

On perturbative quantum field theory with boundary

Z. Bajnok^{1*}, G. Böhm² and G. Takács^{1†}

¹*HAS Theoretical Physics Research Group, Institute for Theoretical Physics,
Eötvös University, H-1117 Budapest Pázmány s. 1/A, Hungary*

²*Research Institute for Particle Physics,
H-1525 Budapest, 114 P.O.B. 49, Hungary*

This work is dedicated to Prof. Zalán Horváth on the occasion of his 60th birthday.

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Abstract

Boundary quantum field theory is investigated in the Lagrangian framework. Models are defined perturbatively around the Neumann boundary condition. The analyticity properties of the Green functions are analyzed: Landau equations, Cutkosky rules together with the Coleman-Norton interpretation are derived. Illustrative examples as well as argument for the equivalence with other perturbative expansions are presented.

1 Introduction

In this paper we study quantum field theory with boundary in perturbation theory. The boundary is a flat hypersurface with a space-like normal vector, i.e. in appropriate inertial coordinates it is just given by constraining one coordinate to take a constant value (conveniently chosen to be zero). The motivation for such a study and most of the explicit examples come from 1 + 1 dimensional field theory, but we present the formalism for an arbitrary number of spacetime dimensions to show its generality.

Our main motivation comes from consideration of integrable boundary QFT in 1+1 dimensions, although integrability is not required for the general formalism to work. In such QFT, it is possible to construct exact scattering amplitudes (S matrices and reflection amplitudes) using the bootstrap procedure. The central idea of the bootstrap is that certain poles of scattering amplitudes give rise to new states (bulk particles or excited boundary states) in the theory, which must be treated as fundamental degrees of freedom in their own right (“nuclear democracy”).

*bajnok@afavant.elte.hu

†takacs@ludens.elte.hu

However, in the general case there are poles which are not assigned to any particle. In the case of bulk theories, bootstrap closure (or consistency) means that all poles are explained in terms of intermediate processes involving particles on-mass-shell, and that the residues can be calculated using the on-shell couplings between the particles. In case of poles associated to bound states, the intermediate state is a one-particle state; the more general processes are associated to so-called Coleman-Thun diagrams [1]. Consistency requires that there exists a consistent choice for the on-shell couplings such that the singular contributions at all the poles can be calculated by the Cutkosky rules (in practice one normally looks only at the most singular terms).

For QFT with boundaries, the respective rules for Coleman-Thun diagrams were generalized by analogy in [2, 3], but there has been no systematic derivation of these rules so far. In the bulk, the usual derivation of singularity positions (Landau equations) and Cutkosky rules uses the analytic properties of covariant perturbation theory [4, 5], although the Cutkosky rules can also be understood as a generalization of unitarity [4].

There have been numerous previous works using perturbation theory for boundary QFTs [6]–[14]. Unfortunately these expansions are usually rather difficult to use to explore the analytic structure, because the propagators do not have simple analytic properties if expressed in terms of the momentum (in some cases even that is impossible). Therefore in Section 2 we propose to use a perturbation expansion around the free (Neumann) boundary condition and a constant bulk field configuration. It is not at all evident that this approach is correct in the general case, and the main difficulty comes from the fact that the vacuum of the boundary QFT may be a non-constant field configuration. As we mentioned above, expanding around such a field configuration would make the investigation of the analytic structure hopeless. Using a toy model, we present evidence in appendix A that a re-summation of our perturbation expansion gives the correct results that are expected from the standard approaches and in particular the tadpole contributions reproduce the effect of the nontrivial background field. As a further support for the correctness of the perturbative expansion, in Appendix B we give a systematic derivation of one-loop counter-terms in boundary sine-Gordon theory that have been used in semiclassical calculations in the literature [15, 16, 17].

We also extend the perturbation theory by taking into consideration fields living on the boundary, which is instrumental for the derivation of the boundary Cutkosky rules. We define asymptotic states, and write down the generalization of LSZ reduction formulae for boundary QFT, which were previously derived in a restricted context by us [18].

In Section 3, using the perturbative formulation, we derive the boundary extension of Landau equations and Coleman-Norton interpretation, while in Section 4 we obtain the boundary Cutkosky rules. In Section 5 we present some explicit examples of singularities in the conjectured scattering amplitudes of boundary sine-Gordon theory. The paper ends by a discussion of the results and the outline of some issues that remain to be investigated.

2 Perturbation theory with a boundary

In the course of analyzing of boundary quantum field theories we follow the same line as presented in [5] for bulk theories. Setting the stage by introducing our conventions, we identify and canonically quantize the free theory. The R matrix is defined via asymptotic states and is related to the Green functions using the boundary analogue of the LSZ formula. Then we introduce Feynman rules for the computation of Green functions, both in coordinate and

momentum space.

2.1 Conventions

The coordinates describing the half space-time are

$$z = (t, \vec{x}, y), \quad \vec{x} = (x^1, \dots, x^{D-1}) \quad , \quad -\infty < t, x^i < \infty, -\infty < y \leq 0$$

We denote the boundary coordinates by $x = (t, \vec{x})$ and use the following abbreviations

$$\int d\vec{x} = \prod_{i=1}^{D-1} \int_{-\infty}^{\infty} dx_i \quad , \quad \int dx = \int d\vec{x} \int_{-\infty}^{\infty} dt \quad , \quad \int dz = \int dx \int_{-\infty}^0 dy$$

$$\vec{\nabla} = (\partial_{x^1}, \dots, \partial_{x^{D-1}})$$

Bulk fields are denoted by $\Phi_\alpha(z)$, while for boundary fields we use $\phi_a(x)$. For simplicity, all the fields are supposed to be real scalars (generalizing to other cases is straightforward). The action is

$$S = \int dz \left\{ \frac{1}{2} \left[(\partial_t \Phi_\alpha)^2 - (\vec{\nabla} \Phi_\alpha)^2 - (\partial_y \Phi_\alpha)^2 - M_\alpha^2 \Phi_\alpha^2 \right] - V(\Phi_\alpha) \right\}$$

$$+ \int dx \left\{ \frac{1}{2} \left[(\partial_t \phi_a)^2 - (\vec{\nabla} \phi_a)^2 - m_a^2 \phi_a^2 \right] - U(\phi_a, \Phi_\alpha(y=0)) \right\} \quad (2.1)$$

where

$$V(\Phi_\alpha) = \sum_{M \geq 3} \sum_{\{\alpha_1, \dots, \alpha_M\}} v_{\alpha_1 \dots \alpha_M} \Phi_{\alpha_1} \dots \Phi_{\alpha_M}$$

describes the bulk interaction, while

$$U(\Phi_\alpha, \phi_a) = \sum_{M, N} \sum_{\{\alpha_1, \dots, \alpha_M\}} \sum_{\{a_1, \dots, a_N\}} u_{\alpha_1 \dots \alpha_M; a_1 \dots a_N} \Phi_{\alpha_1} \dots \Phi_{\alpha_M} \phi_{a_1} \dots \phi_{a_N}$$

contains the bulk-boundary and pure boundary interaction terms (we suppose that it contains no terms with $M = 0$ and $N = 1, 2$).

2.2 Free fields

Here we give a short description of the free theory with $U = V = 0$. Free bulk fields satisfy the equations of motion

$$\left(\partial_t^2 - \vec{\nabla}^2 - \partial_y^2 + M_\alpha^2 \right) \Phi_\alpha(z) = 0 \quad , \quad \partial_y \Phi_\alpha(z)|_{y=0} = 0$$

The field satisfying the boundary condition can be decomposed as

$$\Phi_\alpha(t, \vec{x}, y) = \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \cos(\kappa y) \int \frac{d\vec{k}}{(2\pi)^{D-1}} \exp(i\vec{k} \cdot \vec{x}) \tilde{\Phi}_\alpha(\kappa, \vec{k}, t)$$

$$\tilde{\Phi}_\alpha(\kappa, \vec{k}, t)^\dagger = \tilde{\Phi}_\alpha(\kappa, -\vec{k}, t) \quad , \quad \tilde{\Phi}_\alpha(\kappa, \vec{k}, t) = \tilde{\Phi}_\alpha(-\kappa, \vec{k}, t)$$

Similar decomposition is valid for the canonical momentum $\Pi_\alpha(z) = \partial_t \Phi(z)$. The commutation relations are

$$[\Phi_\alpha(t, \vec{x}, y), \Pi_\beta(t, \vec{x}', y')] = i\delta_{\alpha\beta}\delta(\vec{x} - \vec{x}') (\delta(y - y') + \delta(y + y'))$$

due to the Neumann boundary condition. We can introduce the bulk creation/annihilation operators

$$\begin{aligned} A_\alpha(\kappa, \vec{k}, t) &= i\tilde{\Pi}_\alpha(\kappa, \vec{k}, t) + \Omega_\alpha(\kappa, \vec{k})\tilde{\Phi}_\alpha(\kappa, \vec{k}, t) \\ A_\alpha(\kappa, \vec{k}, t)^\dagger &= -i\tilde{\Pi}_\alpha(\kappa, \vec{k}, t) + \Omega_\alpha(\kappa, \vec{k})\tilde{\Phi}_\alpha(\kappa, \vec{k}, t) \\ \Omega_\alpha(\kappa, \vec{k}) &= \sqrt{\kappa^2 + \vec{k}^2 + M_\alpha^2} \end{aligned}$$

that satisfy

$$[A_\alpha(\kappa, \vec{k}, t), A_\beta(\kappa', \vec{k}', t)^\dagger] = (2\pi)^D 2\Omega_\alpha(\kappa, \vec{k})\delta_{\alpha\beta}\delta(\vec{k} - \vec{k}') (\delta(\kappa - \kappa') + \delta(\kappa + \kappa'))$$

The boundary fields can be quantized in the usual way as free fields living in D dimensional spacetime:

$$\begin{aligned} \phi_a(x) &= \int \frac{d\vec{k}}{(2\pi)^{D-1}} \exp(i\vec{k} \cdot \vec{x}) \tilde{\phi}_a(\vec{k}, t) \quad , \quad \tilde{\phi}_a(\vec{k}, t)^\dagger = \tilde{\phi}_a(-\vec{k}, t) \\ \pi_a = \partial_t \phi_a \quad , \quad [\phi_a(t, \vec{x}), \pi_b(t, \vec{x}')] &= i\delta_{ab}\delta(\vec{x} - \vec{x}') \\ a_b(\vec{k}, t) &= i\tilde{\pi}_b(\vec{k}, t) + \omega_b(\vec{k})\tilde{\phi}_b(\vec{k}, t) \quad , \quad \omega_b(\vec{k}) = \sqrt{\vec{k}^2 + m_b^2} \\ [a_b(\vec{k}, t), a_c(\vec{k}', t)^\dagger] &= (2\pi)^{D-1} 2\omega_b(\vec{k})\delta_{bc}\delta(\vec{k} - \vec{k}') \end{aligned}$$

The (normal ordered) free Hamiltonian can be written as

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \int \frac{d\vec{k}}{(2\pi)^{D-1}} A_\alpha(\kappa, \vec{k}, t)^\dagger A_\alpha(\kappa, \vec{k}, t) + \int \frac{d\vec{k}}{(2\pi)^{D-1}} a_b(\vec{k}, t)^\dagger a_b(\vec{k}, t)$$

and the time development of the modes can be calculated as

$$A_\alpha(\kappa, \vec{k}, t) = e^{-i\Omega_\alpha(\kappa, \vec{k})t} A_\alpha(\kappa, \vec{k}) \quad , \quad a_b(\vec{k}, t) = e^{-i\omega_b(\kappa, \vec{k})t} a_b(\vec{k})$$

The Fock space of free fields can be introduced in the standard way. The vacuum satisfies

$$A_\alpha(\kappa, \vec{k}) |0\rangle = 0 = a_b(\vec{k}) |0\rangle$$

and the space is spanned by the states

$$A_{\alpha_1}(\kappa_1, \vec{k}_1)^\dagger \dots A_{\alpha_M}(\kappa_M, \vec{k}_M)^\dagger a_{b_1}(\vec{k}'_1)^\dagger \dots a_{b_N}(\vec{k}'_N)^\dagger |0\rangle$$

Due to the symmetry $A_\alpha(\kappa, \vec{k}) = A_\alpha(-\kappa, \vec{k})$ we could constrain $\kappa \geq 0$. However, it turns out to be simpler to let κ take general real values, and impose the former symmetry property to account for the presence of the boundary.

The free propagators are

$$g_{ab}(x, x') = \langle 0 | T \phi_a(x) \phi_b(x') | 0 \rangle = i\delta_{ab} \int \frac{dk}{(2\pi)^D} \frac{e^{-ik \cdot (x-x')}}{k^2 - m_a^2 + i\epsilon}$$

where $k = (k_0, \vec{k})$, $\int dk = \int dk_0 \int d\vec{k}$, and

$$\begin{aligned} G_{\alpha\beta}(z, z') &= \langle 0 | T \Phi_\alpha(z) \Phi_\beta(z') | 0 \rangle \\ &= i\delta_{\alpha\beta} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \int \frac{d\vec{k}}{(2\pi)^D} \frac{e^{-i\kappa(x-x')}}{k^2 - \kappa^2 - M_\alpha^2 + i\varepsilon} \left(e^{i\kappa(y-y')} + e^{i\kappa(y+y')} \right) \end{aligned}$$

The bulk propagator contains two terms: the first one describes the direct propagation of the fields, while the other corresponds to reflection on the free boundary. This can also be interpreted as a propagation to the mirror image point at $-y'$, and will play an important role in the formulation of the Feynman rules later.

2.3 Asymptotic states, \mathbf{R} matrix and reduction formulae

We proceed with the usual assumptions: the interaction is switched off adiabatically as $t \rightarrow \pm\infty$. The asymptotic fields are then defined to be free fields Φ_α^{as} and ϕ_a^{as} ($as = in, out$), satisfying

$$\begin{aligned} \lim_{t \rightarrow \mp\infty} \Phi_\alpha(t, \vec{x}, y) - Z_\alpha \Phi_\alpha^{in/out}(t, \vec{x}, y) &= 0 \\ \lim_{t \rightarrow \mp\infty} \phi_a(t, \vec{x}) - Z_a \phi_a^{in/out}(t, \vec{x}) &= 0 \end{aligned}$$

where the Z coefficients, as usual, take care of the canonical normalization of the fields, and the limits are understood in the weak sense (i.e. for any matrix element of the fields).

The modes of the asymptotic fields, which are obtained using the free field mode expansions

$$\begin{aligned} A_\alpha^{as}(\kappa, \vec{k}) &= 2i \int_{-\infty}^0 dy \int d\vec{x} \cos(\kappa y) e^{i(\Omega_\alpha(\kappa, \vec{k})t - \vec{k} \cdot \vec{x})} \overleftrightarrow{\partial}_t \Phi_\alpha^{as}(t, \vec{x}, y) \\ A_\alpha^{as}(\kappa, \vec{k})^\dagger &= -2i \int_{-\infty}^0 dy \int d\vec{x} \cos(\kappa y) e^{-i(\Omega_\alpha(\kappa, \vec{k})t - \vec{k} \cdot \vec{x})} \overleftrightarrow{\partial}_t \Phi_\alpha^{as}(t, \vec{x}, y) \\ a_b^{as}(\vec{k}) &= i \int d\vec{x} e^{i(\omega_b(\vec{k})t - \vec{k} \cdot \vec{x})} \overleftrightarrow{\partial}_t \phi_b^{as}(t, \vec{x}) \\ a_b^{as}(\vec{k})^\dagger &= -i \int d\vec{x} e^{-i(\omega_b(\vec{k})t - \vec{k} \cdot \vec{x})} \overleftrightarrow{\partial}_t \phi_b^{as}(t, \vec{x}) \end{aligned}$$

create asymptotic states as follows:

$$\begin{aligned} \left| \kappa_1, \vec{k}_1, \alpha_1; \dots; \kappa_M, \vec{k}_M, \alpha_M; \vec{k}_1, b_1; \dots; \vec{k}_N, b_N \right\rangle_{as} = \\ A_{\alpha_1}^{as}(\kappa_1, \vec{k}_1)^\dagger \dots A_{\alpha_M}^{as}(\kappa_M, \vec{k}_M)^\dagger a_{b_1}^{as}(\vec{k}_1)^\dagger \dots a_{b_N}^{as}(\vec{k}_N)^\dagger | 0 \rangle . \end{aligned}$$

We assume asymptotic completeness: both the *in* and the *out* states form a complete basis. The unitary transformation between the two is what we call the reflection matrix R (in the usual terminology, reflection matrix means the matrix element of R between a bulk one-particle *in* and a bulk one-particle *out* state). For any given initial and final state, the corresponding matrix element of the R matrix (more precisely: R operator) gives the probability amplitude for the evolution of the initial state $|i\rangle$ at $t = -\infty$ into the final state $|f\rangle$ at $t = +\infty$:

$$R_{fi} = \text{}_{out} \langle f | i \rangle_{in}$$

Now one can proceed to deduce reduction formulae for the matrix elements between asymptotic states. The derivation is essentially the same as in [18], so we only write down two examples of the resulting formulae. Applying the formalism to an incoming bulk particle gives

$$\begin{aligned}
{}_{out} \langle A | \mathcal{O} | \kappa, \vec{k}, \alpha; B \rangle_{in} &= {}_{out} \langle A | \mathcal{O} A_\alpha^{in}(\kappa, \vec{k})^\dagger | B \rangle_{in} = \\
&= \text{disconnected part} + 2i Z_\alpha^{-1/2} \int_{-\infty}^0 dy \int dx \cos(\kappa y) e^{-i(\Omega_\alpha(\kappa, \vec{k})t - \vec{k} \cdot \vec{x})} \times \\
&\quad \left\{ \partial_t^2 - \partial_y^2 - \vec{\nabla}^2 + M_\alpha^2 + \delta(y) \partial_y \right\} {}_{out} \langle A | T \mathcal{O} \Phi_\alpha(t, \vec{x}, y) | B \rangle_{in}
\end{aligned}$$

where \mathcal{O} denotes a general T product of local operators. The reduction formula for an incoming boundary particle created by the a modes are identical to the usual reduction formulae in D space-time dimensions without boundary:

$$\begin{aligned}
{}_{out} \langle A | \mathcal{O} | \vec{k}, b; B \rangle_{in} &= {}_{out} \langle A | \mathcal{O} a_b^{in}(\vec{k})^\dagger | B \rangle_{in} = \\
&= \text{disconnected part} + i Z_a^{-1/2} \int dx e^{-i(\omega_a(\vec{k})t - \vec{k} \cdot \vec{x})} \times \\
&\quad \left\{ \partial_t^2 - \vec{\nabla}^2 + m_a^2 \right\} {}_{out} \langle A | T \mathcal{O} \phi_a(t, \vec{x}) | B \rangle_{in}
\end{aligned}$$

Formulae for outgoing particles can be written in a very similar form.

2.4 Feynman rules in coordinate space

In the interaction picture, the time evolution of the system can be described by the operator

$$\begin{aligned}
U(t) &= T \exp \left\{ -i \int_{-\infty}^t d\tau H_{int}(\tau) \right\} \\
H_{int}(\tau) &= \int_{-\infty}^0 dy \int d\vec{x} V(\Phi_\alpha^{in}(\tau, \vec{x}, y)) + \int d\vec{x} U(\Phi_\alpha^{in}(\tau, \vec{x}, y=0), \phi_a^{in}(t, \vec{x}))
\end{aligned}$$

The R matrix can be expressed as

$$R = U(\infty) = T \exp \left\{ -i \int_{-\infty}^{\infty} d\tau H_{int}(\tau) \right\}$$

and the interacting fields take the form

$$\begin{aligned}
\Phi_\alpha(t, \vec{x}, y) &= U(t)^{-1} \Phi_\alpha^{in}(t, \vec{x}, y) U(t) \\
\phi_a(t, \vec{x}) &= U(t)^{-1} \phi_a^{in}(t, \vec{x}) U(t)
\end{aligned}$$

Using the free field formulae and Wick's theorem, one can readily derive the Feynman rules in coordinate space for the Green's functions

$$\begin{aligned}
G_{\alpha_1 \dots \alpha_m; a_1 \dots a_n}^{m, n}(\vec{x}_1, y_1, t_1 \dots, \vec{x}_m, y_m, t_m; \vec{x}'_1, t'_1, \dots, \vec{x}'_n, t'_n) = \\
\langle 0 | T \Phi_{\alpha_1}(\vec{x}_1, y_1, t_1) \dots \Phi_{\alpha_m}(\vec{x}_m, y_m, t_m) \phi_{a_1}(\vec{x}'_1, t'_1) \dots \phi_{a_n}(\vec{x}'_n, t'_n) | 0 \rangle
\end{aligned}$$

from which the rules for the R matrix can be obtained by applying the reduction formulae. The resulting diagrams contain the following ingredients:

1. Boundary propagator:

$$\frac{a}{t, \vec{x}} \cdots \frac{b}{t', \vec{x}'} = i\delta_{ab} \int \frac{dk}{(2\pi)^D} \frac{e^{-ik \cdot (x-x')}}{k^2 - m_a^2 + i\varepsilon}$$

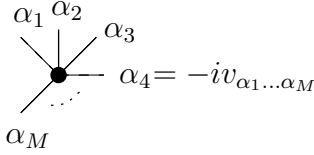
2. Direct bulk propagator

$$\frac{\alpha}{t, y, \vec{x}} \frac{\beta}{t', y', \vec{x}'} = i\delta_{\alpha\beta} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \int \frac{dk}{(2\pi)^D} \frac{e^{-ik \cdot (x-x')}}{k^2 - \kappa^2 - M_\alpha^2 + i\varepsilon} e^{i\kappa(y-y')}$$

3. Reflected bulk propagator

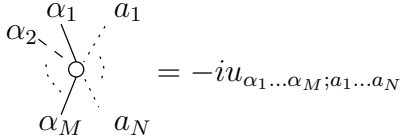
$$\frac{\alpha}{t, y, \vec{x}} \cdots \frac{\beta}{t', y', \vec{x}'} = i\delta_{\alpha\beta} \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \int \frac{dk}{(2\pi)^D} \frac{e^{-ik \cdot (x-x')}}{k^2 - \kappa^2 - m_a^2 + i\varepsilon} e^{i\kappa(y+y')}$$

4. Bulk vertex (it doesn't make a difference whether a leg comes from a direct or reflected bulk propagator)



$$\alpha_4 = -i v_{\alpha_1 \dots \alpha_M}$$

5. Boundary vertex (once again, no distinction between the types of bulk legs)



$$= -i u_{\alpha_1 \dots \alpha_M; a_1 \dots a_N}$$

The vertex positions must be integrated over, for bulk vertices over the bulk (half) space, while for boundary vertices over the boundary. There are also combinatorial factors that follow straightforwardly from counting the number of ways the particular Wick contractions can be made. One has to take care to draw all possible diagrams obtained by putting both the direct and the reflected propagators for each bulk leg or line. In fact, for coordinate space rules we could have used one type of bulk propagator as well: $G_{\alpha\beta}(z, z')$, which consists of the sum of the two pieces. The separation of the bulk propagator into two pieces, however, makes the momentum space formulation much easier.

2.5 Feynman rules in momentum space

We can give the Feynman rules in momentum space by taking a simple Fourier transform of the Green's functions:

$$\begin{aligned} & G_{\alpha_1 \dots \alpha_m; a_1 \dots a_n}^{m,n} \left(\vec{k}_1, \kappa_1, \omega_1, \dots, \vec{k}_m, \kappa_m, \omega_m; \vec{k}'_1, \omega'_1, \dots, \vec{k}'_n, \omega'_n \right) = \\ & \prod_{i=1}^m \left(\int_{-\infty}^{+\infty} dy_i \int dt_i d\vec{x}_i e^{i(\omega_i t_i - \vec{k}_i \cdot \vec{x}_i - \kappa_i y_i)} \right) \prod_{j=1}^n \left(\int dt_j d\vec{x}_j e^{i(\omega'_j t'_j - \vec{k}'_j \cdot \vec{x}'_j)} \right) \times \\ & G_{\alpha_1 \dots \alpha_m; a_1 \dots a_n}^{m,n} \left(\vec{x}_1, y_1, t_1, \dots, \vec{x}_m, y_m, t_m; \vec{x}'_1, t'_1, \dots, \vec{x}'_n, t'_n \right) \end{aligned}$$

(the y integrals are defined by a simple extension of the Green's functions as even functions of y).

1. Boundary propagator:

$$\begin{array}{c} a \\ \cdots \cdots \cdots b \\ \omega, \vec{k} \end{array} = \frac{i\delta_{ab}}{\omega^2 - \vec{k}^2 - m_a^2 + i\varepsilon}$$

2. Direct bulk propagator

$$\begin{array}{c} \alpha \\ \hline \Omega, \kappa, \vec{k} \\ \hline \beta \end{array} = \frac{i\delta_{\alpha\beta}}{\Omega^2 - \vec{k}^2 - \kappa^2 - M_\alpha^2 + i\varepsilon}$$

3. Reflected bulk propagator

$$\begin{array}{c} \alpha \\ \cdots \cdots \cdots \beta \\ \Omega, \kappa, \vec{k} \end{array} = \frac{i\delta_{\alpha\beta}}{\Omega^2 - \vec{k}^2 - \kappa^2 - M_\alpha^2 + i\varepsilon}$$

4. Bulk vertex

$$\begin{array}{c} \alpha_1 \alpha_2 \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \alpha_3 \alpha_4 \\ \cdots \cdots \cdots \\ \alpha_M \end{array} = -iv_{\alpha_1 \dots \alpha_M} (2\pi)^D \delta^{(D)}(\sum k) \pi \delta(\sum' \kappa)$$

The momentum conservation for the D -momenta k is the usual one, but for the κ component, the prime indicates that if two vertices are connected by a reflected bulk propagator, the corresponding κ must be oriented either outgoing or incoming at both vertices. For direct bulk propagators, momentum is conserved as usual: all $D + 1$ components of the momentum are oriented as outgoing at one end and incoming at the other one.

5. Boundary vertex

$$\begin{array}{c} \alpha_1 \quad a_1 \\ \diagdown \quad \diagup \\ \circ \\ \diagup \quad \diagdown \\ \alpha_2 \quad a_2 \\ \cdots \cdots \cdots \\ \alpha_M \quad a_N \end{array} = -iu_{\alpha_1 \dots \alpha_M; a_1 \dots a_N} (2\pi)^D \delta^{(D)}(\sum k)$$

There is no conservation law for the component κ , since the position of boundary vertices is only integrated over along the boundary.

There is also an integration over the momentum of each internal line.

As a result of the above rules, the presence of reflected bulk propagators and boundary vertices breaks κ conservation, as expected from the absence of translational invariance in the direction perpendicular to the boundary.

3 Landau equations

The analyticity properties of the Green functions can be analyzed in the perturbative framework. The Landau equations describe the singularity structure of the various Feynman diagrams which we are going to derive for the boundary theory.

3.1 Derivation of the Landau equations

Consider a Feynman diagram in our generic scalar theory (2.1) with N outer bulk legs with momenta (k_i, κ_i) , $i = 1, \dots, N$ and M outer boundary legs with momenta k_i , $i = N + 1, \dots, M + N$, where k_i are the components of the momenta parallel to the boundary and κ_i are the transverse components (only for bulk legs). Using the fact that D -dimensional Poincare invariance (parallel to the boundary) is unbroken, the resulting amplitude depends only on the invariants $k_i \cdot k_j$ and κ_i . The general Feynman integral can be written

$$G = \int \prod_{i=1}^L \frac{d^D q_i}{(2\pi)^D} \prod_{j=1}^K \frac{d\chi_j}{2\pi} \prod_{r=1}^I (p_r^2 - \pi_r^2 - M_r^2 + i\epsilon)^{-1} \prod_{s=I+1}^{I+J} (p_s^2 - m_s^2 + i\epsilon)^{-1} \quad (3.1)$$

where q_i denote D -dimensional loop momenta for L loops in total, χ_j are the transverse ‘‘loop’’ momenta (see later), (p_r, π_r) are the $D + 1$ -momenta of I internal bulk lines, while p_s are the D -momenta of J internal boundary lines. Momentum conservation can be used to express p and π in terms of q, χ, k and κ (taking into account that a reflected bulk propagator reverses transverse momentum).

We can then introduce Feynman’s parameterization in the usual way

$$G = \int \prod_{i=1}^L \frac{d^D q_i}{(2\pi)^D} \prod_{j=1}^K \frac{d\chi_j}{2\pi} \prod_{r=1}^{I+J} \int_0^1 d\alpha_i \delta\left(\sum \alpha_i - 1\right) \psi(k, \kappa, q, \chi, \alpha)^{-I-J}$$

$$\psi(k, \kappa, q, \chi, \alpha) = \sum_{r=1}^I \alpha_r (p_r^2 - \pi_r^2 - M_r^2 + i\epsilon) + \sum_{s=I+1}^{I+J} \alpha_s (p_s^2 - m_s^2 + i\epsilon)$$

and integrating out the loop momenta q, χ (these integrals are – apart from UV divergences taken care by counter-terms – nonsingular in general, just as in the case of bulk diagrams), we are left with a multiple integral over the α , which can be considered as a function of the invariants $k_i \cdot k_j$ and κ_i .

Following the usual argument, a singularity can occur (in the limit $\epsilon \rightarrow 0$) when the hypercontour in α space is trapped between two singularities of the integrand, or a singularity of the integrand takes place at the boundary of integration. The integrand is singular if

$$\begin{aligned} \alpha_r = 0 \quad \text{or} \quad p_r^2 - \pi_r^2 - M_r^2 = 0 \quad , \quad r = 1, \dots, I \\ \alpha_s = 0 \quad \text{or} \quad p_s^2 - m_s^2 = 0 \quad , \quad s = I + 1, \dots, I + J \end{aligned} \quad (3.2)$$

which are nothing other but straightforward generalizations of the usual Landau equations. Reduced diagrams are defined by shrinking internal lines with $\alpha_i = 0$ to a point, and the remaining lines must be on the mass-shell. A further requirement is that

$$\frac{\partial}{\partial q_i} \left(\sum_{r=1}^I \alpha_r (p_r^2 - \pi_r^2 - M_r^2 + i\epsilon) + \sum_{s=I+1}^{I+J} \alpha_s (p_s^2 - m_s^2 + i\epsilon) \right) = 0 \quad (3.3)$$

and

$$\begin{aligned} \frac{\partial}{\partial \chi_i} \left(\sum_{r=1}^I \alpha_r (p_r^2 - \pi_r^2 - M_r^2 + i\epsilon) + \sum_{s=I+1}^{I+J} \alpha_s (p_s^2 - m_s^2 + i\epsilon) \right) \\ = -\frac{\partial}{\partial \chi_i} \left(\sum_{r=1}^I \alpha_r \pi_r^2 \right) = 0 \quad . \end{aligned} \quad (3.4)$$

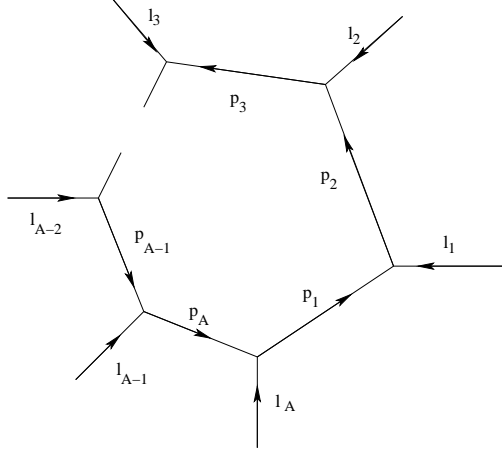


Figure 3.1: Generic form of a closed loop

These equations can be brought into a more elegant form. First concentrate on equation (3.3) that is on the Lorentz invariant part. Consider a closed loop in the Feynman diagram (Fig. 3.1) where the internal momenta are relabeled, the vertices can be of boundary or of bulk type and the various l -s collect all the outer momenta from the loop point of view, but in principle they can be outer (k) or inner (p) or their sums. Using momentum conservation the momentum integration can be evaluated. As a result every p_i , $i = 2 \dots A$ can be expressed in terms of p_1 (which serves as the loop variable) as follows

$$p_2 = p_1 + l_1 ; \quad p_3 = p_2 + l_2 = p_1 + l_1 + l_2 ; \dots \quad p_A = p_1 + \sum_{j=1}^{A-1} l_j \quad (3.5)$$

and the overall momentum conservation $\sum_i l_i = 0$ must hold. Substituting (3.5) into (3.3) for $q_i = p_1$ we obtain

$$\sum_{\text{each loop}} \alpha_i p_i = 0 \quad (3.6)$$

This argument is valid for a D dimensional bulk theory as well as for the D non-transverse component of the momenta in a boundary theory.

Now lets focus on the transverse momentum, that is on (3.4). Consider a loop as before but with bulk vertices only (Fig. 3.2). For bulk propagators we take $\epsilon = 1$ while for the reflected one $\epsilon = -1$. Using momentum conservation the momentum integration can be eliminated successively giving rise to

$$\pi_2 = \epsilon_1 \pi_1 + l_1 ; \quad \pi_3 = \epsilon_2 \pi_2 + l_2 = \epsilon_1 \epsilon_2 \pi_1 + \epsilon_2 l_1 + l_2 ; \dots$$

and in general

$$\pi_i = \prod_{j=1}^i \epsilon_j \pi_1 + \prod_{j=2}^i \epsilon_j l_1 + \dots + \prod_{j=k+1}^i \epsilon_j l_k + \dots + l_i$$

The delta function $\delta(\epsilon_A \pi_A + l_A - \pi_1)$ gives $\pi_1 = \mu_A \pi_1 + \dots$, where we have introduced $\mu_i = \prod_{j=1}^i \epsilon_j$ to encode the parity of the momentum change caused by reflected propagators.

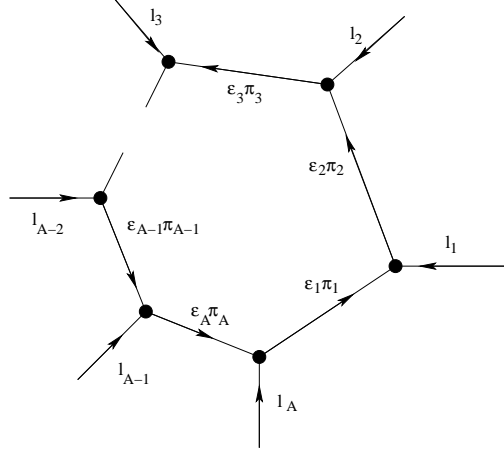


Figure 3.2: Generic form of a closed loop with bulk vertices only

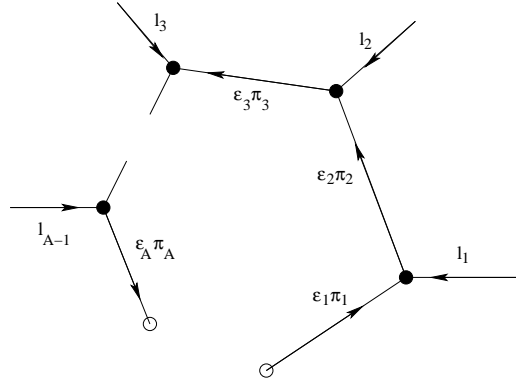


Figure 3.3: Path starting from and ending at a boundary vertex

Clearly if $\mu_A = -1$, that is the loop contains odd number of reflected propagators the delta function eliminates the last integration determining π_1 in terms of the l -s. In this case there is no singularity. If, however, $\mu_A = 1$ (loops with even number of reflected propagator) π_1 is not determined, we have to integrate over it ($\pi_1 = \chi_1$) and we obtain the Landau equation:

$$\sum_{\text{each loop}} \mu_i \alpha_i \pi_i = 0 \quad (3.7)$$

Now consider a path in the Feynman graph starting and ending with boundary vertices but containing bulk vertices only (Fig. 3.3). Using the previous calculation π_A can be expressed in terms of π_1 as before $\pi_A = \mu_{A-1} \pi_1 + \dots$ but now there is no more delta function so π_1 serves as a “loop” variable, χ_1 , for which integration has to be performed. As a result we obtain the following Landau equation:

$$\sum_{\text{each path}} \mu_i \alpha_i \pi_i = 0 \quad (3.8)$$

There is an interpretation of equation (3.6) in the bulk theory in terms of electric network: it is just translated to the usual Kirchoff laws for the D -momenta of the reduced diagram

with “resistances” α_r and “currents” p_r so the “voltage drops” are $\alpha_r p_r$. The generalization for the boundary case: equation (3.7) means that α_r -s are the “resistances” and the π_r -s are the “currents”. Note that there is one current for every component of the $D + 1$ dimensional energy-momentum vector and so for every loop and vertex there are $D+1$ independent Kirchoff laws to write down.

The signs introduced by the reflected propagators can be traced as follows: every reflected propagator carries an “inverter” device that changes the sign of the transverse momentum current, so μ_r keeps track of the actual sign of the current and $\mu_r \alpha_r \pi_r$ are the “voltage drops”. Equation (3.8) means that as far as the transverse momentum current is concerned, all the boundary vertices are on an equipotential surface, which can be considered as the reference point (“earth”). For the components of the energy-momentum that are parallel to the boundary, the Kirchoff laws take the usual form as in the bulk theory.

3.2 Coleman-Norton interpretation

The Landau equations for a generic Feynman graph in the bulk theory are those we have in equation (3.2) and (3.6). Their physical region singularities have $\alpha_i \geq 0$ and contain real momenta p_i . Following Coleman and Norton [19], these equations can be interpreted as the existence of a space-time graph of a process involving classical particles all on the mass shell, all moving forward in time, and interacting only through energy and momentum conserving interactions localized at space-time points. The correspondence is one-to-one and goes as follows: for each internal line draw a vector $\alpha_r p_r$, of length $\alpha_r M_r$. (Lines with $\alpha_r = 0$ are shrunk to a point). For each vertex in the graph a space-time point is associated where the momentum conserving interaction occurs. The consistency of this picture means, that two different paths leading to the same vertex in the Feynman graph define the same space-time point, which is nothing but the equation (3.6).

The Coleman-Norton type interpretation of the boundary Landau equations is the same as in the bulk case: the existence of a space-time graph of a process involving particles all on the mass shell, all moving forward in time, and interacting only through local interactions localized at space-time points where now particles can scatter on the boundary, which reverse the sign of the momenta. The correspondence goes as follows: For each internal bulk line draw a vector $(\alpha_r p_r, \alpha_r \pi_r)$ with length $\alpha_r M_r$ (reflected line in the case of a reflected propagator), while for boundary lines a vector $(\alpha_s p_s)$ of length $\alpha_s m_s$ (lying in the boundary). Bulk vertices are located at space-time bulk points where the momentum preserving interaction occurs. Boundary vertices are located at the boundary points with interactions where the transverse component of the momentum is not conserved. To see that for a space-time diagram the Landau equations are satisfied consider the diagram in Fig. 3.4.

On the left side of the boundary is the real space-time picture of some process, while on the right hand side is its mirror image with respect to the boundary, which is necessary to introduce in order to interpret it in terms of the Landau equations. Instead of drawing a reflected line for a reflected propagator (e.g. from 1 to 2) we consider a straight line from the real world to the mirror one (from 1 to 2') or vice versa. The variable μ keeps track on which side of the boundary we are and to which “world” (real or mirror image) the next line is directed. Consistency of the Coleman-Norton picture means that for two different paths between two interaction point the displacement vectors must sum up to the same value, which is ensured by equation (3.7). Evidently we compare only points that are both on the “real” side of the boundary so it is sufficient to consider loops with even number of reflected propagators.

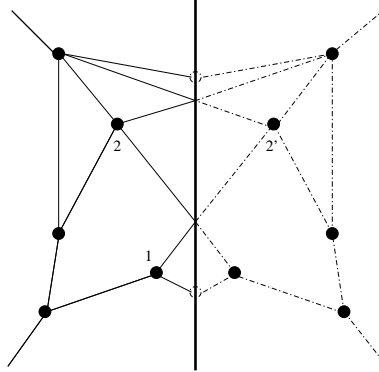


Figure 3.4: Space time diagram of a general on-shell process using the method of images

Another consistency condition is that all the boundary vertices must have the same transverse coordinate, which translates into equation (3.8).

In order to reverse the above argument, we must show that to any solution of the Landau equations, a space-time diagram can be associated. This can be done by drawing the extended diagram first, similarly as we would draw in a bulk theory, then choosing a loop with odd number of reflected propagator and determining the location of the boundary by identifying the endpoints of the closed loop as the mirror image of each other. Clearly such a loop is not closed in the transverse coordinate. Now the Landau equations for the loops with even number of reflected propagators ensure that the location of the boundary does not depend on the choice of the loop with odd number of reflected propagators.

4 Cutkosky rules

4.1 Derivation

A further important ingredient is provided by the boundary analogue of Cutkosky rules to calculate discontinuities across the cuts in the physical region determined by the solutions of the Landau equations. To give the boundary generalization of Cutkosky rules we need a further definition. Note that if we write down a Feynman integral G of the form (3.1) with I bulk internal lines, each bulk line corresponds to either a direct or a reflected propagator so there are actually 2^I distinct diagrams that differ only which bulk lines are taken to be direct or reflected (represented by continuous resp. dashed lines in the associated Feynman diagram). Let $C(G)$ denote the sum of terms the possible assignments of direct/reflected bulk lines. Each individual term can be written in the form (3.1), the difference being the expression of the loop momenta χ in terms of the internal momenta κ due to the different conservation laws obeyed by the two different types of bulk lines.

It is clear that $C(G)$ can have a singularity at a given value of external momenta only if some individual term in the sum does. For an integral of the form (3.1), every singularity corresponds to a solution of the Landau equations (3.2,3.3,3.4). In such a solution there are internal lines which have a corresponding α_r that is nonzero (and so are on the mass shell); these are the ones that make up the reduced diagram corresponding to the given singularity. We remark that since the momentum conservation conditions (3.4) are different for each

individual term in the sum $C(G)$, in general only a few terms give a nonzero contribution to a given singularity.

Let us denote the set of lines which are on-mass-shell at the singularity by G_0 . We restrict ourselves to cases when the graph $G \setminus G_0$, obtained by severing the lines on-mass-shell separates into two disconnected graphs G_1 and G_2 (in the bulk the corresponding singularities are called “normal thresholds”).

Then the rules are that the discontinuity of $C(G)$ (3.1) across the cut corresponding to the singularity can be evaluated by substituting the propagators of the internal lines for which $\alpha_r \neq 0$ as follows:

$$\begin{aligned} \frac{1}{p_r^2 - \pi_r^2 - M_r^2 + i\epsilon} &\rightarrow -2\pi i \delta^{(+)}(p_r^2 - \pi_r^2 - M_r^2) \\ \frac{1}{p_r^2 - m_r^2 + i\epsilon} &\rightarrow -2\pi i \delta^{(+)}(p_r^2 - m_r^2) \end{aligned}$$

change $\epsilon \rightarrow -\epsilon$ for internal lines in one of the components (say G_2 , the other choice just gives the jump in the opposite direction across the cut), and then perform the integration in (3.1). Here $\delta^{(+)}$ denotes a delta function where only the root $p_0 = +\sqrt{p_r^2 + \pi_r^2 + M_r^2}$ or $+\sqrt{p_r^2 + m_r^2}$ is taken into account.

To show this, we generalize a very elegant proof for the bulk Cutkosky rules, originally due to Nakanishi. The bulk proof is spelled out in detail in [5], so we only give the details necessary for its extension to the boundary case. We construct an auxiliary field theory to any given diagram class $C(G)$ in the following way. To each internal bulk line of the diagram we associate a different species of bulk field Φ_r of mass M_r ($r = 1, \dots, I$), similarly to each internal boundary line a boundary field ϕ_s of mass m_s ($s = 1, \dots, J$). To every bulk vertex v with some external bulk line we attach a field $\tilde{\Phi}_v$ of mass \tilde{M}_v , where $\tilde{M}_v^2 = P_v^2$, P_v is the total $(D+1)$ -momentum entering the given vertex from its external legs. Similarly to every boundary vertex u we attach a boundary field $\tilde{\phi}_u$ of appropriate mass \tilde{m}_u , $\tilde{m}_u^2 = p_u^2$ where p_u is the total D -momentum entering the vertex from external bulk and boundary lines. The auxiliary action corresponding to $C(G)$ is defined as

$$\begin{aligned} \mathcal{A}_G = & \int dz \sum_{r=1}^I \frac{1}{2} \left[(\partial \Phi_r)^2 - M_r^2 \Phi_r^2 \right] + \sum'_v \frac{1}{2} \left[(\partial \tilde{\Phi}_v)^2 - \tilde{M}_v^2 \tilde{\Phi}_v^2 \right] + \sum_v \tilde{\Phi}_v \prod_{r \rightarrow v} \Phi_r + \\ & \int dx \sum_{s=1}^J \frac{1}{2} \left[(\partial \phi_s)^2 - m_s^2 \phi_s^2 \right] + \sum'_u \frac{1}{2} \left[(\partial \tilde{\phi}_u)^2 - \tilde{m}_u^2 \tilde{\phi}_u^2 \right] + \sum_u \tilde{\phi}_u \prod_{r \rightarrow u} \Phi_r \prod_{s \rightarrow u} \phi_s \end{aligned}$$

where \sum'_v and \sum'_u mean that the sum goes only for bulk/boundary vertices that have external lines entering them, $\tilde{\Phi}_v$ and $\tilde{\phi}_u$ must be put equal to 1 for vertices which connect only to internal lines, and expressions of the type $r \rightarrow u$ means taking product over internal lines r that connect to vertex u .

The crucial point is that $C(G)$ gives the lowest order contribution to the scattering amplitude with the given external momenta in the auxiliary theory. Let i denote the initial state of this scattering and f the final one (external lines attached to G_1 and G_2 , respectively). Then the transition amplitude for this process can be written

$$\mathcal{T}_{fi} = C(G)$$

However, on-shell processes are unitary, thus we have

$$\mathcal{T}_{fi} - \mathcal{T}_{if}^* = \sum_{G_0} C(G_1) C(G_2)^*$$

where \sum_{G_0} means a phase space summation over the lines involved in G_0 . Following a similar reasoning as in [5] yields exactly the rules spelled out above.

4.2 Comments on the general case

For different topologies, we expect that similar rules hold. The reason is simple: the Cutkosky rules themselves depend only on the analytic structure of propagators. In the perturbation theory around the free (Neumann) boundary conditions we see that the momentum space form of propagators is just the same as in the bulk theory, the only difference comes from the momentum conservation rules which express the internal momenta in terms of the external and loop momenta and from the fact that boundary propagators have vanishing transverse momentum. However, insofar as these expressions are linear (which is true even in the boundary situation), neither the precise form of the conservation rules enters in any derivation of the Cutkosky rules, nor the constraint on the transverse momentum for boundary lines (which is also linear) is relevant. Therefore we can see that the boundary Cutkosky rules given above can be derived for any topology for which a derivation is given in the bulk (e.g. the well-known triangle anomalous threshold diagram, which is not in the class covered by Nakanishi's derivation).

In the above derivation we also supposed that \tilde{M}_v^2 and \tilde{m}_u^2 are positive, which corresponds to singularities in the physical region. Beyond the physical region nontrivial analytic continuation is required similarly to the bulk case. Extension of Cutkosky rules to such singularities is a very complicated issue, tantamount to a generalization of unitarity. In this paper we do not enter into further discussion of singularities in the non-physical region, but simply suppose that an extension of these rules can be worked out.

Finally we note that singular contributions corresponding to the same reduced diagram can be summed up, giving the familiar result that the total contribution (to all orders of perturbation theory) can be obtained by substituting the exact vertex functions for the vertices in the reduced diagram characterizing the singularity (see e.g. Section 2.9. of [4]).

One has to be very careful in deducing the singularities of the Green functions from the analysis of the perturbative series. Sometimes individual terms may have singularities that are cancelled when a sum of terms is taken. Some other times individual terms do not yield the correct singularities at all. It happens when some nonperturbative dynamical property, such as formation of a bound state, is involved. To handle these situations a complete or a partial summation of the perturbative series has to be performed.

4.3 Infrared divergences

As shown in appendix A.2, perturbation theory around the Neumann boundary condition has infrared divergences. These arise from on-shell bulk particles propagating along the boundary as expected from the Landau equations and correspond to a boundary bound state.

However, if there is a quadratic term of the form $\lambda\Phi(y=0)^2/2$ for the corresponding bulk field Φ in the boundary potential V (which is the case in general), then the infrared divergence is nonphysical. There are infinitely many infrared divergent terms with different

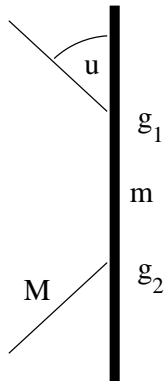


Figure 5.1: Diagram corresponding to an on-shell boundary state

degree of singularity and resummation in the coupling λ shifts the singularity from $k = 0$ to $k = -i\lambda$, and it does not affect the derivation of the physical region Cutkosky rules presented above.

For $\lambda > 0$ the singularity is shifted to the unphysical sheet of the scattering amplitude, so it is physically irrelevant. For $-m < \lambda < 0$, the pole is again shifted away from the region of physical values of the incoming momentum, however, it remains on the physical sheet and appears as a boundary bound state (see A.2). Note that the derivation of the Cutkosky rules given above is perturbative, but as it rests on general principles such as analyticity and unitarity, we expect the rules to extend to singularities corresponding to such nonperturbative intermediate states as well, just as in bulk theories. In fact, the applications presented in the next Section require such an extension of Cutkosky rules to singularities with complex values of incoming momenta, but still in the physical sheet.

Finally, let us remark that the derivation of the Landau equations and the Nakanishi proof for the Cutkosky rules for physical region singularities can be performed in a perturbation theory that starts around a nonzero value of λ . In this case the reflected bulk propagator must be changed to

$$\frac{i}{k_r^2 - \kappa_r^2 - M_r^2 + i\epsilon} \frac{\kappa_r - i\lambda}{\kappa_r + i\lambda}$$

For amplitudes in the physical region, the last factor is just a nonsingular (phase-valued) expression, and so does not affect the derivations in any essential way.

5 Applications

In the following we restrict ourselves to 1 + 1 spacetime dimensions, since the example theory we use is sine-Gordon theory with integrable boundary conditions. The above formalism is of course more general, however, integrable field theories make possible an exact and non-perturbative verification of the general scheme.

5.1 Boundary bound state poles

The simplest application of the Cutkosky rules is when there is a single intermediate line on-shell. If that is a bulk line, then we get back the well-known result for a first order pole corresponding to an intermediate particle in the bulk. If the line in question is a boundary line, we get a discontinuity of the form

$$2\pi g_1 g_2^* \delta^{(+)}(E^2 - m^2)$$

corresponding to the diagram Fig. 5.1, where $E = M \cosh \vartheta$ and $m = M \cos u$ for a pole at imaginary rapidity $\vartheta = iu$.

This reproduces the result of Ghoshal and Zamolodchikov [20], which states that an intermediate on-shell boundary state produces a pole in the reflection factor $R(\vartheta)$ of the form

$$R(\vartheta) \sim \frac{1}{2} \frac{if_1 f_2^*}{\vartheta - iu}$$

where the dimensionless couplings $f_{1,2}$ are related to $g_{1,2}$ by some normalization factors, coming partially from converting the $\delta^{(+)}$ to a function of ϑ and also from relating the reflection factor in terms of the Feynman amplitude, but the precise form of this relation is not important for us here.

5.2 Coleman-Thun diagrams

We present some examples for more complicated intermediate diagrams following Coleman and Thun [1] (see also [2, 3, 22] for the boundary case). To be specific, we consider sine-Gordon theory with integrable boundary conditions. The groundwork for determining its spectrum and scattering amplitudes was laid down in [20, 21], while the bootstrap closure of this theory for general boundary conditions was obtained in [22], following the work in [3].

Before proceeding to the examples, we introduce a useful notation. Let $f(\vartheta)$ be a meromorphic function of one complex variable ϑ . Let us suppose that the Laurent expansion of f around $\vartheta = \vartheta_0$ takes the form

$$f(\vartheta) = A(\vartheta - \vartheta_0)^n + \sum_{k>n} b_k (\vartheta - \vartheta_0)^k \quad , \quad A \neq 0 .$$

We then define

$$f[\vartheta_0] = A .$$

If ϑ_0 is not a pole or a zero of f , then this is just the value of the function. If ϑ_0 is a first order pole, then this is just the residue, while for a first order zero it is the derivative of f at ϑ_0 .

The first family we consider are first order poles independent of the boundary conditions and occur in reflection factors of breathers on any boundary state (ground state and excited states alike). Let us denote the reflection factor of a breather B^k with rapidity ϑ by $R^{(k)}(\vartheta)$ (we omit the specification of the boundary state, as the argument does not depend on it). Then $R^{(k)}$ has a pole at $\vartheta = in\pi/2\lambda$, $n = 1, \dots, k-1$. We pick a pole with a given n and write $k = m + n$. Following [22] this pole can be associated with the scattering process in diagram 5.2 (we recall that an imaginary rapidity difference $\vartheta = iu$ corresponds to a Euclidean angle u in the plane). The diagram obviously gives a first-order pole, since it has 3 propagators (one

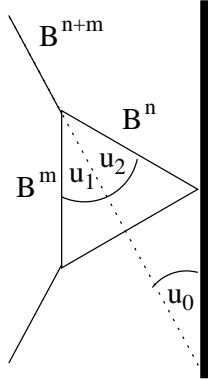


Figure 5.2: Coleman-Thun graph for a boundary-independent pole in the breather reflection factor. The angles are $u_0 = u_1 = n\pi/2\lambda$ and $u_2 = m\pi/2\lambda$.

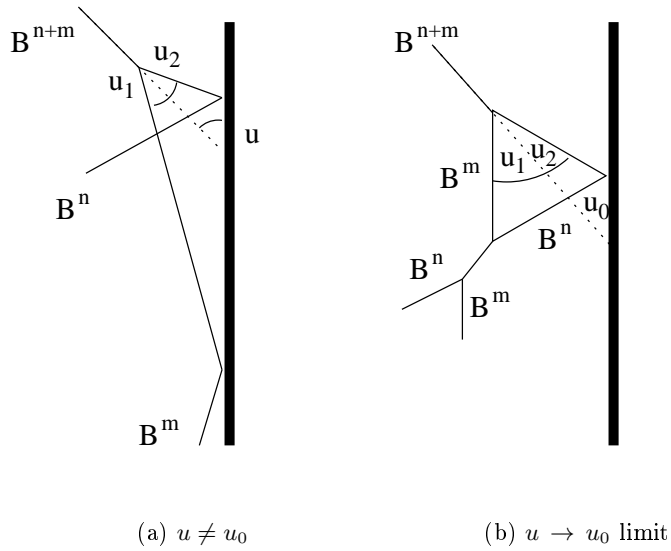


Figure 5.3: Calculating the contribution of the triangle graph using the bootstrap

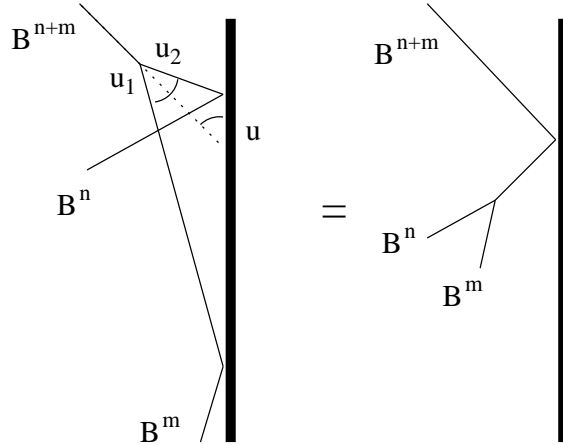


Figure 5.4: Bootstrap relation for $R^{(m+n)}$

$\delta^{(+)}$ factors each), and a loop integral which means 2 integrations, leaving us with a single δ function in the discontinuity which is characteristic of a first order pole.

To establish that this process explains the pole of the reflection factor, we need to calculate its contribution to the residue. The straightforward method would be to write down the Cutkosky rules directly and calculate the integrals, which do give a contribution of the right form. However, the exact coefficient is not easy to obtain as one needs to ensure the right normalization of the reflection amplitude first, and then also extract the breather-breather fusion coupling g_{nm}^{n+m} from the S matrix, once again taking care of all details of normalization.

Instead, we can follow a simpler route, noting that the S matrices and the reflection amplitudes satisfy the bootstrap which entails a vast number of identities. We can tune the incoming rapidity away from $\vartheta = iu_0$, obtaining the process in Fig. 5.3 (a), which, by virtue of factorized scattering, has the amplitude

$$g_{nm}^{n+m} R^{(n)}(\vartheta + iu_1) R^{(m)}(\vartheta - iu_2) S(2\vartheta + i(u_1 - u_2)) \quad , \quad \vartheta = iu$$

Taking $\vartheta \rightarrow iu_2$ we hit a pole in the S matrix. The contribution of this pole reads

$$g_{nm}^{n+m} R^{(n)}(i(u_2 + u_1)) (-1) \frac{1}{2} S[i(u_1 + u_2)]$$

On the other hand, in the same limit the diagram in Fig. 5.3 (a) becomes the one drawn in Fig. 5.3 (b), from which amplitude of the triangle graph Fig. 5.2 can be obtained by dividing with the breather fusion coupling g_{nm}^{n+m} . This means that to explain the pole in the reflection factor of B^{m+n} , the following identity must hold:

$$R^{(m+n)}[iu_2] = -\frac{1}{2} R^{(n)}(i(u_2 + u_1)) S[i(u_1 + u_2)]$$

which can be verified by direct substitution.

It is very interesting to note that this identity is guaranteed by the bootstrap construction of the reflection factor $R^{(m+n)}$, using the relation drawn in Fig. 5.4.

We have also considered two other classes of Coleman-Thun diagrams, using similar arguments. Fig. 5.5 (a) shows a process where an incoming breather splits into two breathers of

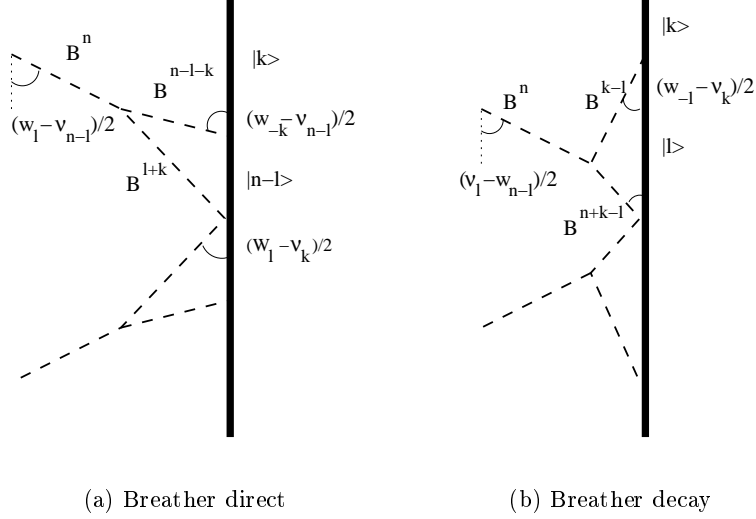


Figure 5.5: Coleman-Thun explanation for boundary dependent poles in breather reflection factors

lower index, one of which excites the boundary state on which the other gets reflected. The process in Fig. 5.5 (b) describes an incoming boundary state decaying by the emission of a breather that fuses with the incoming breather to a higher one, which in turn gets reflected from the boundary. The notations for the boundary states $|n\rangle$ and the fusing angles ν_k, w_l are explained in [18].

Naively, these diagrams would give a second order pole since the number of propagators is 6 (including propagators for boundary states as dictated by the Cutkosky rules), while there are two loop integrals which give the order $6 - 2 \times 2 = 2$. However, the reflection at the middle of these diagrams takes place at a rapidity at which the corresponding amplitude has a first-order zero, which results in a first-order contribution.

Straightforward application of bootstrap principles shows that the contribution of these diagrams to the given residues can be written as

$$\begin{aligned}
 (a) : & \quad -R_{|k\rangle}^{(n-l-k)} [(w_{-k} - \nu_{n-l}) / 2] R_{|n-l\rangle}^{(l+k)} [(w_l - \nu_k) / 2] S^{n-l-k, l+k} [n\pi/2\lambda] \\
 (b) : & \quad -R_{|k\rangle}^{(k-l)} [(w_{-l} - \nu_k) / 2] R_{|l\rangle}^{(n+k-l)} [(\nu_{k-n} - w_{-l}) / 2] S^{n, k-l} [(n+k-l)\pi/2\lambda]
 \end{aligned}$$

If these diagrams are to explain the residues, then the above expressions must be equal to

$$\begin{aligned}
 (a) : & \quad R_{|k\rangle}^{(n)} [(w_l - \nu_{n-l}) / 2] \\
 (b) : & \quad R_{|k\rangle}^{(n)} [(\nu_l - w_{n-l}) / 2]
 \end{aligned}$$

We checked these equalities by direct evaluation. As before, they are also a consequence of the bootstrap relations between the reflection factors and S matrices.

It is quite remarkable that the identities necessary for the diagrams to explain the residues of the pole hold by the bootstrap. This seems to be the case for breather reflection factors in sine-Gordon theory for generic values of the boundary parameters; although we could not check

it for all the poles in the reflection factors of all the breathers on generic excited boundary states due to the very tedious details, it was nevertheless true in each case we examined. For solitons, one does not expect this to be the case as in some diagrams the order of the pole is reduced by a cancellation between pairs of diagrams that are charge conjugate (in terms of soliton charge) analogous to the bulk Coleman-Thun diagrams, and so only the sub-leading term survives, which cannot be calculated so simply as above (in fact, such sub-leading terms have never been calculated in the literature, at least to our knowledge).

There are also known cases when a certain pole in the reflection factor needs a combined explanation in terms of a sum of a Coleman-Thun diagram and a boundary bound state at the same time. For an example in the scaling Lee-Yang model cf. [2]. Such a situation also occurs in boundary sine-Gordon theory with Neumann boundary conditions, where it results from the confluence of two poles and a zero in the appropriate reflection factor as the boundary parameters are tuned to their value at the Neumann boundary condition, and has already been treated in detail in [23].

6 Conclusions

The perturbative approach to boundary QFT around the Neumann boundary condition has some distinct advantages. It is very well suited for investigation of principles of boundary QFT, and for the extension of bulk results like LSZ reduction formulae, Landau equations and Cutkosky rules. For many applications it may be more convenient to start from a different boundary condition (as is usually done in the literature), because that may reduce the number of diagrams (representing a partial re-summation of the perturbative series around the Neumann boundary condition) or eliminate tadpoles in case of a nontrivial vacuum field configuration (see Appendix A).

The main result of this paper is the derivation of the Coleman-Norton interpretation, Cutkosky rules and their application to the boundary bootstrap. Although the rules are derived from perturbation theory, they are actually dependent only on some analyticity properties and unitarity (this is especially obvious for Cutkosky rules, where the proof explicitly relies on unitarity). Therefore one expects that there is a non-perturbative formulation along the lines followed by Eden et al. in the last chapter of their book [4], which would be interesting to work out in detail. Such a formulation, besides being non-perturbative, would also give us an extension of the Cutkosky rules to the non-physical domain and to topologies distinct from the one considered in the proof of Section 4.

Another open issue is to formulate a complete theory of perturbative renormalization with boundaries. In the perturbation theory around the Neumann boundary condition, it seems relatively easy to perform arguments similar to those in bulk theories, e.g. power counting analysis of divergent graphs. We only considered one-loop renormalization in boundary sine-Gordon theory in Appendix B, which was enough for a comparison with semiclassical results in the literature. It would be interesting to have a thorough investigation of renormalizability, proof of locality of counter-terms, investigation of possible anomalies of symmetries and so on. Such an investigation could provide e.g. a proof that the tadpoles indeed restore the correct vacuum expectation value of the field in the generic case (as shown for the toy model in Appendix A) thereby strengthening the derivation of analytic results from perturbation theory.

Finally let us remark that Dirichlet boundary conditions cannot be reached by perturba-

tion theory from the Neumann case, as this would require taking some coupling constants to infinity. However, it is rather trivial to extend the formalism of the paper to this case, which only requires the inclusion of a minus sign in the reflected propagator for fields satisfying Dirichlet boundary conditions. One also needs to require that the corresponding asymptotic fields satisfy Dirichlet boundary conditions, since if one insisted on an asymptotic Neumann boundary condition, that would mean adiabatic switching off for an infinitely strong interaction, which would be a rather odd notion. Instead, taking the asymptotic field to satisfy Dirichlet boundary condition enables one to carry all the formalism over to this case. We only omitted this from the main text to maintain the arguments and the notation as simple as possible.

Acknowledgments

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A A toy model

In our approach, the free field has Neumann boundary conditions, and all the boundary interactions (including their quadratic part) are taken into account perturbatively. To illustrate how this procedure reproduces the interacting boundary conditions, we consider a toy model of a real scalar field Φ . For simplicity, we take 1 + 1 spacetime dimensions (generalization to $D + 1$ is straightforward, see at the end of the section). We take the Lagrangian to be quadratic in the field

$$L = \int_{-\infty}^0 dx \left[\frac{1}{2} (\partial_t \Phi)^2 - \frac{1}{2} (\partial_y \Phi)^2 - \frac{m^2}{2} \Phi^2 \right] - \frac{\lambda}{2} \Phi(y=0, t)^2$$

so the model is exactly solvable. We remark that this toy model has already been used in the literature to illustrate several issues of boundary field theory [24].

A.1 Exact solution

The equations of motion are

$$-\partial_t^2 \Phi + \partial_y^2 \Phi - m^2 \Phi = 0 \quad , \quad \partial_y \Phi|_{y=0} = -\lambda \Phi(y=0, t) \quad (\text{A.1})$$

with the following solution for the modes of the field:

$$f_\kappa(y) = \frac{1}{\sqrt{2\pi(\kappa^2 + \lambda^2)}} \left((\kappa + i\lambda)e^{i\kappa y} + (\kappa - i\lambda)e^{-i\kappa y} \right) \quad .$$

For $\lambda < 0$, there also exists a normalizable mode

$$f_B(y) = \sqrt{2|\lambda|} e^{-\lambda y}$$

The field has the mode expansion

$$\Phi(y, t) = \phi_B(t) f_B(y) + \int_0^\infty d\kappa \phi_\kappa(t) f_\kappa(y) \quad .$$

where dynamics of the modes is

$$\begin{aligned}\frac{d^2\phi_\kappa(t)}{dt^2} &= -(\kappa^2 + m^2)\phi_\kappa(t) \\ \frac{d^2\phi_B(t)}{dt^2} &= -(m^2 - \lambda^2)\phi_B(t)\end{aligned}$$

We remark that periodicity of the boundary mode in time requires $\lambda^2 < m^2$ i.e. $-m < \lambda < 0$ (otherwise the vacuum $\Phi = 0$ is unstable under perturbations in the direction of the boundary mode). The Lagrangian can be written in terms of modes as

$$L = \frac{1}{2} \int_0^\infty d\kappa \left(\dot{\phi}_\kappa(t)^2 - (\kappa^2 + m^2)\phi_\kappa(t)^2 \right) + \frac{1}{2} \left(\dot{\phi}_B(t)^2 - \Omega^2\phi_B(t)^2 \right) \quad , \quad \Omega = \sqrt{m^2 - \lambda^2}$$

We can quantize the theory by introducing the conjugate momenta

$$\pi_\kappa(t) = \dot{\phi}_\kappa(t) \quad , \quad \pi_B(t) = \dot{\phi}_B(t)$$

Creation/annihilation operators can be introduced:

$$\begin{aligned}\phi_\kappa(t) &= \frac{1}{\sqrt{2\omega_\kappa}} \left(a(\kappa)e^{-i\omega_\kappa t} + a^\dagger(\kappa)e^{i\omega_\kappa t} \right) \quad , \quad \omega_\kappa = \sqrt{\kappa^2 + m^2} \\ \phi_B(t) &= \frac{1}{\sqrt{2\Omega}} \left(be^{-i\Omega t} + b^\dagger e^{i\Omega t} \right)\end{aligned}$$

and satisfy

$$\left[a(\kappa), a^\dagger(\kappa') \right] = \delta(\kappa - \kappa') \quad , \quad \left[b, b^\dagger \right] = 1 \quad .$$

For the case $\lambda > 0$ (i.e. no bound states) the propagator takes the form

$$\begin{aligned}\langle 0| T \left(\Phi(y, t)\Phi(y', t') \right) |0\rangle &= \theta(t - t') \langle 0| \Phi(y, t)\Phi(y', t') |0\rangle + \theta(t' - t) \langle 0| \Phi(y', t')\Phi(y, t) |0\rangle \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{d\omega d\kappa}{(2\pi)^2} \frac{i}{\omega^2 - \kappa^2 - m^2 + i\epsilon} \left(e^{i\kappa(y-y') - i\omega(t-t')} \right. \\ &\quad \left. + \frac{\kappa - i\lambda}{\kappa + i\lambda} e^{i\kappa(y+y') - i\omega(t-t')} \right)\end{aligned}$$

from which we can read off the reflection factor

$$R(\kappa) = \frac{\kappa - i\lambda}{\kappa + i\lambda}$$

which has a bound state pole at

$$\kappa = m \sinh \vartheta = -i\lambda$$

where ϑ is the rapidity. Introducing $u = -i\vartheta$

$$\sin u = -\frac{\lambda}{m}$$

For a bound state pole in the physical strip $0 < u < \frac{\pi}{2}$ we need $-m < \lambda < 0$. When this is satisfied, we get an additional term in the propagator. This can be obtained by continuing λ

to negative values or by explicitly adding the contribution of the bound state mode, both of which gives

$$\begin{aligned}
\langle 0|T(\Phi(y,t)\Phi(y',t'))|0\rangle &= \theta(t-t')\langle 0|\Phi(y,t)\Phi(y',t')|0\rangle + \theta(t'-t)\langle 0|\Phi(y,t)\Phi(y',t')|0\rangle \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega d\kappa}{(2\pi)^2}\frac{i}{\omega^2-\kappa^2-m^2+i\epsilon}\left(e^{i\kappa(y-y')-i\omega(t-t')} \right. \\
&\quad \left. + \frac{\kappa-i\lambda}{\kappa+i\lambda}e^{i\kappa(y+y')-i\omega(t-t')}\right) \\
&\quad + \frac{|\lambda|}{\Omega}e^{-\lambda(y+y')}\left(\theta(t-t')e^{-i\Omega(t-t')} + \theta(t'-t)e^{-i\Omega(t'-t)}\right) \quad (\text{A.2})
\end{aligned}$$

The bound state term can also be written in the form

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{i}{\omega^2-\Omega^2+i\epsilon}e^{-i\omega(t-t')}f_B(y)f_B(y')$$

which means that (apart from the y dependence) the boundary bound state behaves as a free boundary field of mass Ω propagating in 0 + 1 dimensions (i.e. a harmonic oscillator) just as we postulated for the boundary degrees of freedom in our general exposition of perturbation theory, except that here the boundary degree of freedom arises as a boundary bound state and not from a separate field introduced in the Lagrangian. As a result, it is not sharply localized to the boundary, but its contribution decreases exponentially away from $y = 0$.

A.2 Boundary perturbation theory

The reflection factor can be expanded in the coupling λ

$$R(\kappa) = 1 + 2\sum_{n=1}^{\infty}\left(-\frac{i\lambda}{\kappa}\right)^n \quad (\text{A.3})$$

We now proceed to show that this is correctly obtained using the perturbation theory introduced above. We suppose that $\lambda > 0$, so that no boundary bound state exists, which would be a non-perturbative effect which perturbation theory is not expected to reproduce.

A.2.1 First order correction

The interacting propagator can be written as

$$\begin{aligned}
G(y,t;y',t') &= \langle 0|T(\Phi(y,t)\Phi(y',t'))\exp\left\{-\frac{i\lambda}{2}\int_{-\infty}^{\infty}d\tau:\Phi(0,\tau)^2:\right\}|0\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i\lambda}{2}\right)^n\langle 0|T(\Phi(y,t)\Phi(y',t'))\prod_{i=1}^n\int_{-\infty}^{\infty}d\tau_i:\Phi(0,\tau_i)^2:|0\rangle
\end{aligned}$$

the first order correction is

$$\begin{aligned}
G^{(1)}(y,t;y',t') &= -\frac{i\lambda}{2}\int_{-\infty}^{\infty}d\tau\langle 0|T(\Phi(y,t)\Phi(y',t')):\Phi(0,\tau)^2:|0\rangle \\
&= -i\lambda\int_{-\infty}^{\infty}d\tau G(y,t,0,\tau)G(y',t',0,\tau)
\end{aligned}$$

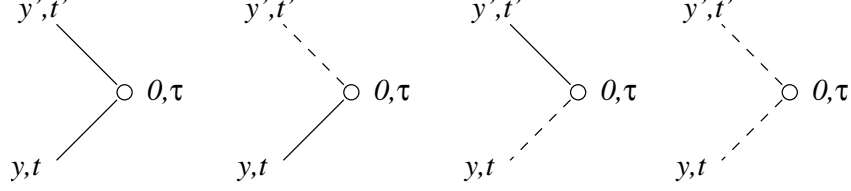


Figure A.1: First order Feynman graphs for the propagator of the toy model

where

$$G(y, t, y', t') = \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega^2 - \kappa^2 - m_a^2 + i\epsilon} \left(e^{i\kappa(y-y')} + e^{i\kappa(y+y')} \right)$$

is the sum of the direct and reflected propagators. This contribution corresponds to the Feynman graphs in Fig. A.1. We can integrate τ out to obtain

$$\int_{-\infty}^{\infty} d\tau G(y, t-\tau)G(y', \tau-t') = 4 \int \frac{d\kappa d\kappa' d\omega}{(2\pi)^3} \frac{i}{\omega^2 - \kappa^2 - m^2 + i\epsilon} \frac{i}{\omega^2 - \kappa'^2 - m^2 + i\epsilon} e^{i\kappa y + i\kappa' y' - i\omega(t-t')}$$

The integral over κ' can be performed using the residue theorem; since $y' < 0$, the contour must be closed in the lower half plane $\Im m \kappa' < 0$. Some further manipulation yields

$$4 \int \frac{d\kappa d\omega}{(2\pi)^2} \frac{1}{2\kappa} \frac{i}{\omega^2 - \kappa^2 - m^2 + i\epsilon} e^{i\kappa(y+y') - i\omega(t-t')}$$

which means that the reflection factor becomes, at this order

$$R(\kappa) = 1 - \frac{2i\lambda}{\kappa}$$

which agrees with (A.3).

A.2.2 Summing up to all orders

At the n th order we get the contribution

$$G^{(n)}(y, t; y', t') = \frac{1}{n!} \left(-\frac{i\lambda}{2} \right)^n \langle 0 | T (\Phi(y, t) \Phi(y', t')) \prod_{i=1}^n \int_{-\infty}^{\infty} d\tau_i : \Phi(0, \tau_i)^2 : | 0 \rangle$$

The relevant graphs are of the form shown in Fig. A.2. The number of contractions can be easily calculated: first one has to decide the order in which the vertices are to be connected starting from one end to the other: this can be done $n!$ ways. Since each vertex has two identical legs, one has an additional factor of 2^n . Collecting all the terms we have

$$\begin{aligned} G^{(n)}(y, t; y', t') &= \frac{2^n n!}{n!} \left(-\frac{i\lambda}{2} \right)^n \int_{-\infty}^{\infty} d\tau_1 \dots \int_{-\infty}^{\infty} d\tau_n G(y, t - \tau_1) \\ &\quad \times \left(\prod_{i=1}^{n-1} G(0, \tau_{i+1} - \tau_i) \right) G(y', \tau_n - t') \\ &= (-i\lambda)^n \int_{-\infty}^{\infty} d\tau_1 \dots \int_{-\infty}^{\infty} d\tau_n G(y, t - \tau_1) \left(\prod_{i=1}^{n-1} G(0, \tau_{i+1} - \tau_i) \right) G(y', \tau_n - t') \end{aligned}$$

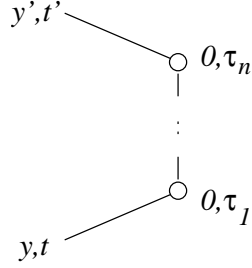


Figure A.2: A typical diagram of order λ^n . To obtain all the diagrams, every line should be allowed to be either a direct bulk (solid) or a reflected bulk (dashed) line, which results in 2^{n+1} terms.

The propagators G are the sum of the direct and reflected piece, so the formula contains 2^{n+1} terms, which turn out to give identical contributions in this case. The τ integrals can be performed one by one, using the same method as for the first-order contribution, with the result:

$$G^{(n)}(y, t; y', t') = \int \frac{d\kappa d\omega}{(2\pi)^2} 2^{n+1} \left(-\frac{i\lambda}{2\kappa}\right)^n \frac{i}{\omega^2 - \kappa^2 - m^2 + i\epsilon} e^{i\kappa(y+y') - i\omega(t-t')} \quad (\text{A.4})$$

Summing up all the contributions, we obtain

$$\begin{aligned} G(y, t; y', t') &= \sum_{n=0}^{\infty} G^{(n)}(y, t; y', t') \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega d\kappa}{(2\pi)^2} \frac{i}{\omega^2 - \kappa^2 - m^2 + i\epsilon} \left(e^{i\kappa(y-y') - i\omega(t-t')} \right. \\ &\quad \left. + R(\kappa) e^{i\kappa(y+y') - i\omega(t-t')} \right) \end{aligned}$$

where

$$R(\kappa) = 1 + 2 \sum_{n=1}^{\infty} \left(-\frac{i\lambda}{\kappa}\right)^n = \frac{\kappa - i\lambda}{\kappa + i\lambda}$$

which is exactly the result we expected. This means that the interacting boundary field, built by summing up the perturbative expansion around the free field with Neumann boundary conditions, does indeed satisfy the correct boundary conditions (A.1).

We remark that this result can be continued analytically to $\lambda < 0$: the pole at $\kappa = i\lambda$ then crosses the contour of the κ integral, and we obtain the result (A.1), which includes the contribution from the boundary bound state. So in this model, although the boundary bound state cannot be obtained by perturbation theory, it can be obtained by analytically continuing the resummed perturbative result from the regime where there is boundary bound state. It is important to resum the perturbation series: to any finite order the nontrivial singularity at $\kappa = -i\lambda$ in the reflection factor is absent.

A.2.3 Concluding remarks

From (A.4) it is apparent that there is an infrared divergence at $\kappa = 0$. This is a general feature of the perturbation theory around the Neumann boundary condition. The perturbative

expansion is a series in λ/κ and so it is convergent only for $\kappa > \lambda$. For long wavelength modes the expansion must be summed up and analytically continued. The physical manifestation of this problem is that $R(\kappa = 0) = -1$, while the Neumann reflection factor is identically $+1$ for any κ . The effective strength of the boundary interaction increases with the wavelength.

To calculate the coordinate space two-point function (A.4) one needs to integrate over all wavelengths and this can only be performed after summing up to all orders. In fact, the proper expression for the two-point function must include an infrared regulator, which can be removed only after summing up the leading behaviour of the function around $\kappa = 0$. As an alternative, one can perform the whole calculation in momentum space, where every contribution is finite.

For any finite order in perturbation theory, the amplitudes display a singularity at $\kappa = 0$, which is in fact a solution to the Landau equations of Section 3.1 with all the bulk internal lines being on-mass-shell, and can be accounted for using the general formalism. Summing up the perturbation series moves this singularity to $\kappa = -i\lambda$, and in the regime $-m < \lambda < 0$ it describes a boundary excited state.

Finally we note that the whole analysis can be generalized to $D + 1$ dimensions. The $D + 1$ -momentum can be written as $(\omega, \vec{k}, \kappa)$ and calculations for any mode with a given value of \vec{k} are isomorphic to a calculation in the $1 + 1$ dimensional case, with the particle mass m replaced by $\sqrt{\vec{k}^2 + m^2}$.

A.3 Extension of the toy model: background fields

We can extend the toy model to include a linear coupling at the boundary. This means that there is a classical background field in the vacuum, which poses an interesting question, since our perturbation theory works in an expansion around the Neumann boundary condition, which has no such fields. The question is whether with an appropriate re-summation in the boundary coupling we can get back to the correct value of the background field. Our Lagrangian is

$$L = \int_{-\infty}^0 dx \left[\frac{1}{2} (\partial_t \Phi)^2 - \frac{1}{2} (\partial_y \Phi)^2 - \frac{m^2}{2} \Phi^2 \right] - \alpha \Phi(y = 0, t) - \frac{\lambda}{2} \Phi(y = 0, t)^2$$

while the equations of motion take the form

$$-\partial_t^2 \Phi + \partial_y^2 \Phi - m^2 \Phi = 0 \quad , \quad \partial_y \Phi|_{y=0} = -\alpha - \lambda \Phi(y = 0, t) \quad (\text{A.5})$$

which has the classical vacuum solution

$$\phi_{vac}(x, t) = -\frac{\alpha}{m + \lambda} e^{mx} \quad (\text{A.6})$$

Let us try to compute this in perturbation theory in both α and λ , i.e. around the Neumann boundary condition. The diagrammatic expression is

$$\langle \Phi(x, t) \rangle = \begin{array}{c} \circ \alpha \\ | \\ \hline \end{array} + \begin{array}{c} \circ \alpha \\ | \\ \circ \lambda \\ | \\ \hline \end{array} + \begin{array}{c} \circ \alpha \\ | \\ \circ \lambda \\ | \\ \circ \lambda \\ | \\ \hline \end{array} + \dots$$

where the diagrams must be understood as classes in which every bulk line can be a direct or a reflected line (see Section 4). There is only a first order contribution in α , while the sum of all orders in λ can be easily seen to give the result of the previous section for the two-point function, with a single α vertex attached to it and integrated over the boundary:

$$\begin{aligned}\langle\Phi(x,t)\rangle &= -i\alpha\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{d\kappa}{2\pi}\frac{i}{\omega^2-\kappa^2-m^2+i\epsilon}e^{-i\omega t}[e^{i\kappa x}+R(\kappa)e^{-i\kappa x}] \\ R(k) &= \frac{\kappa-i\lambda}{\kappa+i\lambda}\end{aligned}$$

The t and ω integrals can be trivially performed, and amount to substituting $\omega = 0$ in the integrand, while the κ integral can be easily performed using the residue theorem with the final result

$$\langle\Phi(x,t)\rangle = -\frac{\alpha}{m+\lambda}e^{mx}$$

which is fully consistent with (A.6).

Obviously, in a general theory if there is a nontrivial vacuum solution to the equations of motion, this means that we are going to have nontrivial tadpole diagrams in the perturbation expansion around the free field with Neumann boundary condition. It would be possible to eliminate tadpoles by expanding in fluctuations around this classical background field, but that would mean space dependent bulk couplings, which would make analytic investigation of perturbation theory extremely difficult. Therefore we choose to expand around the Neumann boundary condition, since re-summation of tadpoles should give us the same result as the background field method, as illustrated in the above example. In the toy model it can be easily seen (just by drawing all diagrams and using the above result for the one-point function) that all the higher correlation functions also get the correct tadpole contributions from perturbation theory.

B One loop renormalization in sine-Gordon theory

In [17] Kormos and Palla considered the semiclassical quantization of the two lowest energy static solutions of the boundary sine-Gordon model:

$$V = \frac{m^2}{\beta^2}\int_{-\infty}^0(1-\cos\beta\Phi) \quad ; \quad U = M_0(1-\cos\frac{\beta}{2}(\Phi-\varphi_0))$$

In their work the semiclassical energy corrections are obtained by summing up the contributions of the oscillators associated to the linearized fluctuations around the static solutions. The appearing standard UV divergences, which are due to the non normal ordered nature of the Lagrangian, are canceled by counter terms: the coupling constants are renormalized $m^2 \rightarrow m^2 + \delta m^2$ and $M_0 \rightarrow M_0 + \delta M_0$. δm^2 is chosen the same as in the bulk theory (obtained in standard perturbation theory), since due to the local nature of the counter term it cannot depend on the presence of the boundary. To determine δM_0 they impose the cancellation of logarithmic divergences. As a result the renormalized quantities of order β^2 take the following form

$$\delta m^2 = -\frac{m^2\beta^2}{4\pi}\int_0^\Lambda\frac{dk}{\sqrt{k^2+m^2}} \quad ; \quad \delta M_0 = -\frac{M_0\beta^2}{8\pi}\int_0^\Lambda\frac{dk}{\sqrt{k^2+m^2}} \quad , \quad (\text{B.1})$$

where Λ is the momentum cutoff.

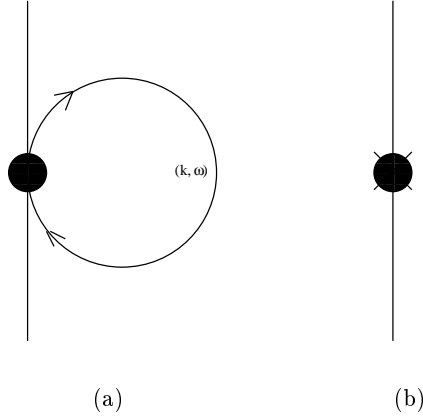


Figure B.1: Bulk divergence (a) and counter-term (b) at one loop

Similar considerations lead to the same counter-terms in the analytically continued sinh-Gordon theory [15, 16]. The aim of this appendix is to derive these conjectured formulae in our systematic perturbative framework. We expand the Lagrangian to the appropriate order in β^2 :

$$V = \frac{m^2}{2}\Phi^2 - \frac{m^2\beta^2}{4!}\Phi^4 \quad , \quad U = \frac{M_0\beta^2}{2 \cdot 2^2}\Phi^2 - \frac{M_0\beta^4}{4! \cdot 2^4}\Phi^4$$

where for simplicity we take $\varphi_0 = 0$. The divergent term at order $m^2\beta^2$ comes from diagram B.1 (a), which is regularized by a momentum cutoff, and is canceled by a counter-term of the form

$$V_{CT} = \frac{\delta m^2}{2}\Phi^2$$

(diagram B.1 (b)).

To compute δm^2 we observe that on diagram B.1 (a) the momentum (k, ω) is incoming and outgoing in the same time, that is it does not contribute to the delta function: the integration is unconstrained (as opposed to the case where the loop propagator is changed to the reflected one, which results in a diagram that is not divergent at all). As a result the delta function factor is the same for both diagrams and from the cancellation of the vertex contribution we have

$$2 \cdot \frac{\delta m^2}{2} = -4 \cdot 3 \cdot \frac{-m^2\beta^2}{4!} \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{i}{\omega^2 - k^2 - m^2 + i\epsilon}$$

(with the combinatorial factors given explicitly). Performing the ω integration we recover the standard result. We have three other divergent diagrams in this order, which can be obtained from B.1 (a) by changing any or both of its external legs to the reflected propagator. Performing the same changes on diagram B.1 (b), however, results a diagram which removes the required divergence.

The divergent diagrams originating from the boundary term are of order $M_0\beta^4$ and are canceled by a counter term of the form

$$U_{CT} = \frac{\delta M_0\beta^2}{2 \cdot 2^2}$$

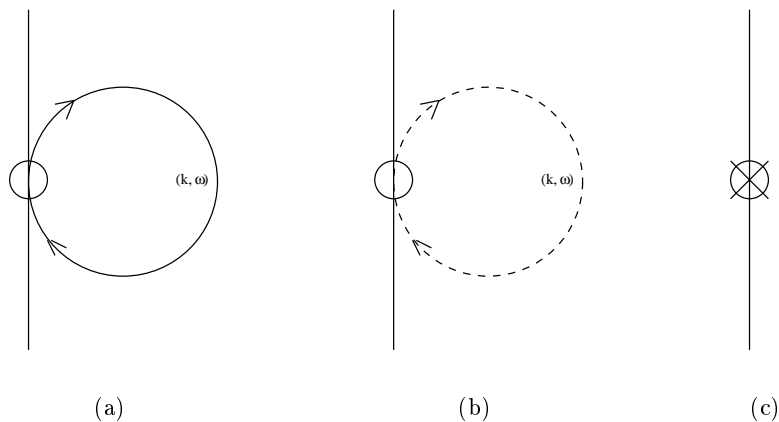


Figure B.2: Divergent contributions (a,b) and counter term (c) for the boundary interaction

The related Feynman graphs are presented on diagram B.2 (a), (b) and (c).

There are two differences compared to the bulk vertices. First of all, β has been replaced by $\frac{\beta}{2}$. More importantly, there is no momentum conservation in the boundary vertices and as a consequence not only diagram (a) but also diagram (b) is divergent, moreover they have the same contribution. Summing up these two terms the counter-term acquires a factor 2 (in addition to $\beta \rightarrow \frac{\beta}{2}$) compared to the bulk computation and confirms the result (B.1).

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