Eötvös Loránd University
Institute of Mathematics

Thesis booklet

Isentropes, Lyapunov exponents and Ergodic averages

Gabriella Keszthelyi

Advisor: Zoltán Buczolich
Professor, Doctor of Sciences

Doctoral School: Mathematics
Director: Tibor Jordán
Professor, Doctor of Sciences

Doctoral Program: Pure Mathematics
Director: András Szűcs
Professor, corresponding member of the Hungarian Academy of Sciences

Eötvös Loránd University

Budapest
2019

1. Introduction

This thesis is based on the following papers: *Equi-topological entropy curves for skew tent maps in the square* [16], *Isentropes and Lyapunov exponents* [17], *Convergence of ergodic averages for many group rotations* [15], all of them are joint works with my advisor Zoltán Buczolich.
In this thesis we investigate particular systems, and their long-term behaviour and complexity. We mostly consider here low dimensional maps, namely interval maps except in the last part, where we generalize from circle rotations to group rotations.
In general we use topological and measure theoretical concepts to understand the systems, the properties we mostly use to describe the system are topological entropy, Lyapunov exponents, invariant measures, ergodicity.

2. Preliminaries

Skew tent maps $T_{\alpha,\beta} : [0,1] \to [0,1]$ are defined by (see Figure 1):

$$T_{\alpha,\beta}(x) = \begin{cases} \frac{\beta}{\alpha}x & \text{if } \ 0 \leq x < \alpha \\ \frac{\beta}{1-\alpha}(1-x) & \text{if } \ \alpha < x \leq 1. \end{cases} \tag{1}$$

The interval map $T_{\alpha,\beta}$ is unimodal, piecewise monotone and of BV. To avoid trivial dynamics we suppose that $0.5 < \beta \leq 1$ and $\alpha \in (1-\beta, \beta)$. (By trivial dynamics we mean that there is either only one attracting fixed point or there is one attracting and one repelling fixed point. In these cases every point is drawn towards the fixed points.) We denote by $\mathcal{U}$ the region of $[0,1]^2$ consisting of (α, β) with nontrivial dynamics. Tents from $\mathcal{U}$ are expanding and have two repelling fixed points which cause the interesting dynamics later.

For $x \in \mathbb{R}$ we put

$$L_{\alpha,\beta}(x) = \frac{\alpha}{\beta}x \text{ and } R_{\alpha,\beta}(x) = \frac{\beta}{1-\alpha}(1-x). \tag{2}$$

Suppose $T = T_{\alpha,\beta}$ is fixed for an $(\alpha,\beta) \in \mathcal{U}$ and $x \in [0,1]$. We define $\underline{I}(x) = \underline{I}_{\alpha,\beta}(x)$ the **itinerary of** x by

(i) $\underline{I}(x)$ is either an infinite sequence of Ls and Rs, or a finite (or empty) sequence of Ls and Rs ended by C. The jth entry of $\underline{I}(x)$ is denoted by $I_j(x)$, $j = 0, 1, \ldots$.

(ii) If $T^j_{\alpha,\beta}(x) \neq \alpha$ for all $j \geq 0$ then $I_j(x) = L$ if $T^j_{\alpha,\beta}(x) < \alpha$ and $I_j(x) = R$ if $T^j_{\alpha,\beta} > \alpha$.

(iii) If $T^k_{\alpha,\beta}(x) = \alpha$ for some k, then if k_0 is the smallest such k and $0 \leq l < k_0$ then $I_{k_0}(x) = C$ and $I_l(x) = L$ if $T^l_{\alpha,\beta}(x) < \alpha$ and $I_l(x) = R$ if $T^l_{\alpha,\beta}(x) > \alpha$.

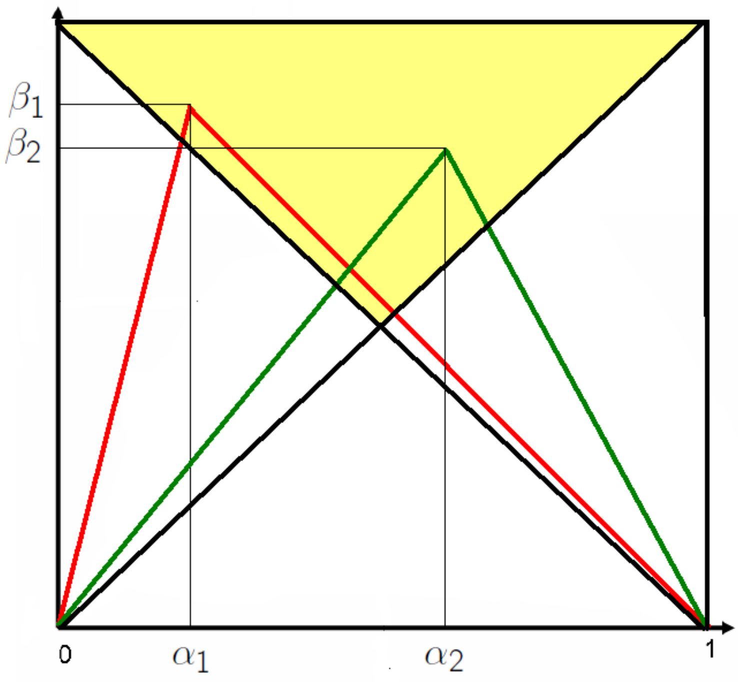

Figure 1: Parameter range $\mathcal{U}$

The **extended itinerary** of $T_{\alpha,\beta}$ denoted by $\underline{I}_E(x) = \underline{I}_{E,\alpha,\beta}(x) \in \{L, R, C\}^{\mathbb{Z}_{\geq 0}}$. If there is no C in $\underline{I}_E(x)$ then $\underline{I}(x) = \underline{I}_E(x)$, if there is a C then $\underline{I}(x)$ is finite, and $\underline{I}_E(x)$ is just the infinite repetition of $\underline{I}(x)$. We denote by $K(\alpha, \beta) = \underline{I}_{\alpha,\beta}(\beta)$ the kneading sequence of $T_{\alpha,\beta}$.

A sequence $\underline{M}$ of symbols L, R, C is called admissible if either is an infinite sequence of Ls and Rs or if $\underline{M}$ is a finite (or empty) sequence of Ls and Rs, followed by C. There is a parity-lexicographical $\prec$-ordering of the kneading sequences:
We set $L \prec C \prec R$.
Let $\underline{A} = A_1A_2\ldots$, $\underline{B} = B_1B_2\ldots$.
Suppose $A_j = B_j$ for all $j < i$, $A_i \prec B_i$ and
the number of Rs up to $j = (i-1)$ is **even** then $\underline{A} \prec \underline{B}$,
the number of Rs up to $j = (i-1)$ is **odd** then $\underline{A} \succ \underline{B}$.

We denote by $\mathfrak{M}$ the class of kneading sequences $K(0.5, \beta)$, $\beta \in (0.5, 1]$, this corresponds to the kneading sequences of $T_{\frac{1}{2},\beta}$ with $\frac{1}{2} < \beta \leq 1$..

By $\mathfrak{M}_\infty$ we denote those kneading sequences in $\mathfrak{M}$ which do not contain C. These are the infinite sequences. On the other hand, $\mathfrak{M}_{<\infty}$ will denote the finite kneading sequences. These are the ones ending with C corresponding to parameter values when the turning point is periodic. In that case we have a Markov partition.

Suppose $K(\alpha, \beta) = \underline{M} \in \mathfrak{M}$. We put $\underline{M}^- = \lim_{x\to\beta-} \underline{I}_{E,\alpha,\beta}(x)$. It is easy to see that $\underline{M}^-$ does not contain C.

Definition 2.1. Let T be a piecewise monotone map. If J is a maximal interval on which $T|_J$ is continuous and monotone, then $T : J \to T(J)$ is called a **lap** of T. The **lapnumber**, $l(T)$ is the number of laps of T.

For these maps for the topological entropy of T, that is $h_{top}(T)$ the following holds.

Proposition 2.2. *Let $T : [0,1] \to [0,1]$ have finitely many laps $l(T)$. Then*

$$h_{top}(T) = \lim_{n\to\infty} \frac{1}{n} \log l(T^n) =$$

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\{\textit{clusters of n-periodic points}\} = \max\{0, \lim_{n\to\infty} \frac{1}{n} \log VT^n\}$$

where two n-periodic points are in the same cluster if they belong to the same lap of T^n and VT^n is the variation of T^n.

3. Tent maps I.

In this section we introduce the equi-kneading curves $\Psi_{\underline{M}}$ and the auxiliary function $\Theta_{\underline{M}}$. The next theorem is a corollary of Theorem 1 of [36].

Theorem 3.1. *For each $\underline{M} \in \mathfrak{M}$ there exist two numbers $\alpha_1(\underline{M}) < \alpha_2(\underline{M})$ and a continuous function $\Psi_{\underline{M}} : (\alpha_1(\underline{M}), \alpha_2(\underline{M})) \to \mathcal{U}$ such that for $(\alpha, \beta) \in \mathcal{U}$ we have $K(\alpha, \beta) = \underline{M}$ if and only if $B = \Psi_{\underline{M}}(\alpha)$. The graphs of the functions $\Psi_{\underline{M}}$ fill up the whole set $\mathcal{U}$. Moreover, $\lim_{\alpha\to\alpha_1(\underline{M})+} \Psi_{\underline{M}}(\alpha) = 1$ if $\underline{M} \succeq RLR^\infty$. If $\underline{M} \prec RLR^\infty$ then the curve $(\alpha, \Psi_{\underline{M}}(\alpha))$ converges to a point on the line segment $\{(\alpha, 1-\alpha) : 0 < \alpha < \frac{1}{2}\}$ as $\alpha \to \alpha_1(\underline{M})+$. If $\underline{M} = RL^\infty$ then $\alpha_1(\underline{M}) = 0$, $\alpha_2(\underline{M}) = 1$ and $\Psi_{\underline{M}}(\alpha) = 1$ for all $\alpha \in (0,1)$. We will refer to $\Psi_{\underline{M}}$s as equi-kneading curves.*

Remark. On these curves the topological entropy $h(\alpha, \Psi_{\underline{M}}(\alpha))$ is constant, therefore we also call them isentropes.
Suppose

$$\underline{M}^- = R\underbrace{L\dots L}_{m_1}R\underbrace{L\dots L}_{m_2}R\underbrace{L\dots L}_{m_3}R\dots. \tag{3}$$

We introduce the notation $\overline{m}_k = m_1 + m_2 + \dots + m_k$.

Theorem 3.2. *Suppose $\underline{M} \in \mathfrak{M}\backslash\{RL^\infty\}$ is given. Then there exists a function $\Theta_{\underline{M}} : \mathcal{U} \to \mathbb{R}$, such that for $(\alpha, \beta) \in \mathcal{U}$ if $K(\alpha, \beta) = \underline{M}$ then $\Theta_{\underline{M}}(\alpha, \beta) = 0$. Moreover,*

$$\Theta_{\underline{M}}(\alpha, \beta) = 1 - \beta + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1-\alpha}{\beta}\right)^k \left(\frac{\alpha}{\beta}\right)^{\overline{m}_k} = \tag{4}$$

$$1 - \beta + \sum_{k=1}^{\infty} \left(\frac{\alpha-1}{\beta}\right)^k \left(\frac{\alpha}{\beta}\right)^{\overline{m}_k}$$

where $m_1 = \overline{m}_1 > 0$, $\overline{m}_k \le \overline{m}_{k+1} \le \overline{m}_k + \overline{m}_1$, $k = 0, 1, \dots$. If $\underline{M} = K(\alpha, \beta) \in \mathfrak{M}_{R,\infty}$ then there exists n such that $\overline{m}_{k+1} = \overline{m}_k$ for $k \ge n$.

This means that $\Psi_{\underline{M}}$ is a subset of $\{(\alpha,\beta)\in\mathcal{U}:\Theta_{\underline{M}}(\alpha,\beta)=0\}$, the zero level set of $\Theta_{\underline{M}}$. Also the isentrope $(\alpha,\Psi_{\underline{M}}(\alpha))$ satisfies the implicit equation $\Theta_{\underline{M}}(\alpha,\Psi_{\underline{M}}(\alpha))=0$. One would like to have the property that the level zero set of $\Theta_{\underline{M}}(\alpha,\beta)=0$ in $\mathcal{U}$ equals the curve of $\Psi_{\underline{M}}$, but it is not exactly true. On the positive side we have the following theorem.

Theorem 3.3. *Suppose $\underline{M}\in\mathfrak{M}\setminus\{RL^\infty\}$ is given, $(\alpha,\beta)\in\mathcal{U}$ and $K(\alpha,\beta)=\widetilde{\underline{M}}\succ\underline{M}$. Then $\Theta_{\underline{M}}(\alpha,\beta)\neq 0$.*

This shows that in the region $\mathcal{U}$ above the curve $(\alpha,\Psi_{\underline{M}}(\alpha))$, $\alpha\in(\alpha_1(\underline{M}),\alpha_2(\underline{M}))$ the auxiliary function $\Theta_{\underline{M}}$ is non-vanishing. Unfortunately, we cannot state the same of under $\Psi_{\underline{M}}$.

Theorem 3.4. *There exists $\underline{M}\in\mathfrak{M}_\infty\setminus\{RL^\infty\}$ such that for some $(\alpha,\beta)\in\mathcal{U}$ we have $\Theta_{\underline{M}}(\alpha,\beta)=0$, but $K(\alpha,\beta)=\widetilde{\underline{M}}\neq\underline{M}$. By Theorem 3.3 we have $K(\alpha,\beta)\prec\underline{M}$. (See a counterexample on Figure 2.)*

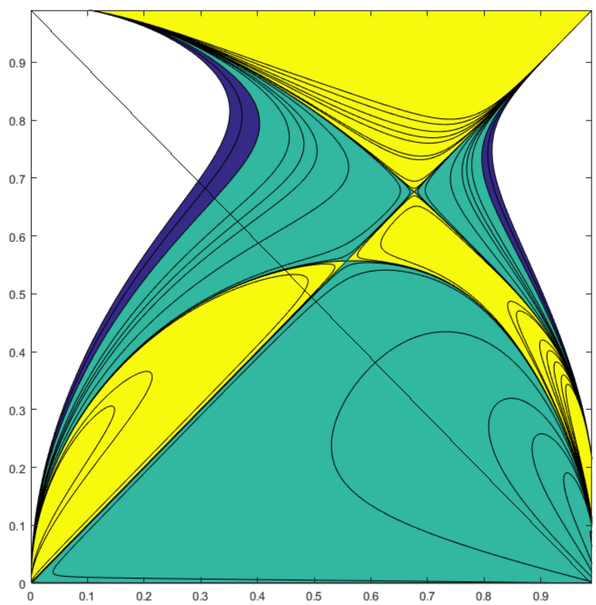

Figure 2: Levels of $\Theta_{\underline{M}}$, $\underline{M}=RLLRLRRLR^\infty$.

Lemma 3.5. *For any $\underline{M}\in\mathfrak{M}\setminus\{RL^\infty\}$ the function $\Theta_{\underline{M}}(\alpha,\beta)$ is infinitely differentiable on $\mathcal{U}$ and also in small neighborhoods of the points $\{(\beta,\beta):\frac{1}{2}<\beta<1\}$.*

M. Misiurewicz asked the question whether the isentropes are perpendicular to the diagonal. Using $\Theta_{\underline{M}}$ we show that the curves $(\alpha,\Psi_{\underline{M}}(\alpha))$ hit the diagonal $\{(\beta,\beta):0.5<\beta<1\}$ almost perpendicularly if (β,β) is close to $(1,1)$ (See Figure 3). The curves $\Psi_{\underline{M}}$ are not necessarily exactly orthogonal to

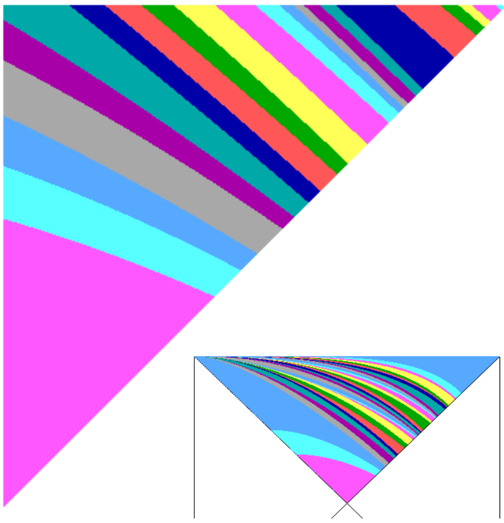

Figure 3: Blow up of equi-kneading curves near $(1/2, 1/2)$

the diagonal, for $\underline{M} = RLLRC$ the curve $(\alpha, \varphi_{\underline{M}}(\alpha))$ is not orthogonal to the diagonal. On the other hand, for $\underline{M} = RLC$ it is.
As we approach the upper right corner of the unit square isentropes are almost perpendicular as the next theorem states.

Theorem 3.6.

$$\lim_{\beta_0 \to 1-} D_\alpha \Psi^{\beta_0}(\beta_0) = -1. \tag{5}$$

Where $D_\alpha \Psi^{\beta_0}$ the derivative of Ψ^{β_0}. This means that the curves Ψ^{β_0} are almost perpendicular to the diagonal $\{(\beta, \beta) : \frac{1}{2} < \beta < 1\}$ at the point (β_0, β_0) when β_0 is close to 1.

4. Tent maps II.

In this section we deal with the differentiability of $\Psi_{\underline{M}}$ and the connection between $\Psi'_{\underline{M}}$ and the Lyapunov exponents.

Definition 4.1. For the skew tent map $T_{\alpha,\beta}$, $(\alpha, \beta) \in \mathcal{U}$ we define the Frobenius-Perron operator $P_{\alpha,\beta} : L^1[0,1] \to L^1[0,1]$ by

$$P_{\alpha,\beta} f(x) = \sum_{z \in \{T^{-1}_{\alpha,\beta}(x)\}} \frac{f(z)}{|T'_{\alpha,\beta}(z)|},$$

which in a more explicit form is

$$P_{\alpha,\beta} f(x) = \frac{\alpha}{\beta} f\left(\frac{\alpha x}{\beta}\right) + \frac{1-\alpha}{\beta} f\left(1 - \frac{1-\alpha}{\beta} x\right) \text{ if } 0 \le x \le \beta, \tag{6}$$

and $P_{\alpha,\beta} f(x) = 0$ if $x > \beta$.

Proposition 4.2. *[7] Let $T : I \to I$ be nonsingular. Then for the Frobenius-Perron operator P_T it holds that $P_T f^* = f^*$ a.e., if and only if the measure $\mu = f^* \cdot \lambda$, defined by $\mu(A) = \int_A f^* d\lambda$, is T-invariant, i.e., if and only if $\mu(T^{-1}A) = \mu(A)$ for all measurable sets A, where $f^* \geq 0$, $f^* \in L^1$ and $\|f^*\|_1 = 1$.*

In other words the invariant density will be the fixed point of $P_{\alpha,\beta}$. We also remind to the definition of the variation of a real function $f : [a, b] \to \mathbb{R}$.

$$Vf = V_{[a,b]}f = \sup_{\mathcal{P}} \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\}$$

where sup is taken for all partitions $\mathcal{P} = \{[x_0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n]\}$ of $[a, b]$. If $V_{[a,b]}f < +\infty$ then f is of bounded variation, BV on $[a, b]$.

Definition 4.3. Suppose $I = [a, b]$, $T : I \to I$. A partition

$$\mathcal{P} = \{[a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n]\}$$

of $[a, b]$ is Markov for T if for any $i = 1, \ldots, n$ the transformation $T|_{(a_{i-1},a_i)}$ is a homeomorphism onto the interior of the connected union of some elements of $\mathcal{P}$, that is onto an interval $(a_{j(i)}, a_{k(i)})$.

Observe that if $T^n_{\alpha,\beta}(\beta) = \alpha$, that is C appears in $K(\alpha, \beta) \in \mathfrak{M}_{<\infty}$ then the partition determined by the points $\{0, \alpha, \beta, T_{\alpha,\beta}(\beta), \ldots, T^{n-1}_{\alpha,\beta}(\beta), 1\}$ provides a Markov partition. If $T_{\alpha,\beta}$ is piecewise expanding and $1/|T'_{\alpha,\beta}|$ is of BV then it admits an absolutely continuous invariant measure, acim whose density is of bounded variation.
If $T : I \to I$ is differentiable, $\log |T|$ is integrable and μ is a T-invariant measure then we can talk about the **Lyapunov** exponent:

$$\Lambda(x) = \lim_{N\to\infty} \frac{1}{N} \log |(T^N)'(x)| =$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log |T'(T^n(x))|.$$

As far as we know in the literature there were two ways to estimate/ approximate Lyapunov exponents of skew tent maps. One method is based on computer programs approximating either γ (which depends on the absolutely invariant measure - acim - $\mu_{\alpha,\beta}$, and $\gamma = \mu_{\alpha,\beta}[0, \alpha]$), or the invariant density $f_{\alpha,\beta}$. The other method, discussed in [3] is based on the fact that if $K(\alpha, \beta) \in \mathfrak{M}_{<\infty}$, that is when the turning point is periodic for $T_{\alpha,\beta}$ then there is a Markov partition for $T_{\alpha,\beta}$. Based on the Markov partition one can obtain a system of linear equations and the solution of this system gives us the invariant density function $f_{\alpha,\beta}$ of the acim $\mu_{\alpha,\beta}$ of $T_{\alpha,\beta}$. Then $\gamma = \mu_{\alpha,\beta}([0, \alpha])$. The drawback of this calculation is that the number of equations is the number of elements in the Markov partition. If $K(\alpha, \beta) \in \mathfrak{M}_\infty$ then there is no Markov partition, but isentropes corresponding to skew tent maps with Markov partition are dense in $\mathcal{U}$. But if we approximate maps $T_{\alpha,\beta}$ with maps of parameters $(\alpha_n, \beta_n) \in \mathfrak{M}_{<\infty}$ then the number of equations tends to infinity.

Our first main theorem in this section is about approximating invariant densities.

Proposition 4.4. *Suppose* $(\alpha_n, \beta_n) \in \mathcal{U}$ *for* $n = 0, 1, \ldots,\quad (\alpha_n, \beta_n) \to (\alpha_0, \beta_0)$ *and* $\mathcal{P}_n = \{[0, \alpha_n], [\alpha_n, 1]\}$. *Suppose that*

$$\begin{aligned} &\forall\, m \geq 1, \exists\, \delta_m > 0 \text{ such that if} \\ &\mathcal{P}_n^{(m)} = \bigvee_{j=0}^{m-1} T_{\alpha_n,\beta_n}^{-j}(\mathcal{P}_n) \text{ then } \min_{I \in \mathcal{P}_n^{(m)}} \lambda(I) \geq \delta_m > 0. \end{aligned} \tag{7}$$

Then:
(A) For any density f *of bounded variation there exists a constant* M *such that for any* n *and* $k = 1, 2, \ldots$

$$V P_{\alpha_n,\beta_n}^k f \leq M.$$

This implies that for any n *there is an invariant density* f_n *of* T_{α_n,β_n} *and the set* $\{f_n\}$ *is a precompact set in* $L^1([0,1], \lambda)$.
(B) Moreover, if $f_{n_k} \to f_0$ *in* L^1 *then* f_0 *is an invariant density for* T_{α_0,β_0}.

The second one is about the connection between the tangents and the Lyapunov exponents:

Proposition 4.5. *Suppose* $(\alpha_0, \beta_0) \in \mathcal{U}$, $\underline{M} = K(\alpha_0, \beta_0) \in \mathfrak{M}_{<\infty}$, *that is there exists a minimal* $n_{\underline{M}} > 1$ *such that* $T_{\alpha_0,\beta_0}^{n_{\underline{M}}}(\beta_0) = \alpha_0$. *Assume that* $\Lambda = \Lambda_{\alpha_0,\beta_0}$ *denotes the Lyapunov exponent of* T_{α_0,β_0} *and* $(\alpha, \Psi_{\underline{M}}(\alpha))$ *is the isentrope satisfying* $\beta_0 = \Psi_{\underline{M}}(\alpha_0)$. *We also suppose that* $\Psi'_{\underline{M}}(\alpha_0)$ *exists, that is the isentrope is differentiable at* α_0. *Then we have the following formula*

$$\Lambda_{\alpha_0,\beta_0} = \Lambda = \gamma \log \frac{\beta_0}{\alpha_0} + (1-\gamma) \log \frac{\beta_0}{1-\alpha_0}, \text{ where } \gamma \text{ satisfies} \tag{8}$$

$$\gamma = \frac{\frac{\Psi'_{\underline{M}}(\alpha_0)}{\beta_0} + \frac{1}{1-\alpha_0}}{\frac{1}{\alpha_0} + \frac{1}{1-\alpha_0}} = \alpha_0(1-\alpha_0)\frac{\Psi'_{\underline{M}}(\alpha_0)}{\beta_0} + \alpha_0. \tag{9}$$

Moreover, if μ *denotes the acim of* T_{α_0,β_0} *then*

$$\gamma = \mu([0, \alpha_0]). \tag{10}$$

Later we actually prove that $\Psi_{\underline{M}}$ are continuously differentiable:

Theorem 4.6. *If* $\underline{M} \in \mathfrak{M}$ *then* $\Psi'_{\underline{M}}$ *exists and is continuous on* $(\alpha_1(\underline{M}), \alpha_2(\underline{M}))$.

One can obtain $\Psi_{\underline{M}}$ by implicit differentiation.

$$\Psi'_{\underline{M}}(\alpha) = -\frac{\partial_1 \Theta_{\underline{M}}(\alpha, \Psi_{\underline{M}}(\alpha))}{\partial_2 \Theta_{\underline{M}}(\alpha, \Psi_{\underline{M}}(\alpha))}, \tag{11}$$

Since the series in (4) converges at an exponential rate if we consider the partial derivatives we also obtain an exponential convergence rate for the partial derivatives and it is very easy to compute/approximate $\Psi'_{\underline{M}}(\alpha)$ and therefore the Lyapunov exponents. Our next theorem says that we can do this in the general case, without having Markov partition.

Theorem 4.7. *Suppose* $(\alpha_0, \beta_0) \in \mathcal{U}$, $\Lambda = \Lambda_{\alpha_0,\beta_0}$ *denotes the Lyapunov exponent of* T_{α_0,β_0} *and* $(\alpha, \Psi_{\underline{M}}(\alpha))$ *is the isentrope satisfying* $\beta_0 = \Psi_{\underline{M}}(\alpha_0)$. *Then* $\Psi'_{\underline{M}}(\alpha_0)$ *exists, moreover* (8) *and* (9) *are satisfied.*

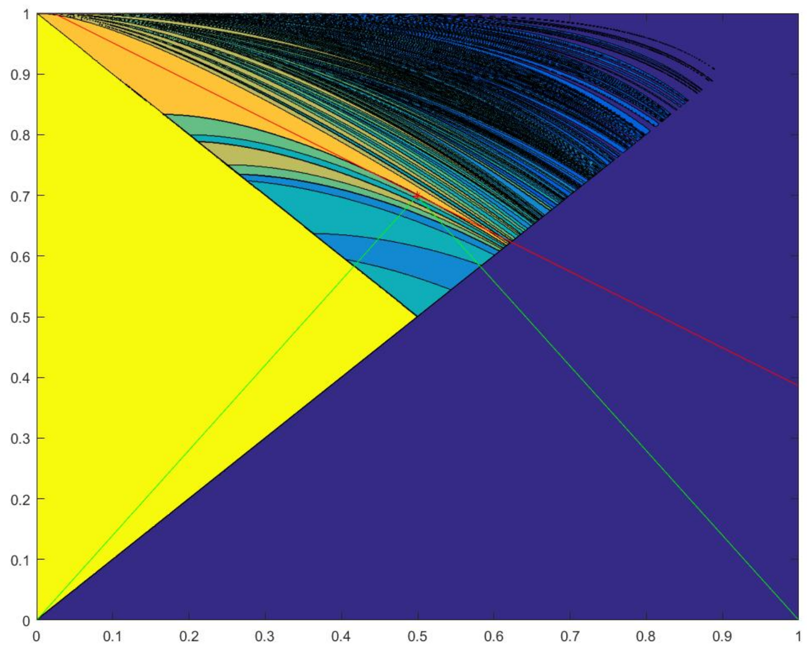

Figure 4: Tangent of $\Psi_{\underline{M}}$, $\underline{M} = K(0.5, 0.7)$, 5000 iterates

5. Convergence of Birkhoff averages for many group rotations

Birkhoff's ergodic theorem says that if we have an ergodic transformation (and measure) and an L^1 function f then the time average and the space average will be the same as N goes to infinity:

$$\frac{1}{N+1}\sum_{i=0}^{N} f(T^i(x)) \to \frac{1}{\mu(X)}\int_X f d\mu = f^* \quad \mu \text{ a.e. } x.$$

The starting point was here whether it is possible to generalize Birkhoff's theorem if we consider measurable functions f which are not necessarily Lebesgue integrable. A major obstruction to the generalization of Birkhoff's theorem to this case is P. Major's following example [32]. There exist $S, T : X \to X$ two conjugate ergodic transformations on a suitable probability space (X, μ) such that

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} f(S^k(x)) = 0 \quad \mu \text{ a.e.}$$

and

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} f(T^k(x)) = a \neq 0 \quad \mu \text{ a.e.}.$$

Later M. Laczkovich raised the question whether in Major's example X can be changed to $\mathbb{T}$ and S, T two different irrational rotations. Let $f : \mathbb{T} \to \mathbb{R}$ be a given measurable function. For an $\alpha \in \mathbb{R}$ put

$$M_N^{\alpha} f(x) = \frac{1}{N+1}\sum_{k=0}^{N} f(x + k\alpha)$$

we will use this particular time average from now on. The answer is yes, Z. Buczolich showed in [12] that for any two independent irrationals α_1, α_2 it is possible to find a measurable $f : \mathbb{T} \to \mathbb{R}$ such that $M_N^{\alpha_1} f(x) \to c_1$ and $M_N^{\alpha_2} f(x) \to c_2 \neq c_1$.

Some of our results also hold for nonconventional ergodic averages. We change (k) in $M_N^{\alpha} f(x)$ for (n_k) which is a strictly monotone increasing sequence of integers, and we consider f on the compact Abelian group G. We investigate some special rotation sets on G equipped with the Haar measure m, where f is measurable:

$$\Gamma_f = \Big\{\alpha \in G : M_N^{\alpha} f(x) \text{ converges for } m \text{ a.e. } x \text{ as } N \to \infty\Big\}.$$

We also introduce some similar but slightly different sets than Γ_f

$$\Gamma_{f,0} = \Big\{\alpha \in G : \frac{f(x + n_k\alpha)}{k} \to 0 \text{ for } \quad m \text{ a.e. } x\Big\} \text{ and}$$
$$\Gamma_{f,b} = \Big\{\alpha \in G : \limsup_{k\to\infty} \frac{|f(x + n_k\alpha)|}{k} < \infty \text{ for } m \text{ a.e. } x\Big\}. \tag{12}$$

It is obvious that $\Gamma_f \subset \Gamma_{f,0} \subset \Gamma_{f,b}$. In [12] it was shown that from $m(\Gamma_{f,0}) > 0$ it follows that $f \in L^1(\mathbb{T})$, when the sequence $n_k = k$ is considered. Our main result in this final section is the following:

Theorem 5.1. *If (n_k) is a strictly monotone increasing sequence of integers and G is a compact, locally connected Abelian group and $f : G \to \mathbb{R}$ is a measurable function then from $m(\Gamma_{f,b}) > 0$ it follows that $f \in L^1(G)$.*

Theorem 5.1 says that if we do not have "too much torsion" in $\widehat{G}$ then from $m(\Gamma_{f,b}) > 0$ it follows that $f \in L^1(G)$. If one considers an individual Z_p (the group of p-adic integers) then its dual group is $Z(p^\infty)$ with all elements of finite order, so still there seems to be "lots of torsion" in the dual group. It is also clear that arithmetic properties of n_k might matter if we consider Z_p. For us it was quite surprising that if one considers ordinary ergodic averages, that is, $n_k = k$ then Z_p behaves like a locally connected group and the following theorem is true.

Theorem 5.2. *Suppose that $n_k = k$, and p is a fixed prime number. We consider $G = Z_p$, the group of p-adic integers. Then for any measurable function $f : G \to \mathbb{R}$ from $m(\Gamma_{f,b}) > 0$ it follows that $f \in L^1(G)$.*

However, such a result does not hold if the dual group $\widehat{G}$ contains "infinitely many multiple torsions". We say that the group G contains **infinitely many multiple torsions** if either there is a prime number p such that G contains a subgroup algebraically isomorphic to the direct sum $(Z/p) \bigoplus (Z/p) \bigoplus \dots$, or there are infinitely many different prime numbers $p_1, p_2, \dots$ such that G contains for any j subgroups of the form $(Z/p_j) \times (Z/p_j)$.

Theorem 5.3. *Suppose that (n_k) is a strictly monotone increasing sequence of integers and G is a compact Abelian group such that its dual group $\widehat{G}$ contains infinitely many multiple torsion. Then there exists a measurable $f \notin L^1(G)$ such that*

$$m(\Gamma_{f,0}) = m(\Gamma_{f,b}) = 1, \text{ where } m \text{ is the Haar-measure on } G. \tag{13}$$

In fact, we show that $\Gamma_{f,0} = \Gamma_{f,b} = G$.

References

[1] I. Assani, Z. Buczolich and D. Mauldin
"An L^1 Counting problem in Ergodic Theory"
J. Anal. Math. **95** (2005), 221–241.

[2] J. Bell
"The Pontryagin duals of $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q}$"
`https://pdfs.semanticscholar.org/bffd/643d972eab01c0b063c34ad1560791899d3a.pdf`

[3] L. Billings and E.M. Bollt
"Probability density functions of some skew tent maps"
Chaos Solitons Fractals **12**, No. 2, 365-376 (2001).

[4] A. B. Blaya and V. J. Lopez
"On the relations between positive Lyapunov exponents, positive entropy, and senstivity for interval maps"
Discrete and continuous dynamical systems Volume 32, Number 2, February 2012 pp. 433–466.

[5] V. Botella-Soler, J. A. Oteo, J. Ros, P. Glendinning
"Families of piecewise linear maps with constant Lyapunov exponent"
`https://arxiv.org/abs/1205.5360`

[6] J. Bourgain, H. Fürstenberg, Y. Katznelson, and D. S. Ornstein
"Pointwise Ergodic Theorems for Arithmetic Sets, with an Appendix"
Publ. Mat. IHES **69** (1989), 5-45.

[7] A. Boyarsky and P. Góra "Laws of chaos: invariant measures and dynamical systems in one dimension"
Boston: Birkhauser, 1997.

[8] Brucks, K., Bruin, H. (2004). "Topics from One-Dimensional Dynamics"
(London Mathematical Society Student Texts). Cambridge: Cambridge University Press.

[9] H. Bruin
"Notes on Ergodic Theory"
`https://www.mat.univie.ac.at/~bruin/ET3.pdf`

[10] H. Bruin and S. van Strien
"On the structure of isentropes of polynomial maps"
Dyn. Syst. **28** no. 3, 381–392 (2013).

[11] Z. Buczolich
"Almost everywhere convergence of ergodic averages"
Real Analysis Exchange. 34 (2009), no. 1, 1–15.

[12] Z. Buczolich,
"Arithmetic averages of rotations of measurable functions"
Ergodic Theory Dynam. Systems **16** (1996), no. 6, 1185–1196.

[13] Z. Buczolich
"Ergodic averages and free $\mathbb{Z}^2$ actions"
Fund. Math., **160**, (1999), 247-254.

[14] Z. Buczolich,
"Non-L^1 functions with rotation sets of Hausdorff dimension one", *Acta Math. Hungar.* **126**, (2010) 23-50.

[15] Z. Buczolich, G. Keszthelyi
"Convergence of ergodic averages for many group rotations"
Ergodic Theory and Dynamical Systems, 36(7), 2107-2120.

[16] Z. Buczolich and G. Keszthelyi
"Equi-topological entropy curves for skew tent maps in the square."
Math. Slovaca, **67**, No. 6, 1577–1594, (2017).

[17] Z. Buczolich, G. Keszthelyi
"Isentropes and Lyapunov exponents" (preprint)
`https://arxiv.org/pdf/1804.01837.pdf`

[18] Z. Buczolich and G. Keszthelyi
"Tangents of Isentropes of skew tent maps in the square." (in preparation).

[19] Z. Buczolich and D. Mauldin
"Divergent Square Averages" *Ann. of Math.* **171**:(3) (2010), 1479-1530., 2010.

[20] Davoud Cheraghi and Tobias Kuna "Dynamical Systems" `http://wwwf.imperial.ac.uk/~dcheragh/Teaching/2016-F-DS-MPE.pdf`

[21] V. Climenhaga and A. Katok
"Measure theory through dynamical eyes"
`https://arxiv.org/pdf/1208.4550.pdf`

[22] P. Collet and J-P Eckmann
"Iterated Maps on the Interval as Dynamical Systems"
Originally published in the series of *Progress in Physics, 1980* Boston: Birkhauser, 2009.

[23] D. Fiebig, U. Fiebig and Z. Niteczki
"Entropy and preimage sets"
Ergodic Theory and Dynamical Systems, 23(6), 1785-1806. (2003)

[24] L. Fuchs,
" Infinite Abelian groups Vol. I."
Pure and Applied Mathematics, Vol. 36 Academic Press, New York-London 1970.

[25] B. Hasselblatt, A. Katok
"Handbook of Dynamical Systems"
Elsevier B. V., 2006,

[26] E. Hewitt and K. A. Ross
"Structure of topological groups, integration theory, group representations"
Second edition. Grundlehren der Mathematischen Wissenschaften Abstract harmonic analysis. Vol. I. **115** Springer-Verlag, Berlin-New York, 1979.

[27] M. Hochman
"Notes on ergodic theory"
`http://math.huji.ac.il/~mhochman/courses/ergodic-theory-2012/notes.final.pdf`

[28] F. Hofbauer, G. Keller
"Quadratic maps without asymptotic measure"
Commun. Math Phys Volume 127, Number 2 (1990), 319-337.

[29] Jonq Juang,Shih-Feng Shieh
"Piecewise linear maps, Liapunov exponents and entropy"
Journal of Mathematical Analysis and Applications Volume 338, Issue 1, 1 February 2008, Pages 358-364.

[30] D. Laiand, and G. Chen
"On statistical properties of the Lyapunov exponent of the generalized skew tent map"
Stochastic Anal. Appl., **20**(2), 375-388 (2002).

[31] D. Lindt, B. Marcus
"An Introduction to Symbolic Dynamics and Coding"
Cambridge University Press, Cambridge, 1995.

[32] P. Major
"A counterexample in ergodic theory"
Acta Sci. Math. (Szeged) 62 (1996), 247–258.

[33] M. C. Mackey and M. Tyran-Kaminska
"Central limit theorem behavior in the skew tent map"
Chaos Solitons Fractals **38**, No. 3, 789-805 (2008).

[34] J. Milnor, W. Thurston
"On iterated maps of the interval"
In: Alexander J.C. (eds) Dynamical Systems. Lecture Notes in Mathematics, vol 1342. (1988) Springer, Berlin, Heidelberg.

[35] J. Milnor and C. Tresser
"On entropy and monotonicity for real cubic maps"
With an appendix by Adrien Douady and Pierrette Sentenac. Comm. Math. Phys. **209***, no. 1, 123–178 (2000).*

[36] M. Misiurewicz and E. Visinescu
"Kneading sequences of skew tent maps"
Ann. Inst. Henri Poincaré, Probab. Stat. **27**, No. 1, 125-140 (1991).

[37] A. Radulescu
"The connected isentropes conjecture in a space of quartic polynomials"
Discrete Contin. Dyn. Syst. **19** no. 1, 139–175 (2007).

[38] J. M. Rosenblatt and M. Wierdl
"Pointwise ergodic theorems via harmonic analysis"
Ergodic theory and its connections with harmonic analysis (Alexandria, 1993), 3–151, London Math. Soc. Lecture Note Ser., **205,** Cambridge Univ. Press, Cambridge, 1995.

[39] W. Rudin,
"Fourier analysis on groups"
Interscience Tracts in Pure and Applied Mathematics, No. 12 Interscience Publishers (a division of John Wiley and Sons), New York-London 1962.

[40] Sylvie Ruette
"Chaos on interval"
University Lecture Series (Book 67) Publisher: American Mathematical Society (March 2, 2017).

[41] B. Saussol, S. Troubetzkoy and S. Vaienti
"Recurrence, dimensions and Lyapunov exponents"
Journal of Statistical Physics 106 (2002) 623-634.

[42] Ya. Sinai and C. Ulcigrai
"Renewal type limit theorem for the Gauss map and continued fractions"
Ergodic Theory Dynam. Systems, **28** (2008), no. 2, 643–655.

[43] Ya. Sinai and C. Ulcigrai
"A limit theorem for Birkhoff Sums of non-integrable functions over rotations"
Probabilistic and Geometric Structures in Dynamics Volume 469, 2008. edited by K. Burns, D. Dolgopyat

[44] W. Srinin
"Measure theoretic aspects of dynamical systems"
`http://www.math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Srinin.pdf`

[45] R. Svetic "A function with locally uncountable rotation set"
Acta. Math. Hungar. 81(4) (1998), 305–314.

[46] "22nd Internet Seminar - Ergodic Theorems" `https://ergodic.mathematik.uni-leipzig.de`

[47] Ana Anušić "Dynamics of unimodal interval maps"
`https://vd.univie.ac.at/fileadmin/user_upload/p_vd/VDS_Mathematics/Minicourse_Anusic_neu.pdf` *Notes for the mini-course at the University of Vienna, May 2017.*