Additive representation functions

DOCTORAL THESIS

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1 Introduction

Let $\mathbb{N}$ denote the set of positive integers, and let $k \geq 2$ be a fixed integer. Let $A = \{a_1, a_2, \ldots\}$ ($a_1 < a_2 < \ldots$) be an infinite sequence of positive integers. For $k \geq 2$ integer and $A \subset \mathbb{N}$, and for $n = 0, 1, 2, \ldots$ let $R_1(A, n, k)$, $R_2(A, n, k)$, $R_3(A, n, k)$ denote the number of solutions of the equations

$$ a_{i_1} + a_{i_2} + \cdots + a_{i_k} = n, \quad a_{i_1} \in A, \ldots, a_{i_k} \in A,$$

and

$$ a_{i_1} + a_{i_2} + \cdots + a_{i_k} = n, \quad a_{i_1} \in A, \ldots, a_{i_k} \in A, \quad a_{i_1} < a_{i_2} < \ldots < a_{i_k},$$

respectively. If $F(n) = O(G(n))$ then we write $F(n) << G(n)$. Put

$$ A(n) = \sum_{a \in A} 1. $$

The research of the additive representation functions began in the 1950's. Starting from a problem of Sidon, P. Erdős proved that there exists a sequence $A \in \mathbb{N}$ so that there are two constants $c_1$ and $c_2$ for which for every $n$

$$ c_1 \log n < R_1(A, n, 2) < c_2 \log n. $$

On the other hand an old conjecture of Erdős states that for no sequence $A$ can we have

$$ \frac{R_1(A, n, 2)}{\log n} \rightarrow c \quad (0 < c < +\infty). $$

There are some related questions in [3] and [12]. These problems led P. Erdős, A. Sárközy and V. T. Sós to study the regularity property and the monotonicity of the function $R_1(A, n, 2)$ see in [6], [7], [8], [9]. In my thesis I study the regularity properties and the monotonicity of the representation...
function $R_1(A, n, k)$ for $k > 2$ integer. I extend and generalize some result of P. Erdős, A. Sárközy and V. T. Sós by using the generator function method and the probabilistic method.

2 The methods we are working with

In my thesis I use the generating function method. We start out from the generating function of the sequence $A$:

$$f(z) = \sum_{a \in A} z^a.$$ 

It is easy to see that

$$f^k(z) = \sum_{n=1}^{\infty} R_1(A, n, k)z^n.$$ 

We use the generating function method to prove the results about the monotonicity. We also use the Hölder - inequality, the Cauchy - inequality and the Parseval - formula. In the next step I tell a few words about the probabilistic method. An important problem in additive number theory is to prove that a sequence with certain properties exists. One of the essential ways to obtain an affirmative answer for such a problem is to use the probabilistic method due to Erdős and Rényi. There is an excellent summary of this method in the Halberstam - Roth book [12]. To show that a sequence with a property $\mathcal{P}$ exists, it suffices to show that a properly defined random sequence satisfies $\mathcal{P}$ with positive probability. Usually the property $\mathcal{P}$ requires that for all sufficiently large $n \in \mathbb{N}$, some relation $\mathcal{P}(n)$ holds. The general strategy to handle this situation is the following. For each $n$ one first shows that $\mathcal{P}(n)$ fails with a small probability, say $p_n$. If $p_n$ is sufficiently small so that $\sum_{n=1}^{\infty} p_n$ converges, then by the Borel - Cantelli lemma, $\mathcal{P}(n)$ holds for all
sufficiently large \( n \) with probability 1. Note that in my proofs the additive representation function is the sum of random variables. However for \( k > 2 \) these variables are not independent. To overcome this trouble in my thesis I apply the theorems of J. H. Kim and V. H. Vu, [15], [25], [26], [27], the Janson inequality [14] and the method of Erdős and Tetali [2], [10].

3 Theses

For \( i = 1, 2, 3 \) we say \( R_i(\mathcal{A}, n, k) \) is monotonous increasing in \( n \) from a certain point on, if there exists an integer \( n_0 \) with

\[
R_i(\mathcal{A}, n + 1, k) \geq R_i(\mathcal{A}, n, k) \quad \text{for} \quad n \geq n_0.
\]

In a series of papers P. Erdős, A. Sárközy and V. T. Sós studied the monotonicity properties of the three representation functions \( R_1(\mathcal{A}, n, 2), R_2(\mathcal{A}, n, 2), R_3(\mathcal{A}, n, 2) \). A. Sárközy proposed the study of the monotonicity of the functions \( R_i(\mathcal{A}, n, k) \) for \( k > 2 \) [2, Problem 5]. He conjectured [3, p. 337] that for any \( k \geq 2 \) integer, if \( R_i(\mathcal{A}, n, k) \) (\( i = 1, 2, 3 \)) is monotonous increasing in \( n \) from a certain point on, then \( A(n) = O(n^{2/k - \varepsilon}) \) cannot hold. In this thesis I prove (see in [18]) the following slightly stronger result on \( R_1(\mathcal{A}, n, k) \):

**Theorem 1.** If \( k \in \mathbb{N}, k \geq 2, \mathcal{A} \subset \mathbb{N} \) and \( R_1(\mathcal{A}, n, k) \) is monotonous increasing in \( n \) from a certain point on, then

\[
A(n) = o\left(\frac{n^{2/k}}{(\log n)^{2/k}}\right)
\]

cannot hold.

Let \( k \geq 2, l \geq 1 \) be a fixed integers. If \( s_0, s_1, s_2, \ldots \) is a given sequence of real numbers, then let \( \Delta_l s_n \) denote the \( l \)-th difference of the sequence
$s_0, s_1, s_2 \ldots$ defined by $\Delta_1 s_n = s_{n+1} - s_n$ and $\Delta_1 s_n = \Delta_1(\Delta_{l-1}s_n)$. It is well-known and it is easy to see by induction that

$$\Delta_1 s_n = \sum_{i=0}^{l}(-1)^{l-i}{l \choose i}s_{n+i}.$$

Let $B(\mathcal{A}, N)$ denote the number of blocks formed by consecutive integers in $\mathcal{A}$ up to $N$, i.e.,

$$B(\mathcal{A}, N) = \sum_{a \leq N, a \in \mathcal{A}} 1.$$

P. Erdős, A. Sárközy and V. T. Sós studied the following problem: what condition is needed to ensure

$$\limsup_{n \to +\infty} |R_1(\mathcal{A}, n + 1, 2) - R_1(\mathcal{A}, n, 2)| = +\infty?$$

They proved in [7] that if $k = 2, l = 1$ and if $\lim_{N \to \infty} \frac{B(\mathcal{A}, N)}{\sqrt{N}} = \infty$, then the above holds. They also proved that their result is nearly sharp.

In [16] I extended their Theorem to any $k > 2$:

**Theorem 2.** If $k \geq 2$ is an integer and $\lim_{N \to \infty} \frac{B(\mathcal{A}, N)}{\sqrt{N}} = \infty$, and $l \leq k$, then $|\Delta_1R_1(\mathcal{A}, n, k)|$ cannot be bounded.

I also proved in [20] that the above result is nearly best possible:

**Theorem 3.** For all $\varepsilon > 0$, there exists an infinite sequence $\mathcal{A}$ such that

(i) $B(\mathcal{A}, N) \gg N^{1/k-\varepsilon}$,

(ii) $R_1(\mathcal{A}, n, k)$ is bounded so that also $\Delta_1R_1(\mathcal{A}, n, k)$ is bounded if $l \leq k$.

In the case $l > k$ I have only a partial result [17]:

**Theorem 4.** If $l \geq 2$ an integer and $\lim_{N \to \infty} \frac{B(\mathcal{A}, N)}{\sqrt{N}} = \infty$, then $|\Delta_1(R_1(\mathcal{A}, n, 2))|$ cannot be bounded.
In [13] G. Horváth extended a theorem of Erdős and Sárközy to any \( k > 2 \) integer. In [19] I proved that his result is nearly best possible by using probabilistic method:

**Theorem 5.** If \( k > 2 \) is a positive integer, \( c_8 \) is a constant large enough in terms \( k \), \( F(n) \) is an arithmetic function satisfying

\[
F(n) > c_8 \log n \quad \text{for} \quad n > n_0,
\]

and there exists a real function \( g(x) \), defined for \( 0 < x < +\infty \), and real numbers \( x_0, n_1 \) and \( c_7, c_9 \) constants such that

(i) \( 0 < g(x) \leq \frac{(\log x)^{1/2}}{x^{1/2}} < 1 \) for \( x \geq x_0 \),

(ii) \( F(n) - k! \sum_{\substack{1 \\ x_1 + x_2 + \ldots + x_k = n}} g(x_1)g(x_2)\ldots g(x_k) \leq c_7(F(n) \log n)^{1/2} \)

for \( n > n_1 \),

then there exists a sequence \( \mathcal{A} \) such that

\[
|R_1(\mathcal{A}, n, k) - F(n)| < c_9(F(n) \log n)^{1/2} \quad \text{for} \quad n > n_2.
\]

We say a set \( \mathcal{A} \) of positive integers is an asymptotic basis of order \( h \) if every large enough positive integer can be represented as the sum of \( h \) terms from \( \mathcal{A} \). In other words \( \mathcal{A} \) is an asymptotic bases of order \( h \) if there exists an \( n_0 \) positive integer such that \( R_3(\mathcal{A}, n, 2) > 0 \) for \( n > n_0 \). A set of positive integers \( \mathcal{A} \) is called Sidon set if all the sums \( a + b \) with \( a \in \mathcal{A}, b \in \mathcal{A}, a \leq b \) are distinct. In other words \( \mathcal{A} \) is a Sidon set if \( R_3(\mathcal{A}, n, 2) \leq 1 \). In [3] and [4] P. Erdős, A. Sárközy and V. T. Sós asked if there exists a Sidon set which is an asymptotic basis of order 3. The problem also appears in [9] (with a typo in it: order 2 is written instead of order 3). In [6] G. Grekos, L. Haddad, C. Helou and J. Pihko proved that a Sidon set cannot be an asymptotic basis
of order 2. Recently J. M. Deshouillers and A. Plagne in [1] constructed a Sidon set which is an asymptotic basis of order at most 7. In this thesis (see also in [21]) I improve on this result by proving:

**Theorem 6.** There exists an asymptotic basis of order 5 which is a Sidon set.

**References**


