of each other.

Table 2.1 shows a typical domain satisfying this condition in which the partition from the definition of a hierarchically cyclic domain is given by $X_1 = \{x_1, x_2, x_3, x_4\}$, $X_2 = \{y_1, y_2, y_3\}$, $X_3 = \{u\}$ and $X_4 = \{z_1, z_2, z_3\}$. Note that the preferences are cyclic permutations of one fixed ordering on $X_1$ and $X_4$. The two different restrictions on $X_2$ are not cyclic permutations of each other; nevertheless, the domain satisfies the defining condition since $\#X_2 \leq 2$.

**Proposition 2.1.** A domain $\mathcal{P}$ satisfies the equal rank difference condition if and only if it is hierarchically cyclic.

**Proof.** It is easily verified that any hierarchically cyclic domain satisfies the equal rank difference condition.

For the proof of Proposition 2.1 we need the following notation. For any $1 \leq i \leq j \leq q = \#X$ let $\succ|_{[i,j]}$ be the restriction of $\succ$ ranging from the $i$th position to the $j$th position of $\succ$, i.e., $\succ|_{[i,j]} = \succ|_{\{x_i, x_{i+1}, \ldots, x_j\}}$ where $x_1 \succ \cdots \succ x_i \succ \cdots \succ x_j \succ \cdots \succ x_q$. In addition, for any $1 \leq i \leq j \leq q$, we define $\mathcal{P}_{[i,j]} := \{\succ|_{[i,j]} : \succ \in \mathcal{P}\}$. Furthermore, for any linear ordering $\succ$ on $X' \subseteq X$ we shall denote by $T_i(\succ)$ the set of the top $i$ alternatives of $\succ$, i.e., $T_i(\succ) = \{x \in X' : rk [x, \succ] \leq i\}$. 

**Table 2.1: A hierarchically cyclic domain**

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It is straightforward to check that a hierarchically cyclic domain satisfies ERD. Hence, we have to prove the converse statement.

**Step 1:** We construct recursively a partition of $X$. Let $i_0 := 0$. To obtain the first partition element $X_1$, we determine the smallest integer $i \in \{i_0 + 1, \ldots, q\}$ satisfying

$$\forall x \in X, \forall \succ, \succ' \in \mathcal{P} : i_0 < rk [x, \succ] \leq i \iff i_0 < rk [x, \succ'] \leq i.$$  \hfill (2.3)

Clearly, at least $q$ satisfies (2.3) and therefore there exists a smallest $i$, denoted by $i_1$, satisfying (2.3). Set $X_1 := \{x \in X : i_0 < rk [x, \succ] \leq i_1\}$ for some $\succ \in \mathcal{P}$. Then, we are finished and the partition consists only of the single set $X_1$. If $X_1 \neq X$, then we proceed inductively to obtain $i_2$ and $X_2$ from (2.3). Repeating this procedure, we get the desired partition $X_1, \ldots, X_r$.

In the following, we only have to consider those sets $X_j$ for which

$$\#\mathcal{P}_{[i_{j-1}+1, i_j]} = \# \{\succ' | x_j : \succ' \in \mathcal{P}\} > 2.$$  \hfill (2.4)

Pick an arbitrary set $X_j$ satisfying (2.4), and set $\mathcal{P}_j := \mathcal{P}_{[i_{j-1}+1, i_j]} = \{\succ_1, \ldots, \succ_{n_j}\}$ and $q_j := i_j - i_{j-1}$. Clearly, $q_j \geq 3$ because of (2.4).

**Step 2:** First, we establish that $\mathcal{P}_j$ contains three preference relations with different top alternatives. Obviously, not all preferences can have the same top alternative, since this would be in contradiction with the construction of $X_j$. Thus, suppose that the preferences in $\mathcal{P}_j$ have two different top alternatives. Without loss of generality we can assume that the first $p \in \{2, \ldots, n_j - 1\}$ preferences have $a \in X_j$ as their top alternative, while the remaining preferences have another alternative $b \in X_j \setminus \{a\}$ as their top alternative. Define

$$Y := \{x \in X_j : \forall k, l \in \{1, \ldots, p\}, \ rk [x, \succ_k] = rk [x, \succ_l]\}.$$  \hfill (2.5)

Clearly, $a \in Y$. Moreover, we must also have $rk [b, \succ_k] = rk [b, \succ_l]$ for all $k, l \in \{1, \ldots, p\}$ by ERD, hence $b \in Y$. Let

$$J := \{k \in \{1, \ldots, q_j\} : \exists y \in Y, k = rk [y, \succ_1]\} \quad \text{and} \quad i^* := \max \{k \in \{1, \ldots, q_j\} : \{1, \ldots, k\} \subseteq J\}.$$

Observe that $i^*$ is well defined, since $\{1\} \subseteq J$.

Clearly, if $i^* = q_j$, then we have $\succ_1 = \cdots = \succ_p$, which cannot be the case, since the preferences $\succ_1, \ldots, \succ_p$ are distinct. Hence, we may assume that $i^* < q_j$. ERD implies that any alternative $z \in X_j \setminus Y$ must be ranked below the $i^*$th position by any preference
relation having \( b \) on top, since \( z \) changes its rank difference to all alternatives in \( T_i \ (\succ_1) = \cdots = T_i (\succ_p) \). Formally,

\[
\forall z \in X_j \setminus Y, \forall l \in \{p+1, \ldots, n_j\} : rk [z, \succ_l] > i^*.
\] (2.6)

Suppose now that none of the alternatives in \( Y \) are ranked lower than \( i^* \) (i.e. suppose that \( \{1, \ldots, i^*\} = J \)). Then, we obtain \( T_i (\succ_1) = \cdots = T_i (\succ_{n_j}) \) by (2.6), which contradicts the construction of \( X_j \).

Thus, there exists an alternative \( y \in Y \) with \( rk [y, \succ_1] > i^* + 1 \) (i.e. \( \{1, \ldots, i^*\} \neq J \)). In this case, we will show that

\[
\forall y \in Y, \forall l \in \{p+1, \ldots, n_j\} : rk [y, \succ_1] > i^* + 1 \Rightarrow rk [y, \succ_l] > i^* \]  

(2.7)

Indeed, suppose that this is not the case, i.e. suppose that \( y \in Y \) is such that \( rk [y, \succ_1] > i^* + 1 \) and \( rk [y, \succ_l] \leq i^* \) for some \( l \in \{p+1, \ldots, n_j\} \). Now pick two alternatives \( u, v \in X_j \setminus Y \) and a preference relation \( \succ_k, k \in \{2, \ldots, p\} \), such that \( rk [u, \succ_1] = rk [v, \succ_k] = i^* + 1 \). If \( u \succ_k y \) or \( v \succ_1 y \), then the pair \( \{u, y\} \) or the pair \( \{v, y\} \), respectively, violates ERD, since \( y \succ_l u \) and \( y \succ_l v \) by (2.6). Similarly, it can be verified that if \( v \succ_k y \succ_k u \) and \( u \succ_1 y \succ_1 v \), then at least one of the pairs \( \{u, v\}, \{v, y\} \) or \( \{u, y\} \) violate ERD by (2.6).

We have thus derived a contradiction, hence (2.7) holds. Together with (2.6), this implies that all alternatives ranked below the \( i^*\)th position in the first \( p \) orderings must also be ranked below the \( i^*\)th position in all remaining orderings. But this means again that \( T_i (\succ_1) = \cdots = T_i (\succ_{n_j}) \), contradicting the definition of \( X_j \). Thus, we must have at least three different top alternatives in \( P_j \).

**Step 3:** We now show that any three top alternatives in \( P_j \) produce a Condorcet cycle. Pick preferences \( \succ_k, \succ_m, \succ_l \in P_j \) having the three different alternatives \( a, b, c \in X_j \), respectively, on top. Without loss of generality we may assume that \( a \succ_k b \succ_k c \). Now it can easily be verified that from the four possibilities

\[
[a \succ_k b \succ_k c, \ b \succ_m a \succ_m c, \ c \succ_l a \succ_l b],
\]

\[
[a \succ_k b \succ_k c, \ b \succ_m c \succ_m a, \ c \succ_l a \succ_l b],
\]

\[
[a \succ_k b \succ_k c, \ b \succ_m a \succ_m c, \ c \succ_l b \succ_l a], \text{ and}
\]

\[
[a \succ_k b \succ_k c, \ b \succ_m c \succ_m a, \ c \succ_l b \succ_l a]
\]

only the Condorcet cycle satisfies ERD.

**Step 4:** We claim that for any three different top alternatives \( a, c \) and \( b \), where \( rk [a, \succ_k] = rk [c, \succ_l] = rk [b, \succ_m] = 1 \), there exists \( t_{k,l,m} \in \{1, \ldots, q_j\} \) such that \( \succ_l \frontset{t_{k,l,m}} \)

\[6\text{Observe that this implies } rk [b, \succ_1] \leq i^*, \text{ since } b \in Y \text{ and } rk [b, \succ_{p+1}] = 1. \]
Chapter 2. Non-Manipulable Domains for the Borda Count

The Borda count that contains \( \succ_{t,k,l,m} \) has the following four properties:

1. \( \succ_{t,k,l,m} \) is a rich Arrovian domain.
2. \( \succ_{t,k,l,m} \) is a rich non-manipulable domain.
3. \( \succ_{t,k,l,m} \) is a rich transitive domain.
4. \( \succ_{t,k,l,m} \) is a rich serial domain.

By Step 3 we can assume that the top elements are ordered in the following way: \( \succ_{t,k} b \succ_{t,k} c, c \succ_{t,k} a \succ_{t,k} b \succ_{t,k} m \succ_{t,k} a \). Take an alternative \( x \) such that \( c \succ_{t,k} x \succ_{t,k} a \). Suppose that \( a \succ_{t,k} x \); this implies \( c \succ_{t,k} x \) by ERD. But then ERD must be violated, since \( x \) cannot maintain its rank difference to both \( a \) and \( b \) in \( \succ_{t,k} \) as well as in \( \succ_{m} \). Hence, we have \( x \succ_{m} a \), and by ERD, the rank difference between \( x \) and \( a \) has to be the same in \( \succ_{t,k} \) as in \( \succ_{m} \). In a similar way one can establish that \( \text{rk} [b, \succ_{t,k}] - \text{rk} [z, \succ_{t,k}] = \text{rk} [b, \succ_{m}] - \text{rk} [z, \succ_{m}] \) for any \( a \succ_{t,k} z \succ_{t,k} b \) and that \( \text{rk} [c, \succ_{m}] - \text{rk} [y, \succ_{m}] = \text{rk} [c, \succ_{k}] - \text{rk} [y, \succ_{k}] \) for any \( b \succ_{m} y \succ_{m} c \).

Next we pick an alternative \( z \) satisfying \( a \succ_{t,k} z \succ_{t,k} b \). Then \( z \) must be ranked below \( a \) in \( \succ_{m} \), since by the above argument, \( \{ w : c \succ_{t,k} w \succ_{t,k} a \} = \{ w : c \succ_{m} w \succ_{m} a \}, \{ w : b \succ_{m} w \succ_{m} c \} = \{ w : b \succ_{t,k} w \succ_{t,k} c \} \), and \( \{ w : a \succ_{t,k} w \succ_{t,k} b \} = \{ w : a \succ_{t,k} w \succ_{t,k} b \} \). Hence, by ERD, \( \text{rk} [c, \succ_{t,k}] - \text{rk} [z, \succ_{t,k}] = \text{rk} [c, \succ_{m}] - \text{rk} [z, \succ_{m}] \). Similarly, \( \text{rk} [a, \succ_{m}] - \text{rk} [y, \succ_{m}] = \text{rk} [a, \succ_{t,k}] - \text{rk} [y, \succ_{t,k}] \) for any \( b \succ_{k} y \succ_{k} c \), and \( \text{rk} [b, \succ_{m}] - \text{rk} [x, \succ_{m}] = \text{rk} [b, \succ_{k}] - \text{rk} [x, \succ_{k}] \) for any \( c \succ_{m} x \succ_{m} a \). Finally, observe that we can choose \( t_{k,1,m} = \text{rk} [b, \succ_{m}] - \text{rk} [a, \succ_{m}] + \text{rk} [a, \succ_{t,k}] - \text{rk} [c, \succ_{t,k}] + \text{rk} [c, \succ_{m}] - \text{rk} [b, \succ_{m}] \).

**Step 5:** Now we can complete the proof. Assume that \( a, b, c, d_{1}, \ldots, d_{n_{1}} \) are the top alternatives of \( \succ_{1}, \ldots, \succ_{n_{1}} \), respectively, where \( a, b \) and \( c \) are pairwise distinct. Apply Step 4 to preferences \( \succ_{1} \), \( \succ_{2} \) and \( \succ_{3} \), and pick another preference relation \( \succ_{m} \in \mathcal{P}_{j} \) arbitrarily.

First, if one of the three first top alternatives, say \( c \), is also the top alternative of \( \succ_{m} \), then Step 3 and ERD imply \( \text{rk} [a, \succ_{m}] - \text{rk} [c, \succ_{m}] = \text{rk} [a, \succ_{3}] - \text{rk} [c, \succ_{3}] \) and \( \text{rk} [b, \succ_{m}] - \text{rk} [c, \succ_{m}] = \text{rk} [b, \succ_{3}] - \text{rk} [c, \succ_{3}] \). Hence, by Step 4 we must have \( \text{rk} [a, \succ_{m}] = \text{rk} [b, \succ_{m}] = \text{rk} [c, \succ_{m}] \).

Second, suppose that \( d_{m} \in X_{j} \) is distinct from \( a, b \) and \( c \). Then it can be easily verified that \( \succ_{2} \mid_{1,1,2,3} \subseteq \mathcal{Z} (\succ_{1} \mid_{1,1,2,3}) \) and \( \succ_{2} \mid_{1,1,2,3} \subseteq \mathcal{Z} (\succ_{1} \mid_{1,1,2,3}) \) implies \( t_{1,2,3} = t_{1,2,m} \).

Thus, in both cases we obtain \( T_{1,2,3} (\succ_{1}) = \cdots = T_{1,2,3} (\succ_{n_{j}}) \). Therefore, we must have \( t_{1,2,m} = q_{j} \) for all \( m \in \{3, \ldots, n_{j}\} \) by the construction of \( X_{j} \). This completes the proof of Proposition 2.1.

As an immediate corollary of Proposition 2.1 and Theorem 2.1 we obtain the following result showing that all rich Arrovian domains for the Borda count are obtained by fixing one preference ordering and including all its cyclic permutations; such domains will henceforth be referred to as cyclic permutation domains. Note that the cyclic permutation domains on three alternatives are precisely the “Condorcet cycles.”

**Theorem 2.2.** For any linear ordering \( \succ \), there is exactly one rich Arrovian domain for the Borda count that contains \( \succ \), namely the cyclic permutation domain \( \mathcal{Z}(\succ) \).
2.4 Non-Manipulable Domains

We now want to ask on what domains the Borda count with tie-breaking rule is non-manipulable. The following result shows that the equal score/rank difference condition is sufficient for non-manipulability.

**Proposition 2.2.** Suppose that the domain $\mathcal{P}$ satisfies the equal score difference condition. Then, any scoring method with any tie-breaking rule is strategy-proof on $\mathcal{P}$.

**Proof.** Take any preference profile $(\succ_1, \ldots, \succ_n)$, and suppose that $x$ is the chosen alternative. Consider any alternative $y$ that voter $i$ prefers to $x$. Since $y$ was not chosen there must exist another voter $j$ such that $x \succ_j y$. By the equal score difference condition, any preference that favours $y$ over $x$ must display the same score difference between these alternatives as $\succ_i$. In particular, voter $i$ cannot change the difference in total scores of $y$ relative to $x$ by reporting a preference that favours $y$ over $x$. Since $y$ is arbitrary this shows that voter $i$ cannot successfully manipulate. $\square$

We now turn to the question of the necessary conditions for non-manipulability. This is a more difficult problem, and we will concentrate on the most interesting case of the Borda count. As already noted, if many conceivable preferences are excluded, strategy-proofness can result simply from the lack of misrepresentation possibilities. We will thus focus in the following on rich domains. Recall that the rich domains satisfying the equal rank difference condition are the cyclic permutation domains. First, we show by example that the Borda count may be non-manipulable also on domains that do not have the form of cyclic permutation domains, provided the tie-breaking rule is appropriately chosen.

**Example 2.2 (Non-manipulability without equal rank difference).** Consider on the universe $X = \{a, b, c, d\}$ the domain $\{\succ_I, \succ_{II}, \succ_{III}, \succ_{IV}\}$, where $a \succ_I b \succ_I c \succ_I d$, $b \succ_{II} a \succ_{II} d \succ_{II} c$, $c \succ_{III} d \succ_{III} a \succ_{III} b$ and $d \succ_{IV} c \succ_{IV} b \succ_{IV} a$. Clearly, this domain is rich and not a cyclic permutation domain. Observe that the equal rank difference condition is only violated by the two pairs $(a, d)$ and $(b, c)$. Hence, manipulation is only possible between alternatives $a$ and $d$, or $b$ and $c$, respectively. In particular, one can easily check that a voter of type $I$, $II$, $III$ or $IV$ can potentially benefit only by reporting type $II$, $I$, $IV$ or $III$, respectively. Note that for any manipulation of this kind, a voter can increase the total score difference only of two alternatives simultaneously over the other two alternatives; moreover, any such change in the score difference is by exactly two units. This property makes the domain rather special.
We will now show that the Borda count is non-manipulable when combined with the tie-breaking rule \( a \tau b \tau c \tau d \). Suppose that a profile with \( n \) voters consists of \( k, l, m \) and \( p \) preferences of types \( \succ_I, \succ_{II}, \succ_{III} \) and \( \succ_{IV} \), respectively. Then, we have
\[
\sum_{i=1}^{n} rk [a, \succ_i] + rk [d, \succ_i] = \sum_{i=1}^{n} rk [b, \succ_i] + rk [c, \succ_i] = 5 (k + l + m + p) .
\] (2.8)

It follows from (2.8) that, if there is to be room for manipulation at all, the total scores of all four alternatives have to be close to each other. Consider the case in which \( a \) was chosen by \( f^B_\tau \); the other cases can be treated analogously. If \( a \) was chosen, then only a voter of type \( III \) might potentially benefit from manipulating (by misreporting to be of type \( IV \)). By the above observations and by the form of the tie-breaking rule, alternative \( d \) could “overtake” \( a \) only if before both received the same total score, or if \( a \) led only by one unit. In the first case, all four alternatives received the same total score by (2.8), while in the latter case \( b \)’s total score was greater or equal to the total score of \( d \), again by (2.8).

Hence, misreporting type \( IV \) either does not change the outcome, or makes \( b \) the winner, which is not beneficial to a type \( III \) voter.

The example shows that on rich domains the equal rank difference condition is not necessary for non-manipulability of the Borda count together with a fixed tie-breaking rule. However, if we require non-manipulability of the Borda count when combined with any tie-breaking rule, the equal rank difference condition re-emerges, as shown by the following result.

**Theorem 2.3.** Suppose that the Borda count is non-manipulable on the rich domain \( \mathcal{P} \) for all tie-breaking rules \( \tau \). Then, \( \mathcal{P} \) satisfies the equal rank difference condition, i.e. \( \mathcal{P} \) is a cyclic permutation domain.

For the proof of Theorem 2.3 we need the following series of lemmas. Given a profile of preferences \((\succ_1, \ldots, \succ_n) \in \mathcal{P}^n \), we say that alternatives \( A \subseteq X \) are indifferent on the top if for all \( a, b \in A \) and all \( c \in X \setminus A \) we have
\[
\sum_{i=1}^{n} rk [a, \succ_i] = \sum_{i=1}^{n} rk [b, \succ_i] < \sum_{i=1}^{n} rk [c, \succ_i] .
\] (2.9)

**Lemma 2.1.** If there exists a preference profile \((\succ_1, \ldots, \succ_n) \in \mathcal{P}^n \) with alternatives \( \{x, y\} \subseteq X \) being indifferent on the top and violating ERD, then there exists a tie-breaking rule such that Borda count is manipulable on \( \mathcal{P} \).
Proof. Suppose that profile $\Pi := (\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ has alternatives $x$ and $y$ violating ERD indifferent on the top. If according to $\Pi$ we have

$$\sum_{i=1}^{n} rk [x, \succ_i] = \sum_{i=1}^{n} rk [y, \succ_i] \geq \sum_{i=1}^{n} rk [c, \succ_i] - 2 (#X - 1)$$

for some $c \in X \setminus \{x, y\}$, then we can take a ‘multiple’ of profile $\Pi$ consisting of $l$ preferences of type $\succ_i$ for each $i$ such that

$$l \sum_{i=1}^{n} rk [x, \succ_i] = l \sum_{i=1}^{n} rk [y, \succ_i] < l \sum_{i=1}^{n} rk [c, \succ_i] - 2 (#X - 1) \quad (2.10)$$

for all $c \in X \setminus \{x, y\}$ and $l$ sufficiently large by $(2.9)$. This ensures that if only one voter reveals another preference relation, either $x$ or $y$ will still be the Borda winning alternative. For notational convenience we will assume that $\Pi = (\succ_1, \ldots, \succ_n)$ already satisfies $(2.10)$.

Since $x$ and $y$ are indifferent on the top, profile $\Pi$ must have voters with preferences $\succ_i$ and $\succ_j$ such that $x \succ_i y$ and $y \succ_j x$. Suppose that there exists another preference $\succ' \in \mathcal{P}$ such that $x \succ' y$ and $rk [y, \succ'] - rk [x, \succ] \neq rk [y, \succ_i] - rk [x, \succ_i]$. Now if $rk [y, \succ'] - rk [x, \succ] > rk [y, \succ_i] - rk [x, \succ_i]$, then, taking a tie-breaking rule selecting $y$ as the winner in case of ties between $x$ and $y$, a voter having preference $\succ_i$ could manipulate by revealing preference $\succ'$. Otherwise, if $rk [y, \succ'] - rk [x, \succ] < rk [y, \succ_i] - rk [x, \succ_i]$, then we take the tie-breaking rule, which selects $x$ as the winner in case of ties between $x$ and $y$. Consider profile $(\succ_1, \ldots, \succ_{i-1}, \succ', \succ_{i+1}, \ldots, \succ_n)$, which has $y$ as the Borda winner. Clearly, voter $i$ can achieve a tie between $x$ and $y$ by revealing $\succ_i$ instead of $\succ'$ and therefore, enforce that $x$ will be chosen, which he prefers to $y$.

Finally, if there does not exist a preference $\succ' \in \mathcal{P}$ such that $x \succ' y$ and $rk [y, \succ'] - rk [x, \succ'] \neq rk [y, \succ_i] - rk [x, \succ_i]$, then there exists a preference $\succ' \in \mathcal{P}$ such that $y \succ' x$ and $rk [x, \succ'] - rk [y, \succ'] \neq rk [x, \succ_i] - rk [y, \succ_i]$, since $x$ and $y$ violate ERD. Hence, to complete the proof we just have to exchange the roles of $x$ and $y$ while repeating the arguments of the previous paragraph. 

The next lemma is a simple corollary to Lemma 2.1.

**Lemma 2.2.** If in a rich domain $\mathcal{P}$ there exists a preference $\succ$ with its top two alternatives violating ERD, then there exists a tie-breaking rule such that Borda count is manipulable on $\mathcal{P}$.

Proof. Let $rk [x, \succ] = 1$ and $rk [y, \succ] = 2$. Since $\mathcal{P}$ is a rich domain, we can find a preference $\succ' \in \mathcal{P}$, which has $y$ as the top alternative. We define $d := rk [x, \succ'] - rk [y, \succ']$. 


Now taking one voter with $\succ'$ and $d$ voters with $\succ$ we obtain a profile that has $\{x, y\}$ indifferent on the top, since $y$ dominates any $z \in X \setminus \{x, y\}$. Now apply Lemma 2.1.

Sometimes the set of alternatives that are indifferent on the top will contain more than two alternatives. In this case the following lemma turns out to be helpful in many cases.

**Lemma 2.3.** Suppose that $\mathcal{P}$ is a rich domain. If there exist two distinct preferences $\succ, \succ' \in \mathcal{P}$ and an alternative $y \in X$ satisfying

- $\operatorname{rk}[y, \succ] \geq 2$

- $\forall x \in X : x \succ y \Rightarrow x \succ' y$,

- $\forall x \in X : x \succ y \Rightarrow \operatorname{rk}[y, \succ] - \operatorname{rk}[x, \succ] \neq \operatorname{rk}[y, \succ'] - \operatorname{rk}[x, \succ']$,

then there exists a tie-breaking rule such that Borda count is manipulable on $\mathcal{P}$.

**Proof.** Let $k = \operatorname{rk}[y, \succ]$. For all $i \in \{1, \ldots, k - 1\}$ we shall denote by $x_i$ the alternative with $\operatorname{rk}[x_i, \succ] = i$. Pick a preference $\succ'' \in \mathcal{P}$ having $y$ as the top alternative. We define values $d_i := \operatorname{rk}[x_i, \succ''] - \operatorname{rk}[y, \succ']$ for all $i \in \{1, \ldots, k - 1\}$. Clearly, we have $\operatorname{rk}[y, \succ] - \operatorname{rk}[x_i, \succ] = k - i$ for all $i \in \{1, \ldots, k - 1\}$. Now let $J := \arg \min_{i \in \{1, \ldots, k - 1\}} \frac{d_i}{k - i}$ and $A := \{x_j \in X : j \in J\}$. Pick an arbitrary $j \in J$. Then it can be verified that a profile consisting of $d_j$ preferences of type $\succ$ and $k - j$ preferences of type $\succ''$ makes alternatives $\{y\} \cup A$ indifferent on the top. In particular, we will take a profile $(\succ_i)_{i=1}^n \in \mathcal{P}^n$ consisting of $ld_j$ preferences of type $\succ$ and $l(k - j)$ preferences of type $\succ''$ for which

$$\sum_{i=1}^n \operatorname{rk}[y, \succ_i] = \sum_{i=1}^n \operatorname{rk}[a, \succ_i] < \sum_{i=1}^n \operatorname{rk}[b, \succ_i] - 2(\#X - 1)$$

is satisfied for all $a \in A$ and all $b \in X \setminus (\{y\} \cup A)$, where $l$ is a sufficiently large positive integer. Thus, we can restrict our attention to alternatives in $\{y\} \cup A$.

We have to deal with two cases. First, suppose that there exists an alternative $a \in A$ such that $\operatorname{rk}[y, \succ] - \operatorname{rk}[a, \succ] < \operatorname{rk}[y, \succ'] - \operatorname{rk}[a, \succ']$. If we select a tie-breaking rule, which prefers $y$ to all alternatives in $A$, then a voter having preference $\succ$ can manipulate by revealing $\succ'$, since he prefers any alternative in $A$ to $y$.

Second, suppose that for all alternatives $a \in A$ we have $\operatorname{rk}[y, \succ] - \operatorname{rk}[a, \succ] \geq \operatorname{rk}[y, \succ'] - \operatorname{rk}[a, \succ']$. If we select a tie-breaking rule, which prefers all alternatives in $A$ to $y$, and consider a profile in which one voter’s preference of type $\succ$ in $(\succ_i)_{i=1}^n$ is replaced by $\succ'$, then this voter with preference $\succ'$ can manipulate by revealing $\succ$, since he prefers any alternative in $A$ to $y$. This completes the proof of the lemma.
Proof of Theorem 2.3. We prove the contraposition of the statement, i.e., if a rich domain \( P \subseteq P_X \) does not satisfy ERD, there exists a tie-breaking rule for which the Borda count is manipulable on \( P \). Hence, suppose that the rich domain \( P \subseteq P_X \) does not satisfy ERD.

**Step 1:** We can assume without loss of generality that the rich domain \( P \) violating ERD consists of exactly \( q \) preferences (recall that \( q = \#X \)). This can be verified as follows. Take an arbitrary rich domain \( P \) violating ERD with \( \#P > q \). Choose \( q \) preferences from \( P \) with different top alternatives, and denote the corresponding domain by \( P_0 \). If \( P_0 \) violates ERD, then we are done. On the other hand, if \( P_0 \) does not violate ERD, then \( P_0 = \mathcal{Z}(\succ) \) for any \( \succ \in P_0 \) by Proposition 1. Consider any preference ordering \( \succ_0 \in P \setminus P_0 \) and replace the preference in \( P_0 \) with the same top alternative as \( \succ_0 \) by the ordering \( \succ_0 \). As is easily verified, the resulting domain violates ERD. Henceforth, we thus assume that \( P = \{ \succ_1, \ldots, \succ_q \} \) is rich and violates ERD.

**Step 2:** We will construct a “chain” of alternatives and preferences. Start with preference \( \succ_1 \) and denote its top alternative by \( x_1 \) and its second ranked alternative by \( x_2 \). Without loss of generality we can assume that \( \succ_2 \) has \( x_2 \) on top. To describe how the procedure goes on suppose that we have already obtained a sequence of distinct alternatives \( x_1, \ldots, x_k \) such that \( r_k[x_i, \succ_i] = 1 \) for all \( i \in \{1, \ldots, k\} \) and \( r_k[x_i, \succ_{i-1}] = 2 \) for all \( i \in \{2, \ldots, k\} \). Now we define \( x_{k+1} \) recursively to be the second ranked alternative of \( \succ_k \). We have found a “chain” if \( x_{k+1} \) equals one of the alternatives \( x_1, \ldots, x_k \). Otherwise, we can suppose without loss of generality that \( x_{k+1} \) is the top alternative of \( \succ_{k+1} \). We iterate the described procedure until we obtain a “chain” of alternatives. Clearly, this procedure terminates in at most \( q \) steps. Thus, we can determine indices \( m, p \in \{1, \ldots, q\} \) such that \( m < p \), \( x_m, \ldots, x_p \) are all distinct, \( r_k[x_i, \succ_i] = 1 \) and \( r_k[x_i+1, \succ_i] = 2 \) for all \( i \in \{m, \ldots, p-1\} \), and \( r_k[x_p, \succ_p] = 1 \) and \( r_k[x_m, \succ_p] = 2 \). In what follows we can assume without loss of generality that \( m = 1 \). Nevertheless we will still denote the length of the chain by \( p \). Furthermore, let \( X' := \{x_1, \ldots, x_p\} \) and \( P_1 := \{\succ_1, \ldots, \succ_p\} \).

**Step 3:** We can manipulate by Lemma 2.2 for some tie-breaking rules if there exists a preference \( \succ_i \in P_1 \) in which the top two alternatives violate ERD. Hence, in what follows we can assume that the top two alternatives of all \( \succ_1, \ldots, \succ_p \) satisfy ERD. But this implies that the top \( p \) alternatives of the preferences in \( P_1 \) follow the pattern shown in Table 2.2. Clearly, if \( p = q \), we cannot have a violation of ERD by Theorem 2.1 hence \( p < q \).

Case (i): Suppose that there exists an alternative \( y \in X \) that is ranked by two distinct preferences \( \succ_i \) and \( \succ_j \) \( (i, j \in \{1, \ldots, p\}) \) at the \( p + 1 \)th position. Then \( y \) violates ERD with all alternatives \( x_1, \ldots, x_p \), since \( P \) is a rich domain and all alternatives \( x_1, \ldots, x_p \) are ranked differently according to \( \succ_i \) and \( \succ_j \) while \( y \) is ranked identically by these two
preferences. Hence, taking \( \succ_i, \succ_j \) and \( y \) we can apply Lemma 2.3.

Case (ii): Suppose that the alternatives \( y_1, \ldots, y_p \in X \) are all distinct and are ranked \( p + 1 \)th by the preferences \( \succ_1, \ldots, \succ_p \), respectively. Let \( Y := \{y_1, \ldots, y_p\} \).

We claim that if there exists an alternative \( y_i \in Y \) and a preference \( \succ_j \in \mathcal{P}_1 \) such that \( rk[y_i, \succ_j] - rk[x_i, \succ_j] \neq p \), then \( y_i \) violates ERD with all alternatives in \( X' \), and manipulation is possible by Lemma 2.3, taking \( \succ_i, \succ_j \) and \( y_i \) as \( \succ, \succ' \) and \( y \), respectively.

We check this claim without loss of generality for alternative \( y_p \).\(^7\) Of course, \( rk[y_p, \succ_p] - rk[x_p, \succ_p] = p \) and therefore, \( rk[y_p, \succ_j] - rk[x_p, \succ_j] \neq p \) implies that \( x_p \) and \( y_p \) violate ERD. Suppose that \( d := rk[y_p, \succ_j] - rk[x_p, \succ_j] < p \). Note that we have \( rk[x_p, \succ_j] = p - j + 1 \) and therefore, it follows that \( rk[y_p, \succ_p] - rk[x_i, \succ_p] = p - i > rk[y_p, \succ_j] - rk[x_i, \succ_j] = d - i \) for all \( i \in \{1, \ldots, j - 1\} \). In addition, for all \( i \in \{j, \ldots, p - 1\} \) we have \( rk[y_p, \succ_p] - rk[x_i, \succ_p] = rk[x_p, \succ_j] - rk[x_i, \succ_j] < rk[y_p, \succ_j] - rk[x_i, \succ_j] \). Now suppose that \( d = rk[y_p, \succ_j] - rk[x_p, \succ_j] > p \). Then clearly, \( rk[y_p, \succ_p] - rk[x_i, \succ_p] = d > p > rk[y_p, \succ_p] - rk[x_i, \succ_p] = p - i \) for all \( i \in \{j, \ldots, p - 1\} \). Furthermore, for all \( i \in \{1, \ldots, j - 1\} \) we have \( rk[y_p, \succ_p] - rk[x_i, \succ_p] = p - i < rk[y_p, \succ_j] - rk[x_i, \succ_j] = d - i \).

Hence, in any case \( y_p \) and \( x_i \) violate ERD for all \( i \in \{1, \ldots, p\} \).

We still have to investigate the case in which for all alternatives \( y_i \in Y \) and for all preferences \( \succ_j \in \mathcal{P}_1 \) we have \( rk[y_i, \succ_j] = rk[x_i, \succ_j] = p \). For this case the first \( p \) preferences are illustrated in Table 2.3. If \( 2p < q \), then we can mimic the arguments given so far for alternatives ranked, by some preference relations in \( \mathcal{P}_1 \), at the \( 2p + 1 \)th position. By

---

\(^7\)If we relabel the alternatives and preferences cyclically, then the claim follows for all the other alternatives \( y_1, \ldots, y_{p-1} \) in the same way.
Table 2.3: Two consecutive full cycles

<p>| | | | | | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>(\succ_1)</td>
<td>(\succ_2)</td>
<td>...</td>
<td>(\succ_{p-1})</td>
<td>(\succ_p)</td>
<td>...</td>
</tr>
<tr>
<td>(x_1)</td>
<td>(x_2)</td>
<td>...</td>
<td>(x_{p-1})</td>
<td>(x_p)</td>
<td>...</td>
</tr>
<tr>
<td>(x_2)</td>
<td>(x_3)</td>
<td>...</td>
<td>(x_p)</td>
<td>(x_1)</td>
<td>...</td>
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<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(x_{p-1})</td>
<td>(x_p)</td>
<td>...</td>
<td>(x_{p-3})</td>
<td>(x_{p-2})</td>
<td>...</td>
</tr>
<tr>
<td>(x_p)</td>
<td>(x_1)</td>
<td>...</td>
<td>(x_{p-2})</td>
<td>(x_{p-1})</td>
<td>...</td>
</tr>
<tr>
<td>(y_1)</td>
<td>(y_2)</td>
<td>...</td>
<td>(y_{p-1})</td>
<td>(y_p)</td>
<td>...</td>
</tr>
<tr>
<td>(y_2)</td>
<td>(y_3)</td>
<td>...</td>
<td>(y_p)</td>
<td>(y_1)</td>
<td>...</td>
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<td>...</td>
</tr>
<tr>
<td>(y_{p-1})</td>
<td>(y_p)</td>
<td>...</td>
<td>(y_{p-3})</td>
<td>(y_{p-2})</td>
<td>...</td>
</tr>
<tr>
<td>(y_p)</td>
<td>(y_1)</td>
<td>...</td>
<td>(y_{p-2})</td>
<td>(y_{p-1})</td>
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</tr>
</tbody>
</table>

Doing so, in a similar way as in case (i), we can derive that manipulation is possible through an appropriately selected tie-breaking rule if an alternative is ranked twice at the \(2p + 1\)th position by some preferences in \(\mathcal{P}_1\). Otherwise, let \(z_i\) be the alternative for which \(rk [z_i, \succ_i] = 2p + 1\) and let \(Z := \{z_1, \ldots, z_p\}\). Now, in an analogous way as in the beginning part of case (ii) one can argue that we can manipulate if there exists an alternative \(z_i \in Z\) such that there exists a preference \(\succ_j \in \mathcal{P}_1\) such that \(rk [z_i, \succ_j] - rk [y_i, \succ_j] \neq p\). The case that remains to be investigated whenever \(3p \leq q\) is illustrated in Table 2.4.

Alternatives \(y_1, \ldots, y_p\) are all top alternatives of a certain preference relation since \(\mathcal{P}\) is a rich domain. We shall denote the set of these preferences by \(\mathcal{P}_2\). Without loss of generality we can assume that \(rk [y_i, \succ_{p+i}] = 1\) for all \(i \in \{1, \ldots, p\}\). Thus, \(\mathcal{P}_2 = \{\succ_{p+1}, \ldots, \succ_{2p}\} \subset \mathcal{P}\). In what follows we have to consider four subcases.

Subcase (a): There exists a preference \(\succ_{p+i} \in \mathcal{P}_2\) that ranks an alternative \(u \in X \setminus (X' \cup Y)\) second, i.e., \(rk [u, \succ_{p+i}] = 2\). Then \(y_i\) and \(u\) violate ERD and \(\mathcal{P}\) is manipulable with respect to an appropriate tie-breaking rule by Lemma 2.2.

Subcase (b): The set of second ranked alternatives of all preferences in \(\mathcal{P}_2\) is a subset
Chapter 2. Non-Manipulable Domains for the Borda Count

Table 2.4: Three consecutive full cycles

<table>
<thead>
<tr>
<th>$\succ_1$</th>
<th>$\ldots$</th>
<th>$\succ_p$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\ldots$</td>
<td>$x_p$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\ldots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$x_p$</td>
<td>$\ldots$</td>
<td>$x_{p-1}$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$\ldots$</td>
<td>$y_p$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\ldots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$y_p$</td>
<td>$\ldots$</td>
<td>$y_{p-1}$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$z_1$</td>
<td>$\ldots$</td>
<td>$z_p$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\ldots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$z_p$</td>
<td>$\ldots$</td>
<td>$z_{p-1}$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

of $Y$. If there exists a preference in $\mathcal{P}_2$ with top two alternatives violating ERD, then we can apply Lemma 2.2. Otherwise, if the top two alternatives of all preferences in $\mathcal{P}_2$ satisfy ERD, then $\mathcal{P}_2$ must have a very special structure, since $y_i$ is ranked just above $y_{i \oplus 1}$ for all $i \in \{1, \ldots, p\}$ whenever $y_i$ is ranked above $y_{i \oplus 1}$\(^8\). Thus, we must have preferences as shown in Table 2.5.

Let

$$A := \left\{ x \in X' : \sum_{i=1}^{p} \text{rk}[x, \succ_{p+i}] \leq \sum_{i=1}^{p} \text{rk}[u, \succ_{p+i}] \text{ for all } u \in X' \right\}.$$

Now pick an alternative $x_i \in A$ and we will make $A \cup Y$ indifferent on the top. Define $d := \left( \sum_{j=1}^{p} \text{rk}[x_i, \succ_{p+j}] \right) - \frac{1}{2}p(p+1)$. For each $\succ \in \mathcal{P}_1$ taking $d$ voters and for each $\succ' \in \mathcal{P}_2$ taking $p^2$ voters we obtain a profile in which all $x \in A$ and all $y \in Y$ are indifferent on the top. In particular, any $x \in A$ beats all alternatives in $X' \setminus A$ and any $y \in Y$ beats all alternatives in $X \setminus (X' \cup Y)$, while alternatives $x \in A$ and $y \in Y$ receive the same Borda

\(^8\)For two integers $k, l \in \{1, \ldots, p\}$, if $k + l \neq p$ and $k + l \neq 2p$, we define $k \oplus l := (k + l) \mod p$, while if $k + l = p$ or $k + l = 2p$, we define $k \oplus l := p$. 
score. Pick an alternative \( x_i \in A \) and consider a voter having preference \( \succ_{i \oplus 1} \). Suppose that the tie-breaking rule prefers \( y_{i \oplus 1} \) to \( x_i \) and \( x_i \) to all other alternatives. Then a voter having preference \( \succ_{i \oplus 1} \) could manipulate by revealing \( \succ_{i \oplus 2} \).

Subcase (c): The set of second ranked alternatives of all preferences in \( P_2 \) is a subset of \( X' \). If there exists a preference \( \succ_{p+j} \in P_2 \) that ranks an alternative \( x_i \) with \( i \neq j \) second, then the top two alternatives of \( \succ_{p+j} \) violate ERD\(^9\) and therefore, by Lemma 2.2 we can find a tie-breaking rule making manipulation possible. Hence, in what follows we can assume that \( rk[x_i, \succ_{p+i}] = 2 \) for all \( i \in \{1, \ldots, p\} \). Since we know that the alternatives in \( X' \) satisfy ERD, \( P_2 \) must have again a very special structure because \( x_i \) is ranked just above \( x_{i \oplus 1} \) for all \( i \in \{1, \ldots, p\} \) whenever \( x_i \) is ranked above \( x_{i \oplus 1} \). Thus, any \( \succ \in P_2 \) must rank

\(^9\)In fact, looking at Table 2.3 it is easy to verify that all pairs \( x_i \) and \( y_j \) \((i \neq j)\) violate ERD, while all pairs \( x_i \in X' \) and \( y_i \in Y \) satisfy ERD on \( P_1 \). For instance, if \( i < j \), then the sequence \( (rk[x_i, \succ_k])_{k=1}^p \) decreases until \( k = i \) and jumps up by \( p-1 \) afterwards, whereas \( (rk[y_i, \succ_k])_{k=1}^p \) still decreases after \( k = i \). Hence, the rank differences between \( x_i \) and \( y_j \) differ according to preferences \( \succ_i \) and \( \succ_{i+1} \). One can argue analogously in case of \( 1 \leq j < i \leq p \).
the alternatives of $X'$ from the 2nd to the $p+1$th position in a cyclic pattern. Therefore, in any preference $\succ_{p+i} \in \mathcal{P}_2$ we have $x_j \succ_{p+i} y_j$ for all $j \in \{1, \ldots, p\} \setminus \{i\}$. Hence, if there exists a preference $\succ_{p+i} \in \mathcal{P}_2$ and a pair of alternatives $x_j, y_j$ ($j \in \{1, \ldots, p\} \setminus \{i\}$) such that $rk[y_j, \succ_{p+i}] - rk[x_j, \succ_{p+i}] \neq p$, then the top two alternatives $y_j$ and $x_j$ of $\succ_{p+j}$ violate ERD and we are done by applying Lemma 2.2.

We still have to investigate the case in which for all preferences $\succ_{p+i} \in \mathcal{P}_2$ and for all pairs of alternatives $x_j, y_j$ ($j \in \{1, \ldots, p\} \setminus \{i\}$) we have $rk[y_j, \succ_{p+i}] - rk[x_j, \succ_{p+i}] = p$. For the case of $p = 3$ we illustrate this case in Table 2.6. Clearly, this case can only occur whenever $2p < q$. Observe that the $p+2$nd positions of each preference in $\mathcal{P}_2$ have to be filled with an alternative from $X \setminus (X' \cup Y)$. Suppose that we have $rk[u, \succ_{p+1}] = p+2$ for an alternative $u \in X \setminus (X' \cup Y)$. Then $u$ violates ERD with all alternatives ranked by $\succ_{p+1}$ above $u$ (i.e., with all alternatives in $X' \cup \{y_1\}$), since $\mathcal{P}$ is a rich domain. More specifically, if $u \neq z_2$, then we can apply Lemma 2.3 with $\succ_{p+1}, \succ_1$ and $u$; while if $u = z_2$, then we can apply Lemma 2.3 with $\succ_{p+1}, \succ_2$ and $u$.

Subcase (d): We still have to investigate the case in which the second ranked alternatives in $\mathcal{P}_2$ come from both $X'$ and $Y$. First, observe that as in subcase (c), if there exists a preference $\succ_{p+j} \in \mathcal{P}_2$ that ranks an alternative $x_i$ with $i \neq j$ 2nd, then $x_i$ and $y_j$ violate ERD and we can apply Lemma 2.2. Hence, in what follows we can assume that if $rk[u, \succ_{p+i}] = 2$ and $u \in X'$, then $u = x_i$.

Second, if there exists a preference $\succ_{p+i} \in \mathcal{P}_2$ that ranks an alternative $y \in Y \setminus \{y_{i+1}\}$
second, then \( y_i \) and \( y \) violate ERD and we are done by Lemma 2.2.

Finally, we can assume that there exists \( \succ_{p+i} \in \mathcal{P}_2 \) such that \( rk[y_{i\oplus 1}, \succ_{p+i}] = 2 \) and \( rk[x_{i\oplus 1}, \succ_{p+(i\oplus 1)}] = 2 \). Now if \( rk[x_{i\oplus 1}, \succ_{p+i}] > 3 \), the top two alternatives \( y_{i\oplus 1} \) and \( x_{i\oplus 1} \) of \( \succ_{p+(i\oplus 1)} \) violate ERD and we are finished by Lemma 2.2. Otherwise, if \( rk[x_{i\oplus 1}, \succ_{p+i}] = 3 \), we can make \( x_{i\oplus 1} \) and \( y_i \) indifferent on the top by taking for all preferences \( \succ \in \mathcal{P}_1 \) two voters each and \( p^2 \) voters with \( \succ_{p+i} \). In particular, \( y_i \) beats any other alternative in \( X \setminus X' \) and \( x_{i\oplus 1} \) beats any other alternative in \( X' \), while \( y_i \) and \( x_{i\oplus 1} \) receive the same Borda score. Since \( x_{i\oplus 1} \) and \( y_i \) violate ERD, we can apply Lemma 2.1. \( \square \)

Table 2.7: The final case of subcase (d)

<table>
<thead>
<tr>
<th>( \succ_1 )</th>
<th>( \succ_2 )</th>
<th>( \succ_3 )</th>
<th>( \succ_4 )</th>
<th>( \succ_5 )</th>
<th>( \succ_6 )</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>( x_3 )</td>
<td>( y_1 )</td>
<td>( y_2 )</td>
<td>( y_3 )</td>
<td>\ldots</td>
</tr>
<tr>
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<td>( x_3 )</td>
<td>( x_1 )</td>
<td>( y_2 )</td>
<td>( x_2 )</td>
<td>\ldots</td>
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<tr>
<td>( x_3 )</td>
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<td>\ldots</td>
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</tr>
<tr>
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<td>( x_3 )</td>
<td>( x_1 )</td>
<td>\ldots</td>
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<tr>
<td>( y_2 )</td>
<td>( y_3 )</td>
<td>( y_1 )</td>
<td>( x_1 )</td>
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<td>( y_3 )</td>
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</tbody>
</table>

The following example shows that the richness assumption in Theorem 2.3 is needed. Consider on \( X = \{a, b, c, d\} \) the domain consisting of the three preference orderings \( abcd \), \( dabc \) and \( dacb \). This domain violates the ERD (in fact even ESD). But the Borda count is non-manipulable with any tie-breaking rule. Indeed, alternatives \( b \) and \( c \) can never win, while there are obviously no manipulation possibilities between alternatives \( a \) and \( d \).

---

\(^{10}\)For \( p = 3 \) and \( i = 1 \) this case is illustrated in Table 2.7.
Chapter 3

Nash implementable domains for the Borda count

A social choice correspondence satisfies Maskin monotonicity if and only if a chosen alternative remains a possible choice whenever in no individual’s ranking its relative position to another alternative decreases. It is well-known that Maskin monotonicity, which we shall henceforth refer to simply as “monotonicity,” is a necessary condition for Nash implementability; moreover, combined with a no veto power condition it is also sufficient provided that there are at least three individuals (Maskin [53]). In this paper, we characterize the preference domains on which the Borda count satisfies monotonicity. Since the Borda count satisfies the no veto power condition whenever there are sufficiently many voters, the result thus also yields the preference domains on which the Borda count is Nash implementable.

The celebrated Muller-Satterthwaite theorem (Muller and Satterthwaite [57]) establishes that, for social choice functions (i.e. single-valued social choice correspondences), monotonicity is equivalent to strategy-proofness, provided that all preference profiles are admissible. By contrast, while strategy-proofness always implies monotonicity, the converse need not be true on restricted domains. In fact, the main result of this chapter provides an illustration of this, showing that there exist preference domains on which the Borda count is monotonic but not strategy-proof when combined with a tie breaking rule.

In Chapter 2 we have characterized the preference domains on which the Borda count (with tie breaking) is strategy-proof. In particular, if all individuals face the same domain restriction, the maximal strategy-proof domains for the Borda count are obtained by fixing one particular ordering of the alternatives and including all its cyclic permutations. We
referred to such domains as *cyclic permutation domains*. In this chapter, we show that monotonicity of the Borda count imposes weaker restrictions and allows one to construct domains on which possibility results emerge in a recursive way from the cyclic permutation domains. The corresponding domains are referred to as *cyclically nested permutation domains*. Specifically, we prove that, under a mild richness condition, the cyclically nested permutation domains are exactly the domains on which the Borda count is monotonic, maintaining the assumption that all individuals face the same domain restriction.

Cyclically nested permutation domains have a more complicated structure than the cyclic permutation domains from which they are recursively constructed. This is the price to be paid when moving from the stronger condition of strategy-proofness to the less demanding condition of monotonicity. In the context of the Borda count, however, monotonicity is a particularly appealing condition since it can be defined in a natural way for social choice *correspondences* (that is, set-valued rules). By contrast, the standard definition of strategy-proofness requires a social choice function. Thus, in order to analyze strategy-proofness, the Borda count has first to be transformed into a social choice function using a tie breaking rule.

Cyclic permutation domains as well as their nested refinements are “small” in the sense that each such domain consists of only as many preference orderings as there are alternatives. More specifically, these domains have the restrictive property that for each alternative and each rank there exists exactly one preference ordering in the domain that has the given alternative at the given rank in the ordering (see Lemma 3.1). Our main characterization result can thus be viewed as the negative finding that the Borda count can be monotonic only on small domains with a very special additional structure.

There is a large literature on domain restrictions in social choice (see Gaertner [34] for a summary). Most contributions in this area, however, have studied majority voting and its generalizations, taking Black's seminal contribution on the notion of single-peaked preferences as the starting point. Some papers, such as Kalai and Muller [44] and Kalai and Ritz [45], have analyzed abstract Arrovian aggregation on restricted domains and obtained characterizations of those domains that admit possibility results. As discussed in Chapter 2, Barbie, Puppe and Tasnádi [11] obtained a characterization of the maximal domains on which the Borda count is strategy-proof. In a similar vein, Sanver [73] characterized the domains on which the plurality rule is strategy-proof, finding that only “trivial” preference domains qualify.

The closest relatives in the literature to the present chapter are Bochet and Storcken [19] and Sanver [72]. Sanver [72] investigates monotonicity of the plurality rule on re-
stricted domains in a model similar to the one used here and shows that the plurality rule can be monotonic only in trivial cases. To the best of our knowledge, Bochet and Storeken [19] is the first paper to study Maskin monotonicity on restricted preference domains in the framework of the abstract social choice model. These authors analyze both maximal strategy-proof and maximal monotonic domains for general social choice functions. However, unlike Sanver [72] and the present chapter in which every individual faces the same preference restriction, Bochet and Storeken [19] consider restrictions of the preference domain of exactly one individual. By consequence, the social choice functions found to satisfy the desired properties of strategy-proofness and monotonicity have a very special hierarchical structure and are in fact “almost” dictatorial.

3.1 Basic Definitions and Statement of Main Result

As in Chapter 2 let \( X \) be a finite universe of social states or social alternatives and let \( q \geq 2 \) be its cardinality. By \( \mathcal{P}_X \) we denote the set of all strict linear orderings (irreflexive, transitive and total binary relations) on \( X \); for simplicity, we will henceforth simply speak of linear orderings, dropping the “strict” qualification. By \( \mathcal{P} \subseteq \mathcal{P}_X \) we denote a generic subdomain of the unrestricted domain \( \mathcal{P}_X \). In contrast to SCFs social choice correspondences are not necessarily single valued.

**Definition 3.1** (Social choice correspondence). A mapping \( f : \bigcup_{n=1}^{\infty} \mathcal{P}^n \rightarrow 2^X \setminus \{\emptyset\} \) that assigns a set of (most preferred) alternatives \( f(\succ_1, \ldots, \succ_n) \in 2^X \setminus \{\emptyset\} \) to each \( n \)-tuple of linear orderings and all \( n \) is called a social choice correspondence (SCC).

Let \( rk[x, \succ] \) denote the rank of alternative \( x \) in the ordering \( \succ \) (i.e. \( rk[x, \succ] = 1 \) if \( x \) is the top alternative in the ranking \( \succ \), \( rk[x, \succ] = 2 \) if \( x \) is second-best, and so on).

**Definition 3.2** (Borda count). The SCC \( f^B \) associated with the Borda count is given as follows: for all \( n \) and all \( \succ_1, \ldots, \succ_n \in \mathcal{P}_X \) we have

\[
x \in f^B(\succ_1, \ldots, \succ_n) \Leftrightarrow \sum_{i=1}^{n} rk[x, \succ_i] \leq \sum_{i=1}^{n} rk[y, \succ_i] \quad \text{for all } y \in X.
\]

\(^1\)There is also a more distantly related literature on monotonic extensions of social choice rules. For instance, the work of Erdem and Sanver [29] is also motivated by the observation that the Borda count, and in fact any scoring method, violates the monotonicity condition on an unrestricted domain. However, the monotonic extensions are again defined on the unrestricted preference domain; therefore, the analysis does not contribute to the question on which preference domains the original (non-extended) social rule would satisfy monotonicity.
We shall denote by \( L(x, \succ) = \{ y \in X \mid x \succ y \} \) the lower contour set and by \( U(x, \succ) = \{ y \in X \mid y \succ x \} \) the upper contour set of the preference \( \succ \) at the alternative \( x \in X \). A SCC \( f \) is called monotonic on \( P \) if for all \( x \in X \), all \( n \) and all \( \succ_1, \ldots, \succ_n, \succ'_1, \ldots, \succ'_n \in P \) we have

\[
[x \in f(\succ_1, \ldots, \succ_n), \ L(x, \succ_i) \subseteq L(x, \succ'_i) \text{ for all } i = 1, \ldots, n] \Rightarrow x \in f(\succ'_1, \ldots, \succ'_n).
\]

We call a domain \( P \) Borda monotonic if \( f_B \) is monotonic on \( P \). Note that any subdomain of a (Borda) monotonic domain is (Borda) monotonic. Given a profile of preferences \((\succ_1, \ldots, \succ_n) \in P^n\), we say that alternatives \( A \subseteq X \) are indifferent on the top if \( A = f_B(\succ_1, \ldots, \succ_n) \).

We will only be interested in preference domains that are minimally rich since without such condition properties such as monotonicity or strategy-proofness can be satisfied in a trivial way. Specifically, we will impose the following condition.

**Definition 3.3 (Minimally rich domain).** A domain \( P \) is called minimally rich if, for any \( x \in X \), there exists (i) \( \succ \in P \) such that \( \text{rk}[x, \succ] = 1 \), and (ii) \( \succ' \in P \) such that \( \text{rk}[x, \succ'] = q \).

Thus, our minimal richness condition requires that each alternative must be (i) most preferred by at least one preference ordering, and (ii) least preferred by some (other) preference ordering. This is slightly stronger than the richness condition used in Chapter 2 which consisted of part (i) only.\(^3\) Part (ii) of the present condition is needed in Lemma 3.4 and in Substep 2B of the proof of Theorem 3.1

**Cyclically nested permutation domains**

We recall that an ordering \( \succ' \) is called a cyclic permutation of \( \succ \) if \( \succ' \) can be obtained from \( \succ \) by sequentially shifting the bottom element to the top while leaving the order between all other alternatives unchanged. The set of all cyclic permutations of a fixed ordering \( \succ \) is denoted by \( Z(\succ) \), which we also call a cyclic permutation domain. In Chapter 2 we have shown that the cyclic permutation domains are exactly the domains on which the Borda count is strategy-proof when combined with any conceivable deterministic

\(^2\)Obviously, every social choice function (i.e. single-valued social choice correspondence) is strategy-proof and monotonic on any domain consisting of only one preference ordering.

\(^3\)Although even weaker than the "minimal" richness condition used here, the condition in [11] is simply called “richness” there.
tie-breaking rule. The cyclic permutation domains serve as the building blocks of the so-called “cyclically nested permutation domains” to be defined presently. The following is the main result of this chapter.

**Theorem 3.1.** A domain is minimally rich and Borda monotonic if and only if it is a cyclically nested permutation domain.

Before giving a formal definition of cyclically nested permutation (henceforth, CNP) domains, we start with an intuitive example illustrating the basic recursive construction of CNP domains. A CNP domain on \( q \) alternatives consists of \( q \) preferences and can therefore be represented by a matrix that collects the preferences in its columns with the best alternative in the first row, the second-best alternative in the second row, and so on. For instance, the matrix in Table 3.1 represents a cyclic permutation domain on a set of three alternatives.

**Table 3.1: Initial step**

\[
\begin{pmatrix}
a & b & c \\
b & c & a \\
c & a & b
\end{pmatrix}
\]

We may now replace the elements of the matrix with different square matrices representing cyclical permutation domains of identical size. For instance, in Table 3.1 we may replace each element with a \( 2 \times 2 \) matrix to obtain the matrix shown in Table 3.2. Thus, we have constructed a new CNP domain on a set of six alternatives. This procedure can repeated any finite number of times, replacing at each step the elements of the given matrix with square matrices of identical size storing different cyclical permutation domains. However, in order to remain within the class of admissible CNP domains, the “replacement mechanism” has to be further restricted, as explained below.

Let us then turn to the formal definition of CNP domains. First, the cyclic permutation domains themselves are called CNP domains of depth 1. Second, we define CNP domains of depth 2, as follows. Assume that \( q = q_1q_2 \), where \( q_1, q_2 \) are two integers greater than 1. Take an arbitrary partition \( X_1, \ldots, X_{q_2} \) of \( X \) into equally sized sets (i.e., \( \#X_i = q_1 \) for all \( i = 1, \ldots, q_2 \)) and let \( X' = \{X_1, \ldots, X_{q_2}\} \). Pick a linear ordering \( \succ' \in \mathcal{P}_{X'} \) and consider

---

\[\text{4}^4\text{Combined with particular, appropriately chosen tie-breaking rules the Borda count can be strategy-proof on a larger class of domains, see Example 2.2.}\]
Table 3.2: A new CNP domain

\[
\begin{pmatrix}
  d & e & f & g & h & i \\
  e & d & g & f & i & h \\
  f & g & h & i & d & e \\
  g & f & i & h & e & d \\
  h & i & d & e & f & g \\
  i & h & e & d & g & f
\end{pmatrix}
\]

the domain \( Z(\succ') \). We now replace each set \( X_i \) with a cyclic permutation domain defined
on the set of alternatives \( X_i \) with cardinality \( q_1 \). For example, if \( q_2 = 3 \) and \( q_1 = 2 \), we
first obtain the domain at the left hand side of Table 3.3 and thereafter the domain at
the right hand side of this table. Note that each factorization of \( q \) into two factors results
in different CNP domains of depth 2; moreover, the order of the factors obviously also
matters.

To formalize the “replacement” mechanism indicated in Table 3.3 pick arbitrary linear
orderings \( \succ'_1 \in \mathcal{P}_{X_1}, \ldots, \succ'_q \in \mathcal{P}_{X_q} \). For each linear ordering \( \succ'' \in Z(\succ') \) on \( X' \) we can
construct a set of preferences \( \mathcal{P}' = \{ \succ''_1, \ldots, \succ''_q \} \) on \( X' \) such that (i) \( \{ \succ''_1|_{X_i}, \ldots, \succ''_q|_{X_i} \} = Z(\succ'_i) \), where \( \succ''_k|_{X_i} \) denotes the restriction of \( \succ''_k \) to \( X_i \), and (ii) \( X_i \succ'' X_j \) implies \( x \succ'_k y \)
for all \( i,j = 1, \ldots, q_2 \), all \( x \in X_i \), all \( y \in X_j \) and all \( k = 1, \ldots, q_1 \). Observe that by
construction any CNP domain of depth 2 on \( X \) consists of exactly \( q = \#X \) preferences.

However, in order to guarantee monotonicity of the Borda count, we must restrict the
admissible replacements by cyclical permutation domains. To see this consider the domain

Table 3.3: Constructing CNP domains

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<th>( \succ'_1 )</th>
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<th>( \succ_5 )</th>
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<td>5</td>
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shown in Table 3.4. Pick a profile \( \Pi \) consisting of one voter of each type. Then \( f^B(\Pi) = \{x_1, x_2, y_1, y_2, z_1, z_2\} \) and monotonicity is violated at alternative \( x_2 \) if, for instance, the voter of type \( \succ_3 \) switches to type \( \succ_6 \). Indeed, while \( x_2 \) improves by two ranks if the voter of type \( \succ_3 \) switches to type \( \succ_6 \), the alternative \( z_1 \) even improves by three ranks and in fact becomes the unique Borda winner.

**Table 3.4: A non-monotonic domain**

<table>
<thead>
<tr>
<th>( \succ_1 )</th>
<th>( \succ_2 )</th>
<th>( \succ_3 )</th>
<th>( \succ_4 )</th>
<th>( \succ_5 )</th>
<th>( \succ_6 )</th>
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</table>

We restrict the admissible replacements by cyclical permutation domains by specifying a set of ordered pairs \((x, y)\) of alternatives that must have the same rank difference in all preference orderings that rank \( x \) above \( y \). For example, in Table 3.3 the rank differences between \( x_1 \) and \( y_1 \) is 2 in all those preferences that rank \( x_1 \) above \( y_1 \); similarly, the rank difference between \( x_1 \) and \( z_1 \) is 4 in all preference orderings that rank \( x_1 \) above \( z_1 \). By contrast, in preference \( \succ_5 \) the alternative \( z_1 \) is one rank above \( x_1 \), while it is 3 ranks above in \( \succ_6 \). Formally, for a domain to qualify as a CNP domain there must exist, for all \( i, j \in \{1, \ldots, q_2\} \) with \( i \neq j \), bijections \( \varphi_{i,j} : X_i \to X_j \) such that, for all \( x \in X_i \), the rank difference between \( x \) and \( \varphi_{i,j}(x) \) is the same in all preference orderings that rank \( x \) above \( \varphi_{i,j}(x) \). To illustrate, consider again the CNP domain on the right hand side of Table 3.3. As is easily verified the required bijections exist in that case; for instance, \( \varphi_{1,3}(x_1) = z_1, \varphi_{1,3}(x_2) = z_2, \varphi_{3,1}(z_1) = x_2 \) and \( \varphi_{3,1}(z_2) = x_1 \). By contrast, for the domain shown in Table 3.4 there exists no bijection with the required properties between the sets \( X_3 = \{z_1, z_2\} \) and \( X_1 = \{x_1, x_2\} \), for instance.

---

5Note that this can happen only across different sets \( X_i \) and \( X_j \); indeed, within the sets \( X_i \), and generally in any cyclical permutation domain, two alternatives that have the same relative position in two different preference orderings must also have the same rank difference.

6Note, in particular, that we do not necessarily require \( \varphi_{i,j} = \varphi_{j,i}^{-1} \).
Now assume that we have defined all CNP domains of depth $n - 1$ and let $q = \prod_{i=1}^{n} q_i$ where $q_1, \ldots, q_n$ are integers greater than 1. The class of CNP domains of depth $n$ is defined as follows. Take an arbitrary partition $X_1, \ldots, X_{q_n}$ of $X$ into equally sized sets (i.e., $q' = \#X_i = q/q_n$ for all $i = 1, \ldots q_n$) and let $X' = \{X_1, \ldots, X_{q_n}\}$. Pick arbitrary CNP domains $\mathcal{P}_1 \subseteq \mathcal{P}_{X_1}, \ldots, \mathcal{P}_{q_n} \subseteq \mathcal{P}_{X_{q_n}}$ with the associated factorization $\prod_{i=1}^{n-1} q_i$, a linear ordering $\succ' \in \mathcal{P}_{X'}$, and consider the domain $Z(\succ')$. For each linear ordering $\succ'' \in Z(\succ')$ on $X'$ construct a set of preferences $\mathcal{P}' = \{\succ''_1, \ldots, \succ''_{q'}\}$ on $X$ such that (i) $\{\succ''_{\phi_i[X_i]} : \phi_i \in \mathcal{P}_i\} = \mathcal{P}_i$, (ii) $X_i \succ'' x_j$ implies $x \succ''_i y$ for all $i, j = 1, \ldots, q_n$, all $x \in X_i$, all $y \in X_j$ and all $k = 1, \ldots, q'$, and (iii) there exist, for all $i, j \in \{1, \ldots, q_n\}$ with $i \neq j$, bijections $\phi_{i,j} : X_i \to X_j$ such that, for all $x \in X_i$, $x$ and $\phi_{i,j}(x)$ have the same rank differences in all preferences of the form $\succ''_k$ that rank $x$ above $\phi_{i,j}(x)$, for all $\succ'' \in Z(\succ')$ and all $k$.

Observe that, by construction, any CNP domain on $X$ consists of exactly $q = \#X$ preferences. Furthermore, one can easily determine the maximal depth of a CNP domain on a given number of alternatives, as follows. Suppose that the prime factorization of $q$ takes the form $q = \prod_{i=1}^{k} p_i^{m_i}$, where $p_i$ are primes and $m_i$ are positive integers for all $i = 1, \ldots, k$; moreover, let $D_q := \sum_{i=1}^{k} m_i$. Then, the maximal depth of a CNP domain on $q$ alternatives is $D_q$. In particular, if $q$ is a prime, only the cyclic permutation domains themselves qualify as CNP domains.

We provide an example of a CNP of depth 3 with $q_1 = 2$, $q_2 = 3$ and $q_3 = 2$ to further illustrate the definition of CNP domains. The first domain is a cyclical permutation domain defined on two sets of alternatives as shown in Table 3.5. Let $X''_1 = \{x_1, x_2, \ldots, x_6\}$, $X''_2 = \{x_7, x_8, \ldots, x_{12}\}$, $\phi_{1,2}(x_i) = x_{i+6}$ for $i = 1, \ldots, 6$, and $\phi_{2,1} = \phi_{1,2}^{-1}$. Next, we replace each set $X''_1$ and $X''_2$ with a CNP domain of depth 2 and associated factorization $2 \cdot 3$. We derive these two CNP domains simultaneously in Table 3.6. Furthermore, we select the partition $X'_1 = \{x_1, x_2\}$, $X'_2 = \{x_3, x_4\}$, $X'_3 = \{x_5, x_6\}$ of $X''_1$ and the partition $X'_4 = \{x_7, x_8\}$, $X'_5 = \{x_9, x_{10}\}$, $X'_6 = \{x_{11}, x_{12}\}$ of $X''_2$.

The sets $X'_1, \ldots, X'_6$ are then replaced by cyclical permutation domains each defined
the bijections

3.7. Finally, inserting the two CNP domains of Table 3.7 into Table 3.5 and employing

In an analogous way, we construct the CNP domain shown at the right hand side of Table 3.7. Finally, inserting the two CNP domains of Table 3.7 into Table 3.5 and employing the bijections \( \varphi_{1.2} \) and \( \varphi_{2.1} \), we obtain the CNP domain of depth 3 shown in Table 3.8.

\[
\begin{array}{cccccc}
\succ'_1 & \succ'_2 & \succ'_3 & \succ'_4 & \succ'_5 & \succ'_6 \\
X'_1 & X'_2 & X'_3 & X'_4 & X'_5 & X'_6 \\
X'_2 & X'_3 & X'_1 & X'_5 & X'_6 & X'_4 \\
X'_3 & X'_1 & X'_2 & X'_6 & X'_4 & X'_5 \\
\end{array}
\]

on the respective sets of two alternatives. Specifically, we replace \( X_i \) by \( \mathcal{Z}(x_{2i-1} \succ x_{2i}) \) for all \( i = 1, \ldots, 6 \), and we choose bijections such that \( \varphi'_{1.2}(x_1) = x_3, \varphi'_{1.2}(x_2) = x_4, \varphi'_{1.3}(x_1) = x_5, \varphi'_{1.3}(x_2) = x_6, \varphi'_{2.3}(x_3) = x_5, \varphi'_{2.3}(x_4) = x_6, \varphi'_{2.1} = (\varphi_{1.2})^{-1}, \varphi'_{3.1} = (\varphi_{1.3})^{-1}, \varphi'_{3.2} = (\varphi_{2.3})^{-1} \). We thus obtain the CNP domain shown at the left hand side of Table 3.7.

We conclude this section with a simple necessary condition for a domain to qualify as a CNP domain.

**Lemma 3.1.** Any CNP domain \( \mathcal{P} \) on \( X \) consists of exactly \( q \) preferences, and for all \( x \in X \) and all \( i \in \{1, \ldots, q\} \) there exists exactly one preference \( \succ_i \in \mathcal{P} \) such that \( \text{rk} [x, \succ_i] = i \).

**Proof.** The statement can be established by induction on the depth of CNP domains. Cyclical permutation domains clearly satisfy the stated property. Assume that the statement holds for all CNP domains of depth \( n - 1 \). Take a CNP domain \( \mathcal{P} \) of depth \( n \) that
Table 3.8: A CNP domain of depth 3

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<th>$\succ_1$</th>
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is constructed from a cyclical permutation domain on $X' = \{X_1, \ldots, X_q\}$ (where the $X_i$ are sets of equal size) and from CNP domains of depth $n - 1$ replacing each set $X_i$. The stated property follows at once from the induction hypothesis for the CNP domains of depth $n - 1$ and the structure of a cyclical permutation domain.

Note that, while any subdomain of a CNP domain is Borda monotonic, no proper subdomain of a CNP domain can be minimally rich by Lemma 3.1. Moreover, we have the following corollary.

**Corollary 3.1.** If $P = \{\succ_1, \ldots, \succ_q\}$ is a CNP domain on $X$, then $f^B(\succ_1, \ldots, \succ_q) = X$, i.e. all alternatives are indifferent on the top if each preference occurs exactly once in a profile.

### 3.2 Characterizing and Detecting CNP Domains

We characterized cyclical permutation domains in Chapter 2 by the equal rank difference condition, which we recall here for minimally rich domains.

**Definition 3.4** (Equal rank difference for minimally rich domains). A minimally rich
domain $\mathcal{P}$ satisfies the equal rank difference (henceforth, ERD) condition if for all $x, y \in X$, all $\succ, \succ' \in \mathcal{P}$

$$(x \succ y \text{ and } x \succ' y) \Rightarrow rk[x, \succ] - rk[y, \succ] = rk[x, \succ'] - rk[y, \succ'].$$

Considering the restricted rank difference condition associated with the bijections $\varphi_{i,j} : X_i \to X_j$ in the recursive definition of CNP domains, an appropriate weakening of the ERD condition leads to a characterization of CNP domains.

**Definition 3.5** (Nested equal rank difference). A domain $\mathcal{P}$ satisfies the nested equal rank difference (henceforth, NERD) condition if there exists a “nested” set system

$$\{X, X_{i_1}, X_{i_2}, \ldots, X_{i_{n-l}}, X_{i_{n-l+1}}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_{n-1}}, X_{i_n} \}_{i_n = 1, \ldots, i_1 = 1}^{q_n; \ldots; q_1}$$

on $X$ such that

1. $q = \prod_{i=1}^{n} q_i$, where $q_i$ are integers greater than 1 for all $i = 1, \ldots, n$;
2. $X = \bigcup_{i=1}^{n} X_i$, $\#X_i = q/q_n$ for all $i = 1, \ldots, q_n$;
   - (a) for all $\succ \in \mathcal{P}$, all $x \in X_i$ and all $y \in X_j$ we have that $x \succ y$ implies $x' \succ y'$ for all $x' \in X_i$ and all $y' \in X_j$;
   - (b) there exist bijections $\varphi_{i,j}^n : X_i \to X_j$ for all $i, j = 1, \ldots, q_n$, $i \neq j$ such that $x \succ y = \varphi_{i,j}^n(x)$ and $x \succ' y \Rightarrow rk[x, \succ] - rk[y, \succ] = rk[x, \succ'] - rk[y, \succ']$
      for all $x \in X_i$ and all $\succ, \succ' \in \mathcal{P}$;
3. for all $l = 2, \ldots, n$ and all $(i_l, i_{l+1}, \ldots, i_n) \in \times_{k=l}^{n} \{1, \ldots, q_k\}$ we have
   - (a) $X_{i_l;i_{l+1},\ldots,i_{n-1},i_n} = \bigcup_{i_{l-1}=1}^{q_{l-1}} X_{i_{l-1};i_l;i_{l+1},\ldots,i_{n-1},i_n}$,
   - $\#X_{i_l;i_{l+1},\ldots,i_{n-1},i_n} = \prod_{k=l}^{n} q_k$ for all $i_{l-1} = 1, \ldots, q_{l-1}$;
   - (b) for all $\succ \in \mathcal{P}$, all $i, j = 1, \ldots, q_{l-1}$, $i \neq j$, all $x \in X_{i_l;i_l,\ldots,i_{n-1},i_n}$ and all $y \in X_{j_l;i_l,\ldots,i_{n-1},i_n}$ we have that $x \succ y$ implies $x' \succ y'$ for all $x' \in X_{i_l;i_l,\ldots,i_{n-1},i_n}$ and $y' \in X_{j_l;i_l,\ldots,i_{n-1},i_n}$;
   - (c) there exists bijections $\varphi_{i,j}^{l-1} : X_{i_l;i_l,\ldots,i_{n-1},i_n} \to X_{j_l;i_l,\ldots,i_{n-1},i_n}$ for all $i, j = 1, \ldots, q_{l-1}$, $i \neq j$ such that $x \succ y = \varphi_{i,j}^{l-1}(x)$ and $x \succ' y \Rightarrow rk[x, \succ] - rk[y, \succ] = rk[x, \succ'] - rk[y, \succ']$
      for all $x \in X_{i_l;i_l,\ldots,i_{n-1},i_n}$ and all $\succ, \succ' \in \mathcal{P}$.
Observe that a minimally rich domain $\mathcal{P}$ satisfies ERD if and only if it satisfies NERD on the nested set system $\{X, \{x_1\}, \{x_2\}, \ldots, \{x_q\}\}$. Thus, ERD is a special case of NERD.

**Proposition 3.1.** $\mathcal{P}$ is a CNP domain if and only if $\mathcal{P}$ is minimally rich and satisfies NERD.

**Proof.** Assume that $\mathcal{P}$ is a CNP domain. We prove that $\mathcal{P}$ satisfies NERD by induction on the depth of CNP domains. By the analysis of Chapter 2, cyclic permutation domains satisfy ERD. Thus, suppose that all CNP domains of depth $n - 1$ satisfy NERD. Pick a CNP domain $\mathcal{P}$ on $X$ of depth $n$ with the associated factorization $q = \prod_{k=1}^{n} q_k$, which already determines the factors for Point 1 in the definition of NERD. By the definition of CNP domains one obtains $\mathcal{P}$ from a cyclical permutation domain on $X' = \{X_1, \ldots, X_q\}$, where the sets of $X'$ partition $X$ into $q_n$ equally sized sets. These $q_n$ sets deliver us the single indexed sets of the required nested set system. It follows from the replacement mechanism of the definition of CNP domains that any preference in $\mathcal{P}$ ranks either any alternative from $X_i \in X'$ higher than any alternative from $X_j \in X'$, or any alternative from $X_i \in X'$ lower than any alternative from $X_j \in X'$; this guarantees Point 2(a) of the NERD conditions. Moreover, each set $X_i \in X'$ has to be replaced with a CNP domain on $X_i$ with the associated factorization $\prod_{k=1}^{n-1} q_k$ such that for fixed bijections $\varphi_{i,j} : X_i \to X_j$ we have that $x \in X_i$ and $\varphi_{i,j}(x)$ maintain their rank differences for all $x \in X_i$ whenever $x$ is ranked above $\varphi_{i,j}(x)$, which ensures Point 2(b) of the definition of NERD. By employing the induction hypothesis, NERD is satisfied by all CNP domains of depth $n - 1$ on $X_i \in X'$. Thus, there exist nested set systems

$$\{X_i, X_{i_{n-1},i}, \ldots, X_{i_2, i_{n-1},i}, X_{i_1, i_2, \ldots, i_{n-1},i}\}_{i_{n-1}=1, \ldots, i_1=1}^{q_n=1, \ldots, q_1}$$

on $X_i$ such that any CNP domain on $X_i \in X'$ satisfies Points 1-3 of the definition of NERD for all $i = 1, \ldots, q_n$. Taking the union of these $q_n$ nested set systems and $X$, we obtain the required nested set system for the domain $\mathcal{P}$.

To prove the converse statement take a minimally rich domain $\mathcal{P}$ satisfying NERD. We show that $\mathcal{P}$ is a CNP domain by induction on $n$ in the definition of NERD. For $n = 1$, NERD boils down to ERD, and therefore $\mathcal{P}$ is a cyclic permutation domain. Assume that a minimally rich domain $\mathcal{P}$ is a CNP domain if it satisfies NERD with $n - 1$, and consider a minimally rich domain $\mathcal{P}$ satisfying NERD for a nested set system with $n$ indices. Point 1 in the definition of NERD delivers us the associated factorization of $\mathcal{P}$ and Point 2 determines the appropriate partition $X' = \{X_1, \ldots, X_q\}$ of $X$. Define the
domain $\mathcal{P}' \subseteq \mathcal{P}_X$, by requiring for all $\succ' \in \mathcal{P}'$ that

$$Y_1 \succ' \ldots \succ' Y_{q_n} \iff \exists \succ \in \mathcal{P}, \forall y_1 \in Y_1, \ldots, \forall y_{q_n} \in Y_{q_n}, y_1 \succ \ldots \succ y_{q_n},$$

where $Y_i \in X'$ for all $i = 1, \ldots, q_n$ and $\bigcup_{i=1}^{q_n} Y_i = X$. Observe that if there exists $y_1 \in Y_1, \ldots, y_{q_n} \in Y_{q_n}$ such that $y_1 \succ \ldots \succ y_{q_n}$, then we must have $y'_1 \succ \ldots \succ y'_{q_n}$ for all $y'_1 \in Y_1, \ldots, y'_{q_n} \in Y_{q_n}$ by Point 2(a). Moreover, the non-emptiness of $\mathcal{P}$ implies the non-emptiness of $\mathcal{P}'$. Since by Point 2(b) there exist bijections $\varphi_{i,j}^n : Y_i \to Y_j$ such that for all $y \in Y_i$ and all $\succeq, \succ' \in \mathcal{P}$ if $y \succ z = \varphi_{i,j}^n(y)$ and $y \succ' z$, then

$$d_{y,z} = rk [y, \succeq] - rk [z, \succeq] = rk [y, \succ'] - rk [z, \succ'],$$

we must have by the definition of $\mathcal{P}'$ that

$$\frac{d_{y,z}}{q/q_n} = rk [Y_i, \succeq^*] - rk [Y_j, \succeq^*] = rk [Y_i, \succeq^{**}] - rk [Y_j, \succeq^{**}]$$

whenever $Y_i \succeq^* Y_j$ and $Y_i \succeq^{**} Y_j$, where $y \in Y_i$ and $z = \varphi_{i,j}^n(y)$. Hence, $\mathcal{P}'$ satisfies ERD, and from the minimal richness condition it follows that $\mathcal{P}' \subseteq \mathcal{P}_X$ has to be a cyclic permutation domain. Moreover, by Point 3 the distinct nested set systems

$$\{X_i, X_{i_{n-1},i}, \ldots, X_{i_2,\ldots,i_{n-1},i}, X_{i_1,i_2,\ldots,i_{n-1},i}\}_{i_{n-1}=1,\ldots,i_1=1}$$

satisfy NERD for all $i = 1, \ldots, q_n$ with the same associated factorization $\prod_{k=1}^{q_n-1} q_k$. This means by the induction hypothesis that $\mathcal{P}_i \subseteq \mathcal{P}_X$, form CNP domains with associated factorizations $\prod_{k=1}^{q_n-1} q_k$ for all $i = 1, \ldots, q_n$. Finally, Point 2 in the definition of NERD assures that $\mathcal{P}$ has to be obtained from the CNP domains on $X_i$ in line with the replacement mechanism specified in the definition of CNP domains, which completes the proof. \hfill \Box

Now we outline a procedure deciding whether a given domain $\mathcal{P}$ on $X$ is a CNP domain.

1. If $#X \neq #\mathcal{P}$, then $\mathcal{P}$ is not a CNP domain.

2. Pick a preference $\succ_1 \in \mathcal{P}$ and label the alternatives so that $rk [x_i, \succ_1] = i$. Next, label the remaining preferences so that $rk [x_1, \succ_i] = i$. If this cannot be done, then $\mathcal{P}$ cannot be a CNP domain by Lemma 3.1.

3. Step 3 aims to determine the associated factorization of $\mathcal{P}$. Let $r = \min \{i \in \{2, \ldots, q\} | rk [x_i, \succ_2] = 1\}$, $n = 1$, $q_1 = r$.

   (a) We have determined the associated factorization if $r = q$. 

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(b) If \( r \) is not a divisor of \( q \), then \( \mathcal{P} \) cannot be a CNP domain.

(c) If \( r \) is a proper divisor of \( q \), then let \( s = \min \{ i \in \{2r, 3r, \ldots, q\} \mid x_i \succ_{r+1} x_1 \} \), increase \( n \) by 1, let \( q_n = s/r \), \( r = s \) and return to Substep (a).

4. Based on the factorization obtained in Step 3, we determine the nested set system by considering preference \( \succ_1 \) for which we have \( x_1 \succ_1 x_2 \succ \ldots \succ x_{q_n} \). First, let \( q' = q/q_n \) and let the single indexed sets be \( X_1 = \{x_1, x_2, \ldots, x_{q'}\} \), \( X_2 = \{x_{q'+1}, x_{q'+2}, \ldots, x_{2q'}\} \), \( \ldots \), \( X_{q_n} = \{x_{q-q'+1}, x_{q-q'+2}, \ldots, x_q\} \). To determine the double indexed sets we have to partition the single indexed sets by taking consecutive sequences of length \( q/(q_{n-1}q_n) \) from the sequence \( x_1, x_2, \ldots, x_q \). We have to proceed in a similar way to obtain the remaining sets. More formally, for all \( k = 0, \ldots, n-1 \), let \( q_k^* = \prod_{l=1}^{k} q_l \) and for all \( k = 1, \ldots, n \), all \( (i_k, \ldots, i_n) \in \times_{l=k}^{n} \{1, \ldots, q_l\} \), let \( j = \left( \sum_{l=k}^{n} (i_l - 1)q_l^* \right) + 1 \) and \( X_{i_k,i_{k+1},\ldots,i_n} = \{x_j, x_{j+1}, \ldots, x_{j+q_n^*-1}\} \).

5. It is straightforward to check whether the NERD conditions 2(a) and 3(b) are satisfied.

6. To find appropriate bijections \( \varphi_{i,j}^{l-1} : X_{i,i_l,\ldots,i_{n-1},i_n} \rightarrow X_{j,j_l,\ldots,j_{n-1},j_n} \) that satisfy NERD conditions 2(b) and 3(c) pick for each \( x \in X_{i,i_l,\ldots,i_{n-1},i_n} \) the preference \( \succ \in \mathcal{P} \) ranking \( x \) on top, then the highest ranked alternative \( y \) out of \( X_{j,j_l,\ldots,j_{n-1},j_n} \) by \( \succ \), and let \( \varphi_{i,j}^{l-1}(x) = y \). If this cannot be done, \( \mathcal{P} \) fails to be a minimally rich domain, and thus, to be a CNP domain. If \( \varphi_{i,j}^{l-1} \) is not a bijection, then \( \mathcal{P} \) cannot be a CNP domain. Otherwise, verify NERD conditions 2(b) and 3(c).

Clearly, only CNP domains are accepted by the above procedure. However, that any CNP domain is accepted by the procedure is less obvious. CNP domains pass Steps 1 and 2 by Lemma 3.1. Assume that we have labeled the preferences of \( \mathcal{P} \) according to Step 2. We verify by induction on \( n \) in the definition of NERD that \( \mathcal{P} \) passes Steps 3-6. If \( n = 1 \), then \( \mathcal{P} \) is a cyclic permutation domain, and thus, \( r = q \) by Step 3. Moreover, Step 4 determines the nested set system \( \{X, \{x_1\}, \ldots, \{x_q\}\} \) associated with cyclic permutation domains. Since \( \mathcal{P} \) satisfies NERD, it also passes Steps 5 and 6. Assume that the procedure works well for nested set systems with \( n-1 \) factors and take a domain \( \mathcal{P} \) needing \( n \) factors. Let \( X' = \{X_1, \ldots, X_{q_n}\} \) be the set of the single labeled sets from the given nested set system and assume that the sets are labeled in a way such that \( X_1 = \{x_1, \ldots, x_{q/q_n}\} \), \( \ldots \), \( X_{q_n} = \{x_{q-q/q_n+1}, \ldots, x_q\} \). Define \( \succ' \in \mathcal{P}_{X'} \) by

\[
X_1 \succ' X_2 \succ' \ldots \succ' X_{q_n} \iff x_1 \succ_{q/q_n} x_1 \succ_{q/q_n+1} \ldots \succ_{q-q/q_n+1} x_{q-q/q_n+1},
\]

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which is well defined by Point 2 of NERD. Moreover, by Point 2 of NERD we obtain $\mathcal{P}$ from $\mathcal{Z}(\succ')$, and therefore, only the alternatives from $X_{q_n}$ are ranked higher than $x_1$ by $\succ_{q/q_n+1}$. Hence, Step 3 determines the last factor $q_n$ correctly. Finally, the induction hypothesis assures that Steps 3-6 work correctly.

3.3 Proper Scoring Methods on CNP Domains

We recall the definition of a scoring method from Chapter 2:\footnote{Let $s : \{1, \ldots, q\} \to \mathbb{R}$ satisfy $s(1) \geq s(2) \geq \ldots \geq s(q)$ and $s(1) > s(q)$. The SCC $f^s$ associated with the scoring method specified by $s$ is given by

$$x \in f^s(\succ_1, \ldots, \succ_n) \iff \sum_{i=1}^n s(\delta k[x, \succ_i]) \geq \sum_{i=1}^n s(\delta k[y, \succ_i]) \ \text{for all } y \in X$$

for all $n$ and all $\succ_1, \ldots, \succ_n \in \mathcal{P}_X$. A scoring method is called \emph{proper} if $s$ is strictly decreasing.

Next we define a nested extension of the Borda count.

\textbf{Definition 3.6 (Nested Borda count).} Assume that there are $n$ given integers $q_k \geq 2$ for all $k = 1, \ldots, n$ such that $q = \prod_{k=1}^n q_k$ and $n$ positive reals $\delta_1, \ldots, \delta_n$ such that $\delta_{i+1} \geq \delta_i q_i$ for all $i = 1, \ldots, n-1$. Let $q_k^* = \prod_{i=1}^k q_i$ for all $k = 0, \ldots, n-1$; and let $j_{i_k, \ldots, i_n} = (\sum_{i=k}^n (i_i - 1)q_i) + 1$ for all $k = 1, \ldots, n$, all $(i_k, \ldots, i_n) \in \times_{l=k}^n \{1, \ldots, q_l\}$. We call a proper scoring method a \emph{Nested Borda count} with the associated factors $q_1, \ldots, q_n$ if the score function $s$ satisfies $s(j_{i_k, \ldots, i_n}) = s(j_{i_{k+1}, \ldots, i_n}) = \delta_k$ for all $i = 1, \ldots, q_k - 1$, all $(i_{k+1}, \ldots, i_n) \in \times_{l=k+1}^n \{1, \ldots, q_l\}$ and all $k = 1, \ldots, n$.

It can be verified that a Nested Borda count boils down to the Borda count if $\delta_{i+1} = \delta_i q_i$ for all $i = 1, \ldots, n-1$ or if $n = 1$. In particular, for any factorization $q = \prod_{k=1}^n q_k$ we obtain the score function of the Borda count if $\delta_1 = 1$ and $\delta_{i+1} = \delta_i q_i$ for all $i = 1, \ldots, n-1$. Informally, the Borda count on $q$ alternatives can be obtained through all factorizations of $q$.

If one considers the nested set system consisting of sets

$$X_{i_k, i_{k+1}, \ldots, i_n} = \{j_{i_k, i_{k+1}, \ldots, i_n}, \ldots, j_{i_k, i_{k+1}, \ldots, i_n} + q_{k-1}^* - 1\}$$

for all $(i_k, \ldots, i_n) \in \times_{l=k}^n \{1, \ldots, q_l\}$ and all $k = 1, \ldots, n$, then any positive integer $j = 1, \ldots, q$ uniquely specifies indices $i_1, i_2, \ldots, i_n$ such that $j \in X_{i_k, i_{k+1}, \ldots, i_n}$ for all $k = 1, \ldots, n$. Taking for all $k = 1, \ldots, n$ the Borda score functions $s_k(i) = q_k + 1 - i$ on $q_k$ alternatives, we
can obtain the score function \( s \) of a Nested Borda count by \( s(j) = (\sum_{k=1}^{n} \delta_k (s_k(i_k) - 1)) + \alpha \), where \( \alpha \in \mathbb{R} \). This is the reason why we refer to the above defined class of scoring methods as Nested Borda counts.

**Proposition 3.2.** Let \( \mathcal{P} \) be a CNP domain with associated factorization \( q = \prod_{k=1}^{n} q_k \). A proper scoring method is monotonic on \( \mathcal{P} \) if and only if it is a Nested Borda count with associated factors \( q_1, \ldots, q_n \).

**Proof.** Since \( \mathcal{P} \) is a CNP domain, \( \mathcal{P} \) satisfies NERD. Thus, we can consider the corresponding nested set system

\[
\{X, X_{i_n}, X_{i_{n-1},i_n}, \ldots, X_{i_2,i_{n-1},i_n}, X_{i_1,i_2,i_{n-1},i_n}\}^{q_1,\ldots,q_n}_{i_n=1,\ldots,i_1=1},
\]

and let \( q_k = \prod_{i=1}^{k} q_i \) for all \( k = 0, \ldots, n - 1 \).

Suppose that the proper scoring method \( s \) is not a Nested Borda count, and therefore, there exists a smallest \( k \in \{1, \ldots, n\} \) such that

\[
d_1 = s(j_{i+1,i_{k+1},\ldots,i_n}) - s(j_{i,i_{k+1},\ldots,i_n}) \neq d_2 = s(j_{i+2,i_{k+1},\ldots,i_n}) - s(j_{i+1,i_{k+2},\ldots,i_n})
\]

for some \( i \in \{1, \ldots, q_k - 2\} \) and \( (i_{k+1}, \ldots, i_n) \in \times_{j=k+1}^{n} \{1, \ldots, q_j\} \). Take a profile \( \Pi \) containing exactly one voter of each type, for instance, let \( \Pi = (\succ_1, \ldots, \succ_q) \). Hence, \( f^s(\Pi) = X \) by Lemma 3.1. By the NERD conditions we can pick two distinct sets

\[
X_{j,i_{k+1},\ldots,i_{n-1},i_n} \quad \text{and} \quad X_{j',i_{k+1},\ldots,i_{n-1},i_n}
\]

for which there exist two sets of preferences \( \mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{P} \) such that \( \#\mathcal{P}_1 = \#\mathcal{P}_2 = q_k^{-1}, \)

\[
\begin{align*}
\hat{j}_{i,k+1},\ldots,i_n &\leq r_k[x,\succ] < \hat{j}_{i+1,k+1},\ldots,i_n, \\
\hat{j}_{i+1,k+1},\ldots,i_n &\leq r_k[y,\succ] < \hat{j}_{i+2,k+1},\ldots,i_n, \\
\hat{j}_{i+1,k+1},\ldots,i_n &\leq r_k[x,\succ'] < \hat{j}_{i+2,k+1},\ldots,i_n, \\
\hat{j}_{i+2,k+1},\ldots,i_n &\leq r_k[y,\succ'] < \hat{j}_{i+2,k+1},\ldots,i_n + q_k^{-1}
\end{align*}
\]

for all \( x \in X_{j,i_{k+1},\ldots,i_{n-1},i_n}, \) all \( y \in X_{j',i_{k+1},\ldots,i_{n-1},i_n}, \) all \( \succ \in \mathcal{P}_1 \) and all \( \succ' \in \mathcal{P}_2 \). We construct profile \( \Pi' \) from profile \( \Pi \) by replacing those voters’ preferences having preferences in \( \mathcal{P}_2 \) by preferences in \( \mathcal{P}_1 \) appropriately. More specifically, if \( \succ \in \mathcal{P}_2 \), then \( \succ \) has to be replaced with the preference \( \succ' \in \mathcal{P}_1 \) satisfying

\[
\succ|X_{j,i_{k+1},\ldots,i_{n-1},i_n} \cup X_{j',i_{k+1},\ldots,i_{n-1},i_n} = \succ'|X_{j,i_{k+1},\ldots,i_{n-1},i_n} \cup X_{j',i_{k+1},\ldots,i_{n-1},i_n},
\]

which can be done by the NERD condition. Moreover, the NERD condition guarantees that profiles \( \Pi \) and \( \Pi' \) satisfy the precondion of monotonicity at any alternative.
Chapter 3. Nash implementionable domains for the Borda count

We only consider case (a) since case (b) can be established in an analogous way. Let $x' = \varphi_{i,j}^{-1}(y)$, $d = s \left( r_k[k, x'] \right) - s \left( r_k[y, x'] \right) = s \left( r_k[x', y'] \right) - s \left( r_k[y, y'] \right)$, $d_1 = s \left( r_k[x, y] \right) - s \left( r_k[y, y'] \right)$ and $d_2 = s \left( r_k[x, y'] \right) - s \left( r_k[y, y'] \right)$. We cannot have $x \triangleright x'$ and $x' \triangleright x$,
since this would violate \( L(x, \succ) \subseteq L(x, \succ') \). Moreover, \( x' \succ x \) and \( x \succ' x' \) cannot be the case, since this would imply \( d_1 < d < d_2 \), which is in contradiction with \( d_1 > d_2 \). The remaining two subcases \( x \succ x' \) and \( x \succ' x' \), and \( x' \succ x \) and \( x' \succ' x \) would imply the non-monotonicity of \( f' \) on a CNP domain of depth \( n - 1 \) and identical factorization as \( s' \) on \( X_i \); a contradiction; which completes the proof of our proposition.

\[ \square \]

**Corollary 3.2.** Within the class of proper scoring methods only the Borda count can be monotonic on all CNP domains.

### 3.4 Some Useful Lemmas

For the proof of Theorem 3.1, we need a series of lemmas some of which are interesting on their own right. If there are \( k \) given preferences \( \succ_1, \ldots, \succ_k \in \mathcal{P} \) and \( k \) given positive integers \( n_1, \ldots, n_k \), then we shall denote by \( \Pi = (n_1 \cdot \succ_1, \ldots, n_k \cdot \succ_k) \) a preference profile in which the first \( n_1 \) voters have preference \( \succ_1 \), the next \( n_2 \) voters have preference \( \succ_2 \) and so on.

**Lemma 3.2.** Suppose that \( \mathcal{P} \) is a minimally rich domain. If there exist two distinct preferences \( \succ, \succ' \in \mathcal{P} \) and two alternatives \( x, y \in X \) satisfying \( r_k[x, \succ] = 1 \), \( r_k[y, \succ] = 2 \) and \( d := r_k[y, \succ'] - r_k[x, \succ'] \geq 2 \), then Borda count violates monotonicity on \( \mathcal{P} \).

**Proof.** Let \( \succ'' \in \mathcal{P} \) be a preference with top alternative \( y \), \( d' = r_k[x, \succ''] - r_k[y, \succ'' \] and \( k = \lceil \frac{2d' + 1}{d + 1} \rceil \). We consider the following two profiles of \( k(d' + 1) - 1 \) individuals: \( \Pi = ((kd' - 1) \cdot \succ, k \cdot \succ'') \) and \( \Pi' = ((kd' - 3) \cdot \succ, 2 \cdot \succ', k \cdot \succ'') \). Observe that the Borda score of \( y \) is greater than that of \( x \) by \( 1 \) in profile \( \Pi \) and since \( y \) dominates the remaining alternatives we have \( f^B(\Pi) = \{ y \} \). For profile \( \Pi' \) the choice of \( k \) assures\(^8\) that \( x \) and \( y \) receive higher Borda scores than any other alternative. Thus, \( f^B(\Pi') = \{ x \} \) by the assumptions imposed on \( \succ \) and \( \succ' \). Finally, the precondition of monotonicity for the alternative \( x \) is satisfied as we switch from \( \Pi' \) to \( \Pi \), but \( y \) becomes the Borda winning alternative in \( \Pi \).

\[ \square \]

**Lemma 3.3.** If \( \mathcal{P} \) is a Borda monotonic minimally rich domain, then for any two preferences in \( \mathcal{P} \) having the same top alternative the second ranked alternatives have to be identical.

\(^7\)In what follows \( \lfloor x \rfloor \) stands for the largest integer not greater than \( x \) and \( \lceil x \rceil \) stands for the smallest integer not less than \( x \).

\(^8\)Any larger integer for \( k \) does the job.
Proof. Suppose there are preferences $\succ, \succ' \in \mathcal{P}$ such that $rk[x, \succ] = 1$, $rk[y, \succ] = 2$, $rk[x, \succ'] = 1$, $rk[z, \succ'] = 2$ and $y \neq z$. Then $rk[y, \succ'] > 2$ and Lemma 3.2 applies. □

Lemma 3.4. If $\mathcal{P}$ is a Borda monotonic minimally rich domain, then

$$\pi(x) = \{y \in X \mid \exists \succ \in \mathcal{P} \text{ such that } rk[x, \succ] = 1 \text{ and } rk[y, \succ] = 2\}$$

defines a one-to-one correspondence (permutation) on $X$.

Proof. The statement is obviously true in case of $q \leq 3$. Therefore, we only have to consider the case of $q > 3$. Suppose that $x$ is ranked first by $\succ \in \mathcal{P}$ and ranked second by $\succ', \succ'' \in \mathcal{P}$. We shall denote the top alternatives of $\succ', \succ'' \in \mathcal{P}$ by $y$ and $z$, respectively. Any $\succ^* \in \mathcal{P} \setminus \{\succ, \succ', \succ''\}$ has to rank $y$ or $z$ lower than $x$; since otherwise, $y$ and $x$ or $z$ and $x$ violate Lemma 3.2. Hence, $\mathcal{P}$ cannot satisfy part (ii) of the minimal richness condition, a contradiction. □

Lemma 3.5. Suppose that $\mathcal{P}$ is a Borda monotonic minimally rich domain. Then we cannot find two distinct preferences $\succ, \succ' \in \mathcal{P}$ and an alternative $y \in X$ such that

- $rk[y, \succ] > 2$
- \begin{align*}
  U(y, \succ) &= U(y, \succ'), \\
  \forall x \in U(y, \succ) : rk[x, \succ] &\neq rk[x, \succ'].
\end{align*}

Proof. Suppose that there exist two distinct preferences $\succ, \succ' \in \mathcal{P}$ and an alternative $y \in X$ such that $d = rk[y, \succ] > 2$, $U(y, \succ) = U(y, \succ')$ and $rk[x, \succ] \neq rk[x, \succ']$ for all $x \in U(y, \succ)$. Let $\succ'' \in \mathcal{P}$ be a preference with top alternative $y$ and $U(y, \succ) = \{x_1, \ldots, x_{d-1}\}$. Observe that $y$ dominates all alternatives in $X \setminus U(y, \succ)$ in all profiles consisting only of preferences $\succ, \succ'$ and $\succ''$. We define values $d_m = 2d - rk[x_m, \succ] - rk[x_m, \succ']$ and $d_m'' = rk[x_m, \succ''] - 1$ for all $m \in \{1, \ldots, d-1\}$. Now let $S = \arg\min_{s \in \{1, \ldots, d-1\}} \frac{d_s'}{d_s''}$ and $A = \{x_s \in X \mid s \in S\}$. For any $s \in S$ it can be verified that a profile consisting of $d_s''$ preferences of type $\succ$, $d_s'$ preferences of type $\succ'$ and $d_s$ preferences of type $\succ''$ makes alternatives $\{y\} \cup A$ indifferent on the top. Let $\Pi = (d_s' \succ, d_s' \succ', d_s \succ'')$. Hence, $f^B(\Pi) = \{y\} \cup A$.

First, if there exists an $s \in S$ such that $x_s$ is ranked higher in $\succ'$ than in $\succ$, then pick an arbitrary alternative $x_m \in A$ achieving the highest rank increase by replacing one voter of type $\succ$ with one voter of type $\succ'$. In this case we construct $\Pi'$ from $\Pi$ by replacing one preference $\succ$ with one preference $\succ'$. It can be checked that $y \notin f^B(\Pi')$,
while \( x_m \in f^B(\Pi') \). Second, if for all \( s \in S \) we have that \( x_s \) is ranked higher in \( \succ \) than in \( \succ' \), then pick an arbitrary alternative \( x_m \in A \) achieving the highest rank decrease from \( \succ \) to \( \succ' \). In this second case we construct \( \Pi' \) from \( \Pi \) by replacing one preference \( \succ' \) with one preference \( \succ \). Again, we have \( y \notin f^B(\Pi') \), while \( x_m \in f^B(\Pi') \). We obtained in both cases a violation of monotonicity at \( y \); a contradiction.

**Lemma 3.6.** Let \( \mathcal{P} \) be a Borda monotonic minimally rich domain, \( X' \subseteq X \) and \( \mathcal{P}' \subseteq \mathcal{P} \). Assume that \( q' := \#X' = \#\mathcal{P}' \) and that the restriction of \( \mathcal{P}' \) to its top \( q' \) alternatives gives a CNP domain on \( X' \). Then for any preference \( \succ \in \mathcal{P} \) there exists a preference \( \succ' \in \mathcal{P}' \) such that the alternatives from \( X' \) must follow each other consecutively in the same order in \( \succ \) as in \( \succ' \).

**Proof.** The restriction of \( \mathcal{P}' \) to its top \( q' \) alternatives equals \( \mathcal{P}'_{|X'} \), which is a CNP domain on \( X' \), by the assumptions of Lemma 3.6. We employ an induction on the depth of the CNP domain on \( X' \). Lemma 3.2 implies that Lemma 3.6 is satisfied whenever \( \mathcal{P}'_{|X'} \) is a CNP domain of depth 1.

Assume that the statement holds true for any CNP domain \( \mathcal{P}'_{|X'} \) of depth less than \( n \). Now let \( \mathcal{P}'_{|X_i} \) be a CNP domain of depth \( n \) and take the partition of \( X' \) into sets \( X_1, \ldots, X_k \) of cardinality \( q'/k \) according to the definition of CNP domains. Then \( \mathcal{P}'_{|X_i} \) are CNP domains of depth \( n-1 \) for all \( i = 1, \ldots, k \). Thus, for all preferences \( \succ \in \mathcal{P} \) and all \( i = 1, \ldots, k \) there exists a preference \( \succ' \in \mathcal{P}' \) such that the alternatives from \( X_i \) must follow each other consecutively in the same order in \( \succ \) as in \( \succ' \) by our induction hypothesis. Pick an arbitrary preference \( \succ' \in \mathcal{P} \setminus \mathcal{P}' \) and suppose that there does not exist a preference \( \succ' \in \mathcal{P}' \) such that the alternatives from \( X' \) must follow each other consecutively in the same order in \( \succ \) as in \( \succ' \). Let \( x_1 \) be the highest ranked \( X' \) alternative by \( \succ \). We can assume without loss of generality that \( x_1 \in X_1 \). We shall denote by \( \succ' \in \mathcal{P}' \) the preference ranking \( x_1 \) on top. We assume for notational convenience that \( \succ' \) ranks \( X_i \) above \( X_{i+1} \) for all \( i = 1, \ldots, k - 1 \). Let \( j \in \{1, \ldots, k\} \) be the largest index such that the alternatives \( \bigcup_{i=1}^{j-1} X_i \) follow each other consecutively in the same order in \( \succ \) as in \( \succ' \). We shall denote by \( x_j \) the highest ranked \( X_j \) alternative in \( \succ' \) and by \( \succ'' \in \mathcal{P}' \) the preference with top alternative \( x_j \). There exists positive integers \( a \) and \( b \) such that profile \( \Pi = (a \cdot \succ', b \cdot \succ'') \) has \( x_j \) and \( U \subseteq \bigcup_{i=1}^{j-1} X_i \) indifferent on the top. We shall denote by \( u \) the lowest ranked alternative from \( U \) by \( \succ' \). Let \( d' = rk [x_j, \succ'] - rk [u, \succ'] \) and \( d = rk [x_j, \succ] - rk [u, \succ] \). We must have \( d' < d \). Let \( c = \left[ \frac{d}{d-d} \right] \). We can assume that \( a > c \), since otherwise, we can take an appropriate multiple of \( a \) and \( b \) to have \( f^B(\Pi) = \{ x_j \} \cup U \) and \( a > c \). Let \( \Pi' = ((a-1) \cdot \succ', b \cdot \succ'') \) and \( \Pi'' = (c \cdot \succ, (a-1-c) \cdot \succ', b \cdot \succ'') \). If \( a \) and \( b \)
were chosen large enough so that no other alternative can interfere, then \( f^B(\Pi') = \{x_j\} \) and \( u \in f^B(\Pi'') \), and therefore, monotonocity is violated at \( u \) by switching from \( \Pi'' \) to \( \Pi' \).

\[ \f_B(\Pi') = \{x_j\} \]

3.5 Proof of Theorem 3.1

Proof. Sufficiency follows from Proposition 3.2, since the Borda count is a Nested Borda count for any factorization of \( q \).

Thus, we just have to prove the necessity of CNP domains for which we need the following notations. For any \( 1 \leq i \leq j \leq q = \|X\| \) let \( \succ_{[i,j]} \) be the restriction of \( \succ \) ranging from the \( i \)th position to the \( j \)th position of \( \succ \), i.e., \( \succ_{[i,j]} \succ \{x_i, x_{i+1}, \ldots, x_j\} \) where \( x_1 \succ \cdots \succ x_i \succ \cdots \succ x_j \succ \cdots \succ x_q \). In addition, for any \( 1 \leq i \leq j \leq q \), we define \( \mathcal{P}_{[i,j]} = \{\succ_{[i,j]} \succ \in \mathcal{P}\} \). Furthermore, for any linear ordering \( \succ \) on \( X' \subseteq X \) we define \( T_i(\succ) = \{x \in X' | rk[x, \succ] \leq i\} \) and \( M_{i,j}(\succ) = \{x \in X' | i \leq rk[x, \succ] \leq j\} \). We divide our proof into several steps.

Step 1: We show that \( \mathcal{P} \) can be partitioned into subdomains having cyclic permutation domains on top.

Lemma 3.4 implies that the top two alternatives determine a permutation \( \pi \) of \( X \). The cycles of permutation \( \pi \) partition \( X \) into sets \( X_1, \ldots, X_p \). We shall denote by \( X' \) an arbitrary set \( X_i \) (\( i = 1, \ldots, p \)), by \( x_1, \ldots, x_m \) its alternatives and by \( \succ_k \in \mathcal{P} \) an arbitrary preference with top alternative \( x_k \) (\( k = 1, \ldots, m \))\footnote{It will turn out that the preference having \( x_k \) on top is unique.} Clearly, \( m \geq 2 \). Let \( \mathcal{P}' = \{\succ_1, \ldots, \succ_m\} \). In what follows we can assume without loss of generality that \( rk[x_k \oplus m_1, \succ_k] = 2 \)\footnote{For two integers \( k, l \in \{1, \ldots, m\} \), if \( k + l \neq m \) and \( k + l \neq 2m \), we define \( k \oplus_m l := (k + l) \mod m \), while if \( k + l = m \) or \( k + l = 2m \), we define \( k \oplus_m l := m \).}

We determine the top \( m \) alternatives of \( \mathcal{P}' \). We must have \( rk[x_k \oplus_m 1, \succ_k] = 3 \) for all \( k = 1, \ldots, m \) by Lemma 3.2 Moreover, it follows from Lemma 3.2 by induction that \( rk[x_k \oplus_m l, \succ_k] = l + 1 \) for all \( l = 1, \ldots, m - 1 \) and all \( k = 1, \ldots, m \). But this implies that the top \( m \) alternatives of the preferences in \( \mathcal{P}' \) follow the pattern shown in Table 3.9 Moreover, the restriction to its top \( m \) alternatives of any preference in \( \mathcal{P} \) with a top alternative from \( X' \) equals the restriction to its top \( m \) alternatives of a preference in \( \mathcal{P}' \). In addition, \( \mathcal{P}' \) prescribes the possible orderings of the alternatives from \( X' \) by any preference in \( \mathcal{P} \) by Lemma 3.6

Clearly, we are finished if \( p = 1 \). Hence, in what follows we will assume that \( p > 1 \).
Step 2: We show that if $\mathcal{P}$ can be partitioned into subdomains having CNP domains on top, then a union of some of these subdomains has a deeper CNP domain on top.

Let $X_1, \ldots, X_p$ be a partition of $X$, $m_i = \#X_i$ and

$$\mathcal{P}^i = \{\succ \in \mathcal{P} \mid \exists x \in X_i \text{ such that } rk[x, \succ] = 1\}$$

for all $i = 1, \ldots, p$. Assume that we have already established that $T_{m_i}(\succ) = X_i$ for all $\succ \in \mathcal{P}^i$ and that $\mathcal{P}|_{X_i}$ are CNP domains on $X_i$ for all $i = 1, \ldots, p$.

We will demonstrate in Step 2 that Borda monotonicity implies the existence of a set of indices $I \subseteq \{1, \ldots, p\}$ such that $\#I \geq 2$ and $\mathcal{P}'_{|Y}$ is a CNP domain on $Y$, where $Y = \cup_{i \in I} X_i$; $\mathcal{P}' = \cup_{i \in I} \mathcal{P}^i$ and $T_{\#Y}(\succ) = Y$ for all $\succ \in \mathcal{P}'$.\footnote{This implies that $m_i = m_j$ for all $i, j \in I$, that the CNP domains $\mathcal{P}^i_{|X_i}$ possess the same factorizations $m_i = \prod_{j=1}^{k} q_j$ for all $i \in I$ and that the factorization associated with $\mathcal{P}'_{|Y}$ is $\left(\prod_{j=1}^{k} q_j\right) \cdot \#I$.}

We can assume without loss of generality that $m_1 \leq m_i$ for all $i = 1, \ldots, p$ and we simply write $m$ for $m_1$. Our proof of Step 2 will require three substeps.

Substep A: We show that the restriction of a subdomain of $\mathcal{P}$ with a CNP domain of size $m$ on top contains from its $m + 1$th to $2m$th positions another CNP domain of the same structure. Moreover, this latter CNP domain is on top of another subdomain of $\mathcal{P}$. This “generalizes” Lemma 3.3 from alternatives to CNP subdomains.

We claim that there exists an $i \in \{2, \ldots, p\}$ such that $\mathcal{P}_{|X_1}$ and $\mathcal{P}^i_{|X_i}$ have identical associated factorizations, and furthermore, $M_{m+1,2m}(\succ) = X_i$ for all $\succ \in \mathcal{P}^1$. In addition, there exists a bijection $\varphi_{1,i} : X_1 \to X_i$ such that $x \in X_1$ and $\varphi_{1,i}(x)$ maintain their rank.
differences in \( P^1 \). The claim of Substep A implies by Lemmas 3.5 and 3.6 that \( P^1_{|X_i} = P^i_{|X_i} \) and \( m_1 = \#P^1_{|X_i} = \#P^i_{|X_i \cup X_i} \). We prove our claim by induction.

**Initial step of Substep A:** We consider a subdomain \( P' \) of \( P^1 \) with a cyclic permutation domain on top. Note that \( P' = P^1 \) if \( P^1_{|X_i} \) is a CNP domain of depth 1. It follows from Lemma 3.5 that there cannot be an alternative \( x \in X \) that is ranked by two distinct preferences \( \succ \) and \( \succ' \) in \( P' \) at the \( m + 1 \)th position. We shall denote the \( n \) distinct alternatives ranked \( m \)th by the preferences in \( P' \) by \( y_1, \ldots, y_n \in X \), the corresponding preferences by \( \succ_1, \ldots, \succ_n \) and the corresponding top alternatives by \( z_1, \ldots, z_n \), respectively. Let \( Y = \{y_1, \ldots, y_n\} \) and \( Z = \{z_1, \ldots, z_n\} \). We can assume without loss of generality that \( P'_{|Z} = Z((\succ^*) \succ) \), where \( z_1 \succ^* z_2 \succ^* \cdots \succ^* z_n \).

We show that \( rk[y_{k+\oplus 1}, \succ_k] = m + 2 \) for all \( k = 1, \ldots, n \). This assures by Lemma 3.6 that the preferences in \( P' \) look like in Table 3.10. For notational convenience we will only

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show that \( rk[y_2, \succ_1] = m + 2 \). We shall denote by \( \succ' \) a preference with top alternative \( y_2 \). Arguing in an even simpler way than in Lemma 3.5 we can find positive integers \( a \) and
b such that $y_2$ together with at least another alternative from set $X_1$ receive the highest Borda scores in profile $\Pi = (a \succ_2, b \succ')$ and the lead of $y_2$ over the alternatives from $X_1 \setminus f^B(\Pi)$ is at least $m$. Let $U \subseteq X_1$ be the set of those alternatives that are ranked higher by $\succ_1$ than by $\succ_2$.\footnote{From the structure of $P_{X_1}$, it follows that $rk[u, \succ_2] - rk[u, \succ_1] = n - 1$ for any $u \in U$ and $rk[x, \succ_1] - rk[x, \succ_2] = 1$ for any $x \in X_1 \setminus U$.}

Suppose that $f^B(\Pi) \cap U \neq \emptyset$. Pick arbitrary alternative $u \in f^B(\Pi) \cap U \neq \emptyset$. Then there exists a $k \in \{1, \ldots, m/n\}$ such that $rk[u, \succ_2] = kn$. We shall denote by $v \in X_1$ the $(k-1)n + 1$th ranked alternative by $\succ_2$.\footnote{Observe that $rk[u, \succ_1] = (k-1)n + 1$ and $rk[v, \succ_1] = (k-1)n + 2$.} Let $d = rk[y_2, \succ_2] - rk[u, \succ_2]$ and $d' = rk[u, \succ'] - rk[y_2, \succ']$. Since $u, y_2 \in f^B(\Pi)$, we must have $ad = bd'$, $u \succ' v$ and by Lemma \ref{lem:3.6} $rk[v, \succ'] = d' + 2$. Let us compare the Borda score of $v$ with that of $u$ in $\Pi$. On the one hand $v$ receives $a(n-1)$ points more than $u$ and on the other hand $u$ receives $b$ points more than $v$. Therefore, we must have

$$a(n-1) \leq b \Leftrightarrow a(n-1) \leq a \frac{d}{d'} \Leftrightarrow d'(n-1) \leq d;$$

a contradiction, since $d < m \leq d'$ and $n \geq 2$ by the Assumptions of Step 2. Thus, $f^B(\Pi) \cap U = \emptyset$.

Let $z$ be the highest ranked alternative from $f^B(\Pi) \setminus \{y_2\}$ by $\succ'$, $\delta = rk[y_2, \succ_1] - rk[y_1, \succ_1]$, $d = rk[y_2, \succ_2] - rk[z, \succ_2]$ and $d' = rk[z, \succ'] - rk[y_2, \succ']$. Observe that $z$ has to be the lowest ranked alternative in $\succ_2$ from set $f^B(\Pi) \setminus \{y_2\}$. Suppose that $\delta \geq 2$, which would mean that $y_2$ does not follow immediately $y_1$ in $\succ_1$. We have to incorporate at least one voter of type $\succ_1$ appropriately in order to obtain a contradiction with $\delta \geq 2$. First, we omit a voter of type $\succ_2$, which makes $y_2$ the single Borda winner with a lead of $d$ over $z$. Second, we compensate this lead by replacing $c = \lceil \frac{d}{\delta - 1} \rceil$ voters of type $\succ_2$ with voters of type $\succ_1$. If $a \leq c$, then by taking an appropriate multiple of $\Pi$, we can ensure that we have more voters of type $\succ_2$ than $c$. Hence, we can assume $a > c$ without loss of generality. Third, we have to take care about not making an alternative $u \in U$ the Borda winning alternative. If $z$ does not lead by $cn$ over alternatives $u \in U$ in $\Pi$, then this can be guaranteed by starting already with an appropriate multiple of $\Pi$.\footnote{More precisely, we should have first defined $c = \lceil \frac{d}{\delta - 1} \rceil$ and $a, b$ afterwards. However, we have followed a different order for expositional reasons.} Again, we can assume without loss of generality that $a$ and $b$ satisfy this latter requirement. Finally, let $\Pi' = (c \succ_1, (a-c-1) \succ_2, b \succ')$ and $\Pi'' = ((a-1) \succ_2, b \succ')$. It can be verified that monotonicity is violated at $z$ by switching from $\Pi'$ to $\Pi''$, since $z \in f^B(\Pi')$ and
\{y_2\} = f(B(\Pi''))$. Thus, we must have $\delta = 1$.

**Induction hypotheses of Substep A:** Assume that we have already obtained a partition $\mathcal{P}^{1,1}, \ldots, \mathcal{P}^{1,t}$ of $\mathcal{P}^1$, disjoint subdomains $\mathcal{P}^{2,1}, \ldots, \mathcal{P}^{2,t} \subseteq \mathcal{P} \setminus \mathcal{P}^1$ with respective top $n = \frac{m}{t}$ alternatives $X^{j,i}$ ($j = 1, 2$ and $i = 1, \ldots, t$) such that $t \geq 2$, $n = \#\mathcal{P}^{1,i} = \#\mathcal{P}^{2,i}$, $\mathcal{P}^{1,i} = \mathcal{P}^{2,i}$ are CNP domains with associated factorizations $n = q_1 \cdot \ldots \cdot q_l$ for all $i = 1, \ldots, t$ and there exist bijections $\varphi : X^{1,i} \to X^{2,i}$ satisfying that $x$ and $\varphi(x)$ maintain their rank differences in $\mathcal{P}^{1,i}$ for all $x \in X^{1,i}$ and all $i = 1, \ldots, t$\footnote{Our initial step assured the existence of a partition with $l = 1$.}. This implies that the factorization associated with $\mathcal{P}^1$ equals $q_1 \cdot \ldots q_{t'} \cdot \ldots q_{t''}$ for some $l'$ and $q_{t+1}, \ldots, q_{t''}$.

**Induction step of Substep A:** Let $r = q_{t+1}$, $h = r/r'$ and $\mathcal{P}^{j,i} = \{\succ^{j,i}_1, \ldots, \succ^{j,i}_n\}$ for all $j = 1, 2$ and all $i = 1, \ldots, t$. We shall denote by $X^{j,i}$ the set of top alternatives of $\mathcal{P}^{j,i}$. If we have labeled the sets $X^{1,i}$ appropriately, then the first $r$ subdomains of $\mathcal{P}^{1}_{|X_1}$ look like in Table 3.11. In what follows we shall focus, for notational convenience, on $\mathcal{P}^{1,1}$ and $\mathcal{P}^{1,2}$.

<table>
<thead>
<tr>
<th>$\mathcal{P}^{1,1}$</th>
<th>$\mathcal{P}^{1,2}$</th>
<th>$\ldots$</th>
<th>$\mathcal{P}^{1,r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X^{1,1}$</td>
<td>$X^{1,2}$</td>
<td>$\ldots$</td>
<td>$X^{1,r}$</td>
</tr>
<tr>
<td>$X^{1,2}$</td>
<td>$X^{1,3}$</td>
<td>$\ldots$</td>
<td>$X^{1,1}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$X^{1,r}$</td>
<td>$X^{1,1}$</td>
<td>$\ldots$</td>
<td>$X^{1,r-1}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$X^{1,t-r+1}$</td>
<td>$X^{1,t-r+2}$</td>
<td>$\ldots$</td>
<td>$X^{1,t}$</td>
</tr>
<tr>
<td>$X^{1,t-r+2}$</td>
<td>$X^{1,t-r+3}$</td>
<td>$\ldots$</td>
<td>$X^{1,t-r+1}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$X^{1,t}$</td>
<td>$X^{1,t-r+1}$</td>
<td>$\ldots$</td>
<td>$X^{1,t-1}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

In addition, we can assume without loss of generality that the alternatives and preferences are labeled in a way that $rk\left(\frac{x^{1,i}}{x^{1,k}_{n+1}}\right) = (i-2)n + 1 + (n-k)$ for all $k = 1, \ldots, n$ and all $i = 2, \ldots, t$ for which $i - 1$ is not divisible by $r$, and otherwise, $rk\left(\frac{x^{1,i}}{x^{1,k}_{n+1}}\right) = (i+r-2)n + 1 + (n-k)$ for all $k = 1, \ldots, n$ and all $i = 1, \ldots, t$.

We shall denote by $y_1, \ldots, y_n$ the $n$ distinct alternatives ranked $m + 1$th by the preferences $\succ^{1,2}_1, \ldots, \succ^{1,2}_n$, respectively. Let $Y = \{y_1, \ldots, y_n\}$. Moreover, for all $i = 1, \ldots, n$ we simply write $\succ_i', \succ_i''$ and $\succ_i^\prime\prime$ for $\succ^{1,1}_i, \succ^{1,2}_i$ and $\succ^{2,2}_i$, respectively.
We can find positive integers \(a\) and \(b\) such that \(Y\) and at least a set of alternatives \(X' \subseteq X_1\) receives the highest Borda score in profile \(\Pi = (a \cdot \succ'_1, \ldots, a \cdot \succ'_n, b \cdot \succ''_1, \ldots, b \cdot \succ''_n)\). Let \(U \subseteq X_1\) be the set of those alternatives that are ranked higher by \(\succ_1\) than by \(\succ'_1\). Observe that \(U = \bigcup_{i=0}^{h} X^{1,ir+1}\).

Suppose that there exists an \(i = 1, \ldots, h\) such that \(u \in f^B(\Pi) \cap X^{1,(i-1)r+1}\). Pick an arbitrary alternative \(y \in Y\). Since \(\{u, y\} \in f^B(\Pi)\), we must have \(a[(h - i)r + 1]n^2 = bd'\), where \(d' = \sum_{i=1}^{n} rk[u, \succ''_i] - rk[y, \succ''_i]\). Take an alternative \(v\) from \(X^{1,(i-1)r+2}\) such that \(d'' = \sum_{i=1}^{n} rk[v, \succ''_i] - rk[y, \succ''_i]\) is as small as possible. Since the Borda score of \(v\) cannot be greater than that of \(u\) in \(\Pi\), we must have \(an(r - 1)n \leq b(d'' - d')\). Therefore,

\[
\frac{bd'}{[(h - i)r + 1]n^2} (r - 1) \leq b(d'' - d') \iff d'(r - 1) \leq (d'' - d') [(h - i)r + 1].
\]

By the hypothesis of Step 2 and by Lemmas 3.5 and 3.6 we must have \(d' \geq mn = tn^2\). The value \(d'' - d'\) would be the largest if \(u\) is ranked higher than \(v\) by any preference \(\succ''_i \in P^{2,2}\). Then the alternatives from \(X^{1,(i-1)r+2}\) must follow immediately the alternatives from \(X^{1,(i-1)r+1}\) in any \(\succ''_i \in P^{2,2}\) by Lemma 3.6. Moreover, we have \(d'' - d' \leq n^2\) by Lemma 3.6 and by the choice of \(v\), which together with equation (3.1) implies

\[
tn^2 \leq d'(r - 1) \leq (d'' - d') [(h - i)r + 1] \leq n^2[(h - i)r + 1].
\]

It follows from these inequalities that \(rh = t \leq [(h - i)r + 1]\), which implies \(ir \leq 1\). Therefore, since \(r, n \geq 2\), \(h \geq 1\) and \(i \geq 1\) we obtained a contradiction and we conclude that \(f^B(\Pi) \cap U = \emptyset\).

Define \(v = \max\{i = 1, \ldots, t \mid X^{1,i} \cap f^B(\Pi) \neq \emptyset\}\) and pick an alternative \(z\) from \(X^{1,v} \cap f^B(\Pi)\). Let \(\succ' \in P^{1,2}\) the preference that ranks \(z\) highest. For notational convenience we can assume that \(\succ' \succ'_n\) and \(\succ'\) ranks \(y_n\) as the highest alternative from \(Y^{1,n}\). Hence, \(z\) is the highest ranked alternative from \(X^{1,v}\) by \(\succ'_n\). Observe that from the way how we labeled the alternatives of \(X_1\) and our assumption on \(\varphi: X^{1,2} \rightarrow X^{2,2} = Y\) it follows that \(rk[y_n, \succ'_k] - rk[x^{1,2}_{n}, \succ'_k] = m\) for all \(k = 1, \ldots, n\). In addition, we can assume for notational convenience that \(\succ'_{k|X^{1,v}} \succ'_{k|X^{1,v}}\) for all \(k = 1, \ldots, n\). We will show that \(rk[y_n, \succ_k] = m + n + 1 + (n - k)\) for all \(k = 1, \ldots, n\). Observe that \(rk[y_n, \succ_k] \geq m + n\), since the shortest sequence of alternatives that must follow an already prescribed order is of length \(n\) and by Lemma 3.5 none of the alternatives of \(Y\) can be ranked \(m + 1\)th by a preference of \(P^{1,1}\). Now take an arbitrary index \(k = 1, \ldots, n\) and let \(\delta_k = rk[y_n, \succ_k] - (m + 1 + n - k)\).
Suppose that $\delta_k > n$. By replacing a preference $\succ'_1$ with $\succ'_n$ in $\Pi$, we can achieve that $f^B(\Pi)$ contains only $y_n$ from $Y$ and only $z$ from $X^{1,v}$. In what follows we shall denote this modified profile by $\Pi$ with a slight abuse of notation. Let $d = rk \{y_n, \succ'_n\} - rk \{z, \succ'_n\}$ and $d' = \sum_{i=1}^{n} rk \{z, \succ''_i\} - rk \{y_n, \succ''_i\}$. Note that $d' \geq mn$ by the assumptions of Step 2 and by Lemmas 3.5 and 3.6. Now we have to incorporate at least one voter of type $\succ'_k$ into $\Pi$ in order to obtain a contradiction with $\delta_k > n$. First, we omit a voter of type $\succ'_1$, which makes $y_n$ the single Borda winner with a lead of $d$ over $z$. Second, we compensate this lead by replacing $c = \left[ \frac{d}{\delta_k-n} \right]$ voters of type $\succ'_k$ with voters of type $\succ'_k$. If $a \leq c$, then by starting with an appropriate multiple of $\Pi$, we can ensure that we have more than $c$ voters of type $\succ'_k$. Hence, we can assume $a > c$ without loss of generality. Third, we have to take care about not making an alternative $u \in U$ the Borda winning alternative. If $z$ does not lead by $cmn$ over alternatives $u \in U$ in $\Pi$, then this can be guaranteed by starting already with an appropriate multiple of $\Pi^{17}$. Thus, we can assume without loss of generality that $a$ and $b$ satisfy this latter requirement. Let $\Pi' = ((a - 2) \cdot \succ'_1, a \cdot \succ'_2, \ldots, a \cdot \succ'_{k-1}, c \cdot \succ'_k, (a - c) \cdot \succ'_k, a \cdot \succ'_{k+1}, \ldots, a \cdot \succ'_{n-1}, (a + 1) \cdot \succ'_n, b \cdot \succ''_1, \ldots, b \cdot \succ''_n)$ and $\Pi'' = ((a - 2) \cdot \succ'_1, a \cdot \succ'_2, \ldots, a \cdot \succ'_{n-1}, (a + 1) \cdot \succ'_n, b \cdot \succ''_1, \ldots, b \cdot \succ''_n)$. It can be verified that monotonicity is violated at $z$ by switching from $\Pi'$ to $\Pi''$, since $\{z\} = f^B(\Pi')$ and $\{y_n\} = f^B(\Pi'')$. Thus, we cannot have $\delta_k > n$.

Suppose that $\delta_k < n$ for some $k$. Then alternative $y_n$ has to be ranked by at least two different preferences of $\mathcal{P}^{1,1}$ at the same position, since $\delta_k \leq n$ for all $i = 1, \ldots, n$. However, this is in contradiction with Lemmas 3.5 and 3.6. Hence, we must have $\delta_k = n$ for all $k = 1, \ldots, n$.

Therefore, since the shortest sequence of alternatives that must follow an already prescribed order is of length $n$ and by Lemma 3.6 none of the alternatives of $Y$ can be ranked $m+1$th by a preference of $\mathcal{P}^{1,1}$, we obtained that $n = \# \mathcal{P}^{1,2}_{\{X_1, \ldots, Y\}}$ and $\mathcal{P}^{1,1}_{\{m+n+1, m+2n\}} = \mathcal{P}^{2,2}_{\{1, n\}}$. Thus, the domain of Table 3.11 extends to a domain as illustrated in Table 3.12. Now Lemma 3.6 implies that the alternatives $\cup_{i=1}^{l+1} Y_i$ must form a CNP domain of depth $l + 1$ with an associated factorization $\prod_{i=1}^{l+1} q_i$. Therefore, our induction works and the induction hypothesis is true for depth $l + 1$. Arriving to $l'$, we see that the claim of Substep A is true, since it follows from Lemma 3.6 that there exists an $i = 2, \ldots, p$ for which $\mathcal{P}^{2,1}, \ldots, \mathcal{P}^{2,t}$ partitions $\mathcal{P}^i$.

Substep B: We “generalize” Lemma 3.4 from alternatives to CNP subdomains on

---

17More precisely, we should have first defined $c = \left[ \frac{d}{\delta_k-n} \right]$ and $a, b$ afterwards. Again, we have followed a different order for expositional reasons.
Table 3.12: Extended domain

<table>
<thead>
<tr>
<th>( P_{1,1} )</th>
<th>( P_{1,2} )</th>
<th>...</th>
<th>( P_{1,r} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_{1,1} )</td>
<td>( X_{1,2} )</td>
<td>...</td>
<td>( X_{1,r} )</td>
</tr>
<tr>
<td>( X_{1,2} )</td>
<td>( X_{1,3} )</td>
<td>...</td>
<td>( X_{1,1} )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( X_{1,r} )</td>
<td>( X_{1,1} )</td>
<td>...</td>
<td>( X_{1,r-1} )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( X_{1,t-r+1} )</td>
<td>( X_{1,t-r+2} )</td>
<td>...</td>
<td>( X_{1,t} )</td>
</tr>
<tr>
<td>( X_{1,t-r+2} )</td>
<td>( X_{1,t-r+3} )</td>
<td>...</td>
<td>( X_{1,t-r+1} )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( X_{1,t} )</td>
<td>( X_{1,t-r+1} )</td>
<td>...</td>
<td>( X_{1,t-1} )</td>
</tr>
<tr>
<td>( Y_1 )</td>
<td>( Y_2 )</td>
<td>...</td>
<td>( Y_r )</td>
</tr>
<tr>
<td>( Y_2 )</td>
<td>( Y_3 )</td>
<td>...</td>
<td>( Y_1 )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Informally, we show that subdomains with two consecutive CNP domains on top form cycles of CNP domains on top.

Substep A implies that there exists an \( I \subseteq \{1, \ldots, p\} \) such that for all \( i \in I \) the subdomains \( P_{i,X_i} \) have all identical factorizations, there exists a \( j \in I \setminus \{i\} \) for which \( M_{m+1,2m}(\succ) = X_j \) for all \( \succ \in P^i \) and there exists a bijection \( \varphi_{i,j} : X_i \rightarrow X_j \) such that \( x \in X_i \) and \( \varphi_{i,j}(x) \) maintain their rank differences in \( P^i \). We shall assume for notational convenience that \( I = \{1, \ldots, r\} \). Hence, there exists a \( \sigma : \{1, \ldots, r\} \rightarrow \{1, \ldots, r\} \) telling us, which set \( X_{\sigma(i)} \) of alternatives must follow immediately the top set \( X_i \) of alternatives for all \( i = 1, \ldots, r \). In Substep B we demonstrate that \( \sigma \) is a bijection. This is clearly the case if \( r = 2 \).

Thus, we can assume that \( r > 2 \). Our proof will be similar to that of Lemmas 3.2-3.4, but we have to replace the alternatives appearing in those proofs with “nested cycles”, which will complicate the argument.

Suppose that \( \sigma \) does not define a bijection. Then there exists \( i, i', i'' \in \{1, \ldots, r\} \) such that \( i \neq i', i \neq i'', i' \neq i'' \), \( \sigma(i') = i \) and \( \sigma(i'') = i \). Moreover, \( m = m_i = m_{i'} = m_{i''} \) by Substep A. For notational convenience let \( X' = \{x_1, \ldots, x_m\} = X_i \), \( Y = \{y_1, \ldots, y_m\} = X_{i'} \), \( Z = \{z_1, \ldots, z_m\} = X_{i''} \) such that \( \varphi_{i,i}(y_l) = x_l \) and \( \varphi_{i',i}(z_l) = x_l \) for all \( l = 1, \ldots, m \). Pick preferences \( \succ_1, \ldots, \succ_m, \succ'_1, \ldots, \succ'_m, \succ''_1, \ldots, \succ''_m \in P \) with respective top alternatives.
If \( r = 3 \), then the alternatives from \( X' \) cannot be lowest ranked alternatives; a contradiction. If \( r > 3 \), then we can assume the existence of a preference \( \succ \in \mathcal{P} \) in which the alternatives from \( Y \) are ranked above the alternatives from \( Z \) and the alternatives from \( Z \) are ranked above the alternatives from \( X' \) by part (ii) of the minimal richness condition. For notational convenience we can assume that

\[
y_l' \succ y_{l'}' \succ y_s' \succ z_t' \succ x_1' \succ \ldots \succ x_m' \\
z_k' \succ z_{k'}' \succ z_{k''} \succ x_1'' \succ \ldots \succ x_m''
\]

and

\[
\ldots \succ y_s \succ \ldots \succ y_{l'} \succ \ldots \succ z_t \succ \ldots \succ z_{t'} \succ \ldots \succ x_1 \succ \ldots \succ x_m
\]

for some \( l, l', k, k', s, s', t, t' \in \{1, \ldots, m\} \).

Let \( J = \arg \min_{j \in \{1, \ldots, m\}} \sum_{u=1}^{m} r_k[y_j, \succ_u] \) and \( Y' = \{y_j \in Y \mid j \in J\} \). Then there exist positive integers \( a \) and \( b \) such that \( b > m^2 + 1 \) and that profile

\[
\Pi = (a \cdot \succ_1, \ldots, a \cdot \succ_m, b \cdot \succ'_1, \ldots, b \cdot \succ'_m),
\]

has alternatives \( Y' \cup X' \) indifferent on top with a lead of at least \((m^2 + 1)q\) over the alternatives from \( X \setminus (X' \cup Y) \). We consider profile

\[
\Pi' = (a \cdot \succ_1, \ldots, a \cdot \succ_m, (b - 1) \cdot \succ'_1, \ldots, (b - 1) \cdot \succ'_m),
\]

in which the top alternatives \( X' \) have a lead of \( m^2 \) over alternatives \( Y' \). To obtain \( \Pi'' \) from \( \Pi' \) we replace \( m^2 + 1 \) preferences of type \( \succ'_j \) with \( m^2 + 1 \) preferences of type \( \succ \). It can be verified that \( f^B(\Pi') = X' \) and \( f^B(\Pi'') = Y' \). Thus, we have a violation of monotonicity at any alternative \( y \in Y' \) if we switch from \( \Pi'' \) to \( \Pi' \).

**Substep C:** What Step 1 showed for alternatives is now established for subdomains with CNP domains on top: the union of some subdomains with CNP domains of identical factorization on top form a subdomain with a deeper CNP domain on top, which completes the proof of Step 2.

Substep B established that the cycles of permutation \( \sigma \) partition \( \{1, \ldots, r\} \) into sets \( I_1, \ldots, I_s \). In what follows we consider, for notational convenience, the case of \( I = I_1 = \{1, \ldots, k\} \) and \( \sigma(1) = 2, \ldots, \sigma(k - 1) = k, \sigma(k) = 1 \). First, in an analogous way to Step 1 we show in Substep C that the cycles formed by alternatives \( X_1, \ldots, X_k \) follow each other in a cyclic pattern in \( \mathcal{P}^1 \cup \ldots \cup \mathcal{P}^k \); that is, for all \( i = 1, \ldots, k \) we have in \( \mathcal{P}^i \) that the alternatives from \( X_{i \in \mathbb{B}_1} \) follow those from \( X_i \), the alternatives from \( X_{i \in \mathbb{B}_2} \) follow
those from $X_{i⊕k\downarrow}$, and so on. Second, we claim that there exist bijections $\tau_{i,j} : X_i \to X_j$ ($i, j = 1, \ldots, k$) such that

$$[x \in X_i, v = \tau_{i,j}(x), x \succ v, x \succ' v] \Rightarrow rk[v, \succ] - rk[x, \succ] = rk[v, \succ'] - rk[x, \succ'] \quad (3.2)$$

for all $\succ, \succ' \in \mathcal{P}^1 \cup \ldots \cup \mathcal{P}^k$ and all $i \neq j$, $i, j = 1, \ldots, k$.

Clearly, both claims are true for the case of $k \leq 2$. Hence, we can assume that $k \geq 3$. We know by Substep B that the first claim of Substep C is true for the top $2m$ alternatives of any preferences in $\mathcal{P}^i$ and that we can define bijections $\tau_{i,i\oplus1} : X_i \to X_{i\oplus1}$ in a way that equation $\text{[3.2]}$ holds true for all $i = 1, \ldots, k$ if we restrict ourselves to the top $2m$ alternatives of $\mathcal{P}^1 \cup \ldots \cup \mathcal{P}^k$.

Our induction hypotheses is that the claim holds true for the top $lm$ alternatives, where $l \in \{2, \ldots, k-1\}$, of any preference in $\mathcal{P}^1 \cup \ldots \cup \mathcal{P}^k$.\footnote{This includes that we have defined bijections $\tau_{i,i\oplus l-1} : X_i \to X_{i\oplus l-1}$ in a way that equation $\text{[3.2]}$ holds true for all $i = 1, \ldots, k$ if we restrict ourselves to the top $lm$ alternatives of $\mathcal{P}^1 \cup \ldots \cup \mathcal{P}^k$.}

For purely notational convenience let $Y = \{y_1, \ldots, y_m\} = X_l$, $Z = \{z_1, \ldots, z_m\} = X_{l+1}$, and $\tau_{l,l+1}(y_i) = z_i$ for all $i = 1, \ldots, m$. We will just consider the case of $z_1$. The other elements of $Z$ can be handled in the same way. We shall denote by $\succ \in \mathcal{P}^1$ the preference that ranks $y_1$ as the highest ranked $Y$ alternative; i.e., $rk[y_1, \succ] = m(l - 1) + 1$. Moreover, let $\succ'$ be a preference ranking $y_1$ on the top and let $\succ'' \succ'$ be a preference ranking $z_1$ on the top. There exists positive integers $a$ and $b$ such that profile $\Pi = (a \cdot \succ', b \cdot \succ'')$ has alternatives $z_1$ and $Y' \subseteq Y$ on the top. Let $y_s = \max\{t \in \{1, \ldots, m\} \mid y_t \in Y'\}$.

Suppose that alternative $z_1$ does not immediately follow $y_m$ in $\succ$; i.e, $rk[z_1, \succ] > ml+1$. Let $\delta = rk[z_1, \succ] - (ml + 1)$, $d = rk[z_1, \succ'] - rk[y_s, \succ']$ and $c = \lceil \frac{d}{\delta} \rceil$. Then considering profiles $\Pi' = ((a - 1) \cdot \succ', b \cdot \succ'')$ and $\Pi'' = (c \cdot \succ, (a - c - 1) \cdot \succ', b \cdot \succ'')$, we can verify that $y_s \in f^{\Pi'}(\Pi'')$ and $f^{\Pi'}(\Pi') = \{z_1\}$ if $a$ and $b$ were selected large enough so that no other alternative can interfere and $a > c + 1$. Monotonicity is now violated at $y_s$ if we switch from $\Pi''$ to $\Pi'$.

It follows from the above defined $\tau_{l,l+1} : X_l \to X_{l+1}$, $\tau_{1,l} : X_1 \to X_l$ and transitivity that we obtained a bijection $\tau_{1,l+1} : X_1 \to X_{l+1}$ in a way that equation $\text{[3.2]}$ holds true for all $l = 1, \ldots, k-1$ if we restrict ourselves to the top $(l+1)m$ alternatives of $\mathcal{P}^1 \cup \ldots \cup \mathcal{P}^k$. One can obtain the remaining bijections in an analogous way.

We conclude that we have constructed the required sets $Y = \cup_{i=1}^k X_i$ and $\mathcal{P}' = \cup_{i=1}^k \mathcal{P}^i$ by induction.

**Step 3**: The partition $X_1, \ldots, X_p$ of $X$ in Step 1 satisfies the requirements of Step 2. Finally, it follows by induction from Step 2 that $\mathcal{P}$ has to be a CNP domain. \qed
Chapter 4

Voting rules in relation to the dictatorial rules

The concept of distance rationalization of voting rules has recently been explored by several authors. Given a notion of consensus and a metric (distance function), a voting rule that is rationalizable chooses the alternative that is closest to being a consensus winner. The seminal work was initiated by Farkas and Nitzan \[30\], who derived the Borda count as the solution of an optimization problem on the set of social choice functions by minimizing the distance from the unanimity principle. Taking other metrics, Nitzan \[63\] obtained the plurality rule among other rules. The approach of minimizing the distance from a set of profiles with a clear winner such as the unanimous winner, the majority winner, or the Condorcet winner has been developed further by \[50\], \[28\], \[5\], \[52\], and \[80\] among others.

All previous works have dealt with the distance rationalizability based on the minimization of the distance to some plausible criterion, such as unanimity or the Condorcet criterion. In contrast, we propose a new alternative, namely, the optimization of the distance to the undesirable dictatorial voting rules, motivated by the classical impossibility results of Arrow \[6\] and Gibbard–Satterthwaite \[38\] \[74\], roughly stating that every voting rule satisfying a subset of reasonable properties leads to dictatorship. In particular, we ask the following question: will we obtain a “good” voting rule if we want to get as close as possible to all dictatorial voting rules or if we get away from the closest dictatorial rule? We investigate this question by employing a quite simple and natural distance function between social choice functions. This chapter is based on our work Bednay, Moskalenko and Tasnádi \[12\].

By getting as close as possible to all dictatorial rules, we are searching for the rules
that minimize the sum of the distances to the dictatorial rules, which is identical to the
set of rules choosing a top alternative of a voter in as many cases as possible. We call
these rules balanced since they represent a kind of compromise between all dictatorial
rules. Using this terminology, we find that the plurality rule and the balanced rule are the
same. Therefore, we consider this as a positive result since the plurality rule is the most
frequently applied one.

By getting away from the closest dictatorial rule, we are searching for the rules that
maximize the distance to the closest dictatorial rule. We refer to these rules as the least
dictatorial rules since in some sense they are the furthest from dictatorship, which emerges
if the collective outcome is determined by a dictatorial rule. In particular, any other rule
in the space of voting rules lies closer to at least one of the dictatorial rules than any of
the least dictatorial rules. We find that our goal results in a quite unpleasant rule, which
we call the reverse-plurality rule, violating properties like unanimity or monotonicity.
Therefore, we consider our second main result as a negative one in the sense that we
obtain an undesirable rule. However, based on our result, from a philosophical point
of view, one could argue that eliminating the ‘dictatorial ingredient’ from voting rules
completely should not be our goal.

Furthermore, we investigate the relationship between minimizing (maximizing) the
sum of distances and minimizing (maximizing) the minimum of distances in our objective
function.

Finally, we would like to mention that based on our approach of measuring the dis-
tance of a voting rule to the dictatorial rules in a follow-up paper Bednay, Moskalenko and
Tasnádi [13] we formulated a non-dictatorship index (NDI). By employing computer sim-
ulations, we estimated the NDIs of some well-known social choice functions (some scoring
rules and Condorcet consistent rules) for 3, 4 and 5 alternatives and up to 100 voters by
generating 1000 random preference profiles, where each profile is selected with the same
probability. We found that among the prominent social choice functions the plurality rule
has the smallest NDI (in line with Proposition [4.1], the Borda count, the Black rule and
the Copeland method follow with approximately identical NDIs, while $k$-approval voting
(for $k = 2$ or $k = 3$) has the highest NDI among the most common social choice functions.
We verified that our NDI behaves consistently to the most straightforward index of ma-
nipulability: the Nitzan–Kelly’s index (Nitzan [62] and Kelly [47]), which determines the
ratio of preference profiles where manipulation is possible to the total number of profiles.
4.1 Distances from the dictatorial rules

As in the previous chapters let $X = \{1, \ldots, q\}$ be the set of alternatives and $N = \{1, \ldots, n\}$ be the set of voters. We shall denote by $P$ the set of all linear orderings on $X$ and by $P^n$ the set of all preference profiles.

**Definition 4.1.** A mapping $f : P^n \rightarrow X$ that selects the winning alternative is called a social choice function, henceforth, SCF.

Note that the SCF in this chapter differs from the one defined in Chapter 2 in that it is defined for a given $n$ though $n$ can take any arbitrary positive value. The difference in these two settings becomes visible if one considers properties not satisfied by an SCF. Anyway, both definitions of an SCF does not allow for possible ties, in which case a fixed tie-breaking rule will be employed. In a slightly more general and definitely more formal way compared with Chapter 2 a tie-breaking rule $\tau : P^n \rightarrow P$ maps preference profiles to linear orderings on $X$, which will be only employed when a formula does not determine a unique winner. If there are more alternatives chosen by a formula ‘almost’ specifying an SCF, then the highest ranked alternative is selected, based on the given tie-breaking rule among tied alternatives. In particular, anonymous tie-breaking rules will play a central role in our analysis.

We will also allow for domain restrictions, since for some preference profiles we may prescribe certain outcomes, which are plausible. Let $S \subseteq P^n$ be a subdomain on which the outcome is already prescribed by some externally chosen principle. Then the values of a SCF have to be specified only on $\overline{S}$, where $\overline{S} = P^n \setminus S$, and therefore we only need to consider SCFs restricted to $\overline{S}$. For instance, for profiles with a Condorcet winner denoted by $S_c$, we may only consider Condorcet consistent SCFs; or for profiles with a majority supported alternative, denoted by $S_m$, we may require that the majority winner should be chosen. We consider the following type of domain restriction.

**Definition 4.2.** A domain restriction $S \subseteq P^n$ is called anonymous if for any bijection $\sigma : N \rightarrow N$ we have for all $(\succ_1, \ldots, \succ_n) \in P^n$ that $(\succ_{\sigma^{-1}(1)}, \ldots, \succ_{\sigma^{-1}(n)}) \in S$ implies $(\succ_1, \ldots, \succ_n) \in S$.

It can be verified that if $S$ is anonymous, then also $\overline{S}$ is anonymous. If $S = \emptyset$, we have the case of an unrestricted domain. It is easy to see that $S_c$ and $S_m$ are anonymous. The introduction of domain restrictions results in a more general framework.

---

1The domain restrictions in this chapter play a different role than those in Chapters 2 and 3. Moreover, they are non-symmetric in the sense that we do not restrict the set of each individual in the same way.
Let $\mathcal{F} = X^P$ be the set of SCFs and $\mathcal{F}^{an} \subset \mathcal{F}$ be the set of anonymous voting rules. The subset of $\mathcal{F}$ consisting of the dictatorial rules will be denoted by $\mathcal{D} = \{d_1, \ldots, d_n\}$, where $d_i$ is the dictatorial rule with voter $i$ as the dictator. In order to define several optimization problems related to dictatorial rules we will employ the following distance function between SCFs:

$$\rho_S(f, g) = \#\{\succ \in S \mid f(\succ) \neq g(\succ)\}, \quad (4.1)$$

where $f, g$ are SCFs and $\rho_S(f, g)$ stands for the number of profiles on which $f$ and $g$ choose different alternatives within $S$. It can be checked that $\rho_S$ specifies a metric over the set of SCFs restricted to $S$. If $S = \emptyset$, we simply write $\rho(f, g)$. Since in case of SCFs we only care about the chosen outcome (and not about a social ranking), and we do not assume any kind of structure on the set of alternatives $X$, it appears natural that we count the number of profiles on which $f$ and $g$ differ.

We specify the set of least dictatorial rules by those ones which are the furthest away from the closest dictatorial rule, which means that we are maximizing the minimum of the distances to the dictators.

**Definition 4.3.** We define the set of *least dictatorial rules* for domain restriction $S$ by

$$\mathcal{F}_{ld}(S) = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \min_{i \in N} \rho_S(f, d_i) \geq \min_{i \in N} \rho_S(f', d_i) \right\}$$

in general and by

$$\mathcal{F}_{ld}^{an}(S) = \left\{ f \in \mathcal{F}^{an} \mid \forall f' \in \mathcal{F}^{an} : \min_{i \in N} \rho_S(f, d_i) \geq \min_{i \in N} \rho_S(f', d_i) \right\}$$

over the set of anonymous voting rules.

When defining least dictatorial rules based on the distance function $\rho_S$, we could have taken the average distance, or equivalently the sum of the distances from the dictators. However, we feel that if we would like to be ‘least dictatorial’, we should be more concerned about the closest dictatorial rule. Nevertheless, we will consider the other possibility at the end of this section and for anonymous SCFs it will turn out that we will obtain the same rules.

An alternative approach to getting as far away from the closest dictator as possible would be getting as close as possible to all dictators at the same time, which could be considered as a kind of neutral or balanced solution with respect to all dictators and, in this sense, as a kind of desirable solution. For simplicity reasons, we will minimize the sum of the distances to the $n$ dictators.
**Definition 4.4.** We define the set of balanced rules for domain restriction $S$ by

$$
\mathcal{F}_b(S) = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \sum_{i \in N} \rho_S(f, d_i) \leq \sum_{i \in N} \rho_S(f', d_i) \right\}
$$

in general and by

$$
\mathcal{F}^a_n(S) = \left\{ f \in \mathcal{F}^a_n \mid \forall f' \in \mathcal{F}^a_n : \sum_{i \in N} \rho_S(f, d_i) \leq \sum_{i \in N} \rho_S(f', d_i) \right\}
$$

over the set of anonymous voting rules.

An equivalent formulation of balanced rules, stating that these rules maximize the number of cases in which a top alternative of a voter is chosen, is derived at the beginning of Section 4.2.

Instead of looking for the rules which are the furthest away from the closest dictatorial rule we could consider the rules which are the closest ones to the furthest dictatorial rule, which means that we are minimizing the maximum of the distances to the dictators.

**Definition 4.5.** We define the set of minmax rules for domain restriction $S$ by

$$
\mathcal{F}_{\text{min max}}(S) = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \max_{i \in N} \rho_S(f, d_i) \leq \max_{i \in N} \rho_S(f', d_i) \right\}
$$

in general and by

$$
\mathcal{F}^a_{\text{min max}}(S) = \left\{ f \in \mathcal{F}^a_n \mid \forall f' \in \mathcal{F}^a_n : \max_{i \in N} \rho_S(f, d_i) \leq \max_{i \in N} \rho_S(f', d_i) \right\}
$$

over the set of anonymous voting rules.

In relation to the definition of balanced rules, we obtain the reverse-balanced rules by getting furthest from all dictators at the same time. In particular, we maximize the sum of the distances to the $n$ dictators.

**Definition 4.6.** We define the set of reverse-balanced rules for domain restriction $S$ by

$$
\mathcal{F}_{rb}(S) = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \sum_{i \in N} \rho_S(f, d_i) \geq \sum_{i \in N} \rho_S(f', d_i) \right\}
$$

in general and by

$$
\mathcal{F}^a_{rb}(S) = \left\{ f \in \mathcal{F}^a_n \mid \forall f' \in \mathcal{F}^a_n : \sum_{i \in N} \rho_S(f, d_i) \geq \sum_{i \in N} \rho_S(f', d_i) \right\}
$$

over the set of anonymous voting rules.
Clearly, there are an infinite number of possibilities to define a set of voting rules based on the distances from individual dictators (for instance, any generalized mean of the individual distances could have been considered). However, we believe that we have chosen the simplest and most natural ones as far as the distance of two alternatives from a set which has no internal structure and the aggregation of individual distances are concerned.

### 4.2 Balanced and least dictatorial SCFs

First, we start with providing a different interpretation of balanced rules. When defining $F_{ud}(S)$, we are looking for SCFs which are in some sense the least dictatorial ones. From an opposite point of view, a SCF that chooses top alternatives of voters in as many cases as possible could result in a desirable SCF. Having this goal in mind, a measure

$$
\mu_S(f, D) = \sum_{\succ \in S} \# \{ i \in N \mid f(\succ) = d_i(\succ) \},
$$

appears as a natural candidate, which we call the measure of conformity.

Introducing the notation $\mu_S(f, g) = \sum_{\succ \in S} 1_{f(\succ)=g(\succ)}$, where $1_{f(\succ)=g(\succ)}$ indicates whether the two chosen alternatives equal, we can obtain the following relationship between $\mu_S$ and $\rho_S$:

$$
\mu_S(f, D) = \sum_{\succ \in S} \sum_{i \in N} 1_{f(\succ)=d_i(\succ)} = \sum_{i \in N} \mu_S(f, d_i) = n \cdot \#S - \sum_{i \in N} \rho_S(f, d_i).
$$

Therefore,

$$
\{ f \in F \mid \forall f' \in F : \mu_S(f, D) \geq \mu_S(f', D) \} = \left\{ f \in F \mid \forall f' \in F : \sum_{i \in N} \rho_S(f, d_i) \leq \sum_{i \in N} \rho_S(f', d_i) \right\},
$$

which means that the set of rules which maximize the number of cases in which a top alternative of a voter is chosen is identical to the set of balanced rules.

The following rule will play a special role:

**Definition 4.7.** The plurality rule $\tilde{f}_\tau$, where $\tau$ is an arbitrary tie-breaking rule, is defined in the following way: If there is a unique alternative, ranked first most often, then that alternative is the chosen one. If not, disregard those alternatives that are not ranked first most often, and select the chosen alternative based on the given tie-breaking rule.
Hence, we have defined a plurality rule with an associated tie-breaking rule. The next proposition shows how the plurality rule relates to the set of balanced rules and the set of minmax rules.

**Proposition 4.1.** Assume that $S$ is an anonymous subdomain of $\mathcal{P}^n$. Then $\tilde{f}_\tau \in \mathcal{F}_b(S)$ and if $\tau$ is anonymous, then $\tilde{f}_\tau \in \mathcal{F}_{\text{min max}}(S)$ is also true. For any $f \in \mathcal{F}^a_b(S)$ and any $g \in \mathcal{F}^a_{\text{min max}}(S)$, there exist tie-breaking rules $\tau$ and $\varphi$, respectively, such that $f = \tilde{f}_\tau$ and $g = \tilde{f}_\varphi$ on $\mathcal{S}$.

**Proof.** By the definition of $\tilde{f}_\tau$ we have
\[
\forall \succ \in \mathcal{P}^n : \# \left\{ i \in N \mid \tilde{f}_{\tau}(\succ) = d_i(\succ) \right\} \geq \# \left\{ i \in N \mid f(\succ) = d_i(\succ) \right\} \quad (4.2)
\]
for any $f \in \mathcal{F}$. Now summing (4.2) over $\mathcal{S}$, we get
\[
\mu_S(\tilde{f}_\tau, D) \geq \mu_S(f, D), \quad (4.3)
\]
from which it follows that $\tilde{f}_\tau \in \mathcal{F}_b$. From now on we assume that $\tau$ is anonymous. Note that (4.3) is equivalent with
\[
\sum_{i \in N} \rho_S(\tilde{f}_\tau, d_i) \leq \sum_{i \in N} \rho_S(f, d_i),
\]
and therefore for any $j \in N$
\[
\rho_S(\tilde{f}_\tau, d_j) = \frac{1}{n} \sum_{i \in N} \rho_S(\tilde{f}_\tau, d_i) \leq \frac{1}{n} \sum_{i \in N} \rho_S(f, d_i) \leq \max_{i \in N} \rho_S(f, d_i) \quad (4.4)
\]
since $\tau$ and $S$ are anonymous and the average is smaller than the maximum; meaning that $\tilde{f}_\tau \in \mathcal{F}_{\text{min max}}(S)$.

For the second statement observe that if $f$ selects for at least one profile in $\mathcal{S}$ an alternative that is not the most times on the top, then the inequality in (4.3) will be strict, and therefore also the inequality in (4.4) will be strict. The tie-breaking rule $\tau$ can be selected in line with $f$.

Since the set of anonymous plurality rules equals both $\mathcal{F}^a_b(S)$ and $\mathcal{F}^a_{\text{min max}}(S)$ by Proposition 4.1 we obtain the following corollary.

**Corollary 4.1.** $\mathcal{F}^a_b(S) = \mathcal{F}^a_{\text{min max}}(S)$.

The following remark clarifies the relationship between $\mathcal{F}_{\text{min max}}(S)$ and $\mathcal{F}_b(S)$. 
Remark 4.1. $\mathcal{F}_{\text{min max}}(\mathcal{S}) \subseteq \mathcal{F}_b(\mathcal{S})$.

Proof. By Proposition 4.1 we know that $\tilde{f}_\tau \in \mathcal{F}_b(\mathcal{S}) \cap \mathcal{F}_{\text{min max}}(\mathcal{S})$ if $\tau$ is anonymous. Assume that $f' \in \mathcal{F}_{\text{min max}}(\mathcal{S})$. Then for any $f \in \mathcal{F}(\mathcal{S})$ and any $j \in N$ we have

$$\rho_S(\tilde{f}_\tau,d_j) = \max_{i \in N} \rho_S(\tilde{f}_\tau,d_i) = \max_{i \in N} \rho_S(f',d_i) \leq \max_{i \in N} \rho_S(f,d_i),$$

where the first equality follows from the anonymity of $\tau$. By $\tilde{f}_\tau \in \mathcal{F}_b(\mathcal{S})$

$$n \cdot \rho_S(\tilde{f}_\tau,d_j) = \sum_{i \in N} \rho_S(\tilde{f}_\tau,d_i) \leq \sum_{i \in N} \rho_S(f',d_i)$$

for any $j \in N$. Combining (4.5) and (4.6), we get

$$\rho_S(\tilde{f}_\tau,d_i) = \rho_S(f',d_j)$$

for any $i, j \in N$, which in turn implies $f' \in \mathcal{F}_b(\mathcal{S})$. $\square$

The next remark points out that we have a proper inclusion in Remark 4.1.

Remark 4.2. $\mathcal{F}_{\text{min max}}(\mathcal{S}) \neq \mathcal{F}_b(\mathcal{S})$.

Proof. Consider the plurality rule, which breaks ties by selecting the most favored alternative of voter 1 from the set of tied alternatives. It can be verified that this rule minimizes the sum of the distances to the dictatorial rules, while it does not minimize the maximum distance to the dictatorial rules. In particular, replacing an anonymous plurality rule with a non-anonymous one does not change the sum of the distances, but may change the maximum of the distances. $\square$

Turning to the reverse-balanced rules, the following rules play a central role:

Definition 4.8. The reverse-plurality rule $f^*_\tau$, where $\tau$ is an arbitrary tie-breaking rule, is defined in the following way: If there is a single alternative, ranked first least often, then that alternative is the chosen one. If not, disregard those alternatives that are not ranked first least often, and select the chosen alternative based on the given tie-breaking rule.

Clearly, the above specified rule can also be just taken on a subset of profiles $\overline{\mathcal{S}}$ in case of a domain restriction $\mathcal{S}$ and any other known rule can be employed on $\mathcal{S}$. It is worth noting that the reverse-plurality rule differs from the anti-plurality rule known in the literature. Though both select the alternatives receiving the fewest number of votes, the former one requests the voters to vote for their most preferred alternative, while the latter one requires that they vote for their least preferred one.
The next proposition shows how the reverse-plurality rules relate to the set of reverse-balanced rules and the set of least dictatorial rules.

**Proposition 4.2.** Assume that $S$ is an anonymous subdomain of $P^n$. Then $f^*_\tau \in F_{rb}(S)$ and if $\tau$ is anonymous, then $f^*_\tau \in F_{ld}(S)$ is also true. For any anonymous $f \in F_{id}(S)$ and any anonymous $g \in F_{rb}(S)$, there exist tie-breaking rules $\tau$ and $\varphi$, respectively, such that $f = f^*_\tau$ and $g = f^*_\varphi$ on $S$.

**Proof.** First, observe that

$$\sum_{i \in N} \rho_S(f, d_i) = \sum_{i \in N} \# \{ \succ \in S \mid f(\succ) \neq d_i(\succ) \}$$

$$= \# \{ (i, \succ) \in N \times S \mid f(\succ) \neq d_i(\succ) \}$$

$$= \sum_{\succ \in S} \# \{ i \in N \mid f(\succ) \neq d_i(\succ) \}$$ \hspace{1cm} (4.7)

for any SCF $f$.

By the definition of $f^*_\tau$ for any $f$ we have

$$\forall \succ \in P^n : \# \{ i \in N \mid f^*_\tau(\succ) \neq d_i(\succ) \} \geq \# \{ i \in N \mid f(\succ) \neq d_i(\succ) \}. \hspace{1cm} (4.8)$$

Now taking the sums over $S$ on both the left hand side and the right hand side of equation (4.8) and then combining it with (4.7), we get

$$\sum_{i \in N} \rho_S(f^*_\tau, d_i) \geq \sum_{i \in N} \rho_S(f, d_i), \hspace{1cm} (4.9)$$

from which it follows that $f^*_\tau \in F_{rb}(S)$. From now on we assume that $\tau$ is anonymous. Furthermore, (4.9) implies for any $j \in N$ that

$$\rho_S(f^*_\tau, d_j) = \frac{1}{n} \sum_{i \in N} \rho_S(f^*_\tau, d_i) \geq \frac{1}{n} \sum_{i \in N} \rho_S(f, d_i) \geq \min_{i \in N} \rho_S(f, d_i) \hspace{1cm} (4.10)$$

since $f^*_\tau$ and $S$ are anonymous and the average is larger than the minimum; meaning that $f^*_\tau \in F_{id}(S)$.

For the second statement observe that if $f$ selects for at least one profile in $S$ an alternative that is not the fewest times on the top, then the inequality in (4.9), and therefore also the inequality in (4.10) will be strict. Finally, an anonymous tie-breaking rule can be chosen in line with $f$. \hfill $\square$

Since the set of anonymous reverse-plurality rules equals both $F_{id}^{an}(S)$ and $F_{rb}^{an}(S)$ by Proposition 4.2 we obtain a similar result to Corollary 4.1.
Corollary 4.2. \( F_{ld}^{an}(S) = F_{rb}^{an}(S) \).

The following two remarks can be established in an analogous way to Remarks 4.1 and 4.2.

Remark 4.3. \( F_{ld}(S) \subseteq F_{rb}(S) \).

Remark 4.4. \( F_{ld}(S) \neq F_{rb}(S) \).

Though \( f^{*}_r \) performs well according to our specification of a least dictatorial rule, as it can be easily verified, over the universal domain it can select a Pareto dominated alternative, never selects a unanimous winner, and violates monotonicity among many other desirable properties. Therefore, we have introduced anonymous domain restrictions so that, for instance, on profiles with a unanimous winner, the unanimous winner should be selected, and we are searching for the least dictatorial rules only over the set of profiles which do not have a unanimous winner. However, Proposition 4.2 shows that even if we fix our choices over an anonymous subset \( S \) of profiles, \( f^{*}_r \) has to be employed over \( \overline{S} \), if we would like to be anonymous and least dictatorial according to our definition.
Part II

Districting
Chapter 5

Axiomatic districting

The districting problem has received considerable attention recently. Much of the recent work has focused on strategic aspects and the incentives induced by different institutional designs on the political parties, legislators and voters (see, among others, Besley and Preston [16], Friedman and Holden [33], Gul and Pesendorfer [39]. Other contributions have looked at the welfare implications of different redistricting policies (e.g. Coate and Knight [26]). There is also a sizable literature on the computational aspects of the districting problem (see, e.g. Puppe and Tasnádi [66], and the references therein, and Ricca, Scozzari and Simeone [71] for a general overview of the operations research literature on the districting problem).

In contrast to these contributions this chapter, which is based on Puppe and Tasnádi [68], takes a normative point of view. We formulate desirable properties (“axioms”), and investigate which districting rules satisfy them. The axiomatic method allows one to endow the vast space of conceivable districting rules with useful additional structure: each combination of desirable properties characterizes a specific class of districting rules, and thereby helps one to assess their respective merits. Furthermore, one may hope that specific combinations of axioms single out a few, perhaps sometimes even a unique districting rule, thus reducing the space of possibilities. Finally, the axiomatic approach may reveal incompatibility of certain axioms by showing that no districting rule can satisfy certain combinations of desirable properties, thereby terminating a futile search.

In a framework with two parties and geographical constraints on the shape of districts, we propose a set of five simple axioms which are motivated by considerations of fairness to voters. The first three axioms restrict the informational basis needed for the construction of a districting. Essentially, they jointly amount to the requirement that the
The only information that may enter a fair districting rule is the number of districts won by the parties. The motivation for such a requirement is that, ultimately, voters care only about outcomes, i.e. the implemented policies, but these outcomes only depend on the distribution of seats in the parliament – through some political decision process that is not explicitly modeled here. Thus, for instance, if two different districtings induce the same seat shares in the parliament, then either none or both should be considered fair since they are indistinguishable in terms of final outcomes. Restricting the informational basis for the assessment of districting rules to the possible seat distributions they imply is also attractive from the viewpoint of managing the complexity of the districting problem, since evidently it greatly simplifies the issue. Our approach is thus “consequentialist” in the sense that the relative merits of a districting are measured only by the possible outcomes it produces. The districting process as such does not matter. We emphasize that the geographical constraints nevertheless play an important role: they enter indirectly in the assessment of districtings since they influence the possible numbers of districts each party can win. For instance, a bias in the seat share in favor of one party may be acceptable if it is forced by the given geographical constraints, but not if it is avoidable by an alternative admissible districting.

Our fourth condition, the “consistency axiom,” requires that an admissible districting should induce admissible sub-districtings on any appropriate subregion. This axiom reflects the normative principle that a “fair” institution must be fair in every part (cf. Balinski and Young’s uniformity principle [8]), or more concretely in our context: a representation of voters via a districting is globally fair only if it is also locally fair. The consistency condition greatly simplifies the internal structure of the admissible districtings, too. The fifth and final condition requires anonymity, i.e. that the districting should be invariant with respect to a re-labeling of parties. In our context, such anonymity requirement has a straightforward normative interpretation in terms of fairness since it amounts to an equal treatment of parties (and voters) ex-ante.

An important conceptual ingredient (and mathematical challenge) of our analysis is the presence of geographical constraints. We model this via an exogenously given collection of admissible districts from which a districting selects a subset that forms a partition of the entire region. We impose one restriction on the collection of admissible districts other than the standard requirement of equal population mass: that it be possible to move from one admissible districting to any other admissible districting via a sequence of intermedi-
This “linkedness” condition is satisfied by a large class of geographies. Except for a technical “no-ties” assumption, no other restriction is imposed on the collection of admissible districts, thus our approach is very general in this respect. In particular, the absence of geographical constraints can be modeled by taking all subregions of equal population mass as the collection of admissible districts (which gives rise to a linked geography). Moreover, restrictions that are frequently imposed on the shape of districts in practice, such as compactness or contiguity, can in principle be incorporated in our approach by an appropriate choice of admissible districts; for an explicit analysis of these and related issues, see e.g. Chambers and Miller [23, 24] and Nagy and Szakál [59].

We prove that on all linked geographies, the first four of our axioms jointly characterize the districting rule which maximizes the number of districts that one party can win, given the distribution of individual votes (the “optimal gerrymandering rule”). Evidently, by generating a maximal number of winning districts for one of the two parties, the optimal gerrymandering rule violates the anonymity condition. As a corollary, we therefore obtain that no districting rule can satisfy all five axioms. The result also suggests that any reasonable districting rule must necessarily be complex: either it has a complex internal structure by violating the consistency principle, or it has to employ a complex informational basis in the sense that it depends on more than the mere number of districts won by each party.

The work closest to ours in the literature is Chambers [21] and [22] who also takes an axiomatic approach. However, one of his central conditions is the requirement that the election outcome be independent of the way districts are formed (“gerrymandering-proofness”), and the main purpose of his analysis is to explore the consequences of this requirement (for a similar approach, see Bervoets and Merlin [14, 15]). By contrast, our focus is precisely on the issue how the districting influences the election outcome, and the aim of our analysis is to structure the vast space of possibilities by means of simple principles. In particular, geographical constraints which are absent in Chambers’ model play an important role in our analysis.

The paper by Landau, Reid and Yershov [49] also addresses the issue of “fair” districting. However, unlike our work their paper is concerned with the question of how to implement a fair solution to the districting problem by letting the parties themselves determine the boundaries of districts. Specifically, these authors propose a protocol similar

\[\text{Since a districting forms a partition of the given region, it is evidently not possible to move from one districting to another districting by changing only one district.}\]
The districting rules that we consider depend among other things on the distribution of votes for each party in the population. One might argue, perhaps on grounds of some “absolute” notion of \textit{ex ante} fairness, that a districting rule must not depend on voters’ party preferences since these can change over time. From this perspective, the districting problem is not really an issue and it would seem that any districting which partitions the population in (roughly) equally sized subgroups should be acceptable. By contrast, in the present paper we are interested in a “relative” or \textit{ex post} notion of fair districting, i.e. in the question of what would constitute an acceptable districting rule given the distribution of the supporters of each party in the population. This question seems particularly important for practical purposes since a districting policy can be successfully implemented only if it receives sufficient support by the \textit{actual} legislative body.

\section{Districting problem}

We assume that parties $A$ and $B$ compete in an electoral system consisting only of single member districts, where the representatives of each district are determined by plurality. The parties as well as the independent bodies face the following districting problem.

\begin{definition}[Districting problem] A \textit{districting problem} is given by the structure $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$, where

\begin{itemize}
  \item the voters are located within a subset $X$ of the plane $\mathbb{R}^2$,
  \item $\mathcal{A}$ is the $\sigma$-algebra on $X$ consisting of all districts that can be formed without geographical or any other type of constraints,
  \item the distribution of voters is given by a measure $\mu$ on $(X, \mathcal{A})$,
  \item the distributions of party $A$ and party $B$ supporters are given by measures $\mu_A$ and $\mu_B$ on $(X, \mathcal{A})$ such that $\mu = \mu_A + \mu_B$,
  \item $t$ is the given number of seats in parliament,
  \item $G \subseteq \mathcal{A}$, also called \textit{geography}, is a collection of admissible districts satisfying $\mu(g) = \mu(X) / t$ and
    \begin{equation}
    \mu_A(g) \neq \mu_B(g)
    \end{equation}
\end{itemize}
\end{definition}
for all \( g \in G \), and admitting a partitioning of \( X \), i.e. there exist mutually disjoint sets \( g'_1, \ldots, g'_t \in G \) such that \( \bigcup_{i=1}^t g'_i = X \).

Condition (5.1) excludes ties in the distribution of party supporters in all admissible districts to avoid the necessity of introducing tie-breaking rules. This condition is satisfied, for instance, if the set of voters is finite, \( \mu, \mu_A, \mu_B \) are the counting measures and the district sizes are odd.

**Definition 5.2** (Districting). A districting for problem \( \Pi = (X, A, \mu, \mu_A, \mu_B, t, G) \) is a subset \( D \subseteq G \) such that \( D \) forms a partition of \( X \) and \( \#D = t \).

We shall denote by \( \delta_A(D) \) and \( \delta_B(D) \) the number of districts won by party \( A \) and party \( B \) under \( D \), respectively. We write \( D_\Pi \) for the set of all districtings of problem \( \Pi \) and let \( \delta_A(D) = \{ \delta_A(D) : D \in D \} \) and \( \delta_B(D) = \{ \delta_B(D) : D \in D \} \) for any \( D \subseteq D_\Pi \).

**Definition 5.3** (Solution). A solution \( F \) associates to each districting problem \( \Pi \) a non-empty set of chosen districtings \( F_\Pi \subseteq D_\Pi \).

### 5.2 Several Solutions

We now present a number of simple solution candidates. The first solution determines the optimal partisan gerrymandering from the viewpoint of party \( A \).

**Definition 5.4** (Optimal solution for \( A \)). The optimal solution \( O^A \) for party \( A \) determines for districting problem \( \Pi = (X, A, \mu, \mu_A, \mu_B, t, G) \) the set of those districtings that maximize the number of winning districts for party \( A \), i.e.

\[
O^A_\Pi = \arg \max_{D \in D_\Pi} \delta_A(D).
\]

Evidently, in the absence of other objectives, \( O^A \) is the solution favored by party \( A \) supporters. The optimal solution \( O^B \) for party \( B \) is defined analogously. If we are referring to an optimal solution \( O \), then we have either \( O^A \) or \( O^B \) in mind.

The next solution minimizes the difference in the number of districts won by the two parties. It has an obvious egalitarian spirit.

**Definition 5.5** (Most equal solution). The solution \( ME \) determines for districting problem \( \Pi = (X, A, \mu, \mu_A, \mu_B, t, G) \) the set of most equal districtings, i.e.

\[
ME_\Pi = \arg \min_{D \in D_\Pi} |\delta_A(D) - \delta_B(D)|.
\] (5.2)
Clearly, depending on the distribution of votes in the population, an equal distribution of seats in the parliament may not be possible. The most equal solution aims to get as close as possible to equality in terms of the number of winning districts for the two parties.

The third solution maximizes the difference in the number of districts won by the two parties. The objective to maximize the winning margin of the ruling party could be motivated, for instance, by the desire to avoid too much political compromise.

**Definition 5.6** (Most unequal solution). The solution $MU$ determines for districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of most unequal districtings, i.e.

$$MU_\Pi = \arg \max_{D \in D_\Pi} |\delta_A(D) - \delta_B(D)| .$$

Fourth, we consider the solution that minimizes partisan bias. It has a clear motivation from the point of view of maximizing representation of the “people’s will” in the sense that the share of the districts won by each party is as close as possible to its share of votes in the population.

**Definition 5.7** (Least biased solution). The solution $LB$ determines for districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of those districtings that minimize the absolute difference between shares in winning districts and shares in votes, i.e.

$$LB_\Pi = \arg \min_{D \in D_\Pi} \left| \frac{\delta_A(D)}{\mu(X)} - \frac{\mu_A(X)}{\mu(X)} \right| = \arg \min_{D \in D_\Pi} \left| \frac{\delta_B(D)}{\mu(X)} - \frac{\mu_B(X)}{\mu(X)} \right| .$$

Finally, we mention the trivial solution that associates to each problem the set of all admissible districtings.

**Definition 5.8** (Complete solution). The complete solution $C$ associates with any districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ the set of all possible districtings $D_\Pi$.

### 5.3 Axioms

In this section, we formulate five simple axioms and argue that each has appeal from a normative (and sometimes also from a pragmatic) point of view.

The case of two districts plays a fundamental role in our analysis. Note that by (5.1) it is not possible that a party can win both districts under one districting and lose both districts under another districting, i.e. if $t = 2$ then $\delta_A(D_\Pi)$ (respectively, $\delta_B(D_\Pi)$) cannot contain both 0 and 2. Our first axiom requires that a solution must in fact be “determinate”
in the two-district case in the sense that it must not leave open the issue whether there is
a draw between the two parties or a victory for one party. In other words, if a solution
chooses a districting that results in a draw between the parties for a given problem it
cannot choose another districting for the same problem that results in a victory for one
party.

**Axiom 5.1** (Two-district determinacy). A solution $F$ satisfies *two-district determinacy* if
for any districting problem $\Pi$ with $t = 2$, the sets $\delta_A(F_\Pi)$ and $\delta_B(F_\Pi)$ are singletons.

The motivation for this axiom stems from our implicit assumption that voters do not
care about the districtings as such, but only about the entailed shares of seats in the
parliament, since it is the latter that influences final outcomes. Any indeterminacy in
the distribution of seats in the parliament potentially influences the outcome and would
thus introduce an element of *arbitrariness* of the final state of affairs. In the two-district
case, such indeterminacy necessarily turns a (unanimous) victory of one party into a draw
between the two parties, or vice versa. Two-district determinacy prevents this to occur.

Evidently, all solutions considered in Section 5.2 with the exception of the complete
solution $C$ satisfy Axiom 5.1. Also observe that on the family of all two-district problems
the most equal solution $ME$ and the least biased solution $LB$ coincide.

Our next axiom requires that a solution behaves “uniformly” on the set of two-district
problems in the sense that the solution must treat different two-district problems in the
same way, provided they admit the same set of possible distributions of the number of
districts won by each party.

**Axiom 5.2** (Two-district uniformity). A solution $F$ satisfies *two-district uniformity* if
for any districting problems $\Pi$ and $\Pi'$ with $t = 2$ such that $\delta_A(D_\Pi) = \delta_A(D_{\Pi'})$ (and therefore
also $\delta_B(D_\Pi) = \delta_B(D_{\Pi'})$) we have $\delta_A(F_\Pi) = \delta_A(F_{\Pi'})$ (and therefore also $\delta_B(F_\Pi) = \delta_B(F_{\Pi'})$).

Even though it is imposed only in the two-district case, Axiom 5.2 is admittedly a
strong requirement. It can be motivated by invoking again the assumption that voters care
about districtings only via their influence on political outcomes. From this perspective,

---

2Observe that overall determinacy, i.e. that $\delta_A(F_\Pi)$ and $\delta_B(F_\Pi)$ be singletons for *every* problem $\Pi$, is a strictly stronger requirement than two-district determinacy; for instance, the least biased solution satisfies two-district determinacy but can easily be shown to violate overall determinacy.

3To verify this, observe that if there exist admissible districtings $D, D' \in D_{\Pi}$ with $\delta_A(D) = 2$ and $\delta_A(D') = 1$, then one must have $0.5 < \mu_A(X)/\mu(X) < 0.75$. Thus, $D'$ must be chosen both by $ME$ and $LB$. 

Axioms 5.2 states that if the possible political outcomes are the same in different two-district problems, then the actual outcome should also be the same. A violation of Axiom 5.2 would mean that characteristics other than the possible distributions of seat shares can influence the solution and hence the final outcome. But if these characteristics play no role in voters’ preferences, it is not clear how one could justify such influence. To illustrate, consider two districting problems $\Pi$ and $\Pi'$ with $\delta_A(D_\Pi) = \delta_A(D_{\Pi'}) = \{1, 2\}$; thus, in either situation there exists one districting under which party $A$ wins both districts and another districting which produces a draw between the two parties. Now assume that party $A$’s share of votes in situation $\Pi$ is in fact larger than its share of votes in situation $\Pi'$, i.e. $\mu_A > \mu'_A$. Couldn’t this give a good reason to select the districting under which $A$ wins both seats in situation $\Pi$ but the districting in which both parties receive one seat in situation $\Pi'$, provided that the difference between $\mu_A$ and $\mu'_A$ is sufficiently large? But then, how large precisely is “sufficiently large”? Is x% enough? And wouldn’t the threshold also have to depend on the absolute level of $\mu_A$? Two-district uniformity answers these question by a very clearcut and simple recommendation: different treatment of different two-district situations, for instance on the grounds that one party has a larger share of votes in one of the situations, is justified only if the difference manifests itself in a difference of the possible number of seats in parliament that the parties can win. Two-district uniformity thus sets a high “threshold” for differential treatment of two-district situations. We emphasize therefore that all candidate solutions presented in Section 5.2 above satisfy this condition; for the least biased solution this follows from Footnote 3 for the other solutions it is evident.

A secondary motivation for Axiom 5.2 is to keep the complexity of a districting solution manageable. Indeed, any influence of characteristics different from the possible seat distribution in parliament – whether derived from the underlying distribution of party supporters or from geographical information – would considerably complicate the definition and implementation of a districting rule.

Our third axiom, imposed on districting problems of any size, has a motivation related to that of the two previous axioms. It states that if a possible districting induces the same distribution of the number of winning districts for each party than some districting chosen by a solution, it must be chosen by this solution as well.

**Axiom 5.3 (Indifference).** A solution $F$ satisfies *indifference* if for any districting problem $\Pi$ we have that $D \in F_\Pi$, $D' \in D_\Pi$, $\delta_A(D) = \delta_A(D')$ and $\delta_B(D) = \delta_B(D')$ implies $D' \in F_\Pi$.

The justification of the indifference axiom is straightforward under the intended notion
of fairness to voters. If voters care only about final outcomes, and if final outcomes only depend on seat shares, then two districtings that entail the same seat distribution in parliament are undistinguishable in terms of final outcomes and have therefore to be treated equally. Evidently, all solutions presented above satisfy this condition.

The following consistency axiom plays a central role in our analysis. It requires that a solution to a problem should also deliver appropriate solutions to specific subproblems. Its spirit is very similar to the uniformity principle in Balinski and Young’s \cite{8} theory of apportionment (“every part of a fair division should be fair”).

Prior to the definition of consistency we have to introduce specific subproblems of a districting problem. For any problem $\Pi$, any $D \in F_\Pi$ and any $D' \subseteq D$, let $Y = \bigcup_{d \in D'} d$ and define the subproblem $\Pi|_Y$ to be $(Y, A|_Y, \mu|_Y, \mu_A|_Y, \mu_B|_Y, \#D', G|_Y)$, where $A|_Y = \{A \cap Y : A \in A\}$, $G|_Y = \{g \in G : g \subseteq Y\}$ and $\mu|_Y, \mu_A|_Y, \mu_B|_Y$ stand for the restrictions of measures $\mu, \mu_A, \mu_B$ to $(Y, A|_Y)$.

**Axiom 5.4** (Consistency). A solution $F$ satisfies consistency if for any districting problem $\Pi$, any $D \in F_\Pi$ and any $D' \subseteq D$ we have for $Y = \bigcup_{d \in D'} d$ that

$$D' \in F_{\Pi|_Y}.$$  

The motivation for imposing the consistency condition in our context is as follows. Most federal countries have both federal and local legislatures, and in many of those countries the same districts are used for both, local and federal elections. The consistency axiom requires that a districting is a global solution, i.e. can be considered “globally fair,” only if it also represents a solution on all appropriate subregions, i.e. is also everywhere “locally fair.”[^4] In other words, consistency forbids to create a globally fair treatment of voters by equilibrating different locally unfair treatments. Moreover, it justifies using the same districts locally and globally – as is common practice in most countries. Finally, consistency may also be of practical value if regions decide to separate, or to increase political independence, since it would allow them to use the same districting as before.

The optimal and complete solutions satisfy consistency. This is evident for the complete solution. To verify it for the optimal solution suppose, by contradiction, that there would exist $D' \subset D \in O^A_{\Pi}$ such that $D' \notin O^A_{\Pi|_Y}$, where $Y = \bigcup_{d \in D'} d$. This would imply $\delta_A(D'' \cup (D \setminus D')) > \delta_A(D)$ for any $D'' \in O^A_{\Pi|_Y}$, a contradiction.

By contrast, the other solutions considered in Section 5.2 violate consistency. This can be verified by considering the districting problem $\Pi$ with $t = 3$ shown in Figure 5.1. It

[^4]: Clearly, this requirement has to be restricted to subregions that are unions of districts, since a given districting does not necessarily induce an admissible sub-districting on other subregions.
consists of 27 voters of which 11 are supporters of party A (indicated by empty circles) and 16 are supporters of party B (indicated by solid circles), and four admissible districtings $D_1 = \{d_1, d_2, d_3\}$, $D_2 = \{d_1, d_4, d_5\}$, $D_3 = \{d_3, d_7, d_8\}$ and $D_4 = \{d_5, d_7, d_9\}$. Note that party A wins two out of the three districts in $D_1$ and $D_2$, respectively, and one of the three districts in $D_3$ and $D_4$, respectively. Consider the solution $ME$ first. Since the difference in the number of winning districts for the two parties is one in all cases, we have $ME_{\Pi} = \{D_1, D_2, D_3, D_4\}$. Consider the districting $D_1 \in ME_{\Pi}$ and $Y = d_1 \cup d_2$. Consistency would require that the districting $\{d_1, d_2\}$ is among the chosen districtings if the solution is applied to the restricted problem on $Y$. But obviously, we have $ME_{\Pi|Y} = \{\{d_7, d_8\}\}$, because the districting $\{d_7, d_8\}$ induces a draw between the winning districts on $Y$ while the districting $\{d_1, d_2\}$ entails two winning districts for party A (and zero districts won by party B). Similarly, $MU$ violates consistency with $D_3 \in MU_{\Pi}$ and $Y = d_7 \cup d_8$ since $MU_{\Pi} = \{D_1, D_2, D_3, D_4\}$ and $MU_{\Pi|Y} = \{\{d_1, d_2\}\}$.

![Figure 5.1: ME, MU and LB violate consistency.](image)

To verify, finally, that also $LB$ violates consistency observe first that $LB_{\Pi} = \{D_3, D_4\}$ in Figure 5.1. Consider $D_4 \in LB_{\Pi}$ and $Y = d_7 \cup d_9$. Consistency would require that the districting $\{d_7, d_9\}$ is among the districtings chosen by the solution on the restricted problem on $Y$. But it is easily seen that $LB_{\Pi|Y} = \{\{d_1, d_4\}\}$, since the districting $\{d_1, d_4\}$ gives rise to a draw between the parties on $Y$ which is closer to their respective relative shares of votes on $Y$. Thus the least biased solution also violates consistency.

Our final axiom expresses a very fundamental principle of fairness and equity in our context, namely the symmetric treatment of parties ex ante.

**Axiom 5.5 (Anonymity).** A solution $F$ satisfies anonymity if exchanging the distributions of party $A$ and party $B$ voters $\mu_A$ and $\mu_B$ does not change the set of chosen districtings: for all districting problems $\Pi = (X, A, \mu, \mu_A, \mu_B, t, G)$,

$$D \in F_{(X, A, \mu, \mu_A, \mu_B, t, G)} \text{ if and only if } D \in F_{(X, A, \mu_B, \mu_A, t, G)}.$$

Note that this can also be interpreted as a requirement of anonymity with respect to voters across different parties; indeed, anonymity with respect to voters of the same party
is already implicit in our definition of a districting problem since only the aggregate mass of parties’ supporters matters and not their identity. It is easily seen that all solutions presented so far with exception of the optimal solution(s) satisfy the anonymity axiom.

In the following we will show that for a large class of geographies no solution can satisfy all five axioms simultaneously. While we consider the anonymity condition to be an indispensable fairness requirement, our proof strategy is to show that the first four axioms characterize the optimal partisan gerrymandering solution $O$. Since this solution evidently violates anonymity the impossibility result follows.

5.4 A Characterization Result and an Impossibility

First, we consider districting problems with only two districts.

**Lemma 5.1.** $F$ satisfies two-district determinacy, two-district uniformity and indifference if and only if $F = O$, $F = ME$ or $F = MU$ for $t = 2$.

*Proof.* Observe that two-district determinacy and two-district uniformity jointly reduce the set of possible districting rules for $t = 2$ to $O$, $ME$ and $MU$ if only the number of winning districts matters (recall that $ME = LB$ on all two-district problems). Now indifference ensures that either all two-to-zero, all one-to-one, or all zero-to-two districtings admissible for problem $\Pi$ have to be selected by solution $F$.

Finally, we have seen that $O$, $ME$ and $MU$ satisfy two-district determinacy, two-district uniformity and indifference, which completes the proof. \qed

Consider a districting problem for $t = 3$ with the 9 admissible districts and the 3 resulting districtings shown in Figure 5.2, in which party $A$ voters are indicated by empty circles and party $B$ voters by solid circles, $\mu$ equals the counting measure on $(X, \mathcal{A})$ and $\mu_A$, $\mu_B$ are given by the respective number of party $A$ and party $B$ voters. It can be verified that, considering the districtings from left to right, we obtain 3 to 0, 2 to 1 and 1 to 2 winning districtings for party $A$, respectively. Thus, e.g. the optimal solution for party $A$ would choose the first districting from the left, while the least biased solution would choose the middle districting.

The geography in the depicted problem is “thin” in the sense that all proper subproblems allow only one possible districting. Therefore, the consistency condition has no bite at all in this problem. In order to make use of the consistency property, we will restrict the family of admissible geographies in the following way.
Figure 5.2: Unlinked districtings.

**Definition 5.9.** The geography $G$ of a problem $\Pi = (X, A, \mu, \mu_A, \mu_B, t, G)$ is linked if for any two possible districtings $D, D' \in D_\Pi$ there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D = D_1$, $\{D_2, \ldots, D_{k-1}\} \subseteq D_\Pi$, $D' = D_k$, and $\#D_i \cap D_{i+1} = t - 2$ for all $i = 1, \ldots, k - 1$.

In Section 5.5 we present a large and natural class of linked geographies, which arise from what we call regular districting problems. In a regular districting problem, $\mu$ is given by some finite measure that is absolutely continuous with respect to the Lebesgue measure, and the admissible districts are the bounded Borel sets whose boundary is a Jordan curve.

While the linkedness condition clearly limits the scope of our analysis, there is no hope in obtaining characterization results of the sort derived here without further assumptions on the family of geographies. Note also that under many specifications of the measure $\mu$ the unrestricted geography which admits all subsets of size $\mu(X)/t$ is linked (for instance, this holds if the set of voters is finite and $\mu$ is the counting measure).

**Proposition 5.1.** If $F$ equals $O^A$ for $t = 2$ and $F$ is consistent and indifferent, then $F = O^A$ for linked geographies.

**Proof.** Consider a districting problem $\Pi = (X, A, \mu, \mu_A, \mu_B, t, G)$ with $t \geq 3$ and suppose that $F_\Pi \neq O^A_\Pi$ but $F$ is consistent and indifferent. Since $F_\Pi$ is not $O^A_\Pi$, there exist $D' \in O^A_\Pi$ and $D \in F_\Pi$ such that $\delta_A(D') > \delta_A(D)$ by indifference. Since $\Pi$ has a linked geography there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D' = D_1$, $\{D_2, \ldots, D_{k-1}\} \subseteq D_\Pi$, $D = D_k$ and $\#D_i \cap D_{i+1} = t - 2$ for all $i = 1, \ldots, k - 1$.

We claim that

$$|\delta_A(D_i) - \delta_A(D_{i+1})| \leq 1$$

(5.5)

for all $i = 1, \ldots, k - 1$, where $D_i$ and $D_{i+1}$ just differ in two districts. To verify (5.5) we shall denote the two pairs of different districts by $d, d', e$ and $e'$, where the first two districts belong to $D_i$ while the latter two to $D_{i+1}$. Observe that $D_i \setminus \{d, d'\} = D_{i+1} \setminus \{e, e'\}$.
by linkedness. Hence,

\[
\delta_A(D_i) - \delta_A(D_{i+1}) = \delta_A(\{d, d'\}) + \delta_A(D_i \setminus \{d, d'\}) - \delta_A(\{e, e'\}) - \\
\delta_A(D_{i+1} \setminus \{e, e'\}) = \delta_A(\{d, d'\}) - \delta_A(\{e, e'\}).
\]

By (5.1) we must have \(|\delta_A(\{d, d'\}) - \delta_A(\{e, e'\})| \leq 1\), which implies, taking (5.6) into consideration, (5.5).

Let \(j^* \in \{2, \ldots, k\}\) be the smallest index such that \(\delta_A(D_{j^*}) = \delta_A(D_k)\). Since \(D_k \in F_{\Pi}\) we have \(D_{j^*} \in F_{\Pi}\) by indifference. Linkedness ensures that \(D_{j^*-1}\) and \(D_{j^*}\) just differ in two districts, which we shall denote by \(d, d', e\) and \(e'\), where the first two districts belong to \(D_{j^*-1}\) while the latter two to \(D_{j^*}\). Furthermore, \(D_{j^*-1} \setminus \{d, d'\} = D_{j^*} \setminus \{e, e'\}\) by linkedness. Let \(Y = d \cup d' = e \cup e'\). Since \(F\) is consistent we have \(\{e, e'\} \in F_{\Pi|Y}\). Our assumption that \(F\) equals \(O^A\) for \(t = 2\) implies \(\delta_A(\{d, d'\}) \leq \delta_A(\{e, e'\})\). If \(j^* = 2\), by consistency

\[
\delta_A(D_1) > \delta_A(D_k) = \delta_A(D_2) = \delta_A(\{e, e'\}) + \delta_A(D_2 \setminus \{e, e'\}) \\
\delta_A(D_1 \setminus \{d, d'\}) = \delta_A(D_1);
\]

a contradiction. Otherwise, suppose that \(j^* > 2\). Then by consistency and the optimality of \(F\) on \(Y\) we must have \(\delta_A(D_{j^*-1}) \leq \delta_A(D_{j^*})\). Moreover, \(\delta_A(D_{j^*-1}) < \delta_A(D_{j^*})\) by the definition of \(j^*\). Then by (5.5) and

\[
\delta_A(D_1) > \delta_A(D_k) = \delta_A(D_{j^*}) > \delta_A(D_{j^*-1})
\]

there exists a \(j' \in \{2, \ldots, j^* - 1\}\) such that \(\delta_A(D_{j'}) = \delta_A(D_k)\). Clearly, \(D_{j'} \in F_{\Pi}\) by indifference, contradicting the definition of \(j^*\). \(\square\)

Since neither the most equal or most unequal solutions satisfy consistency we cannot extend \(ME\) or \(MU\) for \(t = 2\) to arbitrary \(t\) in the manner of Proposition 5.1. However, it might be the case that \(ME\) or \(MU\) for \(t = 2\) can be extended to another consistent solution. The next proposition demonstrates that such an extension does not exist.

**Proposition 5.2.** There does not exist a consistent and indifferent solution \(F\) that equals \(ME\) or \(MU\) for \(t = 2\) even for linked geographies.

\(^5\)We would like to thank Dezső Bednay for suggestions that improved our original proof.
Proof. Suppose that there exists a consistent and indifferent solution $F$ that equals $ME$ for $t = 2$. Consider the districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, 3, G)$, where $X$ consists of 27 voters, $\mathcal{A}$ equals the set of all subsets of $X$, $\mu$ is the counting measure, and $G = \{d_1, \ldots, d_9\}$ is as shown in Figure 5.2 in which party $A$ supporters are indicated by empty circles and party $B$ supporters by solid circles.

![Figure 5.3: ME and MU cannot be extended.]

We can see from Figure 5.2 that the four possible districtings are $D_1 = \{d_1, d_2, d_3\}$, $D_2 = \{d_2, d_4, d_5\}$, $D_3 = \{d_1, d_6, d_7\}$ and $D_4 = \{d_3, d_8, d_9\}$. It can be checked that the given geography is linked. Since $\delta_A(D_1) = 2$ and $\delta_A(D_2) = \delta_A(D_3) = \delta_A(D_4) = 1$ we must have either $\{D_1\} = F_\Pi$, $\{D_2, D_3, D_4\} = F_\Pi$ or $\{D_1, D_2, D_3, D_4\} = F_\Pi$ by indifference. First, consider the cases of $\{D_1\} = F_\Pi$ and $\{D_1, D_2, D_3, D_4\} = F_\Pi$. By consistency we must have $\{d_1, d_2\} \in F(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')$, where $X' = d_1 \cup d_2$, $G' = \{d_1, d_2, d_8, d_9\}$ and $\mathcal{A}'$, $\mu'$, $\mu'_A$, $\mu'_B$ denote the restrictions of $\mathcal{A}$, $\mu$, $\mu_A$, $\mu_B$ to $X'$, respectively. However, $F(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')$ should equal $\{d_8, d_9\}$ since $F = ME$ for $t = 2$; a contradiction. Second, consider the case of $\{D_2, D_3, D_4\} = F_\Pi$ and pick the case of $D_3$. By consistency we must have $\{d_6, d_7\} \in F(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')$, where $X'' = d_6 \cup d_7$, $G'' = \{d_2, d_3, d_6, d_7\}$ and $\mathcal{A}'', \mu'', \mu''_A, \mu''_B$ denote the restrictions of $\mathcal{A}$, $\mu$, $\mu_A$, $\mu_B$ to $X''$, respectively. However, $F(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')$ should equal $\{d_2, d_3\}$ since $F = ME$ for $t = 2$; a contradiction.

Now suppose that there exists a consistent and indifferent solution $F$ that equals $MU$ for $t = 2$. Consider once again the problem shown in Figure 5.2. First, consider the cases of $\{D_1\} = F_\Pi$ and $\{D_1, D_2, D_3, D_4\} = F_\Pi$. By consistency we must have $\{d_1, d_3\} \in F(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')$, where $X' = d_1 \cup d_3$, $G' = \{d_1, d_3, d_4, d_5\}$ and $\mathcal{A}'$, $\mu'$, $\mu'_A$, $\mu'_B$ denote the restrictions of $\mathcal{A}$, $\mu$, $\mu_A$, $\mu_B$ to $X'$, respectively. However, $F(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')$ should equal $\{d_4, d_5\}$ since $F = MU$ for $t = 2$; a contradiction. Second, consider the case of $\{D_2, D_3, D_4\} = F_\Pi$ and pick the case of $D_3$. By consistency we must have $\{d_8, d_9\} \in F(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')$, where $X'' = d_8 \cup d_9$, $G'' = \{d_1, d_2, d_8, d_9\}$ and $\mathcal{A}'', \mu'', \mu''_A, \mu''_B$ denote the restrictions of $\mathcal{A}$, $\mu$, $\mu_A$, $\mu_B$ to $X''$, respectively. However, $F(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')$ should equal $\{d_1, d_2\}$ since $F = MU$ for $t = 2$; a contradiction. □
Our main theorem follows from Lemma 5.1 and Propositions 5.1 and 5.2.

**Theorem 5.1.** The optimal solution $O$ is the only solution that satisfies two-district determinacy, two-district uniformity, indifference and consistency on linked geographies.

We verify, on linked geographies, the tightness of Theorem 5.1, i.e. the independence of the axioms. First, the complete solution only violates two-district determinacy. Second, $ME$, $MU$ and $LB$ just violate consistency.

Third, we investigate indifference. Consider the districting problem $\Pi'$ given by Figure 5.4 in which $X'$ consists of 27 voters, $A'$ equals the set of all subsets of $X'$, $\mu'$ is the counting measure, and $G'$ admit the districts shown in Figure 5.4 where party $A$ supporters are indicated by empty circles and party $B$ supporters by solid circles. Observe that any two consecutive districtings in the sequence $D_1, \ldots, D_4$ only differ in two districts, and therefore, $\Pi'$ has a linked geography. We shall denote by $F$ the solution given by

$$F_{\Pi'} = \begin{cases} \{D_1\} & \text{if } \Pi = \Pi', \\ \Omega^A_{\Pi'} & \text{if } \Pi' \text{ is not a subproblem of } \Pi \text{ and} \\ \{D_4\} \cup \Omega^A_{\Pi'|X \setminus X'} & \text{if } \Pi' \text{ is a subproblem of } \Pi, 
\end{cases}$$

where the voters of problem $\Pi$ are located within $X$ and we say that $\Pi'$ is a subproblem of $\Pi$ if $\Pi' = \Pi|_{X'}$ and $X'$ can be partitioned into three equally sized districts by picking three districts from the geography of problem $\Pi$. It can be verified that $F$ satisfies two-district determinacy, two-district uniformity and consistency. Clearly, $F \neq O^A$ because of $\delta_A(D_1) = 3 > \delta_A(D_4) = 2$ and indifference is violated since otherwise $D_4 \in F_{\Pi}$ should imply $D_2 \in F_{\Pi}$.

Finally, to verify that two-district uniformity cannot be dropped from the list of conditions in Theorem 5.1 we are again considering problem $\Pi'$ from Figure 5.4 and are modifying solution $F$ slightly. We shall denote the two-district subproblem of $\Pi'$ on $X_1 = X' \setminus \{d_3\}$, which consists in choosing either districting $\{d_1, d_2\}$ or $\{d_4, d_5\}$, by $\Pi_1$. 

*Figure 5.4: Indifference is necessary.*
Define $\hat{F}$ as follows,

$$
\hat{F}_\Pi = \begin{cases} 
\{D_2, D_4\} & \text{if } \Pi = \Pi', \\
O^A_\Pi & \text{if } \Pi' \text{ and } \Pi_1 \text{ are not a subproblems of } \Pi \\
\{D_2, D_4\} \cup O^A_{\Pi|X'\setminus X} & \text{if } \Pi' \text{ is a subproblem of } \Pi, \\
\{d_4, d_5\} \cup O^A_{\Pi|X'\setminus X_1} & \text{if } \Pi' \text{ is not a subproblem of } \Pi \text{ but } \\
\Pi_1 \text{ is a subproblem of } \Pi.
\end{cases}
$$

It can be checked that $\hat{F}$ satisfies two-district determinacy, indifference and consistency, but violates two-district uniformity.

Remark 5.1. Two-district determinacy is strictly weaker than overall determinacy\textsuperscript{*} even in the presence of two-district uniformity and consistency.

This can verified by considering the problem $\Pi'$ defined in Figure 5.4 and a slight modification of the construction of solution $F$ described two paragraphs earlier. Denote by $\tilde{F}$ the solution given by

$$
\tilde{F}_\Pi = \begin{cases} 
\{D_1, D_4\} & \text{if } \Pi = \Pi', \\
O^A_\Pi & \text{if } \Pi' \text{ is not a subproblem of } \Pi \text{ and } \\
\{D_1, D_4\} \cup O^A_{\Pi|X'\setminus X} & \text{if } \Pi' \text{ is a subproblem of } \Pi,
\end{cases}
$$

where the voters of problem $\Pi$ are located within $X$. It is easily seen that $\tilde{F}$ satisfies two-district uniformity, consistency and two-district determinacy, but violates overall determinacy.

We obtain the following result as a simple corollary of Theorem 5.1.

**Corollary 5.1.** There does not exist a two-district determinate, two-district uniform, indifferent, consistent and anonymous solution on linked geographies.

### 5.5 Regular Districting Problems

We have already seen examples of linked geographies in Figures 5.1, 5.3 and 5.4. In this section we provide a natural and large class of further examples of districting problems with linked geographies. One might wonder why we need $\sigma$-algebras instead of algebras in this chapter and why we do not give a general class of discrete examples. The reason for this is that we could not produce a general class of linked geographies remaining in the

\textsuperscript{*}For a definition of overall determinacy see Footnote 3.
discrete setting. Akitaya et al. \cite{1} points into the direction why we could face difficulties when considering a discrete setting.\footnote{Though in their model voters can be moved from one district to another in a graph-theoretic setting instead of exchanging an identical number voters between two districts.}

A bounded subset $A$ of $\mathbb{R}^2$ will be called \textit{strictly connected} if its boundary $\partial A$ is a Jordan curve (i.e. a non self-intersecting continuous loop). A subset $A$ of a strictly connected set $B \subseteq \mathbb{R}^2$ \textit{separates} $B$ if $B \setminus A$ is not strictly connected. We call a continuous function $f : X \to \mathbb{R}$ \textit{nowhere constant} if for any $x \in X$ and any neighborhood $N(x)$ of $x$ there exists a $y \in N(x)$ such that $f(x) \neq f(y)$.

\textbf{Definition 5.10} (Regular Districting Problems). A districting problem $\Pi = (X, A, \mu, \mu_A, \mu_B, t, G)$ is called regular if

1. $X$ is a bounded and strictly connected subset of $\mathbb{R}^2$,
2. $A$ equals the set of Borel sets on $X$, i.e. following standard notation $A = \mathcal{B}(X)$,
3. $\mu$ is a finite and absolutely continuous measure on $(X, \mathcal{B}(X))$ with respect to the Lebesgue measure,
4. $G$ consists of all bounded, strictly connected and $\mu(X)/t$ sized subsets lying in $\mathcal{B}(X)$ and satisfying (5.1),
5. there exists a continuous nowhere constant function $f : X \to \mathbb{R}$ such that $\mu_A(C) = \int_C f(\omega)d\mu(\omega)$ for all $C \in \mathcal{B}(X)$, and
6. $\mu_B$ is given by $\mu_B(C) = \mu(C) - \mu_A(C)$ for all $C \in \mathcal{B}(X)$.

The fifth condition is a technical assumption to ensure that the districtings emerging in the proof of Lemma 5.3 below can be selected in a way that they satisfy (5.1). Specifically, we have the following lemma.

\textbf{Lemma 5.2}. \textit{If we have two neighboring\footnote{We call two subsets of the plane neighboring if they share a common boundary of positive length.} bounded, strictly connected and $\mu(X)/t$ sized sets $d, e \in \mathcal{B}(X)$ such that $\mu_A(d) = \mu(d)/2$ (i.e $d$ violates (5.1)), then we can exchange territories between $d$ and $e$ in a way that the two resulting bounded, strictly connected and $\mu(X)/t$ sized sets $d', e' \in \mathcal{B}(X)$ satisfy (5.1).}
Proof. Pick a point \( x \in \partial d \cap \partial e \) from the relative interior of the common boundary of \( d \) and \( e \). Since \( f \) is nowhere constant there exists a \( y \) arbitrarily close to \( x \) in the interior of \( d \) such that \( f(y) \neq f(x) \). Assume that \( f(y) > f(x) \). There exist a neighborhood \( N_{\varepsilon_y}(y) \) of \( y \) and a neighborhood \( N_{\varepsilon_x}(x) \) of \( x \) such that

\[
\forall z \in N_{\varepsilon_y}(y) : \quad f(z) > f(x) + \frac{2}{3}(f(y) - f(x)) \quad \text{and} \\
\forall z \in N_{\varepsilon_x}(x) : \quad f(z) < f(x) + \frac{1}{3}(f(y) - f(x))
\]

by continuity of \( f \).

By establishing a sufficiently thin connection between \( N_{\varepsilon_y}(y) \) and \( N_{\varepsilon_x}(x) \), which shall be assigned to \( e' \), and exchanging a subset of \( N_{\varepsilon_y}(y) \) with a subset of \( N_{\varepsilon_x}(x) \cap e \) in a way such that \( \mu(d) = \mu(d') = \mu(e) = \mu(e') \), we can guarantee that \( \mu_\Pi(d') \neq \mu(d')/2 \).

Finally, the case of \( f(y) < f(x) \) can be handled in an analogous way.

In the following, we write \( D \sim D' \) if \( D, D' \in \mathcal{D}_\Pi \) and there exists a sequence \( D_1, \ldots, D_k \) of districtings such that \( D = D_1 \), \( \{D_2, \ldots, D_{k-1}\} \subseteq \mathcal{D}_\Pi \), \( D' = D_k \) and \( \#D_i \cap D_{i+1} = t - 2 \) for all \( i = 1, \ldots, k - 1 \). It is easily verified that \( \sim \) is an equivalence relation on the set of districtings.

**Lemma 5.3.** The geographies of regular districting problems are linked.

Proof. Linkedness is clearly satisfied if \( t = 1 \) or \( t = 2 \). We show that the linkedness of the geographies of all regular districting problems for \( t \leq n \) implies the linkedness of the geographies of all regular districting problems for \( t = n + 1 \). From this, Lemma 5.3 follows by induction.

Take two arbitrary districtings \( D \) and \( E \) of a districting problem with \( t = n + 1 \). We can pick a district \( d \in D \) such that \( d \) and \( X \) have a non-degenerate curve as a common boundary, i.e. there exists a curve \( C \) of positive length such that \( C \subseteq \partial d \cap \partial X \). We divide our proof into three steps.

**Step 1:** We show that there exists a districting \( D' \sim D \) that contains a district \( d' \) which shares a common boundary of positive length with the boundary of \( X \) and which does not separate \( X \).

If \( d \) itself does not separate \( X \) we are done. Thus, assume that \( d \) separates \( X \). For simplicity, we start with the case in which \( d \) separates \( X \) into only two regions as shown in

\footnote{If \( \mu_A(e) \neq \mu(e)/2 \), then \( \mu_A(e') \neq \mu(e')/2 \) can be guaranteed by exchanging sets of sufficiently small measure \( \mu \) between \( d \) and \( e \). In addition, if \( \mu_A(e) = \mu(e)/2 \) and \( \mu_A(e') = \mu(e')/2 \), then we can repeat the exchange of territories between \( e' \) and \( d' \) to ensure that both sets satisfy (5.1).}
the picture on the left of Figure 5.5. By exchanging territories between the two districts $d$ and $e$, where $e$ is a neighboring district of $d$, as shown in the picture on the left of Figure 5.5, we can arrive at districts $d'$ and $e'$ such that $d'$ does not separate $X$.

More generally, assume that $d$ separates $X$, where the number of strictly disconnected regions of $X \setminus \{d\}$ equals $k \leq n$. We can find a district $e \in D$ and a unique boundary element $x \in \partial e$ such that $x \in \partial d \cap \partial X$ and such that $\partial d$ and $\partial e$ have a common curve of positive length starting from $x$. Hence, one can exchange territories between $d$ and $e$ so that for the resulting new districts $d'$ and $e'$ we have that $d'$ separates $X$ into at most $k - 1$ strictly disconnected regions. Clearly, $D' = (D \setminus \{d, e\}) \cup \{d', e'\} \sim D$. Repeating the described bilateral territorial exchange $k - 1$ times, we thus arrive at a districting $D'$ that contains a district $d'$ which shares a common boundary with $X$ and which does not separate $X$.

By Step 1, we may thus assume that $d \in D$ shares a boundary of positive length with $X$ and does not separate $X$.

**Step 2:** We establish that there exists a districting $E' \sim E$ containing a district $e \in E'$ such that $e$, $d$ and $X$ have a nondegenerate common boundary, $\mu(d \cap e) > 0$ and $d \cup e$ does not separate $X$.

Clearly, there exist a district $e \in E$ possessing a common boundary with $d$ and $X$, and satisfying $\mu(d \cap e) > 0$.

Assume that $e$ separates $X$, where the number of strictly disconnected regions of $X \setminus \{e\}$ equals $k \leq n$ (see Figure 5.6 to the left for a situation with $k = 3$). Then

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10 Both pictures only show the two districts involved in a territorial exchange and not the entire districtings.

11 It might happen that $d'$ or $e'$ violate (5.1) since we only took care of the shapes and sizes of the two districts. However, Lemma 5.2 ensures that through an appropriate territorial exchange between $d'$ and $e'$ we can also ensure (5.1). In what follows we will carry out all territorial exchanges between districts so as to satisfy (5.1) without explicitly mentioning Lemma 5.2 each time.
We can find a district $e' \in E$ with a unique boundary element $x \in \partial e'$ satisfying $x \in d \cap \partial e \cap \partial X$ and that $\partial e \cap \partial e'$ has a common curve of positive length starting from $x$ (as illustrated in the left hand side of Figure 5.6). Hence, one can exchange territories between $e$ and $e'$ so that for the resulting new districts $h$ and $h'$ we have that $d \cap e \subseteq h$, $h$ separates $X$ into at most $k - 1$ strictly disconnected regions (see the right hand side of Figure 5.6). Clearly, $E' = (E \setminus \{e, e'\}) \cup \{h, h'\} \sim E$ and we can repeat the procedure to reduce the number of strictly disconnected regions by replacing $E$ and $e$ with $E'$ and $h$, respectively, until we arrive at a districting $E' \sim E$ containing a district $e'$ that does not separate $X$ and has a common boundary with $d$. Without loss of generality, we can thus replace $e'$ and $E'$ by $e$ and $E$, respectively.

We still have to ensure that $d \cup e$ does not separate $X$. A situation in which $d \cup e$ separates $X$ is shown in the picture on the left hand side of Figure 5.7. In addition, the same picture contains (by the absolute continuity of $\mu$) a possible neighboring district $e'$ to $e$, which is drawn in a way such that $e \cup e'$ does not separate $X$, it covers an area from the separated regions and also an area within $d \cup e$. A possible exchange of territories which reduces the separated area by $d \cup e$ is illustrated in Figure 5.7, where $d \cup h$ separates a smaller area than $d \cup e$.\footnote{District $e'$ in Figure 5.7 is not drawn in the most efficient way in the sense that it is possible to draw

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_6.png}
\caption{Reducing the number of disconnected regions in $E$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_7.png}
\caption{Intertwined districts.}
\end{figure}
strictly connected districts and let \( E' = H \cup \{h, h'\} \). Observe that \( E \setminus \{e\} \sim H \cup \{e'\} \) by the induction hypothesis, \( h \cup h' = e \cup e' \) by construction, and therefore \( E \sim E' \). Replace \( e \) and \( E \) with \( h \) and \( E' \), respectively. After repeating the described territorial exchange finitely many times\(^{13}\) one arrives at a district \( e \) and a districting \( E \) such that \( d \cup e \) does not separate \( X \) and \( e \) still satisfies the other desired properties.

**Step 3:** Since \( d \cup e \) does not separate \( X \) and \( \mu \) is absolutely continuous, there exists a strictly connected set \( h \) such that \( \mu(h) = 2\mu(X)/(n+1) \), \( d \cup e \subset h \), \( d' = h \setminus d \in G \) and \( e' = h \setminus e \in G \) and \( h \) does not separate \( X \) (see Figure 5.8). Let \( H \) be a districting of \( Y = X \setminus h \)

\[ \begin{array}{c}
\text{Figure 5.8: Final step.}
\end{array} \]

into \( n - 1 \) strictly connected districts. Then \( \Pi|_{Y \cup d'} \) and \( \Pi|_{Y \cup e'} \) are regular districting problems, and therefore it follows by the induction hypothesis that \( D \sim H \cup \{d, d'\} \) and \( H \cup \{e, e'\} \sim E \). Clearly, \( \{d, d'\} \sim \{e, e'\} \), which gives \( H \cup \{d, d'\} \sim H \cup \{e, e'\} \). Finally, the statement of Lemma 3 follows from the transitivity of \( \sim \).

\[ \square \]

\(^{13}\)In fact the number of required iterations is at most \( \lceil t\mu(Y) / \mu(X) \rceil + 1 \), where \( Y \) stands for the area “intertwined” by \( d \cup e \).
Chapter 6

The computational complexity of the political districting problem

In the middle of the previous century it was hoped that the problem of gerrymandering could be overcome by computer programs using only data on voters geographic distribution without any statistical information on voters preferences (e.g. Vickrey [79]) and thus determining an ‘unbiased’ districting. The first algorithm finding all districtings with (i) equally sized, (ii) connected, and (iii) compact districts was given by Garfinkel and Nemhauser [36]. The computational difficulty of the problem was clear from the very beginning. Nagel [58] documented in an early survey the computational limitations of automated redistricting by considering the available programs of his time. Altman [3] showed that the problems of achieving any of the three mentioned criteria are NP-hard. Moreover, he also demonstrated that maximizing the number of competitive districts is also NP-hard. Because of the computational difficulty of the problem there is a growing literature on new approaches to finding unbiased districtings (see, for instance, Mehrotra [54], Bozkaya et al. [20], Baçao et al. [7], Chou and Li [25], Ricca and Simeone [69], Ricca et al. [70]). For surveys we refer to Ricca et al. [71], Tasnádi [78], and Kalcsics [46].

Though finding an equally sized districting is already computationally hard, from another point of view it is feared by the public that the continuously increasing computational power makes the problem of carrying out an optimal partisan gerrymandering possible. However, the underlying difficulty of the problem does not allow us to determine an optimal partisan redistricting. Indeed, Altman and McDonald [4] provided evidence that

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1Earlier Hess et al. [41] provided an algorithm striving for similar goals; however, their algorithm did not always obtain optimal solutions.
current computer programs are far away from finding an optimal gerrymandering.

In Section 6.1 we investigate whether a “fair” districting can be achieved by an impartial and independent arbiter or jury. Specifically, a districting will be called (ex post) unbiased if the number of seats won by a party is proportional to its share of votes in the entire population. Based on Puppe and Tasnádi [66] we show that finding an unbiased districting represents an NP-complete problem in the case of geographical constraints. In states with a large population and many districts it can be thus very difficult to find an unbiased districting plan. In case of congressional elections in the United States, striking examples are California with 53 representatives (districts) and Texas with 32 representatives.

In Section 6.2 we establish that a simplified version of the optimal gerrymandering problem is NP-complete based on Puppe and Tasnádi [67]. In Section 6.3 we explicitly add planarity by locating voters in the plane to the geographical constraints following Fleiner, Nagy and Tasnádi [32]. In a recent parallel work Lewenberg and Lev [51] also prove the NP-completeness of optimal gerrymandering in the plane; however, they do not demand equally or almost equally sized districts. In addition, we show that winning an election, i.e. deciding the existence of a districting that guarantees a party a majority of overall seats is also NP-complete. Finally, we bring forward arguments in favor of the computational intractability of determining an optimal partisan districting for real-life problems of modest size in Section 6.4.

6.1 Unbiased districting

We assume that a set of voters has to be partitioned into a given number of equal districts in each of which candidates of two parties, say parties A and B, compete for winning a seat. A district is “won” by a candidate if he/she receives the majority of votes. We shall denote the number of voters by n and the set of voters by \( N = \{1, \ldots, n\} \). Similarly, the number of districts is denoted by d and the set of districts by \( D = \{1, \ldots, d\} \). We assume that \( d \) divides \( n \).

We assume that the voters have deterministic and known party preferences. This is clearly a simplification of reality which, however, allows us to obtain several insightful results. Relaxing these assumptions could be the aim of further research. The voters’ party preferences are summarized by the mapping \( v : N \rightarrow \{A, B\} \) with \( v(j) = A \) interpreted

\(^2\text{For another recent work on the optimal gerrymandering problem we refer to Ito et al. [43].}\)

\(^3\text{This is without much loss of generality, since otherwise we can introduce dummy voters in proportion of the supporters of each party to overcome indivisibilities.}\)
as “voter \( j \) votes for (prefers) party \( A \).” The number of supporters of parties \( A \) and \( B \) are denoted by \( n_A \) and \( n_B \), respectively. Let us assume for simplicity that there exists a positive integer \( k \) such that \( n = d(2k + 1) \). Thus, each district must consist of \( 2k + 1 \) voters and, assuming full participation, each district is won by either party \( A \) or party \( B \). In particular, we exclude in all districts the possibility of a draw.

We introduce the following simple but quite general framework that allows us to incorporate geographical constraints.

**Definition 6.1** (Geography). A set system \( S \subset 2^N \) of \( 2k + 1 \) sized subsets of \( N \) such that there exist appropriately chosen sets \( S_1, \ldots, S_d \in S \) partitioning \( N \) is called a *geography*. A *districting problem* for geography \( S \) is a pair \((N, S)\).

**Definition 6.2** (Districting). For a given geography \( S \subset 2^N \) a mapping \( f : N \to D \) is called a *districting* if \( f^{-1}(i) \in S \) for all \( i \in D \) and \( \cup_{i \in D} f^{-1}(i) = N \).

Observe that if \( S \) consists of all \( 2k + 1 \) sized subsets of \( N \), then we obtain as a special case districting without geographical constraints.

A districting \( f \) and voters’ preferences \( v \) jointly determine the number of districts won by parties \( A \) and \( B \), which we denote by \( F(f, v, A) \) and \( F(f, v, B) \), respectively. Party \( A \) wins the election if \( F(f, v, A) > F(f, v, B) \) and loses the election if \( F(f, v, A) < F(f, v, B) \). The following definition is central to our approach to “fair” districting. In what follows again \( \lfloor x \rfloor \) stands for the largest integer not greater than \( x \) and \( \lceil x \rceil \) stands for the smallest integer not less than \( x \), for any real number \( x \).

**Definition 6.3** (Biasedness). For given voters’ preferences \( v : N \to \{A, B\} \) a districting \( f : N \to D \) is *unbiased* if \( F(f, v, A) = \lfloor d\frac{n_A}{n} \rfloor \) or \( F(f, v, A) = \lceil d\frac{n_A}{n} \rceil \). A districting is *biased* if it is not unbiased.

Thus, a districting plan is unbiased if the number of districts won by each party respects their relative strength in the population as close as possible. Without geographical constraints, an unbiased districting can be found quite easily.

**Proposition 6.1.** An unbiased districting without geographical constraints can be determined in polynomial time, and more specifically, even in linear time.

**Proof.** Fill \( \lfloor d\frac{n_A}{n} \rfloor \) districts with voters of party \( A \), \( \lceil d\frac{n_A}{n} \rceil \) districts with voters of party \( B \) and the remaining district (whenever \( d\frac{n_A}{n} \) is not an integer) with the remaining \( 2k + 1 \) voters.

\[ \square \]
The simple algorithm given in the proof of Proposition 6.1 in particular shows that without geographical constraints an unbiased districting is always feasible. However, this is not always the case in the presence of geographical constraints. We verify this based on the “rectangular country” shown in Figure 6.1. Party A’s supporters are indicated by empty circles and party B’s supporters are indicated by solid circles; it can be verified that $n_A = 290$ and $n_B = 110$. We assume that $k = 2$, i.e., district size is 5, and that therefore $d = 80$ districts have to be formed. Two voters are considered adjacent if they have a common boundary (edge), and a district is connected if two voters living in the same district are “reachable” through a sequence of adjacent voters. We impose the simple restriction on the districting that only connected districts can be formed, which defines a geography $S$ for the rectangular country. Under the distribution of voters’ preferences shown in Figure 6.1 and under the given geographical constraint, party B can win at most 20 districts. To verify this, observe that if a district contains one voter from the left hand side (first 15 columns) of the country, then it cannot be won by Party B. Therefore, winning districts for party B must consist only of voters from the right hand side (last five columns) of the country. Since party B’s proportional share in representatives, resulting in an unbiased districting, would require 22 representatives, an unbiased districting does not exist.

![Figure 6.1: Rectangular country](image)

Our main concern is whether an impartial arbiter or judge can determine an unbiased
districting for a given geography \( S \) on \( N \) from a computational perspective. We establish that even the associated decision problem, i.e. deciding the existence of an unbiased districting, is a computationally intractable NP-complete problem. We call this problem UNBIASED DISTRICTING.

To prove the NP-completeness of UNBIASED DISTRICTING, we shall reduce EXACT COVER BY \( m \)-SETS (\( m \geq 3 \)), a well-known NP-complete problem\(^4\) to UNBIASED DISTRICTING. EXACT COVER BY \( m \)-SETS asks if a given set \( X \) with cardinality \( mq \) possesses an exact cover from a given set system \( C \) of \( m \)-element subsets (henceforth \( m \)-sets) of \( X \) (i.e. \( C_1, \ldots, C_q \in C \) and \( \bigcup_{i=1}^q C_i = X \)), where we can assume that \( \#C \geq q \).

**Theorem 6.1.** UNBIASED DISTRICTING is NP-complete.

*Proof.* First, we verify that the unbiasedness of a districting \( f \) can be checked in polynomial time, and therefore UNBIASED DISTRICTING \( \in \) NP. Assume that the set of party \( A \) voters is represented by \( \{1, 2, \ldots, n_A\} \) and the set of party \( B \) voters by \( \{n_A + 1, \ldots, n\} \). A district of size \( 2k + 1 \) is encoded by a sequence of distinct positive integers not greater than \( n \), a districting \( f \) by a sequence of \( d \) districts, and a geography by \( 2k + 1, n_A, n_B, s = \#S \) and the sequence \( S_1, \ldots, S_s \) of possible districts. The unbiasedness of a given districting \( f \) can be checked by counting the number of winning districts for party \( A \) while reading the encoding of \( f \).

Second, we reduce EXACT COVER BY \( 2k + 1 \)-SETS to UNBIASED DISTRICTING. We start with the motivating example shown in Figure 6.2 to illustrate our construction of a districting problem associated with a given instance of EXACT COVER BY \( 2k + 1 \)-SETS. The empty circles stand for the elements to be covered by disjoint 5-sets (\( k = 2 \)), which we regard as the party \( A \) voters in the districting problem. The given instance of EXACT COVER BY 5-SETS, i.e. the set system \( C \) of 5-sets of party \( A \) supporters, specifies possible districts that are not shown in Figure 6.2 since we allow for arbitrary systems of such sets. The solid circles indicate the voters of party \( B \). We obtain the desired geography \( S \) (on the set of all voters) by adding the sets \( Y_1, \ldots, Y_8 \) and \( Z_1, \ldots, Z_5 \) to \( C \) as shown in Figure 6.2. In the figure, we also see that \( n_A = 15 \) and \( n_B = 25 \), thus an unbiased districting requires exactly 3 winning districts for party \( A \). The crucial observation is that an unbiased districting cannot contain any of the districts \( Y_1, \ldots, Y_8 \). Indeed, among all admissible districts (i.e. those in \( S \)) only the districts in \( C \) are winning districts for party

---

\(^4\)See Garey and Johnson [35, p. 53] for EXACT COVER BY 3-SETS and EXACT COVER BY 4-SETS. The NP-completeness of the EXACT COVER BY \( m \)-SETS for \( m \geq 5 \) can be shown in an analogous way.
A. Moreover, $\mathcal{C}$ contains at most 3 mutually disjoint districts. Therefore, a districting containing a set $Y_i$ cannot contain at the same time 3 districts from $\mathcal{C}$. This shows that an unbiased districting exists if and only if the given instance of EXACT COVER BY 5-SETS has a solution.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.2.png}
\caption{$k = 2$, $d = 8$ and $n = 40$}
\end{figure}

Now let us turn to the general case and take an instance $\mathcal{C}$ on $X$ of EXACT COVER BY $2k + 1$-SETS, where $\#X = (2k + 1)c$ for some integer $c$. The elements of $X$ will be the voters of party $A$, and thus $n_A = \#X$. Let $a = \lceil \frac{(2k+1)c}{k} \rceil$ and $r = (2k+1)c \mod k$ (the remainder of the division of $(2k+1)c$ by $k$). Party $B$ will (by construction) have either $y = a(k+1) + 2k + 1 - r$ voters if $r > 0$ or $y = a(k+1)$ voters if $r = 0$, and we shall denote the associated set of voters by $Y$.

We claim that $y$ is divisible by $2k + 1$. First, we consider the case of $r > 0$. Since $(2k+1)c = ak + r$,

\begin{align*}
y &= a(k + 1) + 2k + 1 - r \\
&= a(k + 1) + 2k + 1 - ((2k + 1)c - ak) \\
&= (2k + 1)(a + 1 - c),
\end{align*}

which proves our claim for $r > 0$. Second, assume that $r = 0$. Since then $(2k+1)c = ak$, we have

\begin{align*}
y &= a(k + 1) \\
&= (2k + 1)c + a \\
&= (2k + 1)c + \frac{c}{k}(2k + 1).
\end{align*}

Now $y$ is divisible by $2k + 1$ because $\gcd(2k + 1, k) = \gcd(k, 1) = 1$ by the Euclidean algorithm and therefore $c$ is divisible by $k$, since all the terms are integers, and hence $y$ is
clearly divisible by $2k + 1$\footnote{gcd stands for the greatest common divisor.}

Next we construct a geography $S$ on $N = X \cup Y$. First, pick a partition $Z_1, \ldots, Z_u$ of $Y$ into $2k + 1$-sets. Second, we partition $X$ into $k$-sets $X_1, \ldots, X_a$ and in addition into an $r$-set $X_{a+1}$ if $r > 0$. Third, we partition $Y$ into $k+1$-sets $Y_1', \ldots, Y_a'$ and in addition into a $2k + 1 - r$-set $Y_{a+1}'$ if $r > 0$. Fourth, match sets $X_1, \ldots, X_a$ with sets $Y_1', \ldots, Y_a'$, respectively, to obtain $2k + 1$-sets $Y_1, \ldots, Y_a$ consisting of $k$ voters of party $A$ and $k + 1$ voters of party $B$. Moreover, match set $X_{a+1}$ with set $Y_{a+1}'$ if $r > 0$, to obtain $2k + 1$-set $Y_{a+1}$ with more voters of party $B$ than party $A$. Let $t = a$ if $r = 0$ and $t = a + 1$ if $r > 0$. Finally, let $S = C \cup \{Y_1, \ldots, Y_t, Z_1, \ldots, Z_u\}$ completing the construction of geography $S$.

Since sets $Y_1, \ldots, Y_t$ determine a districting with district size $2k + 1$ on $N$, we have associated an instance of UNBIASED DISTRICTING with an arbitrary instance of EXACT COVER BY $2k + 1$-SETS.

Because $n_B = y = c'(2k + 1)$ for some positive integer $c'$, party $A$ receives exactly $d^{2A} n = c$ winning districts by an unbiased districting. Remember that the set of winning districts for party $A$ equals $C$ and one can select at most $c$ disjoint districts from $C$. Hence, a districting for geography $S$ is unbiased if and only if it does not contain a set from $Y_1, \ldots, Y_t$, since otherwise party $A$ wins fewer than $c$ districts. Therefore, the necessary and sufficient condition for the existence of an unbiased districting is the existence of an exact cover of $X$ by $2k + 1$-sets from the given set system $C$. Thus, we have reduced EXACT COVER BY $2k + 1$-SETS to UNBIASED DISTRICTING.

Finally, we show that our reduction can be done in polynomial time. Assume that the given instance of EXACT COVER BY $2k + 1$-SETS is given by a sequence $C_1, \ldots, C_v \subseteq X$ of $2k + 1$-sets, where the elements of $X$ are encoded by integers $\{1, 2, \ldots, n_A\}$ and $v \geq c$. Clearly, the input length in integers equals $v(2k + 1)$. Since the reduction produces $t + u$ new $2k + 1$-sets, $2c \leq t \leq 3c$ and $u \leq t \leq 3c$, the required number of computations is linear in $c$ and at most linear in the size of input, which completes the proof.

\vspace{0.5cm}

6.2 Optimal partisan districting

From the viewpoint of an involved political party the optimal policy consists in maximizing the number of districts won by that party (by simple majority, say). This is known as “optimal partisan gerrymandering” or simply optimal partisan districting. In this section we show based on Puppe and Tasnádi \cite{67} that the associated decision problem deciding
whether there exists a districting such that a party wins at least a given number of districts is an NP-complete problem. We employ the notations from the previous section.

First, let us define an optimal districting.

**Definition 6.4** (Optimal districting). For a given problem \((N, S)\) and given voters’ preferences \(v : N \to \{A, B\}\) a districting \(f : N \to D\) is *optimal* for party \(I \in \{A, B\}\) if \(F(f, v, I) \geq F(g, v, I)\) for any districting \(g : N \to D\).

We establish that deciding whether there exists a districting with at least \(m\) winning districts for a party, say party \(A\), is a computationally intractable NP-complete problem; we call this problem WD. In order to prove this, we shall reduce a well-known variant of SET PACKING (henceforth, SP), a proven NP-complete problem (see Garey and Johnson [35, p. 221]), to WD. SP asks whether a given set system \(C\) of subsets of \(X\) such that \(|C| \leq k + 1\) for all \(C \in C\) (with \(k \geq 2\)) possesses at least \(m\) mutually disjoint sets.

**Theorem 6.2.** WD is NP-complete.

**Proof.** Whether a districting \(f\) possesses at least \(m\) winning districts for party \(A\) can be verified easily in polynomial time, and therefore WD \(\in\) NP.

We take an instance of SP for which we can assume without loss of generality that \(X = \bigcup_{C \in C} C\). The elements of the set \(X\) will all be taken to be party \(A\) supporters. First, we associate with an arbitrarily chosen set \(C \in C\) a district \(D_C\) as follows: If \(|C| = j \leq k + 1\), then we add \(k\) new voters for party \(A\) to \(X\), \(k + 1 - j\) new voters for party \(B\) to the set of voters, and define district \(D_C\) a consisting of the party \(A\) supporters from \(C\) and the newly added \(k + (k + 1 - j)\) voters. Clearly, \(D_C\) is a winning district for party \(A\). By carrying out the above procedure, we obtain \(|C|\) districts and a set of voters \(Y\). We illustrate the types of districts that can occur in this manner so far for \(k = 2\) in Figure [6.3](#). As above, party \(A\) supporters are indicated by solid circles.

![Figure 6.3: First step in case \(k = 2\)](image)

Secondly, \(Y\) and the \(2k|Y|\) newly added party \(B\) supporters complete the set of voters \(N\). We partition \(N\) into \(|Y|\) equally sized sets such that each partition element contains
exactly one voter from the set \( Y \), and we include these sets in the geography \( S \). We shall denote the district containing \( y \in Y \) by \( D_y \). Clearly, the above defined partition of \( N \) gives an admissible districting of \( N \) in which party \( B \) wins all districts.

Thirdly, we complete the geography \( S \). Take an arbitrarily chosen set \( C \in \mathcal{C} \) and its associated district \( D_C \) as described in the first step. We partition \( N_C = \cup_{y \in D_C} D_y \) into \( 2k + 1 \) equally sized sets such that one set equals \( D_C \) and the remaining \( 2k \) sets all contain exactly one element from each set \( D_y \setminus \{y\} \) (where \( y \in D_C \)), which gives us districts \( D_i^C \) for \( i = 2, \ldots, 2k + 1 \). We illustrate in Figure 6.4 the districts obtained in this way through our second and third steps for the case \( k = 2 \) and for two given sets \( C, C' \in \mathcal{C} \) with two common elements. The “vertical sets” were derived in our second step, while the “horizontal sets” in our third step. To illustrate the interplay between the vertical and horizontal sets, for example, if \( C \) is contained in a set packing, then \( C' \) cannot be contained in the same set packing; and therefore, turning to winning districts, the derived districting containing \( D_C \) would contain the 5 horizontal districts on the left hand side and the 3 vertical districts on the right hand side.

![Figure 6.4: C = \{y_3, y_4, y_5\} and C' = \{y_4, y_5, y_6\}](image)

Formally, to obtain the districting problem \((N, S)\), let

\[
S = \{D_C\}_{C \in \mathcal{C}} \cup \{D_y\}_{y \in Y} \cup \{D_i^C\}_{i=2, C \in \mathcal{C}}^{2k+1}.
\]

A districting for the geography \( S \) contains at least \( m \) winning districts for party \( A \) if and only if it does contain at least \( m \) sets from \((D_C)_{C \in \mathcal{C}}\), since these sets are exactly the winning sets for party \( A \) by construction of \( S \). Observe that if we can take \( p \geq m \) winning sets for party \( A \), then the districting contains for any winning district \( D = D_C \) of party \( A \) the districts \((D_i^C)_{i=2}^{2k+1}\) by our third step (“horizontal sets”) and for any \( y \in Y \) not contained in a winning district for party \( A \) the associated set \( D_y \) defined by our second
step ("vertical sets"). Therefore, the necessary and sufficient condition for the existence of a districting with at least \( m \) winning districts for party \( A \) is the existence of \( m \) mutually disjoint sets from \( C \). Thus, we have given (since \( k \) is fixed) a polynomial time reduction of SP to WD, which completes the proof.

6.3 Optimal partisan districting on planar geographies

Compared with Sections 6.1 and 6.2 we assume that voters with known party preferences are located in the plane and have to be divided into a given number of almost equally sized districts. The districting problem is defined by the following structure:

Definition 6.5. A districting problem is given by \( \Pi = (X, N, (x_i)_{i \in N}, v, K, D) \), where

- \( X \) is a bounded and strictly connected\(^6\) subset of \( \mathbb{R}^2 \),
- the finite set of voters is denoted by \( N = \{1, \ldots, n\} \),
- the distinct locations of voters are given by \( x_1, \ldots, x_n \in \text{int}(X) \),
- the voters’ party preferences are given by \( v : N \to \{A, B\} \),
- the set of district labels is denoted by \( K = \{1, \ldots, k\} \), where \( \lfloor n/k \rfloor \geq 3 \), and
- \( D \) denotes the finite set of admissible districts consisting of bounded and strictly connected subsets of \( X \) and each of them containing the location of \( \lfloor n/k \rfloor \) or \( \lceil n/k \rceil \) voters, and furthermore,
- we shall assume that based on their locations the \( n \) voters can be partitioned into \( k \) districts \( \{D_1, \ldots, D_k\} \subseteq D \).

Observe that in defining the districting problem, we assumed that obtaining an almost equally sized districting is possible, which can be justified by the fact that finding an admissible districting for real-life problems is possible, while finding a districting satisfying additional requirements such as partisan optimality is difficult. In particular, the staff hired to produce a districting map could always construct a districting map consisting of almost equally sized districts although other properties like partisan optimality are difficult to prove or to confute. Producing a districting with almost equally sized districts, is a tractable problem if there are not too many geographical restrictions since then we can

\(^6\)We call a bounded subset \( A \) of \( \mathbb{R}^2 \) strictly connected if its boundary \( \partial A \) is a closed Jordan curve.
obtain a result by drawing districts from left to right and from top to bottom on a map of a state by keeping the average district size in mind. An initial step for such an algorithm would be, for instance, to order the voters increasingly according to their horizontal or vertical coordinates.

We shall mention that in reality the basic units of a districting problem from which districts have to be created are census blocks or counties rather than voters in order to simplify the problem and at the same time to include natural municipal boundaries. In this case voter preferences \( v : N \to \{A, B\} \) have to be replaced by a function of type \( v' : N' \to [0, 1] \), where \( N' \) stands for the finite set of counties, assigning to each county a fraction of party \( A \) voters. However, our results obtained in this paper can be extended to this more general setting, by allowing the case of almost equally sized counties, for which district outcomes are determined by the number of winning counties for party \( A \), which happens to be the case, for instance, if \( v'(N') = \{\alpha, 1 - \alpha\} \) for a given \( \alpha \in [0, 1/2) \), i.e. the fraction of party \( A \) voters in each county equals either \( \alpha \) or \( 1 - \alpha \), and thus the main result of this paper delivers a worst case scenario for the model with counties as elementary units. Hence, the NP-completeness results in this paper imply the same NP-completeness results within a model with almost equally sized counties and districts, which come closer to the problems handled by gerrymanderers.

Turning back to our districting problem defined on the level of voters, we have to assign each voter to a district.

**Definition 6.6.** An \( f : N \to D \) is a districting for problem \( \Pi \) if there exists a set of districts \( D_1, \ldots, D_k \in D \) such that

- \( f(N) = \{D_1, \ldots, D_k\} \),
- \( \operatorname{int}(D_i) \cap \operatorname{int}(D_j) = \emptyset \) if \( i \neq j \) and \( i, j \in K \),
- \( \{x_i \mid i \in f^{-1}(D_j)\} \subset \operatorname{int}(D_j) \) for any \( j \in K \).

Observe that without loss of generality we do not explicitly require that a districting covers the entire country but just the inhibited areas.

**Definition 6.7.** Two districtings \( f : N \to D \) and \( g : N \to D \) with districts \( D_1, \ldots, D_k \) and \( D'_1, \ldots, D'_k \), respectively, are equivalent if there exists a bijection between the series of sets \( \{x_i \mid i \in f^{-1}(D_1)\}, \ldots, \{x_i \mid i \in f^{-1}(D_k)\} \) and the series of sets \( \{x_i \mid i \in g^{-1}(D'_1)\}, \ldots, \{x_i \mid i \in g^{-1}(D'_k)\} \) such that the respective sets are identical.
Clearly, by defining equivalent districtings we have defined an equivalence relation above the set of districtings for problem II.

A districting \( f \) and voters’ preferences \( v \) determine the number of districts won by parties \( A \) and \( B \), which we denote again by \( F(f, v, A) \) and \( F(f, v, B) \), respectively. If the two parties should receive the same number of votes in a district, its winner is determined by a predefined tie-breaking rule \( \tau : \mathcal{D} \rightarrow \{A, B\} \).

**Definition 6.8.** For a given problem \( \Pi \) and tie-breaking rule \( \tau \) a districting \( f : N \rightarrow \mathcal{D} \) is *optimal* for party \( I \in \{A, B\} \) if \( F(f, v, I) \geq F(g, v, I) \) for any districting \( g : N \rightarrow \mathcal{D} \).

Note that due to the above defined equivalence relation the set of districtings has finitely many equivalence classes, and therefore there exists at least one optimal districting for each party.

We establish that even the decision problem associated with the optimization problem of determining an optimal partisan districting, i.e. deciding for a given districting problem \( \Pi \) whether there exists a districting with at least \( m \) winning districts for a party, say party \( A \), is an NP-complete problem; we call this problem WINNING DISTRICTS. In order to prove this, we shall reduce the INDEPENDENT SET problem on planar cubic\(^7\) graphs, a proven NP-complete problem (see Garey and Johnson [35, p. 195]), to WINNING DISTRICTS. The INDEPENDENT SET problem asks whether a given graph \( G \) has a set of non-neighboring vertices of cardinality not less than \( m \).

**Theorem 6.3.** WINNING DISTRICTS is NP-complete.

*Proof.* Whether a districting possesses at least \( m \) winning districts for party \( A \) can be verified easily in polynomial time, and therefore WINNING DISTRICTS is in NP.

We establish that INDEPENDENT SET on planar cubic graphs reduces to WINNING DISTRICTS. We define the mapping that assigns to an arbitrary planar cubic graph \( G = (V, E) \) a districting problem. We may assume that the graph is embedded in the plane such that all the edges are straight lines and denote the set of their midpoints by \( V_E \). We define \( \varepsilon \) as the minimum of the distances between a point of \( V \cup V_E \) and a non-incident edge. We illustrate the reduction in Figure 6.5. The ‘3-star’ of a vertex \( v \in V \) is the union of the three line segments between \( v \) and the midpoints of the three edges emitting from \( v \).

Let the set of party \( A \) voters be \( V_E \) and with each party \( A \) voter \( M \in V_E \) we associate two party \( B \) voters \( M' \) and \( M'' \) such that \( M', M \) and \( M'' \) lie in this order on the same \(\)\(^7\)A graph is cubic if the degree of each vertex equals 3.
straight line perpendicular to the edge of $M$ and the distance of $M'$ and $M''$ from $M$ is between $\frac{1}{5}\varepsilon$ and $\frac{2}{5}\varepsilon$.

For each midpoint $M \in V_E$ we construct a party $B$ winning district as the $\frac{2}{5}\varepsilon$-neighborhood of $M$. Since each of these districts contains two party $B$ voters and a party $A$ voter, we call them ‘mixed districts’.

We associate with each vertex $v \in V$ a party $A$ winning district as the $\frac{1}{5}\varepsilon$-neighborhood of the 3-star of $v$. Observe that this district contains exactly three voters and they are the midpoints of the edges of $v$ thus we call it ‘$A$-uniform district’.

Consider the set-theoretic difference of the $\frac{2}{5}\varepsilon$-neighborhood and the $\frac{1}{5}\varepsilon$-neighborhood of the 3-star of $v$, i.e. the subset of the plane consisting of the points having distance from the 3-star between $\frac{1}{5}\varepsilon$ and $\frac{2}{5}\varepsilon$. This set contains exactly six voters which are the party $B$ voters corresponding to the midpoints of the edges of $v$. It is straightforward to see that the bisector of any angle defined by the edges at $v$ and the edge different from the sides of that angle divide this set in such a way that each part contains three party $B$ voters. We call these divided parts ‘$B$-uniform districts’.

Now, it is enough to show that the graph $G$ has an independent set of size $m$ if and only if the above defined districting problem has a districting with $m$ party $A$ winning districts.

The ‘if’ part of this claim is obvious since the party $A$ winning districts of a districting are disjoint $A$-uniform districts and they correspond to non-neighboring graph vertices.

For the reverse implication we construct for any given independent set of size $m$ a districting having $m A$ winning districts. Take the $A$-uniform and $B$-uniform districts
corresponding to the vertices of the independent set and for the still uncovered voters take their mixed districts. Clearly, all the voters are covered by a district and it is not hard to see because of the choice of $\varepsilon$ that the chosen districts are disjoint and each contains three voters.

We note that the associated districting problem described above can be obviously determined in polynomial time.

The following easy consequence of Theorem 6.3 has practical importance:

**Theorem 6.4.** The decision problem whether a districting problem $\Pi$ has a districting in which party $A$ gains majority is NP-complete.

**Proof.** Note that all districtings in the proof of Theorem 6.3 have $\frac{3}{2}|V|$ districts, thus there exists a districting with at least $m$ winning districts of party $A$ if and only if the following districting problem extended with dummy voters and districts has a solution in which the $A$ winning districts form a majority. Let us add $\frac{3}{2}|V| - 2m + 1$ extra disjoint $A$ winning districts each containing three extra $A$ voters if $m \leq \frac{3}{2}|V|/2$, otherwise add $2m - \frac{3}{2}|V| - 1$ extra disjoint $B$ winning districts with three extra $B$ voters in each.

**Remark 6.1.** The notion of majority in Theorem 6.4 is irrelevant. The same statement can be proved by analogy for any qualified majority.

### 6.4 A practical approach

Since many NP-complete problems can be solved for real-life instances we would like to point out in this section why it is difficult to find an optimal partisan districting even if only a modest number of districts have to be formed. The number of districts or the number of counties for districting problems can be deceptive because, while the number of districts to be drawn is relatively small, the number of possible districts is already extremely large as we will point in the next two paragraphs.

For example, let us consider the Hungarian Electoral System in which since 2011 Budapest has to be subdivided into 18 electoral districts from a total of 1472 counties, each serving 600-1500 voters. Thus, an average district consists of approximately 82 counties. For simplicity, we model the election map by a 2-dimensional square grid, where every cell represents a county with a given party preference $A$ or $B$. Two cells are connected if they

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8Obviously, the real-life structure is even more complex, the distribution of party $A$ and $B$ voters differ county by county, and there are further restrictions on the set of admissible districts.
share a common edge, so this defines a 4-neighborhood relation on the set of cells.

However, in this simplified structure there is no known formula for the number of possible figures, i.e. districts, formed out of a given number of connected cells, so-called polyominoes, if even orientation matters, they are called fixed polyominoes. It is known that the number of polyominoes grows exponentially. Jensen \[42\] enumerated fixed \(n\)-cell polyominoes up to \(n = 56\) which resulted in \(6.9 \times 10^{31}\) polyominoes for the last case, which equals the number of different shapes that can be formed out of 56 connected squares. This result shows that it is unfeasible to examine all possible cases even for 82 counties on a Budapest scale problem, and therefore in contrast to the knapsack problem the number of districts to be formed in case of a districting problem underestimates the magnitude of the latter problem. Of course, considering possible district shapes is just the first step in arriving to a districting.

Another starting point to obtain a heuristic for gerrymandering, i.e. an algorithm which is not optimal but quick, would be the pack and crack principle. In a similar framework, Puppe and Tasnádi \[67\] showed that not every crack procedure reaches the optimal solution if geographical constraints are present. If the connectivity of the cells is not required, the problem can be easily solved by a simple crack algorithm, which leads to the optimal solution in this special case. The aim of the crack strategy for the beneficiary party is to win the query district with just the least margin, thus weakening the opponent party. In fact, according to this greedy algorithm for a given district size one has to pick just one more cell for party \(A\) than for party \(B\) if the district size is odd. Unfortunately, if we require districts to be connected, it is far from obvious how this greedy approach arrives to a feasible map tiling.

Anyway, Figures 6.6 and 6.7, containing the same gird-like geography with holes (e.g. lakes), show that employing the crack principle in favor of party \(A\) does not result in a party \(A\) optimal districting. In particular, it can be verified that the geography depicted in Figures 6.6 and 6.7 admits just these two feasible districtings from which the crack principle chooses the districting of Figures 6.6 while the party \(A\) optimal districting is shown in Figure 6.7. Figures 6.6 and 6.7 improve on the respective example in Puppe and Tasnádi \[67\] Figure 2] by pointing out that any implementation of the crack principle results for some problems in a non partisan optimal districting.

We still might hope that by a clever combination of packing and cracking we could

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9 In the unlabeled squares we have party \(B\) voters.
10 The numbers close to the districts indicate a possible ordering in which the districts can be chosen based on the crack principle.
obtain a party $A$ optimal districting. The pack and crack principle requires that we draw districts sequentially in a way that the number of wasted votes by party $A$ is decreasing; where in case of a cracked district the number of wasted votes by party $A$ equals the number of party $A$ voters not needed for winning the respective cracked district, while in case of a packed district the number of wasted votes by party $A$ equals the number of party $A$ voters in the respective packed district. However, Figures 6.8 and 6.9 show that the pack and crack principle does not always result in a party $A$ optimal districting since the geography in Figures 6.8 and 6.9 admits just two districtings, the pack and crack principle results in the districting depicted in Figures 6.8 and Figures 6.9 contains the party $A$ optimal districting.
Bibliography


