1 Introduction

The dissertation is devoted to edge-connectivity augmentation of graphs and hypergraphs with an approach different from earlier ones we managed to generalize many known results of this area and we have given simple algorithmic proofs of those old and new results. The algorithms are usually stated in an abstract form and the considered specific edge-connectivity augmentation problems are the applications of these abstract algorithms. Another merit of the dissertation is that through a unified approach we show the connections of some results which were not compared this way before. We consider two notions of edge-connectivity augmentation. The first one is the classical approach where we increase the edge-connectivity of a graph or hypergraph by introducing new edges or hyperedges. This is the topic of the main part of the dissertation (Chapters 2-5). Although the starting structure can sometimes even be a directed graph or hypergraph we emphasize that the new (hyper)edges are always undirected in this thesis. The objective is almost always to minimize the total size of these new (hyper)edges.

On the other hand, in Chapter 6 about source location we consider a different notion of edge-connectivity augmentation. Here the aim is to find a suitable but smallest possible set of nodes $S$ such that contracting $S$ gives the required connectivity.

2 Edge-connectivity augmentation by adding (hyper)edges

Chapter 2 is still an introductory chapter in the thesis. First we show how to formulate the considered edge-connectivity augmentation problems as a covering problem. By covering problem we mean that we are given a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ which will also be called the deficiency function that we want to cover with a graph or hypergraph $G$ which simply means that $d_G(V) \geq p(X)$ has to hold for every $X \subseteq V$ (here $d_G(X)$ is the number of (hyper)edges entering the set $X$). In Section 2.1 we show that the investigated edge-connectivity augmentation problems can indeed be formulated as a covering problem with a suitably chosen deficiency function $p$. This section shows the connection and the main differences between these edge-connectivity augmentation problems based on the properties of their deficiency functions. It is important to note that every considered problem has immediately two fundamentally different versions depending on whether the new hyperedges can be additionally large or they can not (for example we only allow graph-edges). Let us give some definitions. If $p$ is a set function and $X, Y \subseteq V$ are subsets then we consider the following two inequalities

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y).$$

(1)

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X).$$

(-1)

A set $X \subseteq V$ is called $p$-positive if $p(X) > 0$. A pair of sets $X, Y \subseteq V$ is called crossing if $X \cap Y, X - Y, Y - X$ and $V - (X \cup Y)$ are all nonempty sets. The most general class of set functions in the thesis is the class of positively supermodular functions. A function $p$ is called positively supermodular if at least one of (1) and (-1) holds for every pair $X, Y \subseteq V$. 

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Summary of the Ph.D. dissertation

Edge-connectivity augmentation of graphs and hypergraphs

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If we only require that at least one of these two inequations hold for $p$-positive set pairs then $p$ is called positively slow-supersingular. In many cases however the deficiency function has an even more specific property: a set function $p$ is called (positively) crossing supermodular if $p(A) = 0$ holds for every $p$-positive crossing pair $X,Y$. The function $p$ is called (positively) crossing supermodular if $\{A\} \subseteq X \cup Y$. We point out that the adjective "positively" makes algorithmic difficulties but in the considered edge-connectivity augmentation applications this fortunately does not emerge. A set function $p$ is symmetric if $p(X) = p(X \cup Y)$ holds for every $X \subseteq V$. Symmetric crossing supermodular functions are the "simplest" functions that we consider: they emerge in global edge-connectivity augmentation problems. These functions are of course also slow-supersingular which hints necessarily true without the symmetry. An important notion in the symmetrized of a set function $p$ is its symmetrized $p^*$ which is defined as $p^*(X) = \max\{p(X), p(X \cup Y)\}$ for any $X \subseteq V$. Obviously a hypergraph covers $p$ if and only if it covers $p^*$. The global edge-connectivity augmentation problem of directed (more generally mixed) graphs or hypergraphs with unmatched edges or hyperedges) can be formulated as covering a crossing supermodular function. On the other hand the so-called node-to-area edge-connectivity augmentation problem can be considered as covering a crossing supermodular function. We also discuss variants of the local edge-connectivity augmentation problems for symmetric slow-supersingular function to be covered has other special properties. Since the versions of these problems are the main applications of our results in this thesis we formulate these problems explicitly.

**Problem 2.1 (Local edge-connectivity augmentation problem)** Let $H = (V,A)$ be a (hyper-)graph and let $r : V \times V \rightarrow \mathbb{Z}$ be a symmetric edge-connectivity requirement. Our aim is to find a (hyper)graph $H'$ such that $\lambda_{H',H}(u,v) \geq r(u,v)$ for every pair of nodes $u,v$. The global edge-connectivity augmentation problem of hypergraphs in the special case when $\lambda_{H',H}(u,v) = k$ for every pair of nodes $u,v$. Let us define the $\lambda$ function (edge-connectivity) in a more general context.

**Definition 2.2** A mixed hypergraph $H = (V,A)$ is a pair of a finite set $V$ and a family $A$ containing nonempty ordered pairs of subsets of $V$ (the same pair can occur more than once). The elements of $A$ are called hyperedges. For a hyperedge $h = (T,H) \in A$, the set $T$ is called the tail set of $h$, while $H$ is called the head set of $h$. A path between nodes $x$ and $y$ in an alternating sequence of distinct nodes and hyperedges $x = v_0, t_1, h_1, t_2, \ldots, v_n = y$, such that $t_{i+1}$ is a tail node of $h_i$ and $h_i$ is a head node of $v_i$, for all $i$ is termed $\lambda$-path.

More intuitively we can think of an hyperedge as the subset $T \cup H$ of $V$ in which every node has a "role". It is either a head node a tail node or both (head-tail node). An mixed mixed hypergraph (shortly hypergraph) is a special labeled hypergraph in which every node of a hyperedge is a tail node of that hyperedge. On the other hand a path in a mixed hypergraph is simply the following: we start from a node which is the tail of a hyperedge and we jump to a head of a hyperedge from which we can proceed using another hyperedge of which this new node is a tail node (of course we have to take care of the directions). After these preliminaries the main general definition of edge-connectivity needed in this thesis is the following.

**Definition 2.3** Given a mixed hypergraph $H = (V,A)$ and sets $S,T \subseteq V$, the edge-connectivity between $S$ and $T$ denoted by $\lambda_{H}(S,T)$ is the maximum number of arcs (edges) of paths starting in $S$ and ending in $T$. (i. e. $S \neq \emptyset \neq T$) If $\lambda_{H}(S,T) = \infty$ if $S \neq \emptyset \neq T$. Hence we have $\lambda_{H}(S,T) = \lambda_{H}(T,S)$.

This definition specializes to hypergraphs and further to graphs gives back the usual notion of edge-connectivity. With these definitions we formulate the following problems.

**Problem 2.4 (Global edge-connectivity augmentation of mixed hypergraphs)** Let $H = (V,A)$ be a mixed hypergraph, $r : V \rightarrow \mathbb{R}$ be a designated real node, and $\delta_H$ be a nonempty set of edges. Find a (hyper)graph $H' = (V,E)$ such that $\lambda_{H',H}(u,v) \geq k$ and $\lambda_{H',H}(u,v) \geq l$ for any $u,v \in V$.

**Problem 2.5 (Node-to-area edge-connectivity augmentation problem)** Given a (hyper)graph $H = (V,E)$, a collection of subsets $W$ of $V$ and a function $r : V \rightarrow \mathbb{Z}$, our aim is to find a (hyper)graph $H'$ such that $\lambda_{H',H}(W,u,v) \geq r(u,v)$ for any $W \subseteq V$ an $r \in V$.

The variants of these problems are the applications of the results given in Chapter 2-5 of the dissertation. We don’t give the definitions of the deficiency functions belonging to these problems this can be found in the thesis. We did not state the objective function to be optimized in the above problems: the reader can think of the minimum of the total size of the hypergraph $H$ we will speak about other possible objective functions later. Let us also formulate a general covering problem for later references.

**Problem 2.6 (Covering Problem)** Given a symmetric, positive slow-supersingular function $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying $p(0,0) = 0$, find a (hyper)graph $H$ satisfying $\lambda_{H}(0,0) = p(0,0)$ for $\forall x \in \mathbb{R}_p$.

The edge-connectivity augmentation problems investigated in Chapters 2-5 are reformulated into such a covering problem with a positively slow-supersingular deficiency function (possibly with more special properties in the specific problem). Every problem has many versions depending on for example whether the covering hypergraph $H$ can contain arbitrarily large hyperedges or not. We can also consider variants of the problems differing in their objective function. In the minimum version the aim is to minimize the total size of the hypergraph $H$. However, because of the slow-supersingular property, it is more convenient to consider the more general degree-specified version of the problems where we are also given a function $\delta_h : V \rightarrow \mathbb{Z}$ and we are looking for a hypergraph $H$ satisfying this degree-specification at the nodes meaning that $\delta_{\lambda}(v) = \delta_h(v)$ has to be held at every node $v \in V$. The connection between these two versions is given by the contrapositive: if $C(p)$ determined by the “good” degree-specified form

$$C(p) = \{x \in \mathbb{R}^2 : \lambda(0,x) \geq p(x) \} \forall x \in \mathbb{R}_p \geq 0.$$
The integer vectors in $\mathcal{C}(\gamma)$ are called admissible degree-specifications. The properties of the polyhedron $\mathcal{P}(\gamma)$ imply that the total size of a hypergraph covering $p$ is at least $\text{SLR}(\gamma) = \max \sum_{v \in V} c(v) d(v)$, where $c.V \rightarrow \mathbb{R}$ are given node costs.

3 Edge-connectivity augmentation by adding hyperedges

In Chapter 3 we consider the problem of recovering a positively slew-supernodal function by (arbitrarily) hyperedges. Since the size of the hypergraph is not bounded, the solution can be selected in its whole generality, so the solution is due to Krödel [9]. Let us define the following function for a hypergraph $H = (V,E)$:

$$h_0(X) = \{x \in E : x \cap X \neq \emptyset\}.$$ 

We see that the hypergraph $H$ weakly covers the set function $p$ if $h_0(X) \supseteq \gamma$ holds for any $X \subseteq V$. Based on the following two observations we managed to generalize Krödel's results in main directions. The first observation says that Schrijver's supermodular coloring theorem (and its polyhedron $\mathcal{P}(\gamma)$ due to Todd) can in fact be extended to slew-supernodal functions and that this is strongly related to the notion of arc covering defined above.

Theorem 3.1 [9], with Todd Krödel) If $p$ is a monadically slew-supernodal function, $k \geq \max(p(X) : X \subseteq V)$ is an integer and $\gamma \in \mathcal{C}(\gamma) \cap \mathbb{Z}^V$ such that $\gamma(v) \leq k$ holds for every $v \in V$. Let us define the following polyhedron:

$$Q = \{q(p,k,y) = (x \in \mathbb{R}^V : 0 \leq x \leq 1, x(v) = 1 \text{ if } \gamma(v) = k, x(z) \geq 1 \text{ if } p(z) = k, x(z) \geq y(z) - p(z) + 1 \forall z \in V, x \leq \gamma\}.$$ 

Then $Q$ is an integer polyhedron and an integer vector in $Q$ corresponds to a hypergraph in $\mathcal{P}(\gamma)$ that contains every $k$ hyperedge, weakly covers $p$, and satisfies the degree-specification $\gamma$.

The second observation is the following:

Lemma 3.2 [9], with Todd Krödel) If $p : \mathbb{Z} \rightarrow \mathbb{Z}([x] \in \mathbb{Z}^V)$ is a supernodal, monadically slew-supernodal function, $k = \max(p(X) : X \subseteq V)$ and $H$ is a hypergraph which contains every $k$ hyperedge and weakly covers $p$, then $H$ is an integer polyhedron.

Using the known properties of $\gamma$-polyhedrals and their intersections we derive the following corollary: we note that in the following result the number of hyperedges in $H$ is always fixed.

Corollary 3.3 [9], with Todd Krödel) If $p$ is a supernodal, monadically slew-supernodal function then the hypergraph of minimum total cost covering $p$ can be chosen weakly uniform.

4 Edge-connectivity augmentation by adding graph edges

In Chapters 4 and 5 we investigate what happens if we try to cover the deficiency function only with graph edges (and not possible, any). The usual techniques to solve the degree-specification version of such problems is splitting-off, for a given $m \in \mathcal{C}(\gamma) \cap \mathbb{Z}^V$ (admissible degree-specification) and a pair of nodes $u,v$ (asserting that $m(u)$ and $m(v)$ are both positive) we try to include the edge $uv$ in the solution. More formally we substitute the function $m$ and $m'$ by the modified functions $m + m'$ where

$$m' = m - \chi(u) - \chi(v)$$

and

$$m'' = m' - \chi(u)\chi(v).$$

The splitting-off operation is admissible if $m' \in \mathcal{C}(\gamma)$ holds (in other words starting from an admissible degree-specification it creates another admissible degree-specification). The starting point of the results in Lemma 4.1 which is proved with a new approach to splitting-off.

Lemma 4.1 [14], with Todd Krödel) If $p : \mathbb{Z} \rightarrow \mathbb{Z}([x] \in \mathbb{Z}^V)$ is a supernodal, monadically slew-supernodal function, $m \in \mathcal{C}(\gamma) \cap \mathbb{Z}^V$ and $p(X) > 1$ for some set $X \subseteq V$, then there is an admissible splitting-off.

A direct consequence of this lemma is the following generalization of Krödel's above mentioned theorem: we can then instead to cover $p$ with a fixed number of hyperedges here we try to cover it with smallest possible hyperedges:

Corollary 4.2 [14], with Todd Krödel) If $p : \mathbb{Z} \rightarrow \mathbb{Z}([x] \in \mathbb{Z}^V)$ is a supernodal, monadically slew-supernodal function, $m \in \mathcal{C}(\gamma) \cap \mathbb{Z}^V$ then there is a hypergraph if satisfying the degree-specification $m$ and covering $p$ that contains only one large hyperedge.

As further applications Lemma 4.1 we give simple proofs for known results in Section 4.2.2. These proofs are included for didactic reasons: we think that these proofs can be taught well.

Lemma 4.1 suggests the idea of a simple algorithm that performs admissible splitting-offs in a greedy way and when it gets stuck then finishes by adding one large hyperedge to the graph edges found so far. This algorithm (in a more or less modified form) and this specific
greedy approach appears many times but on as well. This approach also motivates the rank requiring versions of the considered problems. This only means that in Problems 2.1, 2.4 and 2.5 we worst that the rank of the hypergraph $H$ is defined should not exceed the rank of the starting (mixed) function $q$ where we defined function in Section 3.5. In Sections 4.3-4.4 we show that the greedy algorithm sketched above applied for these problems (and even for some suitable generalizations) does not give a very bad answer. It will not increase the rank too much. Let us look at these results in more detail.

In Section 4.3 we investigate the case when there is no further admissible splitting-off (i.e., the greedy algorithm is stuck) and our function has some special form. In Section 4.3.1 we assume that the function $p$ is the symmetric version of a positively crossing supermodular function $q$. We make the surprising observation that, after construction of $q$, every subset will have value 1. This implies the question: characterizes the crossing set family $F \subseteq \mathcal{2}^X$ for which $F \cup \{\emptyset\}$ is called crossing if $X \cap \bigcup E \in F$ for every crossing pair $X, Y \in F$ and $\emptyset \notin F \cup \{\emptyset\}$ is empty for a set family $F \subseteq \mathcal{2}^X$.

We consider the following: let $p \in \mathcal{2}^X$ arbitrary and let $(X_1, \ldots, X_t)$ be pairwise disjoint subsets of $V$, for some $t \geq 1$ (or possibly $t = 1$ and $X_1 = \emptyset$). Consider the following family $F_{X_1, \ldots, X_t} = \{X \subseteq V : x \in X \iff x \in X_i$ for some $i \in 1, \ldots, t\}$.

The following theorem is interesting also in itself gives the characterization of the families above.

**Theorem 4.3** (with Tamás Király) Let $F \subseteq \mathcal{2}^X$ be a crossing family with $\emptyset, V \in F$ that satisfies $F \cup \emptyset = F$. Then for any four elements $X, Y, Z, W \in \mathcal{2}^X \setminus \{\emptyset\}$ such that $X \cap Y \neq \emptyset$, $Z \cap W \neq \emptyset$, and $X \cap Z \neq \emptyset$, $Y \cap W \neq \emptyset$, $X \cap Y \neq Z \cap W$. For any such family $F$, $V$ is a maximal crossable set by $F$.

In Section 4.3.2 we investigate the structure of the greedy algorithm applied to the symmetric version of a positively crossing supermodular function $q$. Since covering such a function only with graph edges already includes NP-hard questions (see the node-2-problem in graphs) we need to make further restrictions on the function $q$. The assumption of Hall and Hajnycenko suggests to consider the case when $q = R - \partial q$, where $R$ is a crossing supermodular function that does not take the value 1 and $\partial q$ is the degree function of an arbitrary hypergraph $H_q$. We prove the following lemma about the structure of the greedy algorithm (the node $v \in V$ is called positive if $m(v) > 0$).

**Lemma 4.4** (with Tamás Király) If $Q(q) = X(q)$ is a function of the short form, $y = q - y, m \in 0, \nabla(2^{-l})$, then $e$ is an admissible splitting-off if $m(v) \geq l$ then the $e$ is a hypergraph in $H_q$ that contains more than $l$ positive nodes. Furthermore there is at least one positive node that is avoided by each hypergraph.

As a consequence of this lemma we get the following surprising statement about the rank requiring version of Problem 2.5: the greedy algorithm increases the rank by at most 1 if the starting hypergraph $H_q$ had rank more than 2, but it can increase the rank with 2 if $H_q$ was a graph.

In Section 4.4 we investigate what happens if we apply the greedy algorithm to Problems 2.1, 2.4 and 2.5. First we show that the algorithm applied to the local edge-connectivity augmentation of hypergraphs gives an optimal solution without increasing the rank. We remark that after we have proved this result, it turned out that it was also proved by Ben Cook in [4], in fact by making his proof with some we could give a quite simple proof for this statement. In Section 4.4.2 we consider the global edge-connectivity augmentation of mixed hypergraphs and using the results of Section 4.4.1 we show that the greedy algorithm applied to this problem increases the rank by at most 1. Finally in Section 4.4.3 we define the following generalization of the rank requiring version of Problem 2.5.

**Problem 4.5** Let $q = R - \partial q$ where $R$ is a crossing supermodular function that does not take the value 1, and $\partial q$ is the degree function of an arbitrary hypergraph $H_q$. Red $H_q$ is a hypergraph, $\mathcal{F}$ of minimum total weight covering $q$ such that the rank of $H_q$ does not exceed that of $H_q$.

Then we show that a (simple) modification of the greedy algorithm solves this problem if the rank of $H_q$ is at least 2 (if it is 1 then we need more sophisticated algorithms). This is described in [5]. The result of Section 4.4.3 appeared in [6].

**5 Covering symmetric crossing supermodular functions with graph edges**

If the function satisfies the supermodular inequality for any crossing set pair, then it is even symmetric then covering it with a minimum number of graph edges can be solved in polynomial time. This results is due to Beneserit and Frank. A partition $X = \{X_1, X_2, \ldots, X_t\}$ of $V$ is called a $\mathcal{F}$-full if $\partial q_{X_i} = 0$ for any monotone $f \subseteq 2^X$ of the maximum cardinality of a $\mathcal{F}$-full partition is the dimension of $\mathcal{F}$ and it is denoted by $\dim(\mathcal{F})$.

**Theorem 5.1** (Beneserit and Frank) The minimum number of graph edges covering a symmetric positively crossing supermodular function $q : 2^X \rightarrow \{0, 1\}$ equals $\min\{\dim(H_q), 2^{|V|}/2\}$.

In Section 5.2 we give a relatively simple algorithm for the following theorem. This appeared in [7].

**Problem 5.2** Given a symmetric positively crossing supermodular function $p : 2^V \rightarrow \{0, 1\}$ and a partition $P = \{P_1, P_2, \ldots, P_t\}$ of $V$, we are looking for a graph $G$ covering $p$ that has only edges between different members of the partition $P$.

A special case of this problem the partition constrained global edge-connectivity augmentation of hypergraphs was studied by Beneserit and Frank in [8]. Together with Richard Grappe and Zoltán Németh we managed to handle the more general
For the solution of the minimum weight of the problem consider the
following lower bounds (the solution of the degree-specified version is not detailed in this
summary because of the space limitations).

\[ \phi^*_i = \max \left\{ \sum_{j \in E} \phi_j : \mathcal{F} \text{ is a subpartition of } \{ V_j \} \right\} \text{ for every } i = 1, \ldots, r. \]

Obviously, \( \phi_0 = \max \left\{ \| \mathcal{S} \| (p) / 2, \phi_1, \ldots, \phi_r, \dim(p) - 1 \right\} \) is a lower bound on the minimum number of graph edges covering \( p \) and satisfying the partition constraints. We managed to show algorithmically that this bound can almost always be achieved except in some particular cases (called configurations) when we use one more edge. The definitions of \( \mathcal{C}_G \), \( \mathcal{C}_S \) and \( \mathcal{C}_C \)-configuration appearing in the theorem below have been given here since for space limitations.

**Theorem 5.3** ([12], with Roland Gruppo and Zoltán Szigeti) Let \( \mathcal{G} = (V, E) \) be a symmetric, positively crossing supermodular function and \( \mathcal{P} = (P_1, P_2, \ldots, P_r) \) be a partition of \( V \). The minimum number of graph edges covering \( p \) and satisfying the partition constraints is \( \phi_0 \), either a \( \mathcal{C}_G \), a \( \mathcal{C}_S \), or a \( \mathcal{C}_C \)-configuration exists for \( \mathcal{P} \), where the minimum is \( \phi_0 = 1 \).

If we specialize this theorem for the global edge-connectivity augmentation problem of hypergraphs with partition constrained graph edges then we get the following result (the theorems below the \( \mathcal{C}_G \) and \( \mathcal{C}_C \)-configuration between by specializing the \( \mathcal{C}_G \) and the \( \mathcal{C}_C \)-configuration appearing in the theorem below have been given here since for space limitations. Theorem 5.3’s details are again unverified due to space limitations it is worth to note however that the \( \mathcal{C}_C \)-configuration arises only in the abstract Problem 5.2).

**Theorem 5.4** ([13], with Roland Gruppo and Zoltán Szigeti) Let \( \mathcal{G} = (V, \mathcal{E}) \) be a hypergraph, \( \mathcal{P} \) be a partition of \( V \) and \( \lambda \) an arbitrary positive integer. The minimum number of graph edges between different classes of the partition \( \mathcal{P} \) that result in a k-edge-connected hypergraph when added to \( \mathcal{G} \) is \( \phi_0 + k \), if \( \mathcal{G}_C \) or a \( \mathcal{C}_C \)-configuration exists for \( \mathcal{G} \), and \( \phi_0 \) otherwise, where \( \phi_0 \) is defined as \( \phi_0[X] = k - \| \lambda_0 \| \) for any nonempty \( X \subseteq V \), and \( \phi_0(\emptyset) = \phi_0(\emptyset) = 0 \).

**6 Source Location Problem**

In Chapter 6 we investigate the following hypergraph generalizations of the Source Location Problem.

**Problem 6.1** Given a hypergraph \( H = (V, \mathcal{E}) \), a weight function \( w : V \rightarrow \mathbb{R}_+ \), and a requirement function \( r : V \rightarrow \mathbb{R}_+ \). Find a minimum weight subset of the nodes \( S \subseteq V \) such that \( \lambda_H(V, \emptyset, S, v) \geq r(v) \) for every \( v \in V \).

This problem was investigated in [1] in the case when \( H \) only contains graph edges and it was shown there that this problem is NP-complete in general but if either of \( r \) or \( w \) is constant then the problem can be solved in polynomial time. We also formulate the following abstract form of Problem 6.1 (a set function \( \phi \) is called supermodular if \( \phi \) is symmetric).

**Problem 6.2** Given a supermodular function \( \phi : 2^V \rightarrow \mathbb{R}_+ \), a weight function \( w : V \rightarrow \mathbb{R}_+ \), and a requirement function \( r : V \rightarrow \mathbb{R}_+ \). Find a minimum weight subset of the nodes \( S \subseteq V \) such that

\[ d(X) \geq \max \{ d(V) : v \in X \} \quad \text{for every } X \subseteq V \leq S \quad (4) \]

By generalizing the methods of [1] we show that the abstract Problem 6.2 can be solved if the functions \( r \) and \( w \) are compatible which means that there is an ordering \( v_1, v_2, \ldots, v_n \) of \( V \) such that \( r(v_k) \leq r(v_{k+1}) \leq \cdots \leq r(v_n) \) and \( w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n) \). We show that a simple greedy algorithm solves the abstract Problem 6.2 (with compatible weights and requirements) in polynomial time even if \( d \) does not satisfy the compatibility however in order to implement this algorithm one would need to minimize an interesting supermodular function which is an open problem. On the other hand if \( d \) is also supermodular then this can be solved with standard submodular function minimization techniques. We furthermore show that a little more sophisticated algorithm improves the running time in the case when the requirement function is constant. Specifying these results for Problem 6.2 we get the following application (where \( M(a, m) \) denotes the running time of a maximum flow computation in a graph with \( a \) edges and \( m \) nodes and the total size of a hypergraph \( H = (V, \mathcal{E}) \) is denoted by \( |\mathcal{E}| \).

**Theorem 6.3** ([10]) Problem 6.2 can be solved in \( O(n M(a, m) + |\mathcal{E}|^2) \) time if the functions \( r \) and \( w \) are compatible. The running time can be improved to \( O(n M(a, m) + |\mathcal{E}|) \) if the function \( r \) is constant.

**References**


8

9
The dissertation is based on the following publications:


