Summary of the Ph.D. dissertation

Edge-connectivity augmentation of graphs and hypergraphs

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1 Introduction

The dissertation is devoted to edge-connectivity augmentation of graphs and hypergraphs: with an approach different from earlier ones we managed to generalize many known results of this area, and we have given simple algorithmic proofs of these old and new results. The algorithms are usually stated in an abstract form and the considered specific edge-connectivity augmentation problems are the applications of these abstract algorithms. Another merit of the dissertation is that through a unified approach it shows the connections of some results which were not compared this way before. We consider two notions of edge-connectivity augmentation. The first is the classical approach when we increase the edge-connectivity of a graph or hypergraph by introducing new edges or hyperedges. This is the toppic of the main part of the dissertation (Chapters 2-5). Although the starting structure can sometimes even be a directed graph or hypergraph, we emphasize that the new (hyper)edges are always undirected in this thesis. The objective is almost always to minimize the total size of these new (hyper)edges.

On the other hand, in Chapter 6 about *source location* we consider a different notion of edge-connectivity augmentation. Here the aim is to find a suitable, but smallest possible set of nodes S, such that contracting S gives the required connectivity.

2 Edge-connectivity augmentation by adding (hyper)edges

Chapter 2 is still an introductory chapter in the thesis. First we show how to formulate the considered edge-connectivity augmentation problems as a **covering problem.** By covering problem we mean that we are given a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ (which will also be called the **deficiency function**) that we want to **cover** with a graph or hypergraph G, which simply means that $d_G(X) \geq p(X)$ has to hold for every $X \subseteq V$ (here $d_G(X)$ is the number of (hyper)edges entering the set X). In Section 2.1 we show that the investigated edge-connectivity augmentation problems can indeed be formulated as a covering problem with a suitably chosen deficiency function p. This section shows the connections and the main differences between these edge-connectivity augmentation problems based on the properties of their deficiency functions. It is important to note that every considered problem has immediately two fundamentally different versions, depending on whether the new hyperedges can be arbitrarily large or they can not (for example we only allow graph edges). Let us give some definitions. If p is a set function and $X, Y \subseteq V$ are subsets then we consider the following two inequalities:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y), \qquad (\cap\cup)$$
$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \qquad (-)$$

A set $X \subseteq V$ is called p-positive if $p(X) > 0$. A pair of sets X, Y is called **crossing** if $X \cap Y, X - Y, Y - X$ and $V - (X \cup Y)$ are all nonempty sets. The most general class of set function in the thesis is the class of **positively skew-supermodular functions.** A function p is called **skew-supermodular** if at least one of $(\cap\cup)$ and $(-)$ holds for every pair $X, Y \subseteq V$.

If we only require that at least one of these two inequalities has to hold for p-positive set pairs, then p is called **positively skew-supermodular**. In many cases however the deficiency function has an even more specific property. A set function p is called **(positively) crossing supermodular** if $(\cap\cup)$ holds for every (p-positive) crossing pair X, Y. The function p is called **(positively) crossing negamodular** if $(-)$ holds for every (p-positive) crossing pair X, Y. We point out that the adjective "positively" raises algorithmic difficulties, but in the considered edge-connectivity augmentation applications this fortunately does not emerge. A set function p is **symmetric** if $p(X) = p(V - X)$ holds for every $X \subseteq V$. Symmetric crossing supermodular functions are the "simplest" functions that we consider: they emerge in **global edge-connectivity augmentation problems**. These functions are of course also skew-supermodular which is not necessarily true without the symmetry. An important notion is the **symmetrized** of a set function: if p is any set function then its symmetrized p^s is defined as $p^s(X) = \max\{p(X), p(V - X\}$ for any $X \subseteq V$. Obviously, a hypergraph covers p if and only if it covers p^s. The **global arc-connectivity augmentation problem of directed (more generally mixed) graphs or hypergraphs** (with undirected edges or hyperedges) can be formulated as covering a crossing supermodular function. On the other hand, the so called **node-to-area edge-connectivity augmentation problem** can be considered as covering a crossing negamodular function. We also discuss variants of the **local edge-connectivity augmentation problem**: here the symmetric skew-supermodular function to be covered has other special properties. Since the versions of these problems are the main applications of our results in this thesis, we formulate these problems explicitly.

Problem 2.1 (Local edge-connectivity augmentation problem) *Let H_0 be a (hyper)-graph and let $r : V \times V \to \mathbb{Z}_+$ be a symmetric edge-connectivity requirement. Our aim is to find a (hyper)graph H such that $\lambda_{H_0+H}(u, v) \geq r(u, v)$ for every pair of nodes u, v.*

The **global edge-connectivity augmentation of hypergraphs** is the special case when $r(u, v) = k$ for every pair $u, v \in V$. Let us define the λ function (**edge-connectivity**) in a more general context.

Definition 2.2 *A **mixed hypergraph** $M = (V, \mathcal{A})$ is a pair of a finite set V and a family $\mathcal{A}$ containing nonempty ordered pairs of subsets of V (the same pair can occur more than once). The elements of $\mathcal{A}$ are called **hyperarcs**. For a hyperarc $a = (T_a, H_a) \in \mathcal{A}$, the set T_a is called the **tail set** of a, while H_a is called the **head set** of a. A **path** between nodes s and t is an alternating sequence of distinct nodes and hyperarcs $s = v_0, a_1, v_1, a_2, \ldots, a_k, v_k = t$, such that v_{i-1} is a tail node of a_i and v_i is a head node of a_i for all i between 1 and k.*

More intuitively we can think of a hyperarc a as the subset $T_a \cup H_a$ of V, in which every node has a "role": it is either a **head node**, a **tail node** or even both (**head-tail node**). An undirected hypergraph (shortly hypergraph) is a special mixed hypergraph in which every node of a hyperarc is a head-tail node of that hyperarc. On the other hand, a path in a mixed hypergraph is simply the following: we start from a node which is the tail of a hyperarc, and we jump to a head of this hyperarc, from where we can proceed using another hyperarc of which this new node is a tail node (of course, we have to take care of the repetitions). After

these preliminaries the most general definition of edge-connectivity needed in this thesis is the following.

Definition 2.3 *Given a mixed hypergraph $M = (V, \mathcal{A})$ and sets $S, T \subseteq V$, the **edge-connectivity between** S **and** T denoted by $\lambda_M(S, T)$ is the maximum number of arc-disjoint paths starting in S and ending in T (we say that $\lambda_M(S, T) = \infty$ if $S \cap T \neq \emptyset$). If $s, t \in V$ then let $\lambda_M(s, t) = \lambda_M(\{s\}, \{t\})$.*

This definition specialized to hypergraphs and further to graphs gives back the usual notion of edge-connectivity. With these definitions we formulate the following problems.

Problem 2.4 (Global arc-connectivity augmentation of mixed hypergraphs) *Let $M = (V, \mathcal{A})$ be a mixed hypergraph, $r \in V$ be a designated root node, and k, l be nonnegative integers. Find a (hyper)graph $H = (V, \mathcal{E})$ such that $\lambda_{M+H}(r, v) \geq k$ and $\lambda_{M+H}(v, r) \geq l$ for any $v \in V$.*

Problem 2.5 (Node-to-area connectivity augmentation problem) *Given a (hyper)graph $H_0 = (V, \mathcal{E}_0)$, a collection of subsets $\mathcal{W}$ of V and a function $r : \mathcal{W} \to \mathbb{Z}_+$, our aim is to find a (hyper)graph H such that*

$$\lambda_{H_0+H}(x, W) \geq r(W) \text{ for any } W \in \mathcal{W} \text{ and } x \in V. \quad (1)$$

The variants of these problems are the applications of the results given in Chapter 2-5 of the dissertation. We don't give the definitions of the deficiency functions belonging to these problems, this can be found in the thesis. We did not state the objective function to be optimized in the above problems: the reader can think of the minimization of the total size of the hypergraph H, we will speak about other possible objective functions later. Let us also formulate a general covering problem for later references.

Problem 2.6 (Covering Problem) *Given a symmetric, positively skew-supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ satisfying $p(\emptyset) \leq 0$, find a (hyper)graph H satisfying $d_H(X) \geq p(X)$ for every $X \subseteq V$.*

The edge-connectivity augmentation problems investigated in Chapters 2-5 are reformulated into such a covering problem with a positively skew-supermodular deficiency function (possibly with more special properties in the specific problems). Every problem has many versions depending on, for example, whether the covering hypergraph H can contain arbitrarily large hyperedges or it cannot. We can also consider variants of the problems differing in their objective function. In the **minimum version** the aim is to minimize the total size of the hypergraph H. However, because of the skew-supermodularity it is more convenient to consider the more general **degree-specified version** of the problems, where we are also given a function $m : V \to \mathbb{Z}_+$ and we are looking for a hypergraph H satisfying this degree-specification at the nodes, meaning that $d_H(v) = m(v)$ has to hold at every node $v \in V$. The connection between these two versions is given by the contrapolymatroid $C(p)$ determined by the "good" degree-specifications:

$$C(p) = \{x \in \mathbb{R}^V : x(Z) \geq p(Z)\ \forall Z \subseteq V, x \geq 0\}. \quad (2)$$

The integer vectors in $C(p)$ are called **admissible degree-specifications**. The properties of the polyhedron $C(p)$ imply that the total size of a hypergraph covering p is at least $SLB(p) = \max\{\sum_{X\in\mathcal{X}} p(X) : \mathcal{X}$ is a subpartition of $V\}$. It is also due to the properties of a contrapolymatroid that we can usually handle the so called **minimum node-cost version** of these problems: here the target is to find a hypergraph H minimizing the sum $\sum_{v\in V} c(v)d_H(v)$ where $c : V \to \mathbb{R}_+$ are given node costs.

3 Edge-connectivity augmentation by adding hyperedges

In Chapter 3 we consider the problem of covering a positively skew-supermodular function by (arbitrarily large) hyperedges. Since the size of the hyperedges is not bounded, this problem can be solved in its whole generality, the solution is due to Szigeti [6]. Let us define the following function for a hypergraph $H = (V, \mathcal{E})$:

$$b_H(X) = |\{e \in \mathcal{E} : e \cap X \neq \emptyset\}|.$$

We say that the hypergraph H **weakly covers** the set function p if $b_H(X) \geq p(X)$ holds for any $X \subseteq V$. Based on the following two observations we managed to generalize Szigeti's results in many directions. The first observation says that Schrijver's supermodular colouring theorem (and its polyhedral proof due to Tardos) can in fact be extended to skew-supermodular functions, and that this is strongly related to the notion of weak covering defined above.

Theorem 3.1 ([10], with Tamás Király) *Let p be a positively skew-supermodular function, $k \geq \max\{p(X) : X \subseteq V\}$ an integer and $y \in C(p) \cap \mathbb{Z}^V$ such that $y(v) \leq k$ holds for every $v \in V$. Let us define the following polyhedron:*

$$Q = Q(p,k,y) = \{x \in \mathbb{R}^V : 0 \leq x \leq 1;\ x(v) = 1 \text{ if } y(v) = k;$$
$$x(Z) \geq 1 \text{ if } p(Z) = k;\ x(Z) \leq y(Z) - p(Z) + 1\ \forall Z \subseteq V;\ x \leq y\}.$$

Then Q is an integer g-polymatroid and an integer vector in Q corresponds to a hyperedge in a hypergraph H that contains exactly k hyperedges, weakly covers p and satisfies the degree-specification y.

The second observation is the following.

Lemma 3.2 ([10], with Tamás Király) *If $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is a symmetric, positively skew-supermodular function, $k = \max\{p(X) : X \subseteq V\}$ and H is a hypergraph which contains exactly k hyperedges and weakly covers p then H in fact covers p.*

Using the known properties of g-polymatroids (and their intersections) these observations imply the following corollary (we note that in the following results the number of hyperedges in H is always fewest possible).

Corollary 3.3 ([10], with Tamás Király) *If p is a symmetric, positively skew-supermodular function then the hypergraph of minimum total size covering p can be chosen nearly uniform. If p_1 and p_2 are two such functions satisfying $\max\{p_1(X) : X \subseteq V\} = \max\{p_2(X) : X \subseteq V\} = k$ and $y \in C(p_1) \cap C(p_2)$ is vector satisfying $y(v) \leq k$ for every node $v \in V$ then there exists a hypergraph H containing exactly k hyperedges that covers both p_1 and p_2 and satisfies the degree-specification y.*

If we apply this to problems 2.1, 2.4 and 2.5 we obtain that their optimal solution can be chosen nearly uniform, or we can solve two such problems simultaneously, if the (fairly artificial) condition on the maximum deficiencies holds. Unfortunately we still need this condition, since we show in the thesis that without this condition we obtain NP-complete problems.

4 Edge-connectivity augmentation by adding graph edges

In Chapters 4 and 5 we investigate what happens if we try to cover the deficiency function only with graph edges (with fewest possible, say). The usual technique to solve the degree-specified version of such problems is **splitting-off**: for a given $m \in C(p) \cap \mathbb{Z}^V$ (admissible degree-specification) and a pair of nodes u, v (satisfying that $m(u)$ and $m(v)$ are both positive) we try to include the edge uv in the solution. More formally we substitute the functions p and m by the modified functions p' and m', where

$$m' = m - \chi_{\{u\}} - \chi_{\{v\}} \text{ and } p' = p - d_{(V,\{(uv)\})}. \quad (3)$$

The splitting-off operation is **admissible** if $m' \in C(p')$ holds (in other words, starting from an admissible degree-specification it creates another admissible degree-specification). The starting point of the results is Lemma 4.1 which is proved with a new approach to splitting-off.

Lemma 4.1 ([11], with Tamás Király) *If $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is a symmetric, positively skew-supermodular function, $m \in C(p) \cap \mathbb{Z}^V$ and $p(X) > 1$ for some set $X \subseteq V$ then there is an admissible splitting-off.*

A direct consequence of this lemma is the following new generalization of Szigeti's above mentioned theorem: while so far we tried to cover p with a fewest number of hyperedges, here we try to cover it with smallest possible hyperedges.

Corollary 4.2 ([11], with Tamás Király) *If $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is a symmetric, positively skew-supermodular function, $m \in C(p) \cap \mathbb{Z}^V$ then there exists a hypergraph H satisfying the degree-specification m and covering p that contains only one large hyperedge.*

As further applications of Lemma 4.1 we give simple proofs for known results in Section 4.2.2. These proofs are included for didactic reasons: we think that these proofs can be taught well.

Lemma 4.1 suggests the idea of a simple algorithm that performs admissible splitting-offs in a greedy way, and when it gets stuck then finishes by adding one large hyperedge to the graph edges found so far. This algorithm (in a more or less modified form) and this optimistic greedy approach appears many times later on as well. This approach also motivates the **rank respecting versions** of the considered problems. This only means that in problems 2.1, 2.4 and 2.5 we want that the rank of the hypergraph H in demand should not exceed the rank of the starting (mixed) hypergraph (the objective function is the same). In Sections 4.3-4.4 we show that the greedy algorithm sketched above applied for these problems (and even for some suitable generalizations) does not give a very bad answer: it will not increase the rank too much. Let us look at these results in more detail.

In Section 4.3 we investigate the case when there is no further admissible splitting-off (that is, the greedy algorithm is stuck), and our function has some special form. In Section 4.3.1 we assume that the function p is the symmetrized of a positively crossing supermodular function q. We make the surprising observation that –after contraction of tight sets– every subset will have p value 1. This induces the question: characterize the crossing set families $\mathcal{F} \subseteq 2^V$ for which $\mathcal{F} \cup \mathrm{co}(\mathcal{F}) = 2^V$ (where a family $\mathcal{F} \subseteq 2^V$ is called **crossing** if $X \cap Y, X \cup Y \in \mathcal{F}$ for every crossing pair $X, Y \in \mathcal{F}$, and $\mathrm{co}(\mathcal{F}) = \{X \subseteq V : V - X \in \mathcal{F}\}$ for a set family $\mathcal{F} \subseteq 2^V$). An example of such a family is the following: let $x \in V$ arbitrary and let $X_1, \ldots, X_t$ be pairwise disjoint subsets of $V - x$ for some $t \geq 1$ (possibly $t = 1$ and $X_1 = \emptyset$). Consider the following family:

$$\mathcal{F}_{x,X_1,\ldots,X_t} = \{X \subseteq V : x \in X \text{ or } X \subseteq X_i \text{ for some } i \in 1, \ldots, t\}.$$

The following theorem, which is interesting also in itself, gives the characterisation of the families above.

Theorem 4.3 ([11], with Tamás Király) *Let $\mathcal{F} \subseteq 2^V$ be a crossing family with $\emptyset, V \in \mathcal{F}$ that satisfies $\mathcal{F} \cup \mathrm{co}(\mathcal{F}) = 2^V$. Then either V has exactly four elements and $\mathcal{F} = 2^V \setminus \{\{y, z\}\}$ for some $y \neq z$, $y, z \in V$ or there exists a node $x \in V$ and $X_1, \ldots, X_t$ pairwise disjoint subsets of $V - x$ for some $t \geq 1$ such that either $\mathcal{F}$ or $\mathrm{co}(\mathcal{F})$ is equal to $\mathcal{F}_{x,X_1,\ldots X_t}$ or $\mathcal{F}_{x,X_1,\ldots X_t} \cup \{V - x\}$.*

In Section 4.3.2 we investigate the stuck situation of the greedy algorithm applied to the symmetrized of a crossing negamodular function q. Since covering such a function only with graph edges already includes NP-hard questions (see the node-to-area problem in graphs), we need to make further restrictions on the function q. The assumption of Ishii and Hagiwara suggests to consider the case when $q = R - d_{H_0}$, where R is a crossing negamodular function that does not take the value 1, and d_{H_0} is the degree function of an arbitrary hypergraph H_0. We prove the following lemma about the stuck situation (the node $v \in V$ is called **positive** if $m(v) > 0$).

Lemma 4.4 ([11], with Tamás Király) *If q is a function of the above form, $p = q^s$, $m \in C(p) \cap \mathbb{Z}^V$, there is no admissible splitting-off and $m(V) \geq 5$ then there exists a hyperedge in H_0 that contains more than 1 positive nodes. Furthermore there is at most one positive node that is avoided by such hyperedges.*

As a consequence of this lemma we get the following surprising statement about the rank respecting version of Problem 2.5: the greedy algorithm increases the rank by at most 1 if the

starting hypergraph H_0 had rank more than 2, but it can increase the rank with 2 if H_0 was a graph.

In Section 4.4 we investigate what happens if we apply the greedy algorithm to Problems 2.1, 2.4 and 2.5. First we show that the algorithm applied to the local edge-connectivity augmentation of hypergraphs gives an optimal solution without increasing the rank. We remark that after we have proved this result it turned out that it was also proved by Ben Cosh in [4]: in fact by mixing his proof with ours we could give a relatively simple proof for this statement. In Section 4.4.2 we consider the global arc-connectivity augmentation of mixed hypergraphs and using the results of Section 4.3.1 we show that the greedy algorithm applied to this problem increases the rank by at most 1. Finally in Section 4.4.3 we define the following generalization of the rank respecting version of Problem 2.5.

Problem 4.5 *Let $q = R - d_{H_0}$ where R is a crossing negamodular function that does not take the value 1, and d_{H_0} is the degree function of an arbitrary hypergraph H_0. Find a hypergraph H of minimum total size covering q such that the rank of H does not exceed that of H_0.*

Then we show that a (simple) modification of the greedy algorithm solves this problem if the rank of H_0 is at least 3 (if it is 2, then we need more sophisticated modifications, but this is described in [5]). The result of Section 4.4.3 appeared in [8].

5 Covering symmetric crossing supermodular functions with graph edges

If the function satisfies the supermodular inequality for any crossing set pair, and it is even symmetric, then covering it with a minimum number of graph edges can be solved in polynomial time. This results is due to Benczúr and Frank. A partition $\mathcal{X} = \{X_1, X_2, \ldots, X_t\}$ of V is called p-**full** if $p(\cup_{i \in I} X_i) > 0$ for any nonempty $I \subsetneq \{1, 2, \ldots, t\}$. The maximum cardinality of a p-full partition is the **dimension of** p and is denoted by $\dim(p)$.

Theorem 5.1 (Benczúr and Frank [3]) *The minimum number of graph edges covering a symmetric, positively crossing supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ equals $\max\{\lceil SLB(p)/2 \rceil, \dim(p) - 1\}$.*

In Section 5.3 we give a relatively simple algorithmic proof of this theorem, this appeared in [7]. The simplicity of this proof is demonstrated by the fact that after the suitable modifications we can even solve the partition constrained version of the problem, which is the following.

Problem 5.2 *Given a symmetric, positively crossing supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ and a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of V, we are looking for a graph G covering p that has only edges between different members of the partition $\mathcal{P}$.*

A special case of this problem, the **partition constrained global edge-connectivity augmentation of graphs** was solved by Bang-Jensen, Gabow, Jordán and Szigeti in [2]. Together with Roland Grappe and Zoltán Szigeti we managed to handle the more general

Problem 5.2, too. For the solution of the minimum version of the problem consider the following lower bounds (the solution of the degree-specified version is not detailed in this summary because of the space limitations).

$$\beta_p^i = \max\{\sum_{Y \in \mathcal{F}} p(Y) : \ \mathcal{F} \text{ is a subpartition of } P_i\} \text{ for every } i = 1, \dots r.$$

Obviously, $\phi_p = \max\{\lceil SLB(p)/2 \rceil, \beta_p^1, \dots, \beta_p^r, \dim(p) - 1\}$ is a lower bound on the minimum number of graph edges covering p and satisfying the partition constraints. We managed to show algorithmically that this bound can almost always be achieved, except in some particular cases (called **configurations**), when we need one more edge. The definitions of C_4^*-, C_5^*- and C_6^*-configurations appearing in the theorem below is not given here, again for space limitations.

Theorem 5.3 ([12], with Roland Grappe and Zoltán Szigeti) *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively crossing supermodular function and $\mathcal{P} = \{P_1, P_2, \dots, P_r\}$ be a partition of V. The minimum number of graph edges covering p and satisfying the partition constraints is ϕ_p, unless a C_4^*-, a C_5^*- or a C_6^*-configuration exists for $(p, \mathcal{P})$, when the minimum is $\phi_p + 1$.*

If we specialize this theorem for the **global edge-connectivity augmentation problem of hypergraphs with partition constrained graph edges** then we get the following result (in the theorem below the $\mathcal{C}_4$- and the $\mathcal{C}_6$-configurations can be obtained by specializing the C_4^*- and the C_6^*-configurations appearing in the abstract Theorem 5.3, details are again omitted due to space limitations; it is worth to note however that the C_5^*-configuration arises only in the abstract Problem 5.2).

Theorem 5.4 ([13], with Roland Grappe and Zoltán Szigeti) *Let $H_0 = (V, \mathcal{E}_0)$ be a hypergraph, $\mathcal{P}$ be a partition of V and k be an arbitrary positive integer. The minimum number of graph edges between different classes of the partition $\mathcal{P}$ that result in a k-edge-connected hypergraph when added to H_0 is $\phi_{p_0} + 1$, if a $\mathcal{C}_4$- or a $\mathcal{C}_6$-configuration exists for H_0, and it is ϕ_{p_0} otherwise, where p_0 is defined with $p_0(X) = k - d_{H_0}(X)$ for any nonempty $X \subsetneq V$, and $p_0(\emptyset) = p_0(V) = 0$.*

6 Source Location Problems

In Chapter 6 we investigate the following hypergraphic generalizations of the Source Location Problem.

Problem 6.1 *Given a hypergraph $H = (V, \mathcal{E})$, a weight function $w : V \to \mathbb{R}_+$ and a requirement function $r : V \to \mathbb{R}_+$. Find a mimimum weight subset of the nodes S such that $\lambda_H(S, v) \geq r(v)$ for every $v \in V$.*

This problem was investigated in [1] in the case when H only contains graph edges and it was shown there that this problem is NP-complete in general, but if either of r or w is

constant, then the problem can be solved in polynomial time. We also formulate the following abstract form of Problem 6.1 (a set function d is called **posimodular** if $-d$ is negamodular).

Problem 6.2 *Given a posimodular and submodular function $d : 2^V \to \mathbb{R}_+$, a weight function $w : V \to \mathbb{R}_+$ and a requirement function $r : V \to \mathbb{R}_+$. Find a mimimum weight subset of the nodes S such that*

$$d(X) \geq \max\{r(v) : v \in X\} \text{ for every } X \subseteq V - S. \tag{4}$$

By generalizing the methods of [1] we show that the abstract Problem 6.2 can also be solved if the functions r and w are **compatible**, which means that there is an ordering $v_1, v_2, \dots, v_n$ of V such that $r(v_1) \leq r(v_2) \leq \dots \leq r(v_n)$ and $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$. We show that a simple greedy algorithm solves the abstract Problem 6.2 (with compatible weights and requirements) in polynomial time even if d does not satisfy the submodularity, however in order to implement this algorithm one would need to minimize an intersecting posimodular function, which is an open problem. On the other hand, if d is also submodular then this can be solved with standard submodular function minimization techniques. We furthermore show that a little more sophisticated algorithm improves the running time in the case when the requirement function is constant. Specializing these results for Problem 6.1 we get the following application (where $M(n', m')$ denotes the running time of a maximum flow computation in a graph with n' nodes and m' edges, and the total size of a hypergraph $H = (V, \mathcal{E})$ is denoted by $||\mathcal{E}||$).

Theorem 6.3 ([9]) *Problem 6.1 can be solved in $O(nM(n + |\mathcal{E}|, ||\mathcal{E}||))$ time if the functions r and w are compatible. The running time can be improved to $O(n^2 \log(n) + n||\mathcal{E}||)$ if the function r is constant.*

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