A characterization of Hermitian varieties as codewords

A. Aguglia∗, D. Bartoli†, L. Storme‡, Zs. Weiner§

Abstract

It is known that the Hermitian varieties are codewords in the code defined by the points and hyperplanes of the projective spaces PG(r, q^2). In finite geometry, also quasi-Hermitian varieties are defined. These are sets of points of PG(r, q^2) of the same size as a non-singular Hermitian variety of PG(r, q^2), having the same intersection sizes with the hyperplanes of PG(r, q^2). In the planar case, this reduces to the definition of a unital. A famous result of Blokhuis, Brouwer, and Wilbrink states that every unital in the code of the points and lines of PG(2, q^2) is a Hermitian curve. We prove a similar result for the quasi-Hermitian varieties in PG(3, q^2), q = p^h, as well as in PG(r, q^2), q = p prime, or q = p^2, p prime, and r ≥ 4.

Keywords: Hermitian variety; incidence vector; codes of projective spaces; quasi-Hermitian variety.

MSC: 51E20, 94B05

1 Introduction

Consider the non-singular Hermitian varieties \( \mathcal{H}(r, q^2) \) in PG(r, q^2). A non-singular Hermitian variety \( \mathcal{H}(r, q^2) \) in PG(r, q^2) is the set of absolute points of a Hermitian polarity of PG(r, q^2). Many properties of a non-singular Hermitian variety \( \mathcal{H}(r, q^2) \) in PG(r, q^2) are known. In particular, its size is \( (q^r + 1)(q^r - (-1)^r)/(q^2 - 1) \).
1), and its intersection numbers with the hyperplanes of \( \text{PG}(r, q^2) \) are equal to \((q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})/(q^2 - 1)\), in case the hyperplane is a non-tangent hyperplane to \( \mathcal{H}(r, q^2) \), and equal to \( 1 + q^2(q^{r-1} + (-1)^r)(q^{r-2} - (-1)^r)/(q^2 - 1) \) in case the hyperplane is a tangent hyperplane to \( \mathcal{H}(r, q^2) \); see [16].

Quasi-Hermitian varieties \( \mathcal{V} \) in \( \text{PG}(r, q^2) \) are generalizations of the non-singular Hermitian variety \( \mathcal{H}(r, q^2) \) so that \( \mathcal{V} \) and \( \mathcal{H}(r, q^2) \) have the same size and the same intersection numbers with hyperplanes.

Obviously, a Hermitian variety \( \mathcal{H}(r, q^2) \) can be viewed as a trivial quasi-Hermitian variety and we call \( \mathcal{H}(r, q^2) \) the classical quasi-Hermitian variety of \( \text{PG}(r, q^2) \).

As far as we know, the only known non-classical quasi-Hermitian varieties of \( \text{PG}(r, q^2) \) were constructed in [1, 2, 8, 9, 13, 14].

In [6], it is shown that a unital in \( \text{PG}(2, q^2) \) is a Hermitian curve if and only if it is in the \( \mathbb{F}_p \)-code spanned by the lines of \( \text{PG}(2, q^2) \), with \( q = p^h \), \( p \) prime and \( h \in \mathbb{N} \).

In this article, we prove the following result.

**Theorem 1.1.** A quasi-Hermitian variety \( \mathcal{V} \) of \( \text{PG}(r, q^2) \), with \( r = 3 \) and \( q = p^h > 4 \), or \( r \geq 4, q = p > 4 \), or \( r \geq 4, q = p^2, p > 3 \) prime, is classical if and only if it is in the \( \mathbb{F}_p \)-code spanned by the hyperplanes of \( \text{PG}(r, q^2) \).

Furthermore we consider singular quasi-Hermitian varieties, that is point sets having the same number of points as a singular Hermitian variety \( \mathcal{S} \) and for which each intersection number with respect to hyperplanes is also an intersection number of \( \mathcal{S} \) with respect to hyperplanes. We show that Theorem 1.1 also holds in the case in which \( \mathcal{V} \) is assumed to be a singular quasi-Hermitian variety of \( \text{PG}(r, q^2) \).

## 2 Preliminaries

A subset \( \mathcal{K} \) of \( \text{PG}(r, q^2) \) is a \( k_{n,r,q^2} \) if \( n \) is a fixed integer, with \( 1 \leq n \leq q^2 \), such that:

(i) \( |\mathcal{K}| = k \);

(ii) \( |\ell \cap \mathcal{K}| = 1, n, \) or \( q^2 + 1 \) for each line \( \ell \);

(iii) \( |\ell \cap \mathcal{K}| = n \) for some line \( \ell \).

A point \( P \) of \( \mathcal{K} \) is singular if every line through \( P \) is either a unisecant or a line of \( \mathcal{K} \). The set \( \mathcal{K} \) is called singular or non-singular according as it has singular points or not.

Furthermore, a subset \( \mathcal{K} \) of \( \text{PG}(r, q^2) \) is called regular if
(a) $\mathcal{K}$ is a $k_{n,r,q^2}$;
(b) $3 \leq n \leq q^2 - 1$;
(c) no planar section of $\mathcal{K}$ is the complement of a set of type $(0, q^2 + 1 - n)$.

**Theorem 2.1.** [10, Theorem 19.5.13] Let $\mathcal{K}$ be a $k_{n,3,q^2}$ in $\text{PG}(3, q^2)$, where $q$ is any prime power and $n \neq \frac{1}{2}q^2 + 1$. Suppose furthermore that every point in $\mathcal{K}$ lies on at least one n-secant. Then $n = q + 1$ and $\mathcal{K}$ is a non-singular Hermitian surface.

**Theorem 2.2.** [11, Theorem 23.5.19] If $\mathcal{K}$ is a regular, non-singular $k_{n,r,q^2}$, with $r \geq 4$ and $q > 2$, then $\mathcal{K}$ is a non-singular Hermitian variety.

**Theorem 2.3.** [11, Th. 23.5.1] If $\mathcal{K}$ is a singular $k_{n,3,q^2}$ in $\text{PG}(3, q^2)$ with $3 \leq n \leq q^2 - 1$, then the following holds: $\mathcal{K}$ is $n$ planes through a line or a cone with vertex a point and base $\mathcal{K}'$ a plane section of type

I. a unital;
II. a subplane $\text{PG}(2, q)$;
III. a set of type $(0, n - 1)$ plus an external line;
IV. the complement of a set of type $(0, q^2 + 1 - n)$.

**Theorem 2.4.** [11, Lemma 23.5.2 and Th. 25.5.3] If $\mathcal{K}$ is a singular $k_{n,r,q^2}$ with $r \geq 4$, then the singular points of $\mathcal{K}$ form a subspace $\Pi_d$ of dimension $d$ and one of the following possibilities holds:

1. $d = r - 1$ and $\mathcal{K}$ is a hyperplane;
2. $d = r - 2$ and $\mathcal{K}$ consists of $n > 1$ hyperplanes through $\Pi_d$;
3. $d \leq r - 3$ and $\mathcal{K}$ is equal to a cone $\Pi_d\mathcal{K}'$, with $\pi_d$ as vertex and with $\mathcal{K}$ as base, where $\mathcal{K}'$ is a non singular $k_{n,r-d-1,q^2}$.

**Result 2.5** ([13]). Let $\mathcal{M}$ be a multiset in $\text{PG}(2, q)$, $17 < q$, so that the number of lines intersecting it in not $k \mod p$ points is $\delta$, where $\delta < \frac{2}{3}(q + 1)^2$. Then the number of non $k \mod p$ secants through any point is at most $\frac{\delta}{q+1} + \frac{2\delta}{(q+1)^2}$ or at least $q + 1 - (\frac{\delta}{q+1} + \frac{2\delta}{(q+1)^2})$.

**Property 2.6** ([13]). Let $\mathcal{M}$ be a multiset in $\text{PG}(2, q)$, $q = ph$, where $p$ is prime. Assume that there are $\delta$ lines that intersect $\mathcal{M}$ in not $k \mod p$ points. If through a point there are more than $q/2$ lines intersecting $\mathcal{M}$ in not $k \mod p$ points, then there exists a value $r$ such that the intersection multiplicity of at least $2\frac{\delta}{q+1} + 5$ of these lines with $\mathcal{M}$ is $r$. 

3
Let \( M \) be a multiset in \( \text{PG}(2, q) \), \( 17 < q = p^h \), where \( p \) is prime. Assume that the number of lines intersecting \( M \) in not \( k \pmod{p} \) points is \( \delta \), where \( \delta < (\lfloor \sqrt{q} \rfloor + 1)(q + 1 - \lfloor \sqrt{q} \rfloor) \). Assume furthermore that Property 2.6 holds. Then there exists a multiset \( M' \) with the property that it intersects every line in \( k \pmod{p} \) points and the number of different points in \( (M \cup M') \setminus (M \cap M') \) is exactly \( \lceil \frac{\delta q + 1}{q+1} \rceil \).

Result 2.8 ([15]). Let \( B \) be a proper point set in \( \text{PG}(2, q) \), \( 17 < q \). Suppose that \( B \) is a codeword of the lines of \( \text{PG}(2, q) \). Assume also that \( |B| < (\lfloor \sqrt{q} \rfloor + 1)(q + 1 - \lfloor \sqrt{q} \rfloor) \). Then \( B \) is a linear combination of at most \( \lceil \frac{|B| q + 1}{q+1} \rceil \) lines.

3 Proof of Theorem 1.1

Let \( V \) be the vector space of dimension \( q^{2r} + q^{2(r-1)} + \cdots + q^2 + 1 \) over the prime field \( \mathbb{F}_p \), where the coordinate positions for the vectors in \( V \) correspond to the points of \( \text{PG}(r, q^2) \) in some fixed order. If \( S \) is a subset of points in \( \text{PG}(r, q^2) \) then let \( v^S \) denote the vector in \( V \) with coordinate 1 in the positions corresponding to the points in \( S \) and with coordinate 0 in all other positions; that is \( v^S \) is the characteristic vector of \( S \). Let \( C_p \) denote the subspace of \( V \) spanned by the characteristic vectors of all the hyperplanes in \( \text{PG}(r, q^2) \). This code \( C_p \) is called the linear code of \( \text{PG}(r, p^2) \).

From [12, Theorem 1], we know that the characteristic vector \( v^V \) of a Hermitian variety \( V \in \text{PG}(r, q^2) \) is in \( C_p \). So from now on, we will assume that \( V \) is a quasi-Hermitian variety in \( \text{PG}(r, q^2) \) and \( v^V \in C_p \). In the remainder of this section we will show that \( V \) is a classical Hermitian variety.

The next lemmas hold for \( r \geq 3 \) and for any \( q = p^h \), \( p \) prime, \( h \geq 1 \).

Lemma 3.1. Every line of \( \text{PG}(r, q^2) \), \( q = p^h \), \( p \) prime, \( h \geq 1 \), meets \( V \) in 1 \( \pmod{p} \) points.

Proof. We may express
\[
v^V = v^{H_1} + \cdots + v^{H_t},
\]
where \( H_1, \ldots, H_t \) are (not necessarily distinct) hyperplanes of \( \text{PG}(r, q^2) \). Denote by \( \cdot \) the usual dot product. We get \( v^V \cdot v^V = |V| \equiv 1 \pmod{p} \). On the other hand
\[
v^V \cdot v^V = v^V \cdot (v^{H_1} + \cdots + v^{H_t}) \equiv t \pmod{p},
\]
since every hyperplane of \( \text{PG}(r, q^2) \) meets \( V \) in 1 \( \pmod{p} \) points. Hence, we have \( t \equiv 1 \pmod{p} \). Finally, for a line \( \ell \) of \( \text{PG}(r, q^2) \),
\[
v^\ell \cdot v^V = v^\ell \cdot (v^{H_1} + \cdots + v^{H_t}) \equiv t \pmod{p},
\]
as every line of $\text{PG}(r, q^2)$ meets a hyperplane in 1 or $q^2 + 1$ points. That is, $|\ell \cap V| \equiv 1 \pmod{p}$ and in particular no lines of $\text{PG}(r, q^2)$ are external to $V$. □

**Remark 3.2.** The preceding proof also shows that $V$ is a linear combination of 1 (mod $p$) (not necessarily distinct) hyperplanes, all having coefficient one.

**Lemma 3.3.** For every hyperplane $H$ of $\text{PG}(r, q^2)$, $q = p^h$, $p$ prime, $h \geq 1$, the intersection $H \cap V$ is in the code of points and hyperplanes of $H$ itself.

**Proof.** Let $\Sigma$ denote the set of all hyperplanes of $\text{PG}(r, q^2)$. By assumption,

$$v^V = \sum_{H_i \in \Sigma} \lambda_i v^{H_i}. \quad (1)$$

For every $H \in \Sigma$, let $\pi$ denote a hyperplane of $H$; then $\pi = H_{j_1} \cap \cdots \cap H_{j_{q^2+1}}$, where $H_{j_1}, \ldots, H_{j_{q^2+1}}$ are the hyperplanes of $\text{PG}(r, q^2)$ through $\pi$. We assume $H = H_{j_{q^2+1}}$. For every hyperplane $\pi$ of $H$, we set

$$\lambda_\pi = \sum_{k=1}^{q^2+1} \lambda_{j_k},$$

where $\lambda_{j_k}$ is the coefficient in (1) of $v^{H_{j_k}}$ and $H_{j_k}$ is one of the $q^2 + 1$ hyperplanes through $\pi$.

Now, consider

$$T = \sum_{\pi \in \Sigma'} \lambda_\pi v^\pi, \quad (2)$$

where $\Sigma'$ is the set of all hyperplanes in $H$. We are going to show that

$$T = v^{V \cap H}. \quad (\text{mod } p)$$

In fact, it is clear that at the positions belonging to the points outside of $H$ we see zeros. At a position belonging to a point in $H$, we see the original coefficients of $v^V$ plus $(|\Sigma'| - 1) \lambda_{j_{q^2+1}}$. Note that this last term is 0 (mod $p$), hence $T = v^{V \cap H}$. □

**Corollary 3.4.** For every subspace $S$ of $\text{PG}(r, q^2)$, $q = p^h$, $p$ prime, $h \geq 1$, the intersection $S \cap V$ is in the code of points and hyperplanes of $S$ itself. □

**Remark 3.5.** Lemma 3.3 and Corollary 3.4 are valid for $V$ any set of points in $\text{PG}(r, q^2)$ whose incidence vector belongs to the code of points and hyperplanes of $\text{PG}(r, q^2)$. In particular, it follows that for every plane $\pi$ the intersection $\pi \cap V$ is a codeword of the points and lines of $\pi$, $\pi \cap V$ has size 1 (mod $p$) and so it is a linear combination of 1 (mod $p$) not necessarily distinct lines.
Lemma 3.6. Let $\ell$ be a line of $\text{PG}(r,q^2)$. Then there exists at least one plane through $\ell$ meeting $\mathcal{V}$ in $\delta$ points, with $\delta \leq q^3 + q^2 + q + 1$.

Proof. By way of contradiction, assume that all planes through $\ell$ meet $\mathcal{V}$ in more than $q^3 + q^2 + q + 1$ points. Set $x = |\ell \cap \mathcal{V}|$. We get
\[
\frac{(q^r+1 - (1)^r)(q^r - (1)^r)}{q^2 - 1} > m(q^3 + q^2 + q + 1 - x) + x,
\] (3)
where $m = q^{2(r-2)} + q^{2(r-3)} + \cdots + q^2 + 1$ is the number of planes in $\text{PG}(r,q^2)$ through $\ell$. From (3), we obtain $x > q^2 + 1$, a contradiction. \hfill $\square$

Lemma 3.7. For each line $\ell$ of $\text{PG}(r,q^2)$, $q > 4$ and $q = p^h$, $p$ prime, $h \geq 1$, either $|\ell \cap \mathcal{V}| \leq q + 1$ or $|\ell \cap \mathcal{V}| \geq q^2 - q + 1$.

Proof. Let $\ell$ be a line of $\text{PG}(r,q^2)$ and let $\pi$ be a plane through $\ell$ such that $|\pi \cap \mathcal{V}| \leq q^3 + q^2 + q + 1$; Lemma 3.6 shows that such a plane exists. Set $B = \pi \cap \mathcal{V}$. By Corollary 3.4, $B$ is a codeword of the code of the lines of $\pi$, so we can write it as a linear combination of some lines of $\pi$, that is $\sum \lambda_i v^{e_i}$, where $v^{e_i}$ are the characteristic vectors of the lines $e_i$ in $\pi$.

Let $B^*$ be the multiset consisting of the lines $e_i$, with multiplicity $\lambda_i$, in the dual plane of $\pi$. The weight of the codeword $B$ is at most $q^3 + q^2 + q + 1$, hence in the dual plane this is the number of lines intersecting $B^*$ in not $0 \mod p$ points. Actually, as $B$ is a proper set, we know that each non $0 \mod p$ secant of $B^*$ must be a $1 \mod p$ secant. Using Result 2.5, the number of non $0 \mod p$ secants through any point is at most $q + 1$ or at least $q^2 - q + 1$. In the original plane $\pi$, this means that each line intersects $B$ in either at most $q + 1$ or in at least $q^2 - q + 1$ points. \hfill $\square$

Proposition 3.8. Assume that $\pi$ is a plane of $\text{PG}(r,q^2)$, $q > 4$, and $q = p^h$, $p$ prime, $h \geq 1$, such that $|\pi \cap \mathcal{V}| \leq q^3 + 2q^2$. Furthermore, suppose also that there exists a line $\ell$ meeting $\pi \cap \mathcal{V}$ in at least $q^3 - q + 1$ points, when $q^3 + 1 \leq |\pi \cap \mathcal{V}|$. Then $\pi \cap \mathcal{V}$ is a linear combination of at most $q + 1$ lines, each with weight 1.

Proof. Let $B$ be the point set $\pi \cap \mathcal{V}$. By Corollary 3.4, $B$ is the corresponding point set of a codeword $c$ of lines of $\pi$, that is $c = \sum \lambda_i v^{e_i}$, where lines of $\pi$ are denoted by $e_i$. Let $C^*$ be the multiset in the dual plane containing the dual of each line $e_i$ with multiplicity $\lambda_i$. Clearly the number of lines intersecting $C^*$ in not $0 \mod p$ points is $w(c) = |B|$. Note also, that every line that is not a $0 \mod p$ secant is a $1 \mod p$ secant, as $B$ is a proper point set, hence Property 2.6 trivially holds (with $k = 1$).

Our very first aim is to show that $c$ is a linear combination of at most $q + 3$ different lines. When $|B| < q^3 + 1$, then by Result 2.8 it is a linear combination of at most $q$ different lines.
Next assume that $|B| \geq q^3 + 1$. From the assumption of the proposition, we know that there exists a line $\ell$ meeting $\pi \cap \mathcal{V}$ in at least $q^2 - q + 1$ points and from Lemma 3.7, we also know that each line intersects $B$ in either at most $q + 1$ or in at least $q^2 - q + 1$ points. Hence, if we add the line $\ell$ to $c$ with multiplicity $-1$, we reduce the weight by at least $q^2 - q + 1 - q$ and at most by $q^2 + 1$. If $w(c - v^\ell) < q^3 + 1$, then from the above we know that $c - v^\ell$ is a linear combination of $\left\lceil \frac{w(c - v^\ell)}{q} \right\rceil$ lines. Hence, $c$ is a linear combination of at most $q + 1$ lines. If $w(c - v^\ell) \geq q^3 + 1$, then $w(c) \geq q^3 + q^2 - 2q$ (see above) and so it follows that through any point of $B$, there passes at least one line intersecting $B$ in at least $q^2 - q + 1$ points. This means that we easily find three lines $\ell_1$, $\ell_2$, and $\ell_3$ intersecting $B$ in at least $q^2 - q + 1$ points. Since $w(c) \leq q^3 + 2q^2$, we get that $w(c - v^{\ell_1} - v^{\ell_2} - v^{\ell_3}) \leq q^3 + 2q^2 - 3 \cdot (q^2 - 2q - 2) < q^3 + 1$. Hence, similarly as before, we get that $c$ is a linear combination of at most $q + 3$ lines.

Next we show that each line in the linear combination (that constructs $c$) has weight 1. Take a line $\ell$ which is in the linear combination with coefficient $\lambda \neq 0$. Then there are at least $q^2 + 1 - (q + 2)$ positions, such that the corresponding point is in $\ell$ and the value at that position is $\lambda$. As $B$ is a proper set, this yields that $\lambda = 1$. By Remark 3.5, the number of lines with non-zero multiplicity in the linear combination of $c$ must be $1 \mod (\mod p)$, $p > 2$; hence it can be at most $q + 1$.

**Proposition 3.9.** Assume that $\pi$ is a plane of $\text{PG}(r, q^2)$, $q > 4$, and $q = p^h$, $p$ prime, $h \geq 1$, such that $|\pi \cap \mathcal{V}| \leq q^3 + 2q^2$. Furthermore, suppose that every line meets $\pi \cap \mathcal{V}$ in at most $q + 1$ points. Then $\pi \cap \mathcal{V}$ is a classical unital.

**Proof.** Again let $B = \pi \cap \mathcal{V}$ and first assume that $|B| < q^3 + 1$. Proposition 3.8 shows that $B$ is a linear combination of at most $q + 1$ lines, each with weight 1. But this yields that these lines intersect $B$ in at least $q^2 + 1 - q$ points. So this case cannot occur.

Hence, $q^3 + 1 \leq |B| \leq q^3 + 2q^2$. We are going to prove that there exists at least a tangent line to $B$ in $\pi$. Let $t_i$ be the number of lines meeting $B$ in $i$ points. Set $x = |B|$. Then double counting arguments give the following equations for the integers $t_i$.

\[
\begin{aligned}
\sum_{i=1}^{q+1} t_i &= q^4 + q^2 + 1 \\
\sum_{i=1}^{q+1} i t_i &= x(q^2 + 1) \\
\sum_{i=1}^{q+1} i(i-1)t_i &= x(x-1)
\end{aligned}
\] (4)

Consider $f(x) = \sum_{i=1}^{q+1} (i - 2)(q + 1 - i)t_i$. From (4), we get

\[
f(x) = -x^2 + x[(q^2 + 1)(q + 2) + 1] - 2(q + 1)(q^4 + q^2 + 1).
\]
Since \( f(q^3/2) > 0 \), whereas \( f(q^3 + 1) < 0 \) and \( f(q^3 + 2q^2) < 0 \), it follows that if \( q^3 + 1 \leq x \leq q^3 + 2q^2 \), then \( f(x) < 0 \) and thus \( t_1 \) must be different from zero. Therefore, \( x = q^3 + 1 \) and \( \sum_{i=1}^{q^2+1} (i-1)(q^3 + 1 - i)t_i = 0 \).

Since \((i-1)(q+1-i) > 0\), for \(2 \leq i \leq q\), we obtain \(t_2 = t_3 = \cdots = t_q = 0\), that is, \(B\) is a set of \(q^3+1\) points such that each line is a 1-secant or a \((q+1)\)-secant of \(B\). Namely, \(B\) is a unital and precisely a classical unital since \(B\) is a codeword of \(\pi\) [6].

The above two propositions and Lemma 3.7 imply the following corollary.

**Corollary 3.10.** Assume that \(\pi\) is a plane of \(\text{PG}(r, q^2)\), \(q > 4\) and \(q = p^h\), \(p\) prime, \(h \geq 1\), such that \(|\pi \cap \mathcal{V}| \leq q^3 + 2q^2\). Then \(\pi \cap \mathcal{V}\) is a linear combination of at most \(q + 1\) lines, each with weight 1, or it is a classical unital.  

**Corollary 3.11.** Suppose that \(\pi\) is a plane of \(\text{PG}(r, q^2)\), \(q > 4\) and \(q = p^h\), \(p\) prime, \(h \geq 1\), containing exactly \(q^3 + 1\) points of \(\mathcal{V}\). Then \(\pi \cap \mathcal{V}\) is a classical unital.

**Proof.** Let \(B\) be the point set \(\pi \cap \mathcal{V}\). We know that \(B\) is the support of a codeword of lines of \(\pi\). By Proposition 3.8, if there is a line intersecting \(B\) in at least \(q^2 - q + 1\) points, then \(B\) is a linear combination of at most \(q + 1\) lines, each with multiplicity 1. First of all note that a codeword that is a linear combination of \(q + 1\) lines has weight at least \((q^2 + 1)(q + 1) - 2\binom{q+1}{2}\), that is exactly \(q^3 + 1\). To achieve this, we need that the intersection points of any two lines from a linear combination are all different and the sum of the coefficients of any two lines is zero; which is clearly not the case (as all the coefficients are 1). From Remark 3.5, in this case \(B\) would be a linear combination of at most \(q + 1 - p\) lines and so its weight would be less than \(q^3 + 1\), a contradiction. Hence, there is no line intersecting \(B\) in at least \(q^2 - q + 1\) points, so Proposition 3.9 finishes the proof.  

3.1 Case \(r = 3\)

In \(\text{PG}(3, q^2)\), each plane intersects \(\mathcal{V}\) in either \(q^3 + 1\) or \(q^3 + q^2 + 1\) points since these are the intersection numbers of a quasi-Hermitian variety with a plane of \(\text{PG}(3, q^2)\).

**Lemma 3.12.** Let \(\pi\) be a plane in \(\text{PG}(3, q^2)\) such that \(|\pi \cap \mathcal{V}| = q^3 + q^2 + 1\), then every line in \(\pi\) meets \(\pi \cap \mathcal{V}\) in either 1, \(q + 1\) or \(q^2 + 1\) points.

**Proof.** Set \(C = \pi \cap \mathcal{V}\) and let \(m\) be a line in \(\pi\) such that \(|m \cap C| = s\), with \(s \neq 1\) and \(s \neq q + 1\). Thus, from Corollary 3.11, every plane through \(m\) has to meet \(\mathcal{V}\) in \(q^3 + q^2 + 1\) points and thus

\[|\mathcal{V}| = (q^2 + 1)(q^3 + q^2 + 1 - s) + s,\]
which gives \( s = q^2 + 1 \).

**Proof of Theorem 1.1 (case \( r = 3 \)):** From Corollary 3.11 and Lemma 3.12 it follows that every line in \( \text{PG}(3, q^2) \) meets \( \mathcal{V} \) in either 1, \( q + 1 \), or \( q^2 + 1 \) points. Now, suppose on the contrary that there exists a singular point \( P \) on \( \mathcal{V} \); this means that all lines through \( P \) are either tangents or \((q^2 + 1)\)-secants to \( \mathcal{V} \). Take a plane \( \pi \) which does not contain \( P \). Then \( |\mathcal{V}| = q^2|\pi \cap \mathcal{V}| + 1 \) and since the two possible sizes of the planar sections are \( q^3 + 1 \) or \( q^3 + q^2 + 1 \), we get a contradiction. Thus, every point in \( \mathcal{V} \) lies on at least one \((q + 1)\)-secant and, from Theorem 2.1, we obtain that \( \mathcal{V} \) is a Hermitian surface.

### 3.2 Case \( r \geq 4 \) and \( q = p \)

We first prove the following result.

**Lemma 3.13.** If \( \pi \) is a plane of \( \text{PG}(r, p^2) \), which is not contained in \( \mathcal{V} \), then either

\[
|\pi \cap \mathcal{V}| = p^2 + 1 \quad \text{or} \quad |\pi \cap \mathcal{V}| \geq p^3 + 1.
\]

**Proof.** Let \( \pi \) be a plane of \( \text{PG}(r, p^2) \) and set \( B = \pi \cap \mathcal{V} \). By Remark 3.5, \( B \) is a linear combination of 1 mod (mod \( p \)) not necessarily distinct lines.

If \( |B| < p^3 + 1 \), then by Result 2.5, \( B \) is a linear combination of at most \( p \) distinct lines. This and the previous observation yield that when \( |B| < p^3 + 1 \), then it is the scalar multiple of one line; hence \( |B| = p^2 + 1 \).

**Proposition 3.14.** Let \( \pi \) be a plane of \( \text{PG}(r, p^2) \), such that \( |\pi \cap \mathcal{V}| \leq p^3 + p^2 + p + 1 \). Then \( B = \pi \cap \mathcal{V} \) is either a classical unital or a linear combination of \( p + 1 \) concurrent lines or just one line, each with weight 1.

**Proof.** From Corollary 3.10 we have that \( B \) is either a linear combination of at most \( p + 1 \) lines or a classical unital. In the first case, since \( B \) intersects every line in 1 (mod \( p \)) points and \( B \) is a proper point set, the only possibilities are that \( B \) is a linear combination of \( p + 1 \) concurrent lines or just one line, each with weight 1.

**Proof of Theorem 1.1 (case \( r \geq 4 \), \( q = p \)):** Consider a line \( \ell \) of \( \text{PG}(r, p^2) \) which is not contained in \( \mathcal{V} \). By Lemma 3.6 there is a plane \( \pi \) through \( \ell \) such that \( |\pi \cap \mathcal{V}| \leq q^3 + q^2 + q + 1 \). From Proposition 3.14 we have that \( \ell \) is either a unisecant or a \((p + 1)\)-secant of \( \mathcal{V} \) and we also have that \( \mathcal{V} \) has no plane section of size \((p + 1)(p^2 + 1)\). Finally, it is easy to see like in the previous case \( r = 3 \), that \( \mathcal{V} \) has no singular points, thus \( \mathcal{V} \) turns out to be a Hermitian variety of \( \text{PG}(r, p^2) \) (Theorem 2.2).
3.3 Case \( r \geq 4 \) and \( q = p^2 \)

Assume now that \( V \) is a quasi-Hermitian variety of \( \text{PG}(r, p^4) \), with \( r \geq 4 \).

Lemma 3.17 states that every line contains at most \( p^2 + 1 \) points of \( V \) or at least \( p^4 - p^2 + 1 \) points of \( V \).

**Lemma 3.15.** If \( \ell \) is a line of \( \text{PG}(r, p^4) \), such that \( |\ell \cap V| \geq p^4 - p^2 + 1 \), then \( |\ell \cap V| \geq p^4 - p + 1 \).

**Proof.** Set \( |\ell \cap V| = p^4 - x + 1 \), where \( x \leq p^2 \). It suffices to prove that \( x < p + 2 \). Let \( \pi \) be a plane through \( \ell \) and \( B = \pi \cap V \). Choose \( \pi \) such that \( |B| = |\pi \cap V| \leq p^6 + p^4 + p^2 + 1 \) (Lemma 3.14). Then, by Proposition 3.8, \( B \) is a linear combination of at most \( p^2 + 1 \) lines, each with weight 1. Let \( c \) be the codeword corresponding to \( B \). We observe that \( \ell \) must be one of the lines of \( c \), otherwise \( |B \cap \ell| \leq p^2 + 1 \), which is impossible. Thus if \( P \) is a point in \( \ell \setminus B \), then through \( P \) there pass at least \( p - 1 \) other lines of \( c \). If \( x \geq p + 2 \), then the number of lines necessary to define the codeword \( c \) would be at least \((p + 2)(p - 1) + 1\), a contradiction. \( \square \)

**Lemma 3.16.** For each plane \( \pi \) of \( \text{PG}(r, p^4) \), either \( |\pi \cap V| \leq p^6 + 2p^4 - p^2 - p + 1 \) or \( |\pi \cap V| \geq p^8 - p^5 + p^4 - p + 1 \).

**Proof.** Let \( B = \pi \cap V \), \( x = |B| \), and let \( t_i \) be the number of lines in \( \pi \) meeting \( B \) in \( i \) points. Then, in this case, Equations (2) read

\[
\begin{align*}
\sum_{i=1}^{p^4+1} t_i &= p^8 + p^4 + 1 \\
\sum_{i=1}^{p^4+1} it_i &= x(p^4 + 1) \\
\sum_{i=1}^{p^4+1} i(i-1)t_i &= x(x-1).
\end{align*}
\]

Set \( f(x) = \sum_{i=1}^{p^4+1} (p^2 + 1 - i)(i - (p^4 - p + 1))t_i \). From (5) we obtain

\[
f(x) = -x^2 + [(p^4 + 1)(p^4 + p^2 - p + 1) + 1]x - (p^8 + p^4 + 1)(p^2 + 1)(p^4 - p + 1).
\]

Because of Lemma 3.15, we get \( f(x) \leq 0 \), while \( f(p^6 + 2p^4 - p^2 + 1) > 0 \), \( f(p^8 - p^5 + p^4 - p) > 0 \). This finishes the proof of the lemma. \( \square \)

**Lemma 3.17.** If \( \pi \) is a plane of \( \text{PG}(r, p^4) \), such that \( |\pi \cap V| \geq p^8 - p^5 + p^4 - p + 1 \), then either \( \pi \) is entirely contained in \( V \) or \( \pi \cap V \) consists of \( p^8 - p^5 + p^4 + 1 \) points and it only contains \( i \)-secants, with \( i \in \{1, p^4 - p + 1, p^4 + 1\} \).

**Proof.** Set \( S = \pi \setminus V \). Suppose that there exists some point \( P \in S \). We have the following two possibilities: either each line of the pencil with center at \( P \) is a \((p^4 - p + 1)\)-secant or only one line through \( P \) is an \( i \)-secant, with \( 1 \leq i \leq p^2 + 1 \),
whereas the other $p^i$ lines through $P$ are $(p^4 - p + 1)$-secants. In the former case, when there are no $i$-secants, $1 \leq i \leq p^2 + 1$, each line $\ell$ in $\pi$ either is disjoint from $S$ or it meets $S$ in $p$ points since $\ell$ is a $(p^4 - p + 1)$-secant. This implies that $S$ is a maximal arc and this is impossible for $p \neq 2$ \cite{4,5}.

In the latter case, we observe that the size of $\pi \cap \mathcal{V}$ must be $p^8 - p^5 + p^4 + i$, where $1 \leq i \leq p^2 + 1$. Next, we denote by $t_s$ the number of $s$-secants in $\pi$, where $s \in \{i, p^4 - p + 1, p^4 + 1\}$. We have that

$$
\begin{align*}
\sum_s t_s &= p^8 + p^4 + 1 \\
\sum_s st_s &= (p^4 + 1)(p^8 - p^5 + p^4 + i) \\
\sum_s s(s - 1)t_s &= (p^8 - p^5 + p^4 + i)(p^8 - p^5 + p^4 + i - 1).
\end{align*}
$$

From (6) we get

$$
t_i = \frac{p(p^4 - p - i + 1)(p^5 - i + 1)}{p(p^4 - p - i + 1)(p^4 - i + 1)} = \frac{p^5 - i + 1}{p^4 - i + 1}
$$

and we can see that the only possibility for $t_i$ to be an integer is $ip - p - i + 1 = 0$, that is $i = 1$. For $i = 1$, we get $|B| = p^8 - p^5 + p^4 + 1$.

**Lemma 3.18.** If $\pi$ is a plane of $\text{PG}(r, p^4)$, not contained in $\mathcal{V}$ and which does not contain any $(p^4 - p + 1)$-secant, then $\pi \cap \mathcal{V}$ is either a classical unital or the union of $i$ concurrent lines, with $1 \leq i \leq p^2 + 1$.

**Proof.** Because of Lemmas 3.15, 3.16 and 3.17, the plane $\pi$ meets $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points. Furthermore, each line of $\pi$ which is not contained in $\mathcal{V}$ is an $i$-secant, with $1 \leq i \leq p^2 + 1$ (Lemma 3.15 and the sentence preceding Lemma 3.19). Set $B = \pi \cap \mathcal{V}$. If in $\pi$ there are no $(p^4 + 1)$-secants to $B$, then $|B| \leq p^6 + p^2 + 1$ and by Proposition 3.9 it follows that $B$ is a classical unital.

If there is a $(p^4 + 1)$-secant to $B$ in $\pi$, then arguing as in the proof of Proposition 3.8 we get that $B$ is still a linear combination of $m$ lines, with $m \leq p^2 + 1$. Each of these $m$ lines is a $(p^4 + 1)$-secant to $\mathcal{V}$. In fact if one of these lines, say $v$, was an $s$-secant, with $1 \leq s \leq p^2 + 1$, then through each point $P \in v \setminus B$, there would pass at least $p$ lines of the codeword corresponding to $B$ and hence $B$ would be a linear combination of at least $(p^4 + 1 - s)(p - 1) + 1 > p^2 + 1$ lines, which is impossible.

We are going to prove that these $m$ lines, say $\ell_1, \ldots, \ell_m$, are concurrent. Assume on the contrary that they are not. We can assume that through a point $P \in \ell_n$, there pass at least $p + 1$ lines of our codeword but there is a line $\ell_j$ which does not pass through $P$. Thus through at least $p + 1$ points on $\ell_j$, there are at least $p + 1$ lines of our codeword and thus we find at least $(p + 1)p + 1 > m$ lines of $B$, a contradiction. \hfill \Box
Lemma 3.19. A plane \( \pi \) of \( \text{PG}(r, p^4) \) meeting \( \mathcal{V} \) in at most \( p^6 + 2p^4 - p^2 - p + 1 \) points and containing a \((p^4 - p + 1)\)-secant to \( \mathcal{V} \) has at most \((p^2 + 1)(p^4 - p + 1)\) points.

Proof. Let \( \ell \) be a line of \( \pi \) which is a \((p^4 - p + 1)\)-secant to \( \mathcal{V} \). In this case, \( \pi \cap \mathcal{V} \) is a linear combination of at most \( p^2 + 1 \) lines, each with weight 1 (Proposition 3.8). In particular \( \ell \) is a line of the codeword and hence through each of the missing points of \( \ell \) there are at least \( p \) lines of the codeword corresponding to \( B \). On these \( p \) lines we can see at most \( p^4 - p + 1 \) points.

So let \( \ell_1, \ell_2, \ldots, \ell_p \) be \( p \) lines of the codeword through a point of \( \ell \). Each of these lines contains at most \( p^4 - p + 1 \) points of \( \mathcal{V} \). Thus these \( p \) lines contain together at most \( p(p^4 - p + 1) \) points of \( \mathcal{V} \). Now take any other line of the codeword, say \( e \). If \( e \) goes through the common point of the lines \( \ell_i \), then there is already one point missing from \( e \), so adding \( e \) to our set, we can add at most \( p^4 - p + 1 \) points. If \( e \) does not go through the common point, then it intersects \( \ell_i \) in \( p \) different points. These points either do not belong to the set \( \pi \cap \mathcal{V} \) or they belong to the set \( \pi \cap \mathcal{V} \), but we have already counted them when we counted the points of \( \ell_i \), so again \( e \) can add at most \( p^4 - p + 1 \) points to the set \( \pi \cap \mathcal{V} \). Thus adding the lines of the codewords one by one to \( \ell_i \) and counting the number of points, each time we add only at most \( p^4 - p + 1 \) points to the set \( \pi \cap \mathcal{V} \). Hence, the plane \( \pi \) contains at most \((p^2 + 1)(p^4 - p + 1)\) points of \( \mathcal{V} \).

Lemma 3.20. Let \( \pi \) be a plane of \( \text{PG}(r, p^4) \), containing an \( i \)-secant, \( 1 < i < p^2 + 1 \), to \( \mathcal{V} \). Then \( \pi \cap \mathcal{V} \) is either the union of \( i \) concurrent lines or it is a linear combination of \( p^2 + 1 \) lines (each with weight 1) so that they form a subgeometry of order \( p \), minus \( p \) concurrent lines.

Proof. By Lemmas 3.16, 3.17 and 3.18, \( \pi \) meets \( \mathcal{V} \) in at most \( p^6 + 2p^4 - p^2 - p + 1 \) points and must contain a \((p^4 - p + 1)\)-secant to \( \mathcal{V} \) or \( \pi \cap \mathcal{V} \) is the union of \( i \) concurrent lines. Hence from now on, we assume that \( \pi \) contains a \((p^4 - p + 1)\)-secant. By Result 2.8 such a plane is a linear combination of at most \( p^2 + 1 \) lines. As before each line from the linear combination has weight 1. Note that the above two statements imply that a line of the linear combination will be either a \((p^4 + 1)\)-secant or a \((p^4 - p + 1)\)-secant.

As we have at most \( p^2 + 1 \) lines, the line of a \((p^4 - p + 1)\)-secant must be one of the lines from the linear combination. This also means that through each of the \( p \) missing points of this line, there must pass at least \( p - 1 \) other lines from the linear combination. Hence, we already get \((p - 1)p + 1 \) lines.

In the case in which the linear combination contains exactly \( p^2 - p + 1 \) lines, then from each of these lines there are exactly \( p \) points missing and through each missing point there are exactly \( p \) lines from the linear combination. Hence, the
missing points and these lines form a projective plane of order $p-1$, a contradiction as $p > 3$.

Therefore, as the number of the lines of the linear combination must be $1 \pmod{p}$ and at most $p^2 + 1$, we can assume that the linear combination contains $p^2 + 1$ lines. We are going to prove that through each point of the plane there pass either 0, 1, $p$ or $p + 1$ lines from the linear combination. From earlier arguments, we know that the number of lines through one point $P$ is $0$ or $1 \pmod{p}$. Assume to the contrary that through $P$ there pass at least $p + 2$ of such lines. These $p^2 + 1$ lines forming the linear combination are not concurrent, so there is a line $\ell$ not through $P$. Through each of the intersection points of $\ell$ and a line through $P$, there pass at least $p - 1$ more other lines of the linear combination, so in total we get at least $(p - 1)(p + 2) + 1$ lines forming the linear combination, a contradiction.

Since there are $p^2 + 1$ lines forming the linear combination and through each point of the plane there pass either 0, 1, $p$ or $p + 1$ of these lines, we obtain that on a $(p^4 - p + 1)$-secant there is exactly one point, say $P$, through which there pass exactly $p + 1$ lines from the linear combination and $p$ points, not in the quasi Hermitian variety, through each of which there pass exactly $p$ lines.

If all the $p^2 + 1$ lines forming the linear combination, were $(p^4 - p + 1)$-secants then the number of points through which there pass exactly $p$ lines would be $(p^2 + 1)p/p$. On the other hand, through $P$ there pass $p + 1$ $(p^4 - p + 1)$-secants, hence we already get $(p+1)p$ such points, a contradiction. Thus, there exists a line $m$ of the linear combination that is a $(p^4 + 1)$-secant. From the above arguments, on this line there are exactly $p$ points through each of which there pass exactly $p + 1$ lines, whereas through the rest of the points of the line $m$ there pass no other lines of the linear combination.

Assume that there is a line $m' \neq m$ of the linear combination that is also a $(p^4 + 1)$-secant. Then there is a point $Q$ on $m'$ but not on $m$ through which there pass $p + 1$ lines. This would mean that there are at least $p + 1$ points on $m$, through which there pass more than 2 lines of the linear combination, a contradiction.

Hence, there is exactly one line $m$ of the linear combination that is a $(p^4 + 1)$-secant and all the other lines of the linear combination are $(p^4 - p + 1)$-secants. It is easy to check that the points through which there are more than 2 lines plus the $(p^4 - p + 1)$-secants form a dual affine plane. Hence our lemma follows.

**Lemma 3.21.** There are no $i$-secants to $\mathcal{V}$, with $1 < i < p^2 + 1$.

**Proof.** By Lemma 3.20 if a plane $\pi$ contains an $i$-secant, $1 < i < p^2 + 1$, then $\pi \cap \mathcal{V}$ is a linear combination of either $i$ concurrent lines or lines of an embedded subplane of order $p$ minus $p$ concurrent lines. In the latter case, if $i > 1$ then an $i$-secant is at least a $(p^2 - p + 1)$-secant. Hence, if there is an $i$-secant with $1 < i < p^2 - p + 1$, say $\ell$, we get that for each plane $\alpha$ through $\ell$, $\alpha \cap \mathcal{V}$ is a linear
combination of $i$ concurrent lines. Therefore

$$|\mathcal{V}| = m(ip^4 + 1 - i) + i,$$

(8)

where $m = p^{4(r-2)} + p^{4(r-3)} + \ldots + p^4 + 1$ is the number of planes in $\text{PG}(r, p^4)$ through $\ell$.

Setting $r = 2\sigma + \epsilon$, where $\epsilon = 0$ or $\epsilon = 1$ according to $r$ is even or odd, we can write

$$|\mathcal{V}| = 1 + p^4 + \ldots + p^{4(r-\sigma-1)} + (p^{4(r-\sigma-\epsilon)} + p^{4((r-\sigma-\epsilon)+1)} + \ldots + p^{4(r-1)})p^2$$

Hence, (8) becomes

$$1 + p^4 + \ldots + p^{4(r-\sigma-1)} + (p^{4(r-\sigma-\epsilon)} + p^{4((r-\sigma-\epsilon)+1)} + \ldots + p^{4(r-1)})p^2$$

$$- (p^{4(r-2)} + p^{4(r-3)} + \ldots + p^4 + 1) = ip^{4(r-1)}$$

(9)

Since $\sigma \geq 2$, we see that $p^{4(r-1)}$ does not divide the left hand side of (9), a contradiction.

Thus, there can only be $1$-, $(p^2-p+1)$-, $(p^2+1)$-, $(p^4-p+1)$- or $(p^4+1)$-secants to $\mathcal{V}$. Now, suppose that $\ell$ is a $(p^2-p+1)$-secant to $\mathcal{V}$. Again by Lemma 3.20, each plane through $\ell$ either has $x = (p^2-p+1)p^4 + 1$ or $y = p^2(p^2-p) + p^4 + 1$ points of $\mathcal{V}$. Next, denote by $t_j$ the number of $j$-secant planes through $\ell$ to $\mathcal{V}$. We get

$$\begin{cases} t_x + t_y = m \\ t_x(x - p^2 + p - 1) + t_y(y - p^2 + p - 1) + p^2 - p + 1 = |\mathcal{V}| \end{cases}$$

(10)

Recover the value of $t_y$ from the first equation and substitute it in the second. We obtain

$$(m - t_y)(p^6 - p^5 + p^4 - p^2 + p) + t_y(p^6 + p^4 - p^3 - p^2 + p) + p^2 - p + 1 = |\mathcal{V}|$$

that is,

$$p^3(p^2 - 1)t_y = |\mathcal{V}| - m(p^6 - p^5 + p^4 - p^2 + p) - p^2 - p + 1.$$

It is easy to check that $|\mathcal{V}| - m(p^6 - p^5 + p^4 - p^2 + p) - p^2 + p - 1$ is not divisible by $p + 1$ and hence, $t_y$ turns out not to be an integer, which is impossible. \qed

**Lemma 3.22.** No plane meeting $\mathcal{V}$ in at most $p^6 + 2p^4 - p^2 - p + 1$ points contains a $(p^4 - p + 1)$-secant.

**Proof.** Let $\pi$ be a plane of $\text{PG}(r, p^4)$ such that $|\pi \cap \mathcal{V}| \leq p^6 + 2p^4 - p^2 - p + 1$. It can contain only $1$-, $(p^2+1)$-, $(p^4-p+1)$-, $(p^4+1)$-secants (Lemma 3.15 and Lemma 3.21). If $\pi \cap \mathcal{V}$ contains a $(p^4-p+1)$-secant, we know from Proposition 3.18 that it
is a linear combination of at most $p^2 + 1$ lines, each with weight 1. Suppose that $e$ is a $(p^4 - p + 1)$-secant to $\pi \cap V$. Let $P$ and $Q$ be two missing points of $e$. We know that there must be at least $p - 1$ other lines of the codeword through $P$ and $Q$. Let $f$ and $g$ be two such lines through $Q$. We can find a line, say $m$, of the plane through $P$, that intersects $f$ and $g$ in a point of $V$ and that is not a line of the codeword. Thus $m \cap V \equiv 1 \pmod{p}$. Then $m$ contains at least two points of $V$, but in $P$ it meets at least $p$ lines of the codeword. Hence, $p + 1 \leq |m \cap V| \leq p^2 - p + 1$, and this contradicts Lemma 3.21.

Lemma 3.23. There are no $(p^4 - p + 1)$-secants to $V$.

Proof. If there was a $(p^4 - p + 1)$-secant to $V$, say $\ell$, then, by Lemma 3.22, all the planes through $\ell$ would contain at least $p^8 - p^5 + p^4 + 1$ points of $V$, and thus

$$|V| \geq (p^4(r-2) + p^4(r-3) + \cdots + p^4 + 1)(p^8 - p^5 + p) + p^4 - p + 1,$$

a contradiction.

Proof of Theorem 1.1 (case $r \geq 4$ and $q = p^2$): Consider a line $\ell$ which is not contained in $V$. From the preceding lemmas we have that $\ell$ is either a $1$-secant or a $(p^2 + 1)$-secant of $V$. Furthermore, $V$ has no plane section of size $(p^2 + 1)(p^4 + 1)$. Finally, as in the case $r = 3$, it is easy to see that $V$ has no singular points, thus, by Theorem 2.2, $V$ turns out to be a Hermitian variety of $\text{PG}(r, p^4)$.

4 Singular quasi-Hermitian varieties

In this section, we consider sets having the same behavior with respect to hyperplanes as singular Hermitian varieties.

Definition 4.1. A $d$-singular quasi-Hermitian variety is a subset of points of $\text{PG}(r, q^2)$ having the same number of points and the same intersection sizes with hyperplanes as a singular Hermitian variety with a singular space of dimension $d$.

We prove the following result.

Theorem 4.2. Let $S$ be a $d$-singular quasi-Hermitian variety in $\text{PG}(r, q^2)$. Suppose that either

- $r = 3$, $d = 0$, $q = p^h > 4$, $h \geq 1$, or
- $r \geq 4$, $d \leq r - 3$, $q = p > 4$, or
- $r \geq 4$, $d \leq r - 3$, $q = p^2$, $p > 3$. 

15
Then $S$ is a singular Hermitian variety with a singular space of dimension $d$ if and only if its incidence vector is in the $\mathbb{F}_p$-code spanned by the hyperplanes of $\text{PG}(r,q^2)$.

**Proof.** Let $S$ be a singular Hermitian variety of $\text{PG}(r,q^2)$. The characteristic vector $v^S$ of $S$ is in $C_p$ since [12, Theorem 1] also holds for singular Hermitian varieties. Now assume that $S$ is a $d$-singular quasi-Hermitian variety. As in the non-singular case, by Lemma 3.1 each line of $\text{PG}(r,q^2)$ intersects $S$ in $1 \pmod{p}$ points.

4.1 Case $r = 3$

Suppose that $r = 3$ and therefore $d = 0$. Let $\pi$ be a plane of $\text{PG}(3,q^2)$. In this case $\pi$ meets $S$ in either $q^2 + 1$, or $q^3 + 1$ or $q^3 + q^2 + 1$ points. Therefore Lemma 3.6, Lemma 3.7, Proposition 3.8, and Corollary 3.10 are still valid in the singular case for $r = 3$.

Thus, if $|\pi \cap S| = q^2 + 1$, then Proposition 3.8 implies that $\pi \cap S$ is a line of $\pi$, whereas if $|\pi \cap S| = q^3 + 1$, then Corollary 3.10 gives that $\pi \cap S$ is a classical unital of $\pi$. Now suppose that $|\pi \cap S| = s$ with $s \neq 1, q + 1, q^2 + 1$.

Each plane through $\ell$ must meet $S$ in $q^3 + q^2 + 1$ points and this gives

$$(q^3 + q^2 + 1 - s)(q^2 + 1) + s = q^5 + q^2 + 1$$

that is, $s = q^2 + q + 1$ which is impossible.

Thus each line of $\text{PG}(3,q^2)$ intersects $S$ in either $1$ or, $q + 1$ or, $q^2 + 1$ points and hence $S$ is a $k_{q+1,3,q^2}$. Also, $S$ cannot be non-singular by assumption, hence Theorem 2.3 applies and $S$ turns out to be a cone $\Pi_0 S'$ with $S'$ of type I, II, III or IV as the possible intersection sizes with planes are $q^2 + 1, q^3 + 1, q^3 + q^2 + 1$.

Possibilities II, III, and IV must be excluded, since their sizes cannot be possible. This implies that $S = \Pi_0 \mathcal{H}$, where $\mathcal{H}$ is a non-singular Hermitian curve.

4.2 Case $r \geq 4$

Let $\ell$ be a line of $\text{PG}(r,q^2)$ containing $x < q^2 + 1$ points of $S$. We are going to prove that there exists at least one plane through $\ell$ containing less than $q^3 + q^2 + q + 1$ points of $S$. If we suppose that all the planes through $\ell$ contain at least $q^3 + q^2 + q + 1$ points of $S$, then

$$q^{2(d+1)} \frac{(q^{r-d} - 1^{r-d-1})(q^{r-d-1} - (-1)^{r-d-1})}{q^2 - 1} + q^{2d} + q^{2(d-1)} + \cdots + q^2 + 1 \geq m(q^3 + q^2 + q + 1 - x) + x,$$

16
where $m = q^{2(r-2)} + q^{2(r-3)} + \cdots + q^2 + 1$ is the number of planes through $\ell$ in $\text{PG}(r, q^2)$. We obtain $x > q^2 + 1$, a contradiction.

Therefore, there exists at least one plane through $\ell$ having less than $q^3 + q^2 + q + 1$ points of $S$ and hence Lemma 3.6, Lemma 3.7, Proposition 3.8, and Corollary 3.10 are still valid in this singular case for any $q > 4$.

Next, we are going to prove that $S$ is a $k_{q+1,r,q^2}$, with $q = p^h > 4$, $h = 1, 2$.

**case $q = p$:** Let $\ell$ be a line of $\text{PG}(r, p^2)$. As we have seen there is a plane $\pi$ through $\ell$ such that $|\pi \cap V| \leq p^3 + p^2 + p + 1$. Proposition 3.14 is still valid in this case and thus we have that $\ell$ is either a unisecant or a $(p + 1)$-secant of $S$. Furthermore, we also have that $S$ has no plane section of size $(p + 1)(p^2 + 1)$ and hence $S$ is a regular $k_{p+1,r,p^2}$.

**case $q = p^2$:** We first observe that (8) and (11) hold true in the case in which $V$ is assumed to be a singular quasi-Hermitian variety. This implies that all lemmas stated in the subparagraph 3.3 are valid in our case. Thus, we obtain that $S$ is a $k_{p^2+1,r,p^4}$ and it is straightforward to check that $S$ is also regular.

Finally, in both cases $q = p$ or $q = p^2$, we have that $S$ is a singular $k_{q+1,r,q^2}$ because if $S$ were a non-singular $k_{q+1,r,q^2}$ then, from Theorem 2.2, $S$ would be a non-singular Hermitian variety and this is not possible by our assumptions.

Therefore, by Theorem 2.4 the only possibility is that $S$ is a cone $\Pi_d S'$, with $S'$ a non-singular $k_{q+1,r-d-1,q^2}$. By Lemma 3.3, $S'$ belongs to the code of points and hyperplanes of $\text{PG}(r - d - 1, q^2)$. Since $r - d - 1 \geq 2$, then, by [6] and Theorem 1.1, $S'$ is a non-singular Hermitian variety and, therefore, $S$ is a singular Hermitian variety with a vertex of dimension $d$.

**Acknowledgement.** The research of the second author was partially supported by Ministry for Education, University and Research of Italy (MIUR) (Project PRIN 2012 “Geometrie di Galois e strutture di incidenza” - Prot. N. 2012XZE22K_005) and by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM). The third and fourth author acknowledge the financial support of the Fund for Scientific Research - Flanders (FWO) and the Hungarian Academy of Sciences (HAS) project: Substructures of projective spaces (VS.073.16N).

**References**


