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Mathematical Analysis of Rational Decisions

PhD Tesis

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1. Motivation

The early period of investigation of the rational decisions can be connected with the name of P.A. Samuelson [10, 11, 12] in the thirties and fourties of the last century. He was looking for the answer to the question how the consumption utility function can be recovered from the demand functions. A few years later an analogous problem initiated the research of H.S. Houthakker [5]. In the fifties and sixties the research of J. Arrow [1] and H. Uzawa H. Uzawa [16] is notable. They pointed out that the revealed preference theory touches upon the axiomatic foundation of the entire doctrine of economic rationality.

From the sixties the theory of rational decision has been rapidly developing. M. K. Richter [8, 9] laid the foundation of this development by working out the interpretation of the rationality by the weak and strong axioms of revealed preferences. Further essential results based on the Richter's researches can be connected with the names of S. A. Clark [3], B. Hansson [4], A. K. Sen [13, 14], K. Suzumura [15] and his co-authors W. Bossert and Y. Sprumont [2] etc.

Beyond the decisive role of the revealed rationality in the economics we have to mention that it has strong connections with other scientific fields, too. Among others it has infiltrated not only into psychology (e.g. behavior analysis), sociology (e.g. social welfare analysis), politics (e.g. analysis of voting systems), gallup poll, but it also has appeared in informatics (e.g. search in data bases).

In the literature we have not met any investigation of the question how the rationality depends on the changes of the domain of the choice function. Our work has been focused on this problem.

In this thesis the numbering of the definitions, propositions and corollaries follow the numbering of the dissertation.

2. Choice function and revealed rationality

Let Ω be a final set of alternatives with the cardinality $|\Omega|$ and let 2^Ω denote the powerset of Ω.

Definition 2.1.1. *A subset $\mathfrak{B} \subseteq 2^\Omega \setminus \emptyset$ will be called* optional set system*, if in the decision making the alternatives belonging to the $X \in \mathfrak{B}$ will be evaluated from the same point of view.*

Definition 2.1.2. *The set to set function $C : \mathfrak{B} \to 2^\Omega, \quad C(X) \subseteq X$ will be called a* choice function *given on the subset $\mathfrak{B} \subseteq 2^\Omega$. If $C : \mathfrak{B} \to \mathfrak{B}$, then we say C is* injective.

DEFINITION 2.1.3. *The decision structure given by the triplet* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *will be called* decision mechanism*, if it satisfies the following conditions:*

1. *The cardinality of the set of alternatives* $|\Omega| \geqslant 2$*, the cardinality of the optional set system* $|\mathfrak{B}| \geqslant 1$*;*
2. *The optional set systems* $\mathfrak{B} \subseteq 2^{\Omega} \setminus \emptyset$ *covers the set of alternatives* Ω*, i.e.* $\Omega = \bigcup\limits_{X \in \mathfrak{B}} X$*;*
3. *Any element of* 2^{Ω} *belongs to the optional set system* $\mathfrak{B} \subseteq 2^{\Omega} \setminus \emptyset$ *at most once.*

Moreover, if

4. $\mathfrak{B} = 2^{\Omega} \setminus \emptyset$*, then we speak about* perfect decision mechanism.

If the (perfect) decision mechanism satisfies the condition

5. $C(X) \neq \emptyset \ \ \forall X \in \mathfrak{B}$ *then it is (perfect) real decision mechanism.*

DEFINITION 2.2.1. *The decision mechanism* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *will be called* dominant rational*, shortly* D-rational*, or* D-free of conflict*, if there exists a* P *binary relation on the set of alternatives* Ω *such that for all* $X \in \mathfrak{B}$ *the choice set* $C(X)$ *is equal to the set* $C_P^{\mathrm{D}}(X)$ *of* P*- dominant elements of* X*, i.e.*

$$C(X) = C_P^{\mathrm{D}}(X) = \{x \in X : xPy \ \ \forall y \in X\} \ \ \forall\, X \in \mathfrak{B}.$$

The question when a not necessary perfect decision mechanism will be D-rational has been answered by M. K. Richter [8] defining the relation, which rationalizes the decision mechanism, if it is possible. We have also obtained a similar result but with a different technique.

Let us introduce the following point to set function

$$Z : \Omega \to 2^{\Omega}, \quad x^* \xmapsto{Z} Z(x^*), \text{ where } Z(x^*) = \bigcup\{Y \in \mathfrak{B} : x^* \in C(Y)\}.$$

With the mapping Z we obtain the following statement:

PROPOSITION 2.2.3. *A real decision mechanism* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *is* D*-rational if and only if for all* $x^* \in \Omega$ *and for all* $X \subseteq Z(x^*)$ *the conditions* $X \in \mathfrak{B}$ *and* $x^* \in X$ *imply* $x^* \in C(X)$*.*

To prove the sufficiency part of this proposition we have used the following relation:

DEFINITION 2.2.5. *Let* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *be a real decision mechanism. The relation* P_Z *defined by the graph of the point to set function* Z *will be called the* weak rationalization *revealed by* $\mathfrak{D}$*, i.e.* $xP_Zy \ \Leftrightarrow\ y \in Z(x) \ \ \forall (x, y) \in \Omega \times \Omega$.

Since any point to set function $\mathfrak{f} : \Omega \to 2^{\Omega}$ defines a $P_{\mathfrak{f}}$ binary relation with the equivalence $xP_{\mathfrak{f}}y \ \Leftrightarrow\ y \in \mathfrak{f}(x) \ \ \forall (x, y) \in \Omega \times \Omega$, the function $\mathfrak{f}$ will be called the *point to set function representation* of the relation $P_{\mathfrak{f}}$.

If we consider the relation introduced by M. K. Richter [8] (we will refer to this relation as Richter-relation)

$xRy \Leftrightarrow \exists X \in \mathfrak{B} : x \in C(X), \;\; y \in X,$

then we obtain the following statement:

PROPOSITION 2.2.4. *The weak rationalization P_Z revealed by $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ and the Richter-relation are equivalent, i.e.*

$xP_Zy \;\Leftrightarrow\; xRy \;\; \forall (x, y) \in \Omega \times \Omega,$

or with the other words the mapping Z is the point to set function representation of the Richter-relation.

The mapping Z plays important role not only here, but also in the further sections of the dissertation. Description of several new results and their proofs are based on it.

3. Variability of decision mechanism

Two groups of questions are investigated:

- Under which conditions the revealed preference does not change if we include a new subset of alternatives into the optional set system? Under which conditions do new contradictions not appear in the real decision mechanism after an expansion. In other words, when the rational decision mechanism preserves the rationality or when the irrational decision preserves those and only those conflicts, which have existed in the original real decision mechanism?
- Under which conditions we can delete a subset from the optional set system without changing the revealed Richter-relation? When can a decision aspect be considered to be redundant?

DEFINITION 3.1.1. *Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be a real decision mechanism, R the revealed Richter-relation and let be given the subset $\emptyset \neq X' \in 2^\Omega \setminus \mathfrak{B}$. The decision structure $\mathfrak{D}^+ = (\Omega, \mathfrak{B}^+, C')$ defined by the optional set system and choice function*

$$\mathfrak{B}^+ = \mathfrak{B} \cup \{X'\}, \qquad C'(X) = \begin{cases} C(X), & ha\ X \in \mathfrak{B}, \\ C_R^{\mathrm{D}}(X), & ha\ X \in 2^\Omega \setminus \mathfrak{B}. \end{cases} \tag{3.1}$$

*will be called R-*preserving extension *of $\mathfrak{D}$.*

The name R-preserving extension is reasonable since the following statement is valid:

PROPOSITION 3.1.1. *Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be a real decision mechanism. The Richter-relation R^+ revealed by the decision mechanism $\mathfrak{D}^+ = (\Omega, \mathfrak{B}^+, C')$ defined by (3.1) coincides with the Richter-relation R revealed by the original $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$, i.e. $R^+ = R$.*

However, the decision mechanism which is obtained by the extension given above will not necessary be real decision mechanism. To get a real decision mechanism we have to prove that the requirement $C_R^{\mathrm{D}}(X') \neq \emptyset$ is valid, too. The next proposition gives a necessary and sufficient condition for this requirement to hold.

PROPOSITION 3.1.2. *The real decision mechanism $\mathfrak{D}^+ = (\Omega, \mathfrak{B}^+, C')$ obtained by the extension of $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ with the set X' according to the Definition 3.1.1. will be real decision mechanism if and only if there exists a subset $\emptyset \neq \mathfrak{B}' \subseteq \mathfrak{B}$ of the optional set system such that $X' \subseteq \bigcup_{X \in \mathfrak{B}'} X$ and $X' \bigcap \left(\bigcap_{X \in \mathfrak{B}'} C(X) \right) \neq \emptyset$.*

Though usually several appropriate covering system exist, in the practice it is difficult to find one of them which satisfies the requirement of the proposition. It would be practical to minimize the number of the covering system to be examined. The following statement solves this problem.

PROPOSITION 3.1.3. *Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be a real decision mechanism and $X \in 2^{\Omega} \setminus \emptyset$. $C_R^{\mathrm{D}}(X) \neq \emptyset$ if and only if there exists $x^* \in X$ such that $X \subseteq Z(x^*)$. In this case $x^* \in C_R^{\mathrm{D}}(X)$.*

At the application of Proposition 3.1.3. we have to check the condition of the proposition on at most $|\Omega|$ well defined subsets. Namely, the set system $\mathfrak{B}^*(x^*) = \{Y \in \mathfrak{B} : x^* \in C(Y)\}$ defining the mapping $Z(x^*)$ satisfies the demanded property of the Proposition 3.1.2.

Proposition 3.1.3. guaranties a possible R-preserving extension, namely

PROPOSITION 3.1.4. *Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be a real decision mechanism and let R be the revealed Richter-relation. If there exists an alternative $x^* \in \Omega$ such that the condition $\emptyset \neq Z(x^*) \notin \mathfrak{B}$ holds, then the structure $\mathfrak{D}^+ = (\Omega, \mathfrak{B}^+, C')$ defined by the optional set system $\mathfrak{B}^+ = \mathfrak{B} \cup \{Z(x^*)\}$ and choice function*

$$C(X') = \begin{cases} C(X) & X \in \mathfrak{B} \\ C_R^{\mathrm{D}}(X) & X = Z(x^*) \end{cases}$$

will be a real decision mechanism, too.

If $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ is a (R, D)-rational real decision mechanism, then $\mathfrak{D}^+ = (\Omega, \mathfrak{B}^+, C')$ will also be (R, D)-rational.

More R-preserving extensions can be obtained using the so called set-extension operators. Their definition follows.

DEFINITION 1.2.2. *Let $\mathfrak{B} \subseteq 2^{\Omega}$. An operator* $\mathrm{cl} : \mathfrak{B} \to 2^{\Omega}$, $\mathrm{cl}(\emptyset) = \emptyset$ *will be called*

1. set-extension operator *if it is* extensive, *i.e.* $A \subseteq \mathrm{cl}(A)$;
2. monotone set-extension operator *if it is a set-extension operator and the condition $A \subset B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ fulfills;*
3. idempotent set-extension operator *if it is a set-extension operator and it satisfies the condition* $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

A monotone and idempotent set-extension operator is called closure operator*. A set-extension operator* cl *is* injective*, if* $\mathrm{cl}(A) \in \mathfrak{B}\ \ \forall A \in \mathfrak{B}$*. The set system* $\mathfrak{B}$ *will be called* cl-closed *if* cl *injective.*

Let us introduce the following two operators:

1. $\mathcal{H}_P : 2^\Omega \to 2^\Omega, \qquad X \overset{\mathcal{H}_P}{\longmapsto} \mathcal{H}_P(X) = \{x \in \Omega : \exists y \in X\ \ yPx\},$
   where P is a reflexive binary relation on $\Omega \times \Omega$ (Kortelainen [6]);
2. Let $\mathfrak{f}$ be a point to set function and let us define
   $\mathfrak{Im}_{\mathfrak{f}} : 2^\Omega \setminus \emptyset \to 2^\Omega \setminus \emptyset,\ \ \mathfrak{Im}_{\mathfrak{f}}(X) = \bigcup\limits_{x \in X} \mathfrak{f}(x)$
   using the image set of X by the function $\mathfrak{f}$.

The connection between these operators is formulated by the following

Proposition 3.2.2. *If* $\mathfrak{f}_P$ *is the point to set function representation of a reflexive relation* P*, then* $\mathfrak{Im}_{\mathfrak{f}_P}$ *and* $\mathcal{H}_P$ *are equivalent, i.e.* $\mathfrak{Im}_{\mathfrak{f}_P}(X) = \mathcal{H}_P(X)\ \ \forall X \in 2^\Omega \setminus \emptyset$.

The equivalence of the two operators points out the connection between the Kortelainen's set-extension operator and the decision making based on choice functions.

Otherwise, the equivalence simplifies the proofs of some propositions. Namely, if the role of the relation P is played by the Richter-relation then we obtain the following statements:

Proposition 3.2.6. $\mathcal{H}_R$ *and* $\mathfrak{Im}_Z$ *are monotone set-extension operators, furthermore if the relation* R *is reflexive and transitive, then* $\mathcal{H}_R$ *and* $\mathfrak{Im}_Z$ *are closure operators.*

Further the proofs of some propositions have also been simplified with this equivalence (e.g. Propositions 3.2.3. and 3.2.4.) and new statements for Kortelainen's set-extension have been formulated by it (i.e. Lemma 3.3.1. for the computation of $C_R^{\mathrm{D}}(\mathcal{H}_R(X))$.

Using the set-extension $\mathcal{H}_R$ and $\mathfrak{Im}_Z$ we can define R-preserving extensions of a real decision mechanism as follows:

Proposition 3.2.7. *Let* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *be a real decision mechanism and the revealed Richter-relation be reflexive. If* $\mathcal{H}_R(X) \notin \mathfrak{B}$ *and* $C_R^{\mathrm{D}}(\mathcal{H}_R(X)) \neq \emptyset$ *for some* $X \in \mathfrak{B}$ *then*

$$\mathfrak{D}^+_{\mathcal{H}} = (\Omega, \mathfrak{B} \cup \{\mathcal{H}_R(X)\}, C') \quad \textit{and} \quad \mathfrak{D}^+_Z = (\Omega, \mathfrak{B} \cup \{\mathfrak{Im}_Z(X)\}, C')$$

will be real decision mechanisms, where

$$C(X') = \begin{cases} C(X') & X' \in \mathfrak{B} \\ C_R^{\mathrm{D}}(X') & X' = \mathcal{H}_R(X) = \mathfrak{Im}_Z(X) \end{cases}.$$

If $\mathfrak{D}$ *is a reflexive* (R, D)*-rational real decision mechanism then* $\mathfrak{D}^+_{\mathcal{H}}$ *and* $\mathfrak{D}^+_Z$ *will also be the same ones.*

A modification of an other known set-extension operator (Koshevoy [7]) is suitable to obtain R-preserving extension of a decision mechanism.

Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be a real decision mechanism. The mapping

$$\mathcal{K}_C : \mathfrak{B} \to 2^\Omega \setminus \emptyset, \qquad X \xrightarrow{\mathcal{K}_C} \mathcal{K}_C(X) = \bigcup \{Y \in \mathfrak{B} : C(Y) = C(X)\}, \quad \forall X \in \mathfrak{B}$$

will be called *C-revealed set-extension operator* on $\mathfrak{D}$.

If $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ is a (R, D)-rational real decision mechanism, then the set system

$$\mathfrak{B}_{\mathcal{K}_C} = \mathfrak{B} \cup \{\mathcal{K}_C(X) : \mathcal{K}_C(X) \notin \mathfrak{B},\ X \in \mathfrak{B}\}$$

is the $\mathcal{K}_C$-extension of the optional set system $\mathfrak{B}$, and the decision mechanism $\mathfrak{D}^+_{\mathcal{K}_C} = (\Omega, \mathfrak{B}_{\mathcal{K}_C}, C^{\mathrm{D}}_R)$ will be called the R-preserving $\mathcal{K}_C$-extension of $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$.

We have proved that the optional sets-system $\mathfrak{B}_{\mathcal{K}_C}$ obtained by adding the C-revealed set-extensions of all sets out of $\mathfrak{B}$ will be $\mathcal{K}_C$-closed. For this we had to prove that the operator $\mathcal{K}_C$ is idempotent.

Proposition 3.2.11. *Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be a (R, D)-rational real decision mechanism. Then $\mathcal{K}_C(X) = \mathcal{K}_{C^{\mathrm{D}}_R}(X) = \mathcal{K}_{C^{\mathrm{D}}_R}(\mathcal{K}_{C^{\mathrm{D}}_R}(X)) = \mathcal{K}_{C^{\mathrm{D}}_R}(\mathcal{K}_C(X)) \quad \forall X \in \mathfrak{B}$.*
If the set-extension operator $\mathcal{K}_C$ is injective, then $\mathcal{K}_C(\mathcal{K}_C(X)) = \mathcal{K}_C(X)$.

From this follows

Corollary 3.2.11.1. *The set system $\mathfrak{B}_{\mathcal{K}_C}$ is the narrowest optional set-system for which the set-extension operator $\mathcal{K}_{C^{\mathrm{D}}_R}$ revealed by the (R, D)-rational real decision mechanism $\mathfrak{D}^+_{\mathcal{K}_C} = (\Omega, \mathfrak{B}_{\mathcal{K}_C}, C^{\mathrm{D}}_R)$ is injective.*

In the dissertation it has also been examined what kind of connection is between the set-extensions by the operators $\mathcal{H}_R$ ($\mathfrak{Im}_Z$) and $\mathcal{K}_C$ with respect to the inclusion. We have found that the whole set Ω of alternatives almost always appears between the extensions using the operators mentioned above. This is important because in the real decision processes generally there is no such viewpoint which covers the whole set of alternatives, i.e. Ω does not belong to the optional set system. But the main aim of the decision process is to select from all alternatives the best one(s) for the decision makers. Therefore we can choose from all alternatives by using only the revealed preferences, but we can decide only if the choice is not empty. Further question is whether an (R, D)-rational real decision mechanism could be extended into a perfect one using R-preserving extensions?

Let us introduce two basic concepts for decision mechanism.

Definition 3.3.1. *Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be a (R, D)-rational real decision mechanism, where $\Omega \notin \mathfrak{B}$. We say that the real decision mechanism $\mathfrak{D}$ is* (R, D)-decisive *if $C^{\mathrm{D}}_R(\Omega) \neq \emptyset$.*

Definition 3.3.2. *An (R, D)-rational real decision mechanism $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ will be considered as* (R, D)-stable, *if $C^{\mathrm{D}}_R(X) \neq \emptyset$ for all $X \in 2^\Omega \setminus \emptyset$.*

The decision mechanisms corresponding to these definitions can be characterized by the following propositions:

Proposition 3.3.1. *Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be an (R, D)-rational real decision mechanism such that $\Omega \notin \mathfrak{B}$. This $\mathfrak{D}$ is (R, D)-decisive if and only if there exists an alternative $x^* \in \Omega$ satisfying the condition $Z(x^*) = \Omega$. In this case $x^* \in C_R^{\text{D}}(\Omega)$.*

Proposition 3.3.2. *Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be an (R, D)-rational real decision mechanism. It is (R, D)-stable if and only if for all $X \in 2^\Omega \setminus \emptyset$ there exists $x_X^* \in X$ such that $X \subseteq Z(x_X^*)$.*

Proposition 3.3.3. *If the real decision mechanism $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ (R, D)-stable, then the relation R is complete and R^d is acyclic.*

After the extensions we have examined how we can reduce the optional set system in such a way that the decision mechanism defined by the remaining system would preserve the revealed Richter-relation.

Definition 3.4.1. *Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be a real decision mechanism, R let be the revealed Richter-relation and let $X' \in \mathfrak{B}$. The structure $\mathfrak{D}^- = (\Omega, \mathfrak{B}^-, C')$ defined by the optional set system $\mathfrak{B}^- = \mathfrak{B} \setminus \{X'\}$ and the choice function $C'(X) = C(X)$ for all $X \in \mathfrak{B}^-$ will be called R-*preserving reduction *of the real decision mechanism $\mathfrak{D}$ with the subset X' if $R^- = R$, where R^- is the Richter-relation revealed by the reduced $\mathfrak{D}^-$.*

The R-preserving reduction can be characterized as follows:

Proposition 3.4.2. *Let $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ be a real decision mechanism and choose a subset $X' \in \mathfrak{B}$. The structure $\mathfrak{D}^- = (\Omega, \mathfrak{B}^-, C')$ given in the Definition 3.4.1. will be an R-preserving reduction if and only if the following equivalent conditions are satisfied:*

$$Z(a) = Z^-(a) \ \ \forall a \in C(X') \quad \text{and} \quad \bigcap_{a \in C(X')} Z^-(a) \supseteq X',$$

where $Z^-(a) = \bigcup \{Y \in \mathfrak{B}^- : a \in C(Y)\}$.

4. (P_R,D)-rational decision mechanisms

In the earlier discussion we insisted on preserving the revealed Richter-relation. We have observed that not all the sets not belonging to $\mathfrak{B}$ can be involved in the optional set system, i.e. the decision mechanism is usually not decisive, even less stable. Now we will insist that the choices on the sets of the optional set system should not be changed with a (P, D)-rationality differing from the (R, D)-rationality. It is known (Suzumura [15]) if the decision mechanism has more than one rationalization then they must satisfy the inclusion $R \subseteq P$. It also can occur, that the decision mechanism has no other rationalization but R. These cases are characterized by the following definition and propositions.

Definition 4.1.1. *Let* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *be an* (R, D)*-rational real decision mechanism. The relation* P_R *will be called* extended (R, D)-rationalization *revealed by the real decision mechanism* $\mathfrak{D}$ *if* $R \subset P_R$ *and* $C^{\text{D}}_{P_R}(X) = C^{\text{D}}_R(X) \ \ \forall X \in \mathfrak{B}$. *The set of all extended* (R, D)*-rationalizations revealed by* $\mathfrak{D}$ *will be denoted by* $\mathcal{P}_R$.

Proposition 4.1.1. *Let* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *be an* (R, D)*-rational real decision mechanism.* $\mathcal{P}_R = \emptyset$ *if the optional set system* $\mathfrak{B}$ *satisfies the following conditions:*

1. $\mathfrak{B}$ *contains all subsets of two elements of* Ω*;*
2. *For all* $a \in \Omega$ *there exists a subset* $X \in \mathfrak{B}$ *such that* $a \in C(X)$.

Proposition 4.1.2. *Let us assume that the* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *is an* (R, D)*-rational real decision mechanism and between the revealed Richter- and Samuelson-relations the strict* $R \subset S^d$ *inclusion holds. Then* $S^d \in \mathcal{P}_R$. *Here the revealed Samuelson-relation is defined as follows:* $xSy \Leftrightarrow \exists X \in \mathfrak{B} : x \in C(X),\ y \in X \setminus C(X)$.

For the next statement we need two auxiliary definitions:

Definition 4.1.2. *We say that the relation* P_{sc} *on* $\Omega\times\Omega$ *is the* strict complementary part *of the relation* P *if*

$$xP_{\text{sc}}y \Leftrightarrow x\overline{P}y \ \text{ and } \ \exists Y \in \mathfrak{B} : x, y \in Y, x \notin C^{\text{D}}_P(Y), \ \text{de } xPz\ \forall z \in Y \setminus \{y\}.$$

Definition 4.1.3. *The relation* Q^{ab} *is the* atomization *of* P *belonging to the pair of alternatives* $(a, b) \in \Omega \times \Omega$ *if and only if*

$$xQ^{ab}y \Leftrightarrow x = a, y = b \text{ and } aPb.$$

Let $\mathcal{Q}(P)$ *denote the set of all atomizations of the relation* P.

Proposition 4.1.3. *Let* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *be an* (R, D)*-rational real decision mechanism. The relation* P_R *is an extended* (R, D)*-rationalization revealed by* $\mathfrak{D}$ *if and only if there exists a finite chain of relations* $\{P^{(i)},\ i = 0, \ldots, k\}$ $k \geqslant 1$ *such that:*

1. $P^{(0)} = R;\ \ P^{(i-1)} \subset P^{(i)},\ i = 1, \ldots, k;\ \ P^{(k)} = P_R;$
2. *For all* $i = 1, \ldots, k$ *there exists a pair of alternatives* $(a_i, b_i) \in \Omega \times \Omega$ *such that* $P^{(i)} \cap \overline{P^{(i-1)}} = Q^{a_i b_i}_{i-1} \in \mathcal{Q}_{i-1}$, *where* $\mathcal{Q}_{i-1} = \mathcal{Q}(\overline{P^{(i-1)}} \cap \overline{P^{(i-1)}_{\text{sc}}})$ *and* $P^{(i-1)}_{\text{sc}}$ *is the strict complementary part of* $P^{(i-1)}$.

The length k *of the chain will be called* distance *between the relations* R *and* P_R *and it will be denoted by* $d(R, P_R)$.

In most decision problems neither the (R, D)-decisiveness, nor the (R, D)-stability are guaranteed. We have also discussed, whether there is any extension of the revealed preference R, under which the choices will be preserved on the sets of $\mathfrak{B}$, but the revealed choices from the sets beyond $\mathfrak{B}$ will not be empty. This discussion has required to generalize the concepts of decisiveness and stability for the extended (R, D)-rationality, too.

DEFINITION 4.2.1. *Let* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *be an* (R, D)*-rational real decision mechanism and* $\Omega \notin \mathfrak{B}$. *We say, that the real decision mechanism* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *is* weakly (R, D)-decisive, *if it is not* (R, D)*-decisive, but there exists an extended* (R, D)*-rationalization* $P_R \supset R$ *such that* $C^{\mathrm{D}}_{P_R}(\Omega) \neq \emptyset$ *and the distance* $d(R, P_R)$ *is minimal.*

DEFINITION 4.2.2. *Let* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *be an* (R, D)*-rational real decision mechanism.* $\mathfrak{D}$ *will be called* weakly (R, D)-stable *if it is not* (R, D)*-stable, but there exists an extended* (R, D)*-rationalization* $P_R \supset R$ *such that* $C^{\mathrm{D}}_{P_R}(X) \neq \emptyset$ *for all* $X \in 2^{\Omega} \setminus \emptyset$ *and the distance* $d(R, P_R)$ *is minimal.*

The following propositions give necessary and sufficient conditions under which the conditions of the weakly decisiveness and the weakly stability hold.

PROPOSITION 4.2.1. *Let* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *be an* (R, D)*-rational real decision mechanism. Let* $X \in (2^{\Omega} \setminus \emptyset) \setminus \mathfrak{B}$. *An extended* (R, D)*-rationalization* $P_R \supset R$ *for which the distance* $d(R, P_R)$ *is minimal and* $C^{\mathrm{D}}_{P_R}(X) \neq \emptyset$ *exists if and only if, there is an alternative* $x^* \in X$ *such that one of the following conditions holds:*

1. $X \subseteq Z(x^*) = \Omega$ *and* $\exists (x, y) \in \Omega \times \Omega$ *such that* $x \neq x^*$, $x\overline{R}y$ *and* $x\overline{R_{\mathrm{sc}}}y$;
2. $X \subseteq Z(x^*)$, $\Omega \setminus Z(x^*) \neq \emptyset$ *and there exists* $y \in \Omega \setminus Z(x^*)$ *for which* $x^*\overline{R}y$ *and* $x^*\overline{R_{\mathrm{sc}}}y$;
3. $X \setminus Z(x^*) \neq \emptyset$ *and in any case of* $Y \in \mathfrak{B}$ *if* $x^* \in Y$ *either* $Y \subseteq Z(x^*)$ *or* $(Y \setminus Z(x^*)) \setminus (X \setminus Z(x^*)) \neq \emptyset$ *fulfills.*

We can obtain as special cases of this proposition the following statements with the notion $\mathfrak{B}(x) = \{Y \in \mathfrak{B} : x \in Y\}$.

PROPOSITION 4.2.2. *The* (R, D)*-rational real decision mechanism* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *is weakly* (R, D)*-decisive if and only if there exists* $x^* \in \Omega$ *such that for all* $Y \in \mathfrak{B}(x^*)$ *we have* $Y \subseteq Z(x^*)$.

PROPOSITION 4.2.3. *The* (R, D)*-rational* $\mathfrak{D} = (\Omega, \mathfrak{B}, C)$ *is weakly* (R, D)*-stable if and only if for all* $X \in (2^{\Omega} \setminus \emptyset) \setminus \mathfrak{B}$ *there exists* $x^*_X \in X$, *such that for all* $Y \in \mathfrak{B}(x^*_X)$ *one of the following condition holds:*

1. $Y \subseteq Z(x^*_X)$;
2. $(Y \setminus Z(x^*_X)) \setminus (X \setminus Z(x^*)) \neq \emptyset$.

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[$B3$] Bodó B.,A kiválasztási függvény racionalitása és racionalizálhatósága az opcionális halmazrendszer függvényében, (On the rationality and rationalizability of the choice function with respect to the optional set system), *Alkalmazott Matematikai Lapok*, **25** (2008), 47-59.