Operator Preconditioning in Hilbert Space

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Theses of PhD Dissertation

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1 Introduction

The numerical solution of linear elliptic partial differential equations consists of two main steps: discretization and iteration, where generally some conjugate gradient method is used for solving the finite element discretization of the problem. However, when for elliptic problems the discretization parameter tends to zero, the required number of iterations for a prescribed tolerance tends to infinity. The remedy is suitable preconditioning. This can rely on the functional analytic background of the corresponding elliptic operators, which means that the preconditioning process takes place on the operator level. That is, we look for a suitable preconditioning operator for the operator equation, which is close to the original one in some sense, and use its discretization as a preconditioning matrix. Here we use the generalized conjugate gradient, least square methods (GCG-LS(s)) and the conjugate gradient normal (CGN) algorithm. These algorithms, the coincidence of the full GCG-LS and the truncated GCG-LS(0) methods and the related convergence theorems are discussed in [1, 2, 3, 4].

2 Preliminaries and short summary of the results

A general theory has been developed for such preconditioning using the notion of equivalent operators, which has been introduced and investigated in the aspect of linear convergence in [6]. As a further step, mesh independent superlinear convergence has been proved for the GCG-LS method for elliptic equations with homogeneous Dirichlet and mixed boundary conditions under FEM discretization, with severe restrictions on the coefficients (see [3, 7]). In [4] the notion of compact-equivalence has been introduced, summarized in [5], and superlinear mesh independence has been proved for the CGN algorithm without any restrictions (with the exception of the usual smoothness and coercivity conditions).

Based on these papers, we have compared the relation between the known theoretical convergence estimate and the numerical results and we have shown that the convergence rate remains valid even in cases not covered by the theory (cf. [14, 15]). Then we have extended the scope of the theoretical results to cases that have not been considered before: first we have dealt with symmetric preconditioning for elliptic systems using the compact normal operator framework and the GCG-LS algorithms (see [9, 12]). Then we have considered equations with nonhomogeneous mixed boundary conditions using operator pairs and the notion of compact-equivalence with the CGN method (cf. [13]). In contrast with finite element discretizations which fits in naturally with the Hilbert space background, there is no such abstract background for finite difference discretization, only a case-by-case study is possible. We have investigated a special model problem (see [11]) and we have derived a convergence estimate analogous to the finite element case. In [10] we have shown that the use of nonsymmetric preconditioners is more advantageous for singularly perturbed problems than symmetric preconditioning. Finally we have applied these results to nonlinear elliptic and time-dependent problems (cf. [8, 13]).
3 Summary of the applied methods

3.1 Compact-equivalence and the convergence of the CGN algorithm

Let $H$ be a real Hilbert space and consider the operator equation

$$Lu = g$$

(1)

with a linear unbounded operator $L$ in $H$, where $g \in H$ is given. We would like to consider its preconditioned form in weak sense in an energy space of a suitable symmetric operator $S$. The set of $S$-bounded and $S$-coercive operators is denoted by $BC_S(H)$ (see [4]).

**Definition 1.** (cf. [4]) For a given operator $L \in BC_S(H)$, we call $u \in H_S$ the weak solution of equation (1) if

$$\langle L_Su, v \rangle_S = \langle g, v \rangle \quad \forall \ v \in H_S,$$

(2)

where $L_S \in B(H_S)$ represents the unique extension of the bounded bilinear form $(u, v) \mapsto \langle Lu, v \rangle$ from $D(L)$ to $H_S$.

**Definition 2.** (cf. [4]) The operators $L, K \in BC_S(H)$ are compact-equivalent in $H_S$ if $L_S = \mu K_S + Q_S$ for some constant $\mu > 0$ and compact operator $Q_S \in B(H_S)$.

As an important special case, we can consider compact-equivalence with $\mu = 1$ for the operators $S$ and $L \in BC_S(H)$. Then

$$L_S = I + Q_S$$

(3)

holds with some compact operator $Q_S$.

Let us consider the operator equation (1) where $L \in BC_S(H)$, $g \in H$ and $u \in H_S$ is the weak solution defined in (2). To solve it numerically, let $V_h = \text{span}\{\varphi_1, \ldots, \varphi_n\} \subset H_S$ be a finite dimensional subspace of dimension $n$ and $L_h = \{(L_S\varphi_i, \varphi_j)_S\}_{i,j=1}^n, g_h = \{(g, \varphi_j)\}_{j=1}^n$. Then the discrete solution $u_h \in V_h$ is $u_h = \sum_{i=1}^n c_i \varphi_i$, where $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ is the solution of the linear system $L_h c = g_h$, which is the discretized form of (2). Now assume that $L$ and $S$ are compact-equivalent with $\mu = 1$, i.e. relation (3) holds. If $S$ is used as a preconditioner, then the discretized form of the operator decomposition (3) becomes $L_h = S_h + Q_h$, and the corresponding preconditioned form of $L_h c = g_h$ has the form

$$(I_h + S_h^{-1}Q_h) c = S_h^{-1}g_h,$$

(4)

where $S_h = \{(\varphi_i, \varphi_j)_S\}_{i,j=1}^n, Q_h = \{(Q_S\varphi_i, \varphi_j)_S\}_{i,j=1}^n$.

**Theorem 3.** (cf. [4, Thm. 4.1]) Assume that $L \in BC_S(H)$, $L$ and $S$ are compact-equivalent with $\mu = 1$, i.e. (3) holds. Then the CGN algorithm for system (4) yields

$$\frac{\|r_k\|_{S_h}}{\|r_0\|_{S_h}}^{1/k} \leq \frac{2}{m^2} \left( \frac{1}{k} \sum_{i=1}^k (|\lambda_i(Q_S^* + Q_S)| + \lambda_i(Q_S^*Q_S)) \right)^{k \to \infty} 0,$$

(5)
where \( r_k \) is the residual vector; \( m > 0 \) comes from the \( S \)-coercivity of \( L \) and the right-hand side is independent of the subspace \( V_h \).

### 3.2 The compact normal operator approach and the convergence of the GCG-LS algorithm

Let \( H \) be a complex Hilbert space and consider the operator equation (1) with an unbounded linear operator \( L : D \subset H \rightarrow H \) defined on a dense domain \( D \), and with some \( g \in H \). Equation (1) is assumed to satisfy the following

**Assumptions 4.**

(i) The operator \( L \) is decomposed in \( L = S + Q \) on its domain \( D \) where \( S \) is a self-adjoint operator in \( H \);

(ii) \( S \) is a strongly positive operator, i.e. \( \exists \ p > 0 \) such that \( \langle Su, u \rangle \geq p \|u\|^2 \quad \forall u \in D \);

(iii) there exists \( \varrho > 0 \) such that \( \Re \langle Lu, u \rangle \geq \varrho \langle Su, u \rangle \quad \forall u \in D \);

(iv) the operator \( Q \) can be extended to the energy space \( H_S \), and then \( S^{-1}Q \) is assumed to be a compact normal operator on \( H_S \).

Now we replace equation (1) by its preconditioned form

\[
S^{-1}Lu = f \equiv S^{-1}g \iff (I + S^{-1}Q)u = f \equiv S^{-1}g.
\]

**Theorem 5.** (cf. [3, Thm. 3]) Let Assumptions 4 hold. Then the GCG-LS algorithm applied for equation (6) in \( H_S \) yields for all \( k \in \mathbb{N} \)

\[
\left( \frac{\|e_k\|_{L_h}}{\|e_0\|_{L_h}} \right)^{1/k} \leq \frac{2}{\varrho} \left( \frac{1}{k} \sum_{i=1}^{k} |\lambda_i(S^{-1}Q)| \right) \xrightarrow{k \to \infty} 0,
\]

where \( e_k = u_k - u^* \) is the error vector; \( \lambda_k(S^{-1}Q) \ (k \in \mathbb{N}) \) are the ordered eigenvalues of the compact normal operator \( S^{-1}Q \).

Equation (1) can be solved numerically by using Galerkin discretization. Let us consider the finite dimensional subspace \( V_h = \text{span}\{\varphi_1, \ldots, \varphi_n\} \subset H_S \) of dimension \( n \) and \( S_h = \{\langle \varphi_i, \varphi_j \rangle_S \}_{i,j=1}^{n} \), \( Q_h = \{\langle Q\varphi_i, \varphi_j \rangle \}_{i,j=1}^{n} \), \( g_h = \{\langle g, \varphi_j \rangle \}_{j=1}^{n} \). Then the discrete solution \( u_h \in V_h \) is \( u_h = \sum_{i=1}^{n} c_i \varphi_i \), where \( c = (c_1, \ldots, c_n) \in \mathbb{R}^n \) is the solution of the linear system \( L_h c = g_h \), where \( L_h = S_h + Q_h \). If the operator \( S \) is used as a preconditioner, then the discretized form of the preconditioned operator equation (6) becomes

\[
(I_h + S_h^{-1}Q_h) c = S_h^{-1}g_h.
\]

**Theorem 6.** (cf. [3, Cor. 4]) Suppose that \( H \) is a complex separable Hilbert space, Assumptions 4 are satisfied and the matrix \( S_h^{-1}Q_h \) is \( S_h \)-normal. Then the GCG-LS algorithm for system (8) yields

\[
\left( \frac{\|e_k\|_{L_h}}{\|e_0\|_{L_h}} \right)^{1/k} \leq \frac{2}{\varrho} \left( \frac{1}{k} \sum_{i=1}^{k} |\lambda_i(S^{-1}Q)| \right) \xrightarrow{k \to \infty} 0,
\]

3
where the right-hand side is independent of the subspace $V_h$.

4 Main results

4.1 Equations with homogeneous boundary conditions

Using the theoretical background of Subsection 3.2 for elliptic convection-diffusion problems with homogeneous Dirichlet and mixed boundary conditions, first we have investigated the relation between the known theoretical convergence estimate and the numerical results. We have confirmed the mesh independent superlinear convergence property of the GCG-LS($0$) when symmetric part preconditioning has been applied to the FEM discretization of the boundary value problem. We have shown that the convergence rate remains valid even in cases not covered by the theory, i.e. when another symmetric operator is used as a preconditioner, not only the symmetric part of the operator. The numerical computations have also yielded better results than the theoretical estimate (9).

4.2 Equations with nonhomogeneous mixed boundary conditions

Let us consider the elliptic boundary value problem

$$
-\text{div}(A \nabla u) + b \cdot \nabla u + cu = g \\
\left. u \right|_{\Gamma_D} = 0, \quad \left. \frac{\partial u}{\partial \nu} + \alpha u \right|_{\Gamma_N} = \gamma
$$

satisfying the following assumptions:

Assumptions 7. (i) $\Omega \subset \mathbb{R}^d$ is a bounded piecewise $C^1$ domain; $\Gamma_D, \Gamma_N$ are disjoint open measurable subparts of $\partial \Omega$ such that $\partial \Omega = \Gamma_D \cup \Gamma_N$;

(ii) $A \in L^\infty(\overline{\Omega}, \mathbb{R}^{d \times d})$ and for all $x \in \overline{\Omega}$ the matrix $A(x)$ is symmetric; further, $b \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$, $\alpha \in L^\infty(\Gamma_N)$;

(iii) we have the coercivity properties

$$\exists \ p > 0 \text{ such that } A(x)\xi \cdot \xi \geq p |\xi|^2 \quad \forall \ x \in \overline{\Omega}, \ \xi \in \mathbb{R}^d,$$

$$\hat{c} := c - \frac{1}{2} \text{div} b \geq 0 \text{ in } \Omega, \quad \hat{\alpha} := \alpha + \frac{1}{2}(b \cdot \nu) \geq 0 \text{ on } \Gamma_N;$$

(iv) either $\Gamma_D \neq \emptyset$, or $\hat{c}$ or $\hat{\alpha}$ has a positive lower bound.

The definition of the operator $L$ which corresponds to equation (10) has to be understood as a pair of operators: one acts on $\Omega$ and the other one acts on the Neumann boundary. Formally we have

$$L \equiv \begin{pmatrix} M & P \\ P & \end{pmatrix}$$

$$L \begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} Mu \\ P\eta \end{pmatrix} = \begin{pmatrix} -\text{div}(A \nabla u) + b \cdot \nabla u + cu \\ \frac{\partial u}{\partial \nu} + \alpha u \end{pmatrix} \bigg|_{\Gamma_N}. \quad (13)$$
Let us define a symmetric elliptic operator on the same domain in an analogous way:

\[
S \equiv \begin{pmatrix} N \\ R \end{pmatrix}, \quad S \begin{pmatrix} u \\ \eta \end{pmatrix} = \begin{pmatrix} Nu \\ R\eta \end{pmatrix} = \begin{pmatrix} -\text{div}(G \nabla u) + \sigma u \\ \frac{\partial \eta}{\partial \nu} + \beta \eta |_{\Gamma_N} \end{pmatrix}
\] (14)

satisfying similar assumptions as of \( L \):

**Assumptions 8.**

(i) substituting \( G \) for \( A \), \( \Omega \), \( \Gamma_D \), \( \Gamma_N \) and \( G \) satisfy Assumptions 7;

(ii) \( \sigma \in L^\infty(\Omega) \), \( \sigma \geq 0 \), \( \beta \in L^\infty(\Gamma_N) \), \( \beta \geq 0 \); further, if \( \Gamma_D \neq \emptyset \), then \( \sigma \) or \( \beta \) has a positive lower bound.

Let us consider the differential equation (10) with given functions \( g \in L^2(\Omega) \), \( \gamma \in L^2(\Gamma_N) \). We are interested in solving the analogous operator equation

\[
L \begin{pmatrix} u \\ u |_{\Gamma_N} \end{pmatrix} = \begin{pmatrix} g \\ \gamma \end{pmatrix},
\] (15)

which is the appropriately modified version of the operator equation (1). Here the Hilbert space \( H \) is defined as the product space \( H = L^2(\Omega) \times L^2(\Gamma_N) \).

**Theorem 9.** If Assumptions 7-8 hold, then the operator \( L \) is \( S \)-bounded and \( S \)-coercive in \( H \), i.e. \( L \in BC_S(L^2(\Omega) \times L^2(\Gamma_N)) \).

Consider again the differential equation (10) with the corresponding operator \( L \) in (13) and preconditioner \( S \) in (14) and assume that \( A = G \), then it follows from [4] that \( L \) and \( S \) are compact-equivalent with \( \mu = 1 \), thus (3) holds. On the discrete level – with the notations of Subsection 3.1 – the finite element solution \( u_h \in V_h \) is obtained by solving the linear system \( L_h c = d_h \), where

\[
(L_h)_{ij} = \int_{\Omega} (A \nabla \varphi_i \cdot \nabla \varphi_j + (b \cdot \nabla \varphi_j) \varphi_i + c \varphi_i \varphi_j) + \int_{\Gamma_N} \alpha \varphi_i \varphi_j, \quad (d_h)_j = \int_{\Omega} g \varphi_j + \int_{\Gamma_N} \gamma \varphi_j.
\]

Let us take the symmetric operator \( S \) and introduce its stiffness matrix in \( H_S \) as

\[
(S_h)_{ij} = \langle \varphi_i, \varphi_j \rangle_S = \int_{\Omega} (G \nabla \varphi_i \cdot \nabla \varphi_j + \sigma \varphi_i \varphi_j) + \int_{\Gamma_N} \beta \varphi_i \varphi_j,
\]

and consider the preconditioned equation

\[
(I_h + S_h^{-1} Q_h) c = S_h^{-1} d_h,
\] (16)

where \( L_h \) and \( S_h \) come from the elliptic operators \( L \) and \( S \) and \( Q_h = L_h - S_h \). In this case the operator \( Q_S \) is defined as

\[
\langle Q_S \begin{pmatrix} u \\ u |_{\Gamma_N} \end{pmatrix}, \begin{pmatrix} v \\ v |_{\Gamma_N} \end{pmatrix} \rangle_S = \int_{\Omega} ((b \cdot \nabla u) v + (c - \sigma) u v) + \int_{\Gamma_N} (\alpha - \beta) u v.
\] (17)
Theorem 10. With Assumptions 7-8 and \( A = G \), the CGN algorithm for system (16) yields

\[
\left( \frac{\| r_k \|_{S_h}}{\| r_0 \|_{S_h}} \right)^{1/k} \leq \frac{2}{m^2} \left( \frac{1}{k} \sum_{i=1}^{k} (|\lambda_i(Q^*_S + Q_S)| + \lambda_i(Q^*_SQ_S)) \right) \rightarrow_{k \to \infty} 0,
\]

where \( m > 0 \) comes from the \( S \)-coercivity of \( L \) in Theorem 9.

4.3 Finite difference discretization for a model problem

Let us consider a special model problem which has been analysed in [16] in the context of linear convergence. The convection-diffusion problem

\[
Lu \equiv -\Delta u + b \cdot \nabla u + cu = g
\]

\[
u|_{\Gamma_D} = 0
\]

is posed on the unit square \( \Omega : = [0, 1]^2 \subset \mathbb{R}^2 \) with constant coefficients \( b = (b_1, b_2) \in \mathbb{R}^2 \) and \( c \in \mathbb{R} \). We assume \( c \geq 0 \), then the usual coercivity condition \( c - \frac{1}{2} \text{div} b \geq 0 \) holds. The FDM discretization of (18) on a given grid \( \omega_h \) leads to a linear algebraic system \( L_h u_h = g_h \). Our goal is to solve this equation by iteration, applying the preconditioned GCG-LS method. The proposed preconditioner is obtained via a symmetric preconditioning operator

\[
S u := -\Delta u + \sigma u \quad \text{for} \quad u|_{\partial \Omega} = 0,
\]

where \( \sigma \in \mathbb{R} \), \( \sigma \geq 0 \): namely, the matrix \( S_h \) is defined as the FDM discretization of the operator \( S \) on the same grid \( \omega_h \). The preconditioned form of the discretized system is \( S_h^{-1}L_h u_h = f_h \equiv S_h^{-1}g_h \). Here we are interested in the superlinear convergence property of the preconditioned GCG-LS, where the operators \( L, S \) are replaced by the matrices \( L_h, S_h \), respectively.

Let \( \omega_h \) be a uniform grid on \([0, 1]^2\), \( b_1, b_2 \geq 0 \) and let us define upwind or centered differencing for the first order and centered differencing for the second order derivatives, respectively. Denote by \( n \) the number of interior gridpoints in each direction, and by \( h = 1/(n+1) \) the grid parameter. Let \( L_h, S_h \) and \( Q_h \) denote the matrices corresponding to the discretizations of \( L \), \( S \) and \( Q = L - S \), respectively.

Proposition 11. Let us consider problem (18) with a convection term \( b = (b, b) \), where \( b \in \mathbb{R}^+ \) is arbitrary, and let \( \sigma := c \) in (19), i.e. \( S \) is the symmetric part of \( L \). Then, using either centered or backward differencing, the eigenvalues \( \lambda_{jm}(S_h^{-1}Q_h) \) satisfy

\[
\frac{1}{k^2} \sum_{j,m=1}^{k} |\lambda_{jm}(S_h^{-1}Q_h)| \leq \frac{2\sqrt{2b}}{k^2} \sum_{j,m=1}^{\frac{k}{2}+1} \frac{1}{\sqrt{\sigma + 4m^2 + 4j^2}} \quad (k = 1, \ldots, n),
\]

where the sequence on the right-hand side is independent of \( h \) and tends to 0 as \( k \to \infty \).

Inequality (20) shows that the sequence of the error quotients \((\| e_k \|_{L_h}/\| e_0 \|_{L_h})^{1/k}\) can be estimated in a mesh uniform superlinear way, analogously to estimate (9) for the finite element case.
4.4 Extension of the theory to systems

Let us consider systems of the form

\[
- \text{div}(K_i \nabla u_i) + b_i \cdot \nabla u_i + \sum_{j=1}^{\ell} V_{ij} u_j = g_i \quad (i = 1, \ldots, \ell) \\
\left. u_i \right|_{\partial \Omega} = 0
\]

(21)

satisfying the following assumptions.

Assumptions 12.  
(i) The bounded domain $\Omega \subset \mathbb{R}^d$ is $C^2$-diffeomorphic to a convex domain;
(ii) for all $i, j = 1, \ldots, \ell$ the functions $K_i \in C^1(\Omega)$, $V_{ij} \in L^\infty(\Omega)$ and $b_i \in C^1(\Omega)^d$;
(iii) there exists $m > 0$ such that $K_i \geq m$ holds for all $i = 1, \ldots, \ell$;
(iv) letting $V = \{V_{ij}\}_{i,j=1}^\ell$, the coercivity property $\lambda_{\min}(V + V^T) - \max_{1 \leq i \leq \ell} \text{div} b_i \geq 0$ holds pointwise on $\Omega$, where $\lambda_{\min}$ denotes the smallest eigenvalue;
(v) $g_i \in L^2(\Omega)$ for all $i = 1, \ldots, \ell$.

For brevity, we write (21) – using vector notations – as

\[
Lu \equiv - \text{div}(K \nabla u) + b \cdot \nabla u + Vu = g \\
\left. u \right|_{\partial \Omega} = 0
\]

(22)

For the numerical solution of system (22), one usually considers its FEM discretization, which leads to a linear algebraic system $L h c = g_h$. This can be solved by the CGM using some suitable preconditioner. Here we consider preconditioners based on the following preconditioning operator. Letting $\sigma_i \in L^\infty(\Omega)$, $\sigma_i \geq 0$ be suitable functions and

\[
S_i u_i := - \text{div}(K_i \nabla u_i) + \sigma_i u_i \quad (i = 1, \ldots, \ell)
\]

(23)

for $\left. u_i \right|_{\partial \Omega} = 0$, and define the $\ell$-tuple of independent elliptic operators

\[
S u = \begin{pmatrix} S_1 u_1 \\ \vdots \\ S_\ell u_\ell \end{pmatrix}
\]

(24)

We have proved mesh independent superlinear convergence of the preconditioned CGM in the framework of compact normal operators in Hilbert space. This has been achieved in two steps: on the theoretical level, the preconditioned form of system (22)

\[
S^{-1}Lu = f \equiv S^{-1}g
\]

(25)

has been considered and we have proved that the CGM converges superlinearly in the Sobolev space $H_0^1(\Omega)^\ell$. Based on this, on the practically relevant discrete level we have considered the
preconditioned form

\[ S_h^{-1}L_h c = f_h \equiv S_h^{-1}g_h \]  

(26)

of the algebraic system \( L_h c = g_h \), where \( S_h \) denotes the discretization of \( S \) in the same FEM subspace as for \( L_h \), and we have proved that the superlinear convergence of the CGM is mesh independent, i.e. independent of the considered FEM subspace. On both levels the full GCG-LS and the truncated GCG-LS(0) algorithms has been considered, and the results have been proved under certain special assumptions that ensure the normality of the preconditioned operator in the corresponding Sobolev space. First we have considered symmetric part preconditioning. The symmetric part of \( L \) falls into the type (23) coordinatewise if and only if

\[ V_{ij} = -V_{ji} \quad (i \neq j), \quad \text{and } \sigma_i \text{ in (23) is chosen as } \sigma_i = V_{ii} - \frac{1}{2} (\text{div } b_i). \]  

(27)

Theorem 13. Under Assumptions 12 and condition (27), the preconditioned truncated GCG-LS(0) algorithm for system (21) with the preconditioning operator (23)-(24) converges superlinearly in the space \( H^1_0(\Omega)^\ell \) according to the estimate (7).

Assumptions 14.  
(i) For all \( i = 1, \ldots, \ell \), \( K_i \equiv K \in \mathbb{R} \), \( \sigma_i \equiv \sigma \in \mathbb{R} \) and \( b_i \equiv b \in \mathbb{R}^d \);  
(ii) \( V \in \mathbb{R}^{\ell \times \ell} \) is a normal matrix.

Theorem 15. Under Assumptions 12 and 14, the preconditioned full GCG-LS algorithm for system (21) with the preconditioning operator (23)-(24) converges superlinearly in the space \( H^1_0(\Omega)^\ell \) according to the estimate (7).

Corollary 16. Let Assumptions 12 hold. Consider the FEM discretization of system (21), using the stiffness matrix of (24) as preconditioner, under one of the following conditions:

(a) the requirements in (27) hold, \( V_h \subset H^1_0(\Omega)^\ell \) is an arbitrary FEM subspace and the truncated GCG-LS(0) algorithm is used (here the \( S_h \)-normality of \( S_h^{-1}Q_h \) automatically holds);  
(b) Assumptions 14 hold, \( V_h \subset H^1_0(\Omega)^\ell \) is a FEM subspace for which the matrix \( S_h^{-1}Q_h \) is \( S_h \)-normal, and the full GCG-LS is used.

Then the mesh independent superlinear convergence estimate (9) is valid.

4.5 Systems with nonhomogeneous mixed boundary conditions

We have applied the operator pair approach for elliptic systems of the form

\[
\begin{aligned}
-\text{div}(A_i \nabla u_i) + b_i \cdot \nabla u_i + \sum_{j=1}^{\ell} V_{ij} u_j &= g_i \\
\left. u_i \right|_{\Gamma_D} &= 0, \\
\left. \frac{\partial u_i}{\partial n} + \alpha_i u_i \right|_{\Gamma_N} &= \gamma_i
\end{aligned}
\]  

(28)

\( i = 1, \ldots, \ell \)
satisfying the combination of Assumptions 7 and 12, where the corresponding operator \( L \) and the preconditioner \( S \) are defined as an \( \ell \)-tuple of operator pairs:

\[
L = (L_1, \ldots, L_\ell) = \left( \begin{pmatrix} M_1 \\ P_1 \end{pmatrix}, \ldots, \begin{pmatrix} M_\ell \\ P_\ell \end{pmatrix} \right), \quad S = (S_1, \ldots, S_\ell) = \left( \begin{pmatrix} N_1 \\ R_1 \end{pmatrix}, \ldots, \begin{pmatrix} N_\ell \\ R_\ell \end{pmatrix} \right),
\]

satisfying similar conditions as in Assumptions 7-8 and the operator pairs are defined analogously to (13)-(14). As in (15) for a single equation, we look for the weak solution of the operator equation

\[
L \begin{pmatrix} u \\ u|_{\Gamma_N} \end{pmatrix} = \begin{pmatrix} g \\ \gamma \end{pmatrix}. \tag{29}
\]

Extending the results for equations to systems, it is easy to verify that for \( G_i = A_i \) \((i = 1, \ldots, \ell)\) the operators \( L \) and \( S \) are compact-equivalent with \( \mu = 1 \), i.e. \( L_S = I + Q_S \) holds in \( H_S \) with some compact operator \( Q_S \). Now let us consider the discrete equation \( L_h c = d_h \) and its preconditioned form

\[
S_h^{-1} L_h c = (I_h + S_h^{-1} Q_h) c = S_h^{-1} d_h, \tag{30}
\]

where \( L_h \) and \( S_h \) come from the elliptic operators \( L \) and \( S \), \( Q_h = L_h - S_h \). Symmetric part preconditioning can also be considered, analogously to the previous subsection. When \( S \) is not the symmetric part of \( L \), then \( Q_S \in B(H_S) \) can be defined as the sum of similar operators corresponding to (17). Now the conditions of Theorem 3 are satisfied, thus the CGN algorithm provides a mesh independent superlinear convergence result.

**Corollary 17.** With suitable combination of Assumptions 7-8 and \( A_i = G_i \) \((i = 1, \ldots, \ell)\), the CGN algorithm for the system (30) yields

\[
\left( \frac{\|r_k\|_{S_h}}{\|r_0\|_{S_h}} \right)^{1/k} \leq \frac{2}{m^2} \left( \frac{1}{k} \sum_{j=1}^{k} \left( |\lambda_j(Q_S + Q_S)| + \lambda_j(Q_S Q_S) \right) \right) \xrightarrow{k \to \infty} 0.
\]

**References**


