Maximum scattered $\mathbb{F}_q$-linear sets of $\text{PG}(1, q^4)$

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Abstract

There are two known families of maximum scattered $\mathbb{F}_q$-linear sets in $\text{PG}(1, q^t)$: the linear sets of pseudoregulus type and for $t \geq 4$ the scattered linear sets found by Lunardon and Polverino. For $t = 4$ we show that these are the only maximum scattered $\mathbb{F}_q$-linear sets and we describe the orbits of these linear sets under the groups $\text{PGL}(2, q^4)$ and $\text{PΓL}(2, q^4)$.

1 Introduction

Recent investigations on linear sets in a finite projective line $\text{PG}(1, q^t)$ of rank $t$ concerned: the hypersurface obtained from the linear sets of pseudoregulus type by applying field reduction [12]; a geometric characterization of the linear sets of pseudoregulus type [9]; a characterization of the clubs, that is, the linear sets of rank $r$ with a point of weight $r - 1$ [13]; a generalization of clubs in order to construct KM-arcs [10]; a condition for the equivalence of two linear sets [8, 18]; the definition and study of the class of a linear set in order to study their equivalence [7]; a construction method which yields MRD-codes from maximum scattered linear sets of $\text{PG}(1, q^t)$ [17]. Furthermore, the linear sets in $\text{PG}(1, q^t)$ coincide with the so-called splashes of subgeometries [13]. The results of such investigations make it reasonable to attempt to classify the linear sets in $\text{PG}(1, q^t)$ of rank $t$ for small $t$.

A point in $\text{PG}(1, q^t)$ is the $\mathbb{F}_{q^t}$-span $\langle \mathbf{v} \rangle_{\mathbb{F}_{q^t}}$ of a nonzero vector $\mathbf{v}$ in a two-dimensional vector space, say $W$, over $\mathbb{F}_{q^t}$. If $U$ is a subspace over $\mathbb{F}_q$ of

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If \(b\) is not true if \(t > \frac{q^r - 1}{q - 1}\), respectively, then \(L_U\) is scattered. Equivalently, \(L_U\) is scattered if and only if all its points have weight one. A scattered \(\mathbb{F}_q\)-linear set of rank \(t\) in \(\text{PG}(1, q^4)\) is maximum scattered. An example of maximum scattered \(\mathbb{F}_q\)-linear set in \(\text{PG}(1, q^4)\) is \(L_V\) with \(V = \{(u, u^q) : u \in \mathbb{F}_{q^4}\}\). Any subset of \(\text{PG}(1, q^4)\) projectively equivalent to this \(L_V\) is called linear set of pseudoregulus type. See [9] for a geometric description, and [7] or the survey [16] for further background on linear sets. Note that for any \(\varphi \in \Gamma L(2, q^4)\) with related collineation \(\tilde{\varphi} \in \text{PGL}(2, q^4)\) and any \(\mathbb{F}_q\)-linear set \(L_U\), \(L_U\varphi = (L_U)^{\tilde{\varphi}}\). In [7, Theorem 4.5] it is proved that if \(t = 4\) and \(L_U\) has maximum field of linearity \(\mathbb{F}_q\), that is, \(L_U\) is not an \(\mathbb{F}_q\)-linear set for \(s > 1\), then any linear set in the same orbit of \(L_U\) under the action of \(\text{PGL}(2, q^4)\) is of type \(L_{U\varphi}\) with \(\varphi \in \Gamma L(2, q^4)\). Note that this is not true if \(t > 4\). In [14], Lunardon and Polverino construct a class of maximum scattered linear sets:

**Theorem 1.1 ([14]).** Let \(q\) be a prime power, \(t \geq 4\) an integer, \(b \in \mathbb{F}_{q^4}\) such that the norm \(N_{q^4/q}(b)\) of \(b\) over \(\mathbb{F}_q\) is distinct from one, and

\[
U(b, t) = \{(u, bu^q + u^{q^t - 1}) : u \in \mathbb{F}_{q^4}\}. \tag{1}
\]

If \(b \neq 0\) then \(L_{U(b, t)}\) is a maximum scattered \(\mathbb{F}_q\)-linear set in \(\text{PG}(1, q^4)\) and if \(q > 3\), then it is not of pseudoregulus type.

It can be directly seen that \(L_{U(0, t)}\) is maximum scattered of pseudoregulus type. For \(t = 4\), Theorem [14] can be extended to \(q = 3\), as it can be checked by using the package \text{FinInG} of \text{GAP} [3]. In the following \(t = 4\) is assumed. For all \(b \in \mathbb{F}_{q^4}\), define

\[
U(b) = U(b, 4) = \{(x, bx^q + x^{q^3}) : x \in \mathbb{F}_{q^4}\}. \tag{2}
\]

In section [2] it is shown that \(N_{q^4/q}(b) \neq 1\) is a necessary condition to obtain scattered linear sets of \(\text{PG}(1, q^4)\) and the case \(N_{q^4/q}(b) = 1\) is dealt with. In this case, \(L_{U(b)}\) contains either one or \(q + 1\) points of weight two, and the remaining points have weight one.

The main result in section [3] is that if \(L\) is a maximum scattered linear set in \(\text{PG}(1, q^4)\), then \(L\) is projectively equivalent to \(L_{U(b)}\) for some \(b \in \mathbb{F}_{q^4}\) with \(N_{q^4/q}(b) \neq 1\) (cf. Theorem [3, 4]).
In section 4 the orbits of the $\mathbb{F}_q$-linear sets of rank four in $\text{PG}(1,q^4)$ of type $L_{U(b)}$, under the actions of both $\text{PGL}(2,q^4)$ and $\text{PFl}(2,q^4)$, are completely characterized. Such orbits only depend on the norm $b^{q^2+1}$ of $b$ over $\mathbb{F}_q^2$. In particular, $\text{PG}(1,q^4)$ contains precisely $q(q-1)/2$ maximum scattered linear sets up to projective equivalence (Theorem 4.5), one of them is of pseudoregulus type, the others are as in Theorem 1.1.

2 Classification

This section is devoted to the classification of all $L_{U(b)}$ for $b \in \mathbb{F}_q^4$, where $U(b)$ is as in (2).

Theorem 2.1. For $b \in \mathbb{F}_q^4$ the following holds.

1. If $N_{q^4/q}(b) \neq 1$, then $L_{U(b)}$ is scattered.

2. If $N_{q^4/q}(b) = 1$, then $L_{U(b)}$ has a unique point with weight two, the point $\langle (1,0) \rangle_{\mathbb{F}_q^4}$, and all other with weight one.

3. If $N_{q^4/q}(b) \neq 1$ and $N_{q^4/q}(b) = 1$, then $L_{U(b)}$ has $q + 1$ points with weight two and all other with weight one.

Proof. Put $f_b(x) = bx^q + x^q$. For $x \in \mathbb{F}_q^4$ the point $P_x := \langle (x, f_b(x)) \rangle_{\mathbb{F}_q^4}$ of $L_{U(b)}$ has weight more than one if and only if there exists $y \in \mathbb{F}_q^4$ and $\lambda \in \mathbb{F}_q^4 \setminus \mathbb{F}_q$ such that $\lambda x, f_b(x) = (y, f_b(y))$. This holds if and only if

$$\lambda bx^q + \lambda x^q - \lambda^q bx^q - \lambda^q x^q = 0.$$  \hspace{1cm} (3)

For a given $x$ the solutions in $\lambda$ of (3) form an $\mathbb{F}_q$-subspace whom rank equals to the weight of the point $P_x$. Since $q$-polynomials over $\mathbb{F}_q^4$ of rank 1 are of the form $\alpha \text{Tr}_{q^4/q}(\beta x) \in \mathbb{F}_q^4[x]$, it is clear that the kernel of the $\mathbb{F}_q^4$-linear map in the variable $\lambda$ at the left-hand side of (3) has dimension at most two and hence the weight of each point of $L_{U(b)}$ is at most two. If $(\lambda, x)$ is a solution of (3) for some $\lambda \in \mathbb{F}_q^4$ and $x \in \mathbb{F}_q^4$, then $(\lambda', x')$ is also a solution for each $\lambda' \in \langle 1, \lambda \rangle_{\mathbb{F}_q^4}$ and $x' \in \langle x \rangle_{\mathbb{F}_q^4}$ and hence for each $\mu \in \mathbb{F}_q^4$ if $P_x$ has weight two, then $P_{\mu x} := \langle (\mu x, f_b(\mu x)) \rangle_{\mathbb{F}_q^4}$ has weight two as well. Note that $P_{\mu x} = \langle (1, \mu^{q^2-1}(bx^{q-1} + x^{q^2-1})) \rangle_{\mathbb{F}_q^4}$ and hence if $P_x \neq \langle (1,0) \rangle_{\mathbb{F}_q^4}$ has weight two, then $\{P_{\mu x} : \mu \in \mathbb{F}_q^4\}$ is a set of $q + 1$ distinct points with weight 2.

The function $f_b(x)$ is not $\mathbb{F}_q^2$-linear and hence the maximum field of linearity of $L_{U(b)}$ is $\mathbb{F}_q$. It follows (cf. [7, Proposition 2.2]) that $L_{U(b)}$ has
at least one point with weight one, say \((\langle x_0, f_b(x_0) \rangle)_{\mathbb{F}_q^4}\). Then the line of \(AG(2, q^4)\) with equation \(x_0Y = f_b(x_0)X\) meets the graph of \(f_b(x)\), that is, \(\{(x, f_b(x)): x \in \mathbb{F}_{q^4}\}\), in exactly \(q\) points. It follows from [1, 2], see also [3], that the number of directions determined by \(f_b(x)\) is at least \(q^3 + 1\), and hence also \(|L_{U(b)}| \geq q^3 + 1\). Denote by \(w_1\) and \(w_2\) the number of points of \(L_{U(b)}\) with weight one and two, respectively. Then

\[
w_1 + w_2 = |L_{U(b)}| \geq q^3 + 1, \tag{4}
\]

\[
w_1(q - 1) + w_2(q^2 - 1) = q^4 - 1. \tag{5}
\]

Subtracting (4) \((q - 1)\)-times from (5) gives \(w_2(q^2 - q) \leq q^3 - q\) and hence \(w_2 \leq q + 1\). At this point it is clear that in \(L_{U(b)}\) there is either one point with weight two, the point \(\langle (1, 0) \rangle_{\mathbb{F}_q^4}\), or there are exactly \(q + 1\) of them and \(\langle (1, 0) \rangle_{\mathbb{F}_q^4}\) is not one of them.

If \(N_{q^4/q}(b) \neq 1\), then Theorem 1.1 states that \(L_{U(b)}\) is scattered. We show that \(\langle (1, 0) \rangle_{\mathbb{F}_{q^4}}\) has weight two if and only if \(N_{q^4/q^2}(b) = 1\). Note that the weight of this point is the dimension of the kernel of \(f_b(x)\). If \(f_b(x) = 0\) for some \(x \in \mathbb{F}_{q^4}^*\), then \(b = -x^{q^4-q}\) and hence, by taking \((q^2 + 1)\)-th powers at both sides, \(N_{q^4/q^2}(b) = 1\). On the other hand, if \(N_{q^4/q^2}(b) = 1\), then \(b = w^{q^2-1}\) for some \(w \in \mathbb{F}_{q^4}^*\). Let \(\varepsilon\) be a non-zero element of \(\mathbb{F}_{q^4}\) such that \(\varepsilon^{q^2} + \varepsilon = 0\). Then it is easy to check that the kernel of \(f_b(x)\) is \(\langle (\varepsilon w)^{q^3} \rangle_{\mathbb{F}_{q^2}}\) which has dimension two over \(\mathbb{F}_q\) and hence \(\langle (1, 0) \rangle_{\mathbb{F}_{q^4}}\) has weight two.

It remains to prove that if \(N_{q^4/q}(b) = 1\) and \(N_{q^4/q^2}(b) \neq 1\), then there is at least one point (hence precisely \(q + 1\) points) of weight two. After rearranging in (3), we obtain

\[
(\lambda - \lambda^q)q^3 - 1 = bx^{q^2-q^3}. \tag{6}
\]

By taking \((q^2 + 1)\)-th powers on both sides we can eliminate \(x\), obtaining

\[
(\lambda - \lambda^q)(q^3q^{q-1}q^{q-1}) = (\lambda - \lambda^q)(q^3q^{q-1}q^{q-1}) = b^{q^2+1}. \tag{7}
\]

It is clear that we can find \(\lambda \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q\) satisfying (7) if and only if there exists \(\varepsilon \in \mathbb{F}_{q^4}^*\) such that

\[
(\lambda - \varepsilon^q)^{q^2-1}/b = \varepsilon^{q^2-1}. \tag{8}
\]

Then \(x \in \langle \varepsilon^q \rangle_{\mathbb{F}_{q^2}}\) with \(y = \lambda x\) satisfies our initial conditions in (3).
Now use $N_{q^4/q}(b) = 1$ and put $b = \mu q^{-1}$ for some $\mu \in F_{q^4}^*$. Then (7) can be written as
\[
\left(\frac{\lambda - \lambda q}{\mu}\right)^{(q-1)(q^2+1)} = 1.
\] (9)

We can solve (9) if and only if there exists $\delta \in F_{q^4}^*$ such that
\[
\left(\frac{\lambda - \lambda q}{\mu}\right)^{q-1} = \delta^{q^2-1},
\] (10)
or, equivalently,
\[
\left\langle \frac{\lambda - \lambda q}{\mu} \right\rangle_F = \langle \delta^{q^2+1} \rangle_F.
\] (11)

Now we will continue in PG($F_{q^4}, F_q$) = PG($3, q$). At the left-hand side of (11) we can see a point of the hyperplane $H_\mu$ defined as
\[
H_\mu = \{\langle z \rangle_F : \text{Tr}_{q^4/q}(\mu z) = 0\},
\]
while on the right-hand side we can see a point of the elliptic quadric $Q$ defined as
\[
Q = \{\langle z \rangle_F : z^{(q-1)(q^2+1)} = 1\}.
\]

For a proof that $Q$ is an elliptic quadric see [5, Theorem 3.2]. Since $Q \cap H_\mu \neq \emptyset$ it follows that we can always find $\lambda \in F_{q^4} \setminus F_q$ satisfying (10) and hence $LU(b)$ is not scattered.

\[\square\]

Remark 2.2. The linear sets in Theorem 2.1 are of sizes $q^3 + q^2 + q + 1$, $q^3 + q^2 + 1$, or $q^3 + 1$. The linear set associated with $\{(x, \text{Tr}_{q^4/q}(x)) : x \in F_{q^4}\}$ is of size $q^3 + 1$ as well. As it turns out from [2] the projective line PG($1, q^4$) also contains $F_q$-linear sets of size $q^3 + q^2 - q + 1$.

3 The canonical form

In this section $L$ denotes a maximum scattered $F_q$-linear set in PG($1, q^4$), not of pseudoregulus type. In particular, this implies $q > 2$. By [15], $L$ is a projection $p_L(\Sigma)$, where the vertex $\ell$ is a line and $\Sigma$ is a $q$-order canonical subgeometry\footnote{Let PG($V, F_q$) = PG($n-1, q^4$), let $U$ be an $n$-dimensional $F_q$-vector subspace of $V$, and $\Sigma = \{(u)_{F_q} : u \in U \setminus \{0\}\}$. If $\langle \Sigma \rangle = \text{PG}(n-1, q^4)$, then $\Sigma$ is a ($q$-order) canonical subgeometry of PG($n-1, q^4$). Here and in the following, angle brackets (−) without a subscript denote projective span in PG($n-1, q^4$), that is, PG($3, q^4$) in our case.} in PG($3, q^4$), with $\ell \cap \Sigma = \emptyset$. The axis of the projection
is immaterial and can be chosen by convenience. Let $\sigma$ be a generator of the subgroup of order four of $\text{PGL}(4, q^4)$ fixing pointwise $\Sigma$. Let $M$ be a $k$-dimensional subspace of $\text{PG}(3, q^4)$. We say that $M$ is a subspace of $\Sigma$ if $M \cap \Sigma$ is a $k$-dimensional subspace of $\Sigma$, which happens exactly when $M^\sigma = M$.

**Proposition 3.1.** Let $\Sigma'$ be the unique $q^2$-order canonical subgeometry of $\text{PG}(3, q^4)$ containing $\Sigma$, that is, the set of all points fixed by $\sigma^2$. Then the intersection of $\ell$ and $\Sigma'$ is empty.

**Proof.** Assume the contrary, that is, there exists a point $P$ in $\ell \cap \Sigma'$. Then $P^{\sigma^2} = P$, the subspace $\ell_P = \langle P, P^{\sigma^2} \rangle$ is a line, and satisfies $\ell_P^{\sigma^2} = \ell_P$, whence $\ell_P$ is a line of $\Sigma$. This implies that $p_\ell(\ell_P)$ is a point, and $L$ is not scattered.

Let $K$ and $K'$ be the Klein quadrics representing – via the Plücker embedding $\wp$ – the lines of $\Sigma$ and $\Sigma'$. In order to precisely define $\wp$, take coordinates in $\text{PG}(3, q^4)$ such that $\Sigma$ (resp. $\Sigma'$) is the set of all points with coordinates rational over $\mathbb{F}_q$ (resp. $\mathbb{F}_{q^2}$), and define the image $\wp$ of any line $\Gamma$ through minors of order two in the usual way. Then $K = K' \cap \text{PG}(5, q)$ by considering $\text{PG}(5, q)$ as a subset of $\text{PG}(5, q^2)$. The only nontrivial element of the subgroup of $\text{PGL}(6, q^2)$ fixing $\text{PG}(5, q)$ pointwise is

\begin{equation}
\tau : ( (x_0, x_1, x_2, x_3, x_4, x_5) )_{\mathbb{F}_{q^2}} \mapsto ( (x_0^q, x_1^q, x_2^q, x_3^q, x_4^q, x_5^q) )_{\mathbb{F}_{q^2}}.
\end{equation}

Then $K_2 = K$, and $\sigma^2 = \wp \tau$.

**Proposition 3.2.** Let $S$ be a solid in $\text{PG}(5, q^2)$ such that (i) $S \cap K' \cong Q^{-}(3, q^2)$, (ii) $S \cap \mathcal{K} = \emptyset$. Then $S \cap S^\tau \cap K'$ is a set of two distinct points forming an orbit of $\tau$.

**Proof.** If $\dim(S \cap S^\tau) \geq 2$, then $S \cap S^\tau$ contains a plane of $\text{PG}(5, q)$. Each plane of $\text{PG}(5, q)$ meets $\mathcal{K}$ in at least one point of $\text{PG}(5, q)$, contradicting (ii). Then $r = S \cap S^\tau$ is a line fixed by $\tau$, so it is a line of $\text{PG}(5, q)$. This $r$ is external to the Klein quadric $\mathcal{K}$ by (ii), hence it meets $K'$ in two points. Since both of $K'$ and $r$ are fixed by $\tau$ the assertion follows.

**Proposition 3.3.** There is a line $r$ in $\text{PG}(3, q^4)$, such that $r$ and $r^\sigma$ are skew lines both meeting $\ell$, and $r^{\sigma^2} = r$.

**Proof.** Let $\Sigma$ and $\Sigma'$ be as in Proposition 3.1. Since $\ell \cap \Sigma' = \emptyset$, $\ell$ defines a regular (Desarguesian) spread $\mathcal{F}$ of $\Sigma'$. The lines of $\mathcal{F}$ are all lines $\langle P, P^{\sigma^2} \rangle \cap \Sigma'$ where $P \in \ell$. The image $\mathcal{F}^\wp$ under the Plücker embedding of $\mathcal{F}$ is an
elliptic quadric \( S \cap K' \cong Q^-(3, q^2) \) in \( \text{PG}(5, q^2) \), \( S \) a solid. Since \( \mathbb{L} \) is scattered, there is no line of \( \mathcal{F} \) fixed by \( \sigma \), whence \( S \cap K = \emptyset \). Then the assertion follows from Proposition 3.2.

**Theorem 3.4.** Any maximum scattered linear \( \mathbb{F}_q \)-linear set in \( \text{PG}(1, q^4) \) is projectively equivalent to \( L_{U(b)} \) for some \( b \in \mathbb{F}_q^* \), \( N_{q^4/q}(b) \neq 1 \).

**Proof.** The set \( L_{U(0)} \) is a linear set of pseudoregulus type. Now assume that \( L = p_\ell(\Sigma) \) is maximum scattered, not of pseudoregulus type. Coordinates \( X_0, X_1, X_2, X_3 \) in \( \text{PG}(3, q^4) \) can be chosen such that

\[
\Sigma = \{ (u, u^q, u^{q^2}, u^{q^3}) : u \in \mathbb{F}_q^* \},
\]

and a generator of the subgroup of \( \text{PGL}(4, q^4) \) fixing \( \Sigma \) pointwise is

\[
\sigma : \langle (x_0, x_1, x_2, x_3) \rangle_{\mathbb{F}_q^4} \mapsto \langle (x_0^q, x_1^q, x_2^q) \rangle_{\mathbb{F}_q^4}.
\]

Define \( C = \ell \cap r \), where \( r \) is as in Proposition 3.3. The points \( C \) and \( C^{q^2} \) lie on \( r \), as well as the points \( C^\sigma \) and \( C^{q\sigma} \) lie on \( r^\sigma \). By Proposition 3.1, \( C \neq C^{q^2} \) and \( C^\sigma \neq C^{q^2} \). This implies \( \ell \subset \langle C, C^\sigma, C^{q\sigma} \rangle \), and \( \langle C, C^\sigma, C^{q^2}, C^{q^3} \rangle = \text{PG}(3, q^4) \). Since the stabilizer of \( \Sigma \) in \( \text{PGL}(4, q^4) \) acts transitively on the points \( C \) of \( \text{PG}(3, q^4) \) such that \( \langle C, C^\sigma, C^{q^2}, C^{q^3} \rangle = \text{PG}(3, q^4) \) [3, Proposition 3.1], it may be assumed that \( C = \langle (0, 0, 1, 0) \rangle_{\mathbb{F}_q^4} \), whence

\[
\ell = \langle (0, 0, 1, 0), (0, a, 0, -b) \rangle_{\mathbb{F}_q^4},
\]

for some \( a, b \in \mathbb{F}_q^* \), not both of them zero. If \( a = 0 \), then \( \mathbb{L} \) is of pseudoregulus type [3, Theorem 2.3], so \( a = 1 \) may be assumed. For any point \( P_u = \langle (u, u^q, u^{q^2}, u^{q^3}) \rangle_{\mathbb{F}_q^4} \) in \( \Sigma \), the plane containing \( \ell \) and \( P_u \) has coordinates \( [u^3 + bu^q, -bu, 0, -u] \), and this leads to the desired form for the coordinates of \( \mathbb{L} \). \( \square 

**4 Orbits**

Analogously to the definition of the \( \Gamma L \)-class of linear sets (cf. Definition 2.4 in [7]) we define the \( GL \)-class, which will be needed to study \( \text{PGL}(2, q^4) \)-equivalence. Note that for any scattered \( \mathbb{F}_q \)-linear set the maximum field of linearity is \( \mathbb{F}_q \).

**Definition 4.1.** Let \( L_U \) be an \( \mathbb{F}_q \)-linear set of \( \text{PG}(1, q^4) \) of rank \( t \) with maximum field of linearity \( \mathbb{F}_q \). We say that \( L_U \) is of \( \Gamma L \)-class \( s \) (resp.
GL-class $s$ if $s$ is the largest integer such that there exist $\mathbb{F}_q$-subspaces $U_1$, $U_2$, $\ldots$, $U_s$ of $\mathbb{F}_q^2$ with $L_{U_i} = L_U$ for $i \in \{1, 2, \ldots, s\}$ and there is no $\varphi \in \Gamma L(2, q^4)$ such that $U_i = U_j^q$ for each $i \neq j$, $i, j \in \{1, 2, \ldots, s\}$.

The first part of the following result is [7, Theorem 4.5], while the second part follows from its proof. We briefly summarize the main steps of the proof from [7].

**Theorem 4.2.** [7, Theorem 4.5] Each $\mathbb{F}_q$-linear set of rank four in $\text{PG}(1, q^4)$, with maximum field of linearity $\mathbb{F}_q$, is of $GL$-class one. More precisely, if $L_U = L_V$ for some 4 dimensional $\mathbb{F}_q$-subspaces $U$, $V$ of $\mathbb{F}_q^2$, then there exists $\varphi \in \Gamma L(2, q^4)$ such that $U^\varphi = V$. Also, $\varphi$ can be chosen such that it has companion automorphism either the identity, or $x \mapsto x^{q^2}$.

**Proof.** Assume $L_U = L_V$. We may assume $\langle (0, 1) \rangle_{\mathbb{F}_q} \notin L_U$. Then $U = U_f = \{(x, f(x)) : x \in \mathbb{F}_q^2\}$ and $V = V_g = \{(x, g(x)) : x \in \mathbb{F}_q^4\}$ for some $q$-polynomials $f$ and $g$ over $\mathbb{F}_q^2$. By [7, Proposition 4.2], either $g(x) = f(\lambda x)/\lambda$, or $g(x) = \hat{f}(\lambda x)/\lambda$ for some $\lambda \in \mathbb{F}_q^*$, where here $\hat{f}$ denotes the adjoint map of $f$ with respect to the bilinear form $\langle x, y \rangle := \text{Tr}_{q^4/q}(xy)$. The $\mathbb{F}_q^2$-linear map $v \mapsto \lambda v$ maps $U_g$ to one of $U_f$, or $U_f$. In the proof of [7, Theorem 4.5], a $\kappa \in \Gamma L(2, q^4)$ with companion automorphism the identity, or $x \mapsto x^{q^2}$ is determined such that $U_f^\kappa = U_f$. \hfill $\square$

**Theorem 4.3.** For any $b \in \mathbb{F}_q^4$, $L_{U(b)}$ is of GL-class one.

**Proof.** By Theorem 4.2 if $L_{U(b)} = L_V$, then there exists $\varphi \in \Gamma L(2, q^4)$ such that $U(b)^\varphi = V$ and the companion automorphism of $\varphi$ is $x \mapsto x^{q^2}$, or the identity. In order to prove the statement it is enough to show that $U(b)$ and $U(b)^{q^2} = \{(x^{q^2}, y^{q^2}) : (x, y) \in U(b)\}$ lie on the same orbit of $\Gamma L(2, q^4)$. If $b = 0$, then $U(b) = U(b)^{q^2}$. If $b \neq 0$, then for any $u \in \mathbb{F}_q^4$,

$$
\begin{pmatrix}
 b^{q^3} & 0 \\
 0 & b^{q^2}
\end{pmatrix}
\begin{pmatrix}
 u \\
 bu^{q^3} + u^{q^3}
\end{pmatrix}
= 
\begin{pmatrix}
 b^{q^3} & 0 \\
 0 & b^{q^2}
\end{pmatrix}
\begin{pmatrix}
 b^{q^3} & 0 \\
 0 & b^{q^2}
\end{pmatrix}
^{q^2} = 
\begin{pmatrix}
 v \\
 bv^{q^3} + v^{q^3}
\end{pmatrix}
^{q^2},
$$

with $v = b^{q^3}u^{q^2}$.

\hfill $\square$

**Corollary 4.4.** Let $b, c \in \mathbb{F}_q^4$. The linear sets $L_{U(b)}$ and $L_{U(c)}$ are projectively equivalent if and only if $U(b)$ and $U(c)$ are in the same orbit under the action of $GL(2, q^4)$.\hfill $\square$
Proof. The “if” part is obvious, so assume that $L^\kappa_U(b) = L_U(c)$ where $\kappa \in \text{GL}(2, q^4)$. Then $L_U(\kappa) = L_U(c)$ and by Theorem 4.3 there is $\kappa' \in \text{GL}(2, q^4)$ such that $U(b)\kappa' = U(c)$.

It follows that in order to classify the $\mathbb{F}_q$-linear sets $L_U(b)$ up to $\text{PGL}(2, q^4)$ and $\text{PTL}(2, q^4)$-equivalence, it is enough to determine the orbits of the subspaces $U(b)$ under the actions of $\text{GL}(2, q^4)$ and $\text{PGL}(2, q^4)$.

Theorem 4.5. Let $q$ be a power of a prime $p$.

(i) For any $b, c \in \mathbb{F}_{q^4}$, $L_U(b)$ and $L_U(c)$ are equivalent up to an element of $\text{PTL}(2, q^4)$ if and only if $c^{q^2 + 1} = b^{q^2 + 1}$ for some integer $s \geq 0$.

(ii) For any $b, c \in \mathbb{F}_{q^4}$, the linear sets $L_U(b)$ and $L_U(c)$ are projectively equivalent if and only if $c^{q^2 + 1} = b^{q^2 + 1}$ or $c^{q^2 + 1} = b^{-q(q^2 + 1)}$.

(iii) All linear sets described in 2. of Theorem 2.1 are projectively equivalent.

(iv) There are precisely $q(q-1)/2$ distinct linear sets up to projective equivalence in the family described in 1. of Theorem 2.1 and these are the only maximum scattered linear sets of $\text{PG}(1, q^4)$.

(v) There are precisely $q$ distinct linear sets up to projective equivalence in the family described in 3. of Theorem 2.1.

Proof. Take $b \in \mathbb{F}_{q^4}^*$. If $L_U(b)$ is not scattered, then it clearly cannot be equivalent to $L_U(0)$ (the scattered linear set of pseudoregulus type), while if $L_U(b)$ is scattered, then it follows from Theorem 1.1 (and from a computer search when $q = 3$) that $U(b)$ and $U(0)$ yield projectively inequivalent linear sets. Since the automorphic collineations $(x, y) \mapsto (x^{p^k}, y^{p^k})$ fix $U(0)$, it also follows that $L_U(0)$ and $L_U(b)$ lie on different orbits of $\text{PTL}(2, q^4)$. Thus (i) and (ii) are true when one of $b$ or $c$ is zero, so from now on we may assume $b \neq 0$ and $c \neq 0$.

The sets $L_U(b)$ and $L_U(c)$ are equivalent up to elements of $\text{PTL}(2, q^4)$ if and only for some $\psi = p^k$, $k \in \mathbb{N}$ and some $A, B, C, D \in \mathbb{F}_{q^4}$ such that $AD - BC \neq 0$ the following holds:

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u^\psi \\ u^{\psi q} + u^{\psi q^3} \end{pmatrix} : u \in \mathbb{F}_{q^4} \right\} = \left\{ \begin{pmatrix} v \\ cv^{q} + v^{q^3} \end{pmatrix} : v \in \mathbb{F}_{q^4} \right\}. \quad (15)$$

Furthermore, by Corollary 4.4, $L_U(b)$ and $L_U(c)$ are projectively equivalent if, and only if, (15) has a solution with $\psi = 1$. This leads to a polynomial in
$u^\psi$ of degree at most $q^3$ which is identically zero. Equating its coefficients to zero,
\[
\begin{align*}
Aq^3 - D &= 0 \\
Bq^4c + Bq^3 &= 0 \\
A^3c - Db^\psi &= 0 \\
Bq^3c + Bq^3b^\psi - C &= 0.
\end{align*}
\]

(16)

Assume that $L_{U(b)}$ and $L_{U(c)}$ are in the same orbit of $\text{PGL}(2, q^4)$, and take $\psi = 1$ in case they are also projectively equivalent. If $D \neq 0$, then the first and third equations imply $b^\psi = Dq^2 - C$ and so $c^d + 1 = b^\psi(q^2+1)$.

If $D = 0$, then $BC \neq 0$; from the second equation, $(b^\psi q)q^2 + 1 = 1$, hence $c^d + 1 = b^{-\psi q(q^2+1)}$. This proves the only if parts of (i) and (ii).

Conversely, if $c^d + 1 = b^\psi(q^2+1)$ for some $s \in \mathbb{N}$, then $b^\psi c^{-1} = \delta q^2 - 1$ for some $\delta \in \mathbb{F}^*_q$. The quadruple $A = \delta q$, $B = C = 0$, $D = \delta$ with $\psi = p^s$ is a solution of (16) with $AD - BC \neq 0$. This proves the if part of (i) when $c^d + 1 =$ $b^\psi(q^2+1)$ and the if part of (ii) when $c^d + 1 =$ $b^q + 1$. If $b^q + 1 = c^d + 1 = 1$, i.e. when $U(b)$ and $U(c)$ define linear sets described in 2. of Theorem 2.1, then the above condition holds, thus (iii) follows. From now on we may assume $b^q + 1 \neq 1$ and $c^d + 1 \neq 1$.

Assume $c^d + 1 = b^{-\psi q(q^2+1)}$ for some $s \in \mathbb{N}$, i.e. $b^\psi c^{-1} = \varepsilon q^2 - 1$ for some $\varepsilon \in \mathbb{F}^*_q$. Define $\psi = p^s q^3$. A $\rho \in \mathbb{F}^*_q$ exists such that $\rho^q - 1 = -1$. Take $A = D = 0$, $B = (\rho \varepsilon q^3$, $C = \varepsilon \rho c(1 - b^\psi(2q+1))$. If $C = 0$, then $b^q + 1 = 1$, a contradiction. So $AD - BC \neq 0$ and (16) has a solution. If $p^s = q$, then $\psi = 1$, hence in this case $L_{U(b)}$ and $L_{U(c)}$ are projectively equivalent. This finishes the proofs of (i) and (ii).

Now we prove (iv). Note that $N_{q^4}/q(b) = (b^q + 1)^{q+1}$ for any $b \in \mathbb{F}_q$, therefore, $L_{U(b)}$ is a maximum scattered $\mathbb{F}_q$-linear set not of pseudoregulus type if, and only if, $b^q + 1$ is an element of the set
\[
S = \{x \in \mathbb{F}_{q^2} : x^{q+1} \neq 1\}.
\]

The orbits of point sets of type $L_{U(b)}$, $b \neq 0$, under the action of $\text{PGL}(2, q^4)$ are as many as the pairs $\{x, x^{-q}\}$ of elements in $S$. Since all such pairs are made of distinct elements, adding one for the linear set of pseudoregulus type, one obtains
\[
1 + \frac{q^2 - q - 2}{2} = \frac{q(q - 1)}{2}.
\]

Finally we prove (v). $L_{U(b)}$ is an $\mathbb{F}_q$-linear set described in 3. of Theorem 2.1 if, and only if, $b^q + 1$ is an element of the set
\[
Z = \{x \in \mathbb{F}_{q^2} \setminus \{1\} : x^{q+1} = 1\}.
\]
The orbits of point sets of this type under the action of \( \text{PGL}(2, q^4) \) are as many as the pairs \( \{x, x^{-q}\} \) of elements in \( Z \). Since for each \( x \in Z \) we have \( x = x^{-q} \), this number is \( q \).

Remark 4.6. The number of orbits of maximum scattered linear sets under the action of \( \text{PGL}(2, q^4) \) depends on the exponent \( e \) in \( q = p^e \). A general formula is not provided here. For \( e = 1 \) each orbit which does not arise from the linear set of pseudoregulus type is related to two or four norms over \( \mathbb{F}_{q^2} \), according to whether \( N_{q^4/q^2}(b) \in \mathbb{F}_q \setminus \{0, 1, -1\} \) or not. This leads (including now the linear set of pseudoregulus type) to a total number of \((q^2 - 1)/4\) orbits for odd \( q \).

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