Problems in infinite graph theory with finite counterpart

Ph.D. Thesis

Attila Joó

MTA-ELTE Egerváry Research Group on Combinatorial Optimization

Supervisor: Gábor Sági, Ph.D.
Senior research fellow, Alfréd Rényi Institute of Mathematics and Associate Professor, Department of Algebra, Budapest University of Technology and Economics

Member of the Hungarian Academy of Sciences

Eötvös Loránd University
Faculty of Science
Institute of Mathematics
Mathematical Doctoral School
Director: prof. Miklós Laczkovich, D.Sc.
Member of the Hungarian Academy of Sciences

Doctoral Program: Pure Mathematics
Director: prof. András Szűcs, D.Sc.
Member of the Hungarian Academy of Sciences

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Introduction

Infinite graph theory has a great tradition in Hungary. It was one of the favourite topics of Pál Erdős, the most unique personality and mind in the history of Hungarian mathematics. Looking for infinite generalization of theorems in finite graph theory was a usual starting point of the works of Erdős in this field (there is an excellent survey in [21]). For more recent investigations in the topic we refer to the works of R. Aharoni, R. Diestel, C. Thomassen and S. Shelah.

Sometimes a theorem about finite graphs fails for obvious reasons for infinite graphs or remains true (maybe under some reasonable restrictions) but with essentially the same proof. In the dissertation we are focusing on generalizations where the proofs need new ideas well beyond the proof of the finite counterpart. One of the most remarkable theorem of this kind is the celebrated infinite generalisation of Menger’s theorem [2] by R. Aharoni and E. Berger (conjectured originally by Erdős). The proof takes sixty pages and it is a result of hard work of decades although Menger’s original theorem is a part of the basic graph theory education and its proof is not put out a single page.

Each of the seven chapters of the dissertation is one of our results in infinite graph theory. They are not based on each other. The common motive is that they are motivated by phenomena from finite graph theory.
Notation and Preliminaries

Mostly we use the standard notation of set theory, graph theory and matroid theory. We apply some fundamental facts about infinite matroids. Majority of them are well-known for finite matroids and have the same proof in the infinite case.

0.1 Set theory

We use some basic set theoretic notation. For the power set of $X$, we write $\mathcal{P}(X)$. We denote the symmetric difference of the sets $X$ and $Y$ by $X \triangle Y$. The variables $\alpha, \beta, \gamma, \xi, \zeta$ always stand for ordinals and $\kappa, \lambda$ for cardinals. We denote the smallest infinite cardinal (i.e. the set of the natural numbers) by $\omega$. If $\kappa$ is a cardinal, then $\kappa^+$ is its successor cardinal. The restriction of a function $F$ to some $X \subseteq \text{dom}(F)$ is denoted by $F|_X$, and $F[X]$ stands for $\text{ran}(F|_X)$. We use the abbreviation $B - x + y$ for the set $(B \setminus \{x\}) \cup \{y\}$. We write $\bigcup \mathcal{X}$ (\bigcap \mathcal{X}) for the union (intersection) of the elements of $\mathcal{X}$.

0.2 Graphs and digraphs

The graphs $G = (V, E)$ and digraphs (directed graphs) $D = (V, A)$ in the dissertation may have multiple edges. Loops will be irrelevant in all of the results. If the edge $e$ goes from $u$ to $v$, then $\text{tail}(e) = u$ and $\text{head}(e) = v$. We use the abbreviation $D - e$ for $(V, A \setminus \{e\})$. For $X \subseteq V$ let $\text{span}_D(X)$ be the set of those edges of $D$ whose heads and tails are contained in $X$ and let $D[X] = \ldots$
The set of the ingoing and outgoing edges of \( X \subseteq V \) are denoted by \( \text{in}_D(X) \) and \( \text{out}_D(X) \), respectively (in the undirected case we use just \( \text{out}_G(X) \)) and let \( \text{cut}_D(X) = \text{in}_D(X) \cup \text{out}_D(X) \). For a singleton \( \{v\} \) we write \( \text{in}_D(v) \) instead of \( \text{in}_D(\{v\}) \) and we use this kind of abbreviation in connection with singletons in the case of other set-functions as well.

In a path we do not allow repetition of vertices (we say walk if we allow them) and in the context of digraphs paths assumed to be directed. Furthermore paths are finite unless we state explicitly otherwise. We may define directed paths by the corresponding vertex sequence (if parallel edges do not appear there or it does not matter which we use). This sequence determines an ordering \( <_P \) an \( V(P) \).

We denote by \( \text{start}(P) \) the \( <_P \)-smallest and by \( \text{end}(P) \) the \( <_P \)-largest vertex of a path \( P \). For \( u <_P v \), the subdigraph of \( P \) induced by the elements of the interval \([u, v]\) is denoted by \( P[u, v] \) and called the segment of \( P \) from \( u \) to \( v \). The initial segments of \( P \) are the segments in the form \( P[\text{start}(P), v] \). We define terminal segments similarly. If the last edge \( e \) of a path \( P \) is identical to the first edge of a path \( Q \) and \( V(P) \cap V(Q) = \{\text{tail}(e), \text{head}(e)\} \), then \( (V(P) \cup V(Q), A(P) \cup A(Q)) \) is a path that we call the concatenation of \( P \) and \( Q \). We say that the path \( P \) goes from \( X \) to \( Y \) (or shortly \( P \) is an \( X \rightarrow Y \) path) if \( \text{start}(P) \in X \) and \( \text{end}(P) \in Y \).

A path \( P \) goes strictly from \( X \) to \( Y \) if exactly the first vertex of \( P \) is in \( X \) and exactly the last one is in \( Y \). In this situation, we say, that \( P \) is a strict \( X \rightarrow Y \) path. We call a digraph \( D \) connected if for all \( u, v \in V(D) \) there is a \( u \rightarrow v \) path in \( D \). We call a digraph weakly connected if its underlying undirected graph is connected. The weakly connected components of a digraph are the connected components of its underlying undirected graph with the original orientations. For \( u \neq v \) we denote by \( \lambda_D(u, v) \) the local edge-connectivity from \( u \) to \( v \) in \( D \) i.e.

\[
\lambda_D(u, v) = \min\{|A'| : A' \subseteq A, \text{ there is no path from } u \text{ to } v \text{ in } (V, A \setminus A')\}.
\]

Let us write \( \text{to}_D(X) \) for the set of those vertices from which \( X \) is reachable by a directed path in \( D \). The length of a path \( P \) is the number of its edges. A path may consist of a single vertex in which case it is a trivial path. For a system \( \mathcal{P} \) of paths, let \( A(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} A(P) \) and we denote by \( A_{\text{last}}(\mathcal{P}) \) the set of the last edges of the paths in \( \mathcal{P} \).
A digraph $D$ is called a **branching** if it is a directed forest in which every vertex is reachable by a unique path from $X := \{v \in V(D) : |\text{in}_D(v)| = 0\}$. This $X$ is the **root set** of the branching. If this root set is the singleton $\{s\}$, then it is an **$s$-arborecence**.

### 0.3 Infinite matroids

There was several attempts to extend the notion of matroid by allowing infinite ground sets but keeping the concept of duality. Finally in [6] the authors achieved this goal which made possible the intensive development of the field. In this subsection we give the notation and the facts that we will use in connection with matroids.

The pair $\mathcal{M} = (S, \mathcal{I})$ is a matroid if $\mathcal{I} \subseteq \mathcal{P}(S)$ and it satisfies the following axioms.

1. $\emptyset \in \mathcal{I}$,

2. $I \subseteq I' \in \mathcal{I}$ implies $I \in \mathcal{I}$,

3. if $B$ is a $\subseteq$-maximal element of $\mathcal{I}$ and $I \in \mathcal{I}$ is not maximal, then there is an $i \in B \setminus I$ such that $(I \cup \{i\}) \in \mathcal{I}$,

4. if $I \in \mathcal{I}$ and $I \subseteq X \subseteq S$, then the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a $\subseteq$-maximal element.

The elements of $\mathcal{I}$ are called **independent** sets and the other subsets of $S$ are the **dependent** sets. $\mathcal{M}$ is a **free matroid** if $\mathcal{I} = \mathcal{P}(S)$. The $\subseteq$-maximal independent sets are the **bases** of the matroid (axiom 4 with $I := \emptyset$ and $X := S$ ensures that each matroid has at least one base). In notation, sometimes we will not distinguish the matroid from its ground set unless it would lead misunderstanding.

**0.3.1 Fact.** If $B_1$ and $B_2$ are bases of the same matroid and $B_1 \setminus B_2$ is finite, then $|B_1 \setminus B_2| = |B_2 \setminus B_1|$. 

3
It implies that if there is a finite base, then all the bases are finite and have the same size \( r(M) \) which is called the **rank** of \( M \). ZFC alone is not able to decide if the bases of a fixed matroid have necessarily the same cardinality. The Generalized Continuum Hypothesis decides the question affirmatively (as shown by D. A. Higgs in [14]) but it is false under some other set theoretic assumptions (proved by N. Bowler and S. Geschke in [5]). Hence if there is no finite base, then the rank is simply defined to be \( \infty \). For \( S' \subseteq S \), the pair \( (S', I \cap \mathcal{P}(S')) \) is a matroid, it is the submatroid of \( M \) that we get by **restriction** to \( S' \). For \( S' \subseteq S \) we denote by \( r(S') \) the rank of the submatroid (corresponding to) \( S' \). A \( \subseteq \)-minimal dependent set is called a **circuit**.

**0.3.2 Fact.** A set \( S' \subseteq S \) is dependent iff it contains a circuit (which is not straightforward for an infinite \( S' \ ).

**0.3.3 Fact.** The relation \( \{ \langle x, y \rangle \in S \times S : \exists C \text{ circuit with } x, y \in C \} \) is transitive.

By adding the diagonals, we may extend the relation above to an equivalence relation. The equivalence classes are called the **components** of the matroid. A matroid is called **finitary** if all of its circuits are finite. In these matroids, an infinite set is independent if and only if all of its finite subsets are independent, in fact this property characterize the finitary matroids.

**0.3.4 Fact.** Fix a base \( B \) of (the submatroid corresponding to) \( S' \subseteq S \). Then the subsets \( I \) of \( S\setminus S' \), for which \( (I \cup B) \in I \), forms a matroid on \( S\setminus S' \) and it does not depend on the choice of \( B \). It is the submatroid of \( M \) that we get by **contracting** \( S' \).

For \( S', S'' \subseteq S \), we may restrict first \( M \) to \( S' \cup S'' \) and in the resulting matroid contract \( S'' \). In this case, we denote the resulting submatroid by \( S'/S'' \) (if \( M \) is clear from the context).

If \( M_\xi = (S_\xi, \mathcal{I}_\xi) \ (\xi < \kappa) \) are matroids with pairwise disjoint ground sets, then the direct sum \( M := \bigoplus_{\xi < \kappa} M_\xi \) of the matroids \( \{M_\xi\}_{\xi < \kappa} \) is the matroid on \( \bigcup_{\xi < \kappa} S_\xi \) where \( I \in \mathcal{I}_M \) if \( (I \cap S_\xi) \in \mathcal{I}_\xi \) for all \( \xi < \kappa \). Every matroid is the direct sum of its components.
An element $i \in S$ is called a **loop** if $\{i\}$ is a circuit. We denote by $\text{span}(S')$ the union of $S'$ and the loops of $S/S'$.

**0.3.5 Fact.** $\text{span}$ is a closure operator.

**0.3.6 Fact** (weak circuit elimination). If $C_1, C_2$ are circuits with $i \in C_1 \cap C_2$, then $C_1 \cap C_2 - i$ contains a circuit.

**0.3.7 Corollary.** If $i \in \text{span}(I) \setminus I$ for some independent set $I$, then there is a unique circuit $C \subseteq I \cup \{i\}$. Necessarily $i \in C$ since $I$ is independent.

For $i \in \text{span}(I)$, let us define $C(i, I) = \begin{cases} 
\{i\} & \text{if } i \in I, \\
\text{the unique circuit } C \text{ above} & \text{if } i \notin I.
\end{cases}
$

**0.3.8 Fact.** If $B$ is a base and $i \in S \setminus B$, then for any $j \in C(i, B)$ the set $B - j + i$ is a base again.

**0.3.9 Corollary.** If $I$ is independent and $i \in I \cap \text{span}(J)$ for some $J \subseteq S$, then there is some $j \in J$ ($j = i$ is allowed) such that $I - i + j$ is independent. Furthermore $i$ and $j$ are in the same component of the matroid and if $I$ is a base, then Fact 0.3.1 ensures that $I - i + j$ is a base as well.

It is worth mentioning that despite of its young age the theory of infinite matroids is extremely rich and deep although we need in the dissertation just the basics above from it. One can find a detailed survey about the theory of infinite matroids in the Habilitation thesis of N. Bowler [4].
Chapter 1

Infinite generalization of Gomory-Hu trees

1.1 Introduction

Let $G = (V, E)$ be a countable connected simple graph and let $c : E \rightarrow \mathbb{R}_+ \setminus \{0\}$ be a weight-function, then $(V, E, c)$ is called a weighted graph. We call the subsets of $V$ cuts. We say $X$ is an $u - v$ cut for some $u \neq v \in V$ if $u \in X$ and $v \notin X$. A cut $X$ separates $u$ and $v$ if $X$ is either a $u - v$ or a $v - u$ cut. Let $d_c(X) = \sum_{e \in \text{out}_G(X)} c(e)$ and let $\lambda_c(u, v) := \inf\{d_c(X) : X \text{ is a } u - v \text{ cut} \}$ for $u \neq v \in V$. A cut $X$ is an optimal $u - v$ cut if it is a $u - v$ cut with $d_c(X) = \lambda_c(u, v)$. A cut $X$ is optimal if it is an optimal $u - v$ cut for some $u \neq v \in V$. The weighted graph $(V, E, c)$ is finitely separable if $\lambda_c$ has just finite values. A tree $T = (V, F)$ is a Gomory-Hu tree for $(V, E, c)$ if for all $u \neq v \in V$ there is an $e \in F$ such that the fundamental cuts corresponding to $e$ (i.e. the vertex sets of the components of $T - e$) separate optimally $u$ and $v$ in $(V, E, c)$. Gomory and Hu proved in [12] that for all finite weighted graph there exists a Gomory-Hu tree. It has several interesting consequences. For example the function $\lambda_c$ may have at most $n - 1$ different values instead of $\binom{n}{2}$ (where $n$ is the number of the vertices) and there is at least two optimal cuts that consist of a single vertex, namely the leafs of the Gomory-Hu tree (unless the graph is
trivial). In this chapter we extend their theorem for infinite weighted graphs with finite total weight. Note that, the strict positivity of \( c \) and the connectedness of \( G \) are not real restrictions since throwing away edges \( e \) with \( c(e) = 0 \) has no effect on the values of the cuts and one can construct Gomory-Hu trees component-wise and join them to a Gomory-Hu tree. Furthermore, if the sum of the weights is finite, then the weighted graph must be countable.

In a more abstract folklore version of the Gomory-Hu theorem there is no graph, one just has a finite set \( V \) and a function \( b : \mathcal{P}(V) \to \mathbb{R}_+ \) which is symmetric (\( b(X) = b(V \setminus X) \)) and submodular i.e.

\[
b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \text{ if } X, Y \subseteq V.
\]

Let \( \lambda_b(u, v) = \inf \{b(X) : X \text{ a } u-v \text{ cut} \} \) (this definition makes sense for infinite \( V \) as well). In this case there exists an abstract **Gomory-Hu tree** with respect to \( b \) in the following sense. There is a tree \( T \) on the vertex set \( V \) in such a way that for every \( u \neq v \in V \) there is some \( e \in E(T) \) such that for a fundamental cut \( X \) corresponding to \( e \), we have \( b(X) = \lambda_b(u, v) \).

### 1.2 Preparations

Let \((V, E, c)\) be a weighted graph.

#### 1.2.1 Proposition

\[
d_c(X) + d_c(Y) \geq d_c(X \cup Y) + d_c(X \cap Y) \text{ for all } X, Y \subseteq V.
\]

**Proof:** If edge \( e \) goes between \( X \setminus Y \) and \( Y \setminus X \), then it contributes \( 2c(e) \) to the left side and 0 to the right side of the inequality. The contribution of any other type of edge is the same for both sides. \( \blacksquare \)

For a sequence \((X_n)\) let

\[
\liminf X_n = \bigcup_{m=0}^{\infty} \bigcap_{n=m}^{\infty} X_n
\]

\[
\limsup X_n = \bigcap_{m=0}^{\infty} \bigcup_{n=m}^{\infty} X_n.
\]
If \( \lim \inf X_n = \lim \sup X_n \), then we denote this set by \( \lim X_n \) and call \((X_n)\) convergent.

1.2.2 Claim.

1. If \((X_n)\) is a convergent sequence of cuts, then \( d_c(\lim X_n) \leq \lim \inf d_c(X_n) \).

2. In addition, if \( \sum_{e \in E} c(e) < \infty \), then \( \lim d_c(X_n) \) exists and \( \lim d_c(X_n) = d_c(\lim X_n) \) holds.

Proof: It is routine to check that \( \text{out}_G(\lim X_n) = \text{out}_G(X_n) \) holds. Consider the discrete measure space \((E, \mathcal{P}(E), \tilde{c})\) where \( \tilde{c}(F) = \sum_{e \in F} c(e) \) for \( F \subseteq E \).

By applying the Fatou lemma to the characteristic functions of the sets \( \text{out}_G(X_n) \) we obtain

\[
d_c(\lim(X_n)) = \tilde{c}(\text{out}_G(\lim X_n)) = \tilde{c}(\lim \text{out}_G(X_n)) = \\
\tilde{c}(\lim \inf \text{out}_G(X_n)) \leq \lim \inf \tilde{c}(\text{out}_G(X_n)) = \lim \inf d_c(X_n).
\]

At the statement 2 we have \( \tilde{c}(E) < \infty \), thus the constant 1 function is integrable. Therefore by using Lebesgue theorem to the characteristic functions of the sets \( \text{out}_G(X_n) \) we obtain

\[
d_c(\lim X_n) = \tilde{c}(\lim \text{out}_G(X_n)) = \lim \tilde{c}(\text{out}_G(X_n)) = \lim d_c(X_n).
\]

Let us formulate our main result in the following more abstract way.

1.2.3 Theorem. Let \( V \) be a nonempty countable set and let \( b : \mathcal{P}(V) \to \mathbb{R}_+ \cup \{\infty\} \) such that

0. \( b(X) = 0 \iff X \in \{\emptyset, V\} \),

1. \( b(X) = b(V \setminus X) \) for \( X \subseteq V \), (\( b \) is symmetric)

2. \( b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \) for \( X, Y \subseteq V \), (\( b \) is submodular)

3. if \((X_n)\) is a nested sequence of cuts, then \( b(\lim X_n) = \lim b(X_n) \) (\( b \) is monotone-continuous)
4. $\lambda_b$ has only finite values. ($b$ finitely separate)

Then there exists an abstract Gomory-Hu tree with respect to $b$.

1.2.4 Remark. Properties 1,2 imply that for any $X,Y$ we also have

$$b(X) + b(Y) \geq b(X \setminus Y) + b(Y \setminus X).$$

Observe that if $\sum_{e \in E} c(e) < \infty$ holds, then $b := d_c$ satisfy the properties above. Hence as a special case of Theorem 1.2.3 we obtain:

1.2.5 Corollary. Every weighted graph with $\sum_{e \in E} c(e) < \infty$ admits a Gomory-Hu tree.

Consider the following weakening of 3.

3’ if $(X_n)$ is a nested sequence of cuts, then $b(\lim X_n) \leq \lim \inf b(X_n)$.

If we do not assume $\sum_{e \in E} c(e) < \infty$ and we demand just $(V, E, c)$ to be finitely separable, then Claim 1.2.2 ensures that $b := d_c$ still satisfy this weaker condition (see Claim 1.2.2/1). We will see by a counterexample that in this case one can not guarantee the existence of a Gomory-Hu tree. Even so, the next theorem provides something similar but weaker. A system of sets is called laminar if any two members of it are either disjoint or $\subseteq$-comparable.

1.2.6 Theorem. If $b$ satisfies conditions 0,1,2,3’,4, then there is a laminar system $\mathcal{L}^*$ of optimal cuts such that any pair from $V$ is separated optimally by some element of $\mathcal{L}^*$.

Proof:

1.2.7 Claim. For any $u \neq v \in V$ there exists an $u - v$ cut $X^*$ with $b(X^*) = \lambda^0(u,v)$.

Proof: Let $u,v$ be fix. The error of the sequence $(X_n)$ of $u - v$ cuts is

$$\sum_{n=0}^{\infty} (b(X_n) - \lambda(u,v)).$$
It is enough to prove the existence of a nested sequence \((Y_n)\) of \(u - v\) cuts with finite error. Indeed, from the finiteness of the error it follows that \(\lim b(Y_n) = \lambda(u, v)\), hence by property 3’

\[
\lambda_b(u, v) \leq b \left( \bigcap_{n=0}^{\infty} Y_n \right) \leq \lim inf b(Y_n) = \lim b(Y_n) = \lambda_b(u, v).
\]

1.2.8 Proposition. For any sequence \((X_n)\) with finite error there is another sequence \((Z_n)\) with less or equal error such that \(Z_0 \supseteq \bigcup_{n=1}^{\infty} Z_n\).

Proof: Replace in the sequence \((X_n)\) the member \(X_0\) by \(X_0 \cup X_1\) and the member \(X_1\) by \(X_1 \cap X_0\). By submodularity the error of the new sequence \((X^1_n)\) is less or equal. Then replace \(X^1_0 = X_0 \cup X_1\) by \(X^2_0 := X^1_0 \cup X^1_1 = X_0 \cup X_1 \cup X_2\) and replace \(X^1_2\) by \(X^2_2 := X^2_0 \cap X^2_1 = X_2 \cap (X_0 \cup X_1)\). In general let

\[
X^m_{n+1} = \begin{cases} 
X^m_0 \cup X^m_{m+1} & \text{if } n = 0 \\
X^m_0 \cap X^m_{m+1} & \text{if } n = m + 1 \\
X^m_n & \text{otherwise.}
\end{cases}
\]

Finally we claim that the following “limit” of these sequences is appropriate.

\[
Z_0 := \bigcup_{n=0}^{\infty} X_n \\
Z_{n+1} := X_{n+1} \cap \left( \bigcup_{i=0}^{n} X_i \right).
\]

For

\[
S_m := \sum_{n=0}^{\infty} (b(X^m_n) - \lambda(u, v))
\]

\((S_m)\) is a non-negative decreasing sequence thus it has a limit \(S\) i.e.

\[
S = \lim_{m \to \infty} \sum_{n=0}^{\infty} (b(X^m_n) - \lambda(u, v)).
\]

Consider the counting measure on \(\omega\) and apply Fatou lemma:
\[
S = \liminf_m \sum_{n=0}^\infty (b(X^m_n) - \lambda(u,v)) \\
\geq \sum_{n=0}^\infty (\liminf_m b(X^m_n) - \lambda(u,v)) \\
= \liminf_m b\left(\bigcup_{i=0}^m X_i\right) - \lambda(u,v) + \sum_{n=1}^\infty (b(Z_n) - \lambda(u,v)) \\
\geq \sum_{n=0}^\infty (b(Z_n) - \lambda(u,v)).
\]

At the last step we applied property 3’ ($S_0 + \lambda(u,v)$ is an obvious upper bound for \{b(X^m_0)\}_{m<\omega}). Hence the error of (Z_n) is smaller or equal than the error of the earlier sequences. ■

Let $X_n$ be a $u - v$ cut with $b(X_n) - \lambda(u,v) \leq 1/2^{n+1}$. Then the error of $(X_n)$ is at most 1. Apply Proposition 1.2.8 with $(X_n)$ to obtain $(X^1_n)$ and let $Y_0 = X^1_0$. Use Proposition 1.2.8 to the terminal segment of $(X^1_n)$ consists all but the 0-th element (this sequence has error at most $1 - (b(Y_0) - \lambda_0(u,v))$) to obtain $(X^2_n)$ and let $Y_1 = X^2_0$. By continuing the process recursively we build up a desired nested $(Y_n)$ with error at most 1. ■ ■

1.2.9 Remark. One can observe in the proof above that if $(X_n)$ is a sequence of $u - v$ cuts with finite error, then simply $\bigcup_{n=0}^\infty X_n$ and $\bigcap_{n=0}^\infty X_n$ are optimal $u - v$ cuts.

1.2.10 Proposition. The intersection and the union of (even infinitely many) optimal $u - v$ cuts is an optimal $u - v$ cut.

Proof: Let $X$ and $Y$ be optimal $u - v$ cuts. On the one hand, $b(X) \leq b(X \cup Y)$ and $b(Y) \leq b(X \cap Y)$ hold since $X \cup Y$ and $X \cap Y$ are $u - v$ cuts. Thus

\[
b(X) + b(Y) \leq b(X \cup Y) + b(X \cap Y).
\]

On the other hand,

\[
b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y)
\]
by submodularity. Hence equality holds and therefore \( b(X) = b(X \cup Y) \) and \( b(Y) = b(X \cap Y) \) since \( b(X), b(Y) < \infty \) because of property 4. By induction we know the statement for finitely many optimal \( u-v \) cuts. Consider an infinite family \( \mathcal{X} \) of optimal \( u-v \) cuts. Let \( V = \{v_n\}_{n<\omega} \) and let \( X'_n \in \mathcal{X} \) with \( v_n \notin X'_n \) if \( v_n \notin \bigcap \mathcal{X} \) and an arbitrary element of \( \mathcal{X} \) otherwise. Then \( X_n := \bigcap_{m=0}^n X'_m \) is an optimal \( u-v \) cut again and \( \bigcap_{n=0}^\infty X_n = \bigcap \mathcal{X} \) as well by property 3'.

1.2.11 Corollary. There is a \( \subseteq \)-smallest (largest) optimal \( u-v \) cut \( X_{u,v} \) which is the intersection (union) of all optimal \( u-v \) cuts.

1.2.12 Claim. Let \( X \) be an optimal \( s-t \) cut and let \( Y \) be an optimal \( u-v \) cut.

1. Assume \( X \) is a \( u-v \) cut. Then \( Y \cup X \) is an optimal \( u-v \) cut if \( t \notin Y \) and \( Y \cap X \) is an optimal \( u-v \) cut if \( t \in Y \).

2. Assume \( X \) is a \( v-u \) cut. Then \( Y \cup (V \setminus X) \) is an optimal \( u-v \) cut if \( s \notin Y \) and \( Y \setminus X \) is an optimal \( u-v \) cut if \( s \in Y \).

3. Assume \( u, v \in X \). Then \( Y \cap X \) is an optimal \( u-v \) cut if \( t \notin Y \) and \( Y \cup (V \setminus X) \) is an optimal \( u-v \) cut if \( t \in Y \).

4. Assume \( u, v \notin X \). Then \( Y \setminus X \) is an optimal \( u-v \) cut if \( s \notin Y \) and \( Y \cup X \) is an optimal \( u-v \) cut if \( s \in Y \).

Proof: It is enough to prove 1 and 3 since by replacing \( X \) with the optimal \( t-s \) cut \( V \setminus X \) in them we obtain 2 and 4 respectively. To prove 1 assume first that \( t \notin Y \). Since \( X \cup Y \) is a \( s-t \) cut and \( X \cap Y \) is a \( u-v \) cut we have \( b(X \cup Y) \geq b(X) \) and \( b(X \cap Y) \geq b(Y) \). Combining this with submodularity we get

\[
b(X) + b(Y) \geq b(X \cup Y) + b(X \cap Y) \geq b(X) + b(Y),
\]

thus equality must hold in both inequalities. If \( t \in Y \) and \( s \in Y \), then \( X \cup Y \) is a \( u-v \) cut and \( X \cap Y \) is a \( s-t \) cut; therefore by arguing similarly as above we obtain that \( X \cup Y \) must be an optimal \( u-v \) cut. Finally if \( t \in Y \) and \( s \notin Y \), then on the one hand \( Y \) separates \( t \) and \( s \) and \( X \) does this optimally therefore \( b(X) \leq b(Y) \), on the other hand \( Y \) is an optimal \( u-v \) cut and \( X \) is an \( u-v \) cut again.
cut hence \( b(Y) \leq b(X) \). Thus \( b(X) = b(Y) \) therefore \( X \) and \( Y \) both are optimal \( u - v \) cuts hence by Proposition 1.2.10 \( X \cup Y \) and \( X \cap Y \) as well. The proof of 3 is similar. ■

**1.2.13 Corollary.** If \( X \) is an optimal \( s - t \) cut and \( u \neq v \in X \), then either \( X_{u,v} \subseteq X \) or \( X_{v,u} \subseteq X \) (where \( X_{x,y} \) stands for the \( \subseteq \)-smallest optimal \( x - y \) cut).

**Proof:** If \( X_{u,v} \subseteq X \), then we are done. Assume \( X_{u,v} \not\subseteq X \). By the minimality of \( X_{u,v} \) the \( u - v \) cut \( X_{u,v} \cap X \) cannot be optimal therefore by Claim 1.2.12/3 \( X_{u,v} \cap (V \setminus X) \) is an optimal \( u - v \) cut. But then \( V \setminus [X_{u,v} \cup (V \setminus X)] = X \setminus X_{u,v} \) is an optimal \( v - u \) cut therefore we obtain \( X_{v,u} \subseteq (X \setminus X_{u,v}) \subseteq X \).

Theorem 1.2.6 follows immediately from the next lemma (actually we need the lemma just with finite \( \mathcal{L} \)).

**1.2.14 Lemma.** If \( \mathcal{L} \) is a laminar system of optimal cuts and \( u \neq v \in V \), then there is a cut \( X^* \) for which \( \mathcal{L} \cup \{X^*\} \) is laminar and \( X^* \) separates optimally \( u \) and \( v \).

**Proof:** Let us partition \( \mathcal{L} \) into four parts \( \mathcal{L}_{u,v} := \{X \in \mathcal{L} : u \in X \land v \notin X \} \), we define \( \mathcal{L}_{v,u}, \mathcal{L}_{u,v} \) and \( \mathcal{L}_{v,u} \) similarly. If \( X_{u,v} \subseteq \hat{X} \) for some \( \hat{X} \in \mathcal{L}_{u,v} \), then \( \{X_{u,v}\} \cup \mathcal{L}_{u,v} \cup \mathcal{L}_{v,u} \) is laminar. Suppose that we have no such an \( \hat{X} \) not even if we interchange \( u \) and \( v \). By Corollary 1.2.13 we know that for all \( W \in \mathcal{L}_{u,v} \) either \( X_{u,v} \subseteq W \) or \( X_{v,u} \subseteq W \). Hence by symmetry we may assume that \( \bigcap \mathcal{L}_{u,v} \supseteq X_{u,v} \).

We will show that \( \{X_{u,v}\} \cup \mathcal{L}_{u,v} \cup \mathcal{L}_{v,u} \) is laminar in this case as well. Let \( X \in \mathcal{L}_{v,u} \) be arbitrary. Then \( X_{v,u} \not\subseteq X \) otherwise \( \hat{X} := X \) would be a bound. But then \( X_{v,u} \cap X \) cannot be an optimal \( v - u \) cut by the minimality of \( X_{v,u} \). Therefore by Claim 1.2.12/1 we know that \( X_{v,u} \cup X \) is an optimal \( v - u \) cut and hence \( V \setminus (X_{v,u} \cup X) \) is an optimal \( u - v \) cut. Thus \( V \setminus (X_{v,u} \cup X) \supseteq X_{u,v} \) from which \( X \cap X_{u,v} = \emptyset \) follows.

Thus we may suppose that \( \{X_{u,v}\} \cup \mathcal{L}_{u,v} \cup \mathcal{L}_{v,u} \) is laminar. If for some \( Y \in \mathcal{L}_{u,v} \) the set \( \{X_{u,v}, Y\} \) is not laminar, then the cut \( X_{u,v} \cap Y \) may not be an optimal \( u - v \) cut because of the definition of \( X_{u,v} \). But then \( X_{u,v} \cup Y \) is an optimal \( u - v \) cut by Claim 1.2.12/1. Let \( \mathcal{Y} := \{Y : Y \in \mathcal{L}_{u,v} \land \{X_{u,v}, Y\} \text{ is not laminar} \} \).
The set \( \{X_{u,v} \cup Y : Y \in \mathcal{Y}\} \) consists of optimal \( u - v \) cuts and totally ordered by \( \subseteq \). By taking a cofinal sequence of type at most \( \omega \) and applying 3' we obtain that \( X_0 := \bigcup \mathcal{Y} \) is an optimal \( u - v \) cut. Note that \( \{X_0\} \cup (\mathcal{L} \setminus \mathcal{L}_{u,v}) \) is laminar. For each \( Z \in \mathcal{L}_{u,v} \) fix some \( s_Z, t_Z \) such that \( Z \) is an optimal \( s_Z - t_Z \) cut. We claim that if for such a \( Z \) the pair \( \{X_0, Z\} \) is not laminar, then \( \{X_{u,v}, Z\} \) is not laminar as well. Indeed, consider just the construction of \( X_0 \) and the fact that if for a cut \( Y \in \mathcal{L}_{u,v} \) we have \( Y \cap Z \neq \emptyset \), then \( Z \subseteq Y \) by the laminarity. Let

\[
Z := \{Z \in \mathcal{L}_{u,v} : \{X_0, Z\} \text{ is not laminar}\}.
\]

For \( Z \in Z \) we know that \( \{X_{u,v}, Z\} \) is not laminar. By the definition of \( X_{u,v} \) the cut \( X_{u,v} \setminus Z \) may not be an optimal \( u - v \) cut hence by Claim 1.2.12/4 it follows, that \( s_Z \in X_{u,v} (\subseteq X_0) \). Finally by Claim 1.2.12/4 we may take \( X^* = X \cup \bigcup Z \) by adding countably many elements of \( Z \) with union \( \bigcup Z \) one by one to \( X \) and taking limit. \( \blacksquare \)

### 1.3 A counterexample

In the previous section we obtained (as a special case of Theorem 1.2.6) the existence of a laminar system of optimal cuts for countably infinite finitely separable weighted graphs which elements separate any vertex pair optimally. In this section we show by an example that one cannot guarantee the existence of a Gomory-Hu tree as well without further assumptions. Let \( V = \{v_n\}_{n \leq \omega} \) and let \( E = \{v_n v_{n+1}, v_n v_{n+1}\}_{n < \omega} \). Finally \( c(v_n v_{n+1}) := 1 \) for all \( n < \omega \) and with the notation \( e_n = v_n v_{n+1} \)

\[
c(e_n) := \begin{cases} 
2 & \text{if } n = 0 \\
c(e_{n-1}) + n + 1 & \text{if } n > 0.
\end{cases}
\]
1.3.1 Claim. If $n < m < \omega$, then $\{v_0, v_1, \ldots, v_n\} =: V_n$ is the only optimal $v_n - v_m$ cut.

Proof: Pick an optimal $v_n - v_m$ cut $X$. Since $d_c(V_n) < c(e_k)$ whenever $k > n$, a cut $X$ may not separate the end vertices of such an $e_k$. Then $v_\omega \notin X$ otherwise $d_c(X) = \infty$. Thus we have $X \subseteq V_n$. Suppose, for seeking a contradiction, that $v_l \notin X$ for some $l < n$ and $l$ is the largest such an index. Then

$$d_c(X) - d_c(V_n) \geq c(e_l) - l - 1 > 0,$$

which contradicts to the optimality of $X$. ■

1.3.2 Claim. $G$ has no Gomory-Hu tree.

Proof: Assume, to the contrary, that $T$ is a Gomory-Hu tree of $G$. For all $e \in E(T)$ pick the fundamental cut $X_e$ that corresponds to $e$ and does not contain $v_\omega$. On the one hand, $\mathcal{L} := \{X_e\}_{e \in E(T)}$ is a laminar system of optimal cuts that contains at least one $\subseteq$-maximal element (if $e$ incident with $v_\omega$ in $T$, then $X_e$ is a maximal element). On the other hand, $\mathcal{L} = \{V_n\}_{n \in \omega}$ since the optimal cuts are unique up to complementation and the additional condition “does not contain $v_\omega$” makes them unique. This is a contradiction since $(V_n)$ is a strictly increasing sequence. ■

1.3.3 Remark. One can obtain also a locally finite counterexample by some easy modification of our counterexample above.
1.4 Existence of an abstract Gomory-Hu tree

In this section we prove our main result (which is Theorem 1.2.3). It will be convenient to use the following equivalent but formally weaker definition of Gomory-Hu trees.

1.4.1 Claim. \( T = (V, F) \) is a Gomory-Hu tree with respect to \( b \) if for all \( uv \in F \) the fundamental cuts corresponding to \( uv \) in \( T \) separate optimally \( u \) and \( v \).

Proof: Let \( u \neq v \in V \) be arbitrary and let \( v_1, v_2, \ldots, v_m \) be the vertices of the unique \( u - v \) path in \( T \) numbered in the path order.

1.4.2 Proposition. For all pairwise distinct \( u, v, w \in V \) we have:

\[
\lambda_b(u, w) \geq \min\{\lambda_b(u, v), \lambda_b(v, w)\}.
\]

Proof: It follows from the fact that if a cut separates \( u \) and \( w \), then it separates either \( u \) and \( v \) or \( v \) and \( w \) as well. \( \blacksquare \)

On one hand by applying the Proposition above repeatedly we obtain

\[
\lambda_b(u, v) \geq \min\{\lambda_b(v_i, v_{i+1}) : 1 \leq i < m\} =: \lambda_b(v_{i_0}, v_{i_0+1}) \text{ for some } 1 \leq i_0 < m.
\]

On the other hand, the fundamental cuts corresponding to the edge \( v_{i_0}v_{i_0+1} \) separates \( u \) and \( v \) and have value \( \lambda_b(v_{i_0}, v_{i_0+1}) \) by assumption. Thus

\[
\lambda_b(u, v) \leq \lambda_b(v_{i_0}, v_{i_0+1}).
\]

Hence equality holds and the fundamental cuts corresponding to \( v_{i_0}v_{i_0+1} \in F \) are optimal cuts between \( u \) and \( v \). \( \blacksquare \)

A sequence \((X_n)\) of optimal cuts is \textbf{essential} if all of its members separate optimally a vertex pair that the earlier members do not.

1.4.3 Lemma. If \((X_n)\) is a \( \subseteq \)-monotone sequence of optimal cuts and \( \lim X_n =: X \notin \{\emptyset, V\} \), then \((X_n)\) has no essential subsequence.
Proof: Assume, to the contrary, that \((X_n)\) is a counterexample. By symmetry we may suppose that \((X_n)\) is increasing. By trimming \((X_n)\) we may assume that it is essential witnessed by \(s_n, t_n\) i.e. \(X_n\) is an optimal \(s_n - t_n\) cut but \(X_m\) is not whenever \(m < n < \omega\).

1.4.4 Claim. \(t_n \notin X\) holds for all large enough \(n < \omega\).

Proof: Suppose that \(t_n \in X\) for infinitely many \(n\). By the monotone-continuity the numerical sequence \((b(X_n))\) converges to \(b(X) > 0\), thus \(b(X_n) \geq b(X)/2 > 0\) for large enough \(n\). On the one hand, \(b(X \setminus X_n) \to 0\) since \((X \setminus X_n) \to \emptyset\) monotonously. On the other hand, \(X \setminus X_n\) is a \(t_n - s_n\) cut for infinitely many \(n\) because of the indirect assumption and for such an \(n\)

\[
b(X \setminus X_n) \geq b(X_n) \geq b(X)/2 > 0
\]

which is a contradiction. ■

By trimming \((X_n)\), we may assume that \(t_n \notin X\) for all \(n < \omega\). It implies that \(b(X_n) \leq b(X_{n+1})\) for each \(n < \omega\) because \(X_{n+1}\) is an \(s_n - t_n\) cut and \(X_n\) is an optimal \(s_n - t_n\) cut. But then \(s_{n+1} \notin X_n\) for all \(n < \omega\), otherwise \(X_n\) would be at least as good \(s_{n+1} - t_{n+1}\) cut as the optimal one but \(X_{n+1}\) is the first optimal \(s_{n+1} - t_{n+1}\) cut of the sequence by the choice of \((X_n)\).

1.4.5 Claim. \(X_n\) is an optimal \(s_n - s_{n+1}\) cut for all \(n < \omega\).

Proof: By Corollary 1.2.13, there is an \(Y \in \{X_{s_n, s_{n+1}}, X_{s_{n+1}, s_n}\}\) for which \(Y \subseteq X_{n+1}\). If \(b(Y) < b(X_n)\) would hold, then \(Y\) would be either a better \(s_n - t_n\) cut than \(X_n\) or a better \(s_{n+1} - t_{n+1}\) cut than \(X_{n+1}\); which both are impossible. ■

Since \(s_n \in X\) for all \(n < \omega\) the Claim above contradicts to Claim 1.4.4 with the choices \(s_n := s_n\) and \(t_n := s_{n+1}\). ■ ■

Take an optimal cut \(X\). For \(u \neq v \in X\) let \(u \prec_X v\) if \(X_{u,v} \nsubseteq X\).

1.4.6 Claim. The relation \(\prec_X\) is a strict partial ordering on \(X\).
Proof: It is irreflexive by definition. For transitivity assume \( u \prec_X v \prec_X w \). If \( u = w \), then we have \( u \prec_X v \) and \( v \prec_X u \) which contradicts to Corollary 1.2.13. Thus we may assume that \( u, v, w \) are pairwise distinct. Suppose, to contrary, that \( u \prec_X w \) does not hold i.e. \( X_{u,w} \subseteq X \). Assume first that \( v \in X_{u,w} \). By Corollary 1.2.13, either \( X_{u,v} \subseteq X_{u,w} \) or \( X_{v,u} \subseteq X_{u,w} \). Since \( u \prec_X v \), necessarily \( X_{v,u} \subseteq X_{u,w} \). But then \( X_{u,w} \) and \( X_{v,u} \) are both \( v-w \) cuts and

\[
\lambda_b(v, w) \geq \min\{\lambda_b(v, u), \lambda_b(u, w)\} = \min\{b(X_{v,u}), b(X_{u,w})\}
\]

shows that one of them is optimal which contradicts to \( v \prec_X w \).

Hence \( v \notin X_{u,w} \) holds. \( X_{u,w} \) is not an optimal \( u-v \) cut since \( u \prec_X v \). Therefore \( b(X_{u,w}) > b(X_{v,u}) \). (Note that \( X_{v,u} \subseteq X \) by \( u \prec_X v \) and by Corollary 1.2.13). Hence \( w \notin X_{v,u} \) otherwise \( X_{v,u} \) would be a better cut between \( w \) and \( u \) than the optimal. On the other hand, \( X_{v,u} \) is not an optimal \( v-w \) cut since \( v \prec_X w \) hence \( X_{w,v} \subseteq X \) and \( b(X_{w,v}) < b(X_{v,u}) \) hold. Necessarily \( u \in X_{w,v} \), otherwise \( X_{w,v} \) separates better \( w \) and \( u \) than \( X_{u,w} \), but then \( X_{w,v} \) separates better \( u \) and \( v \) than \( X_{v,u} \), which is a contradiction. \( \blacksquare \)

1.4.7 Lemma. If \( X \) is an optimal \( s-t \) cut, then \( X \) has a \( \prec_X \)-minimal element \( s' \). For all such an \( s' \), cut \( X \) it is an optimal \( s'-t \) cut.

Proof: Let \( A = \{ x \in X : \lambda_b(x, t) = \lambda_b(s, t) \} \) and \( B := \{ y \in X : \lambda_b(y, t) < \lambda_b(s, t) \} \). Then \( A \cup B \) is a partition of \( X \). Note that \( A \neq \emptyset \) since \( s \in A \).

1.4.8 Proposition. For all \( x \in A \) and \( y \in B : x \prec_X y \) holds.

Proof: If \( x \in A \) and \( y \in B \), then \( \lambda_b(x, y) < \lambda_b(s, t) (= \lambda_b(x, t)) \), otherwise

\[
\lambda_b(y, t) \geq \min\{\lambda_b(x, y), \lambda_b(x, t)\} = \lambda_b(x, t) = \lambda_b(s, t)
\]

contradicts to \( y \in B \). Therefore if \( X_{x,y} \subseteq X \) would hold, then (since \( X_{x,y} \) is a \( x-t \) cut)

\[
\lambda_b(x, t) \leq \lambda_b(x, y) < \lambda_b(s, t)
\]

which is impossible since \( x \in A \). \( \blacksquare \)
By Proposition 1.4.8, it is enough to find a minimal element for the poset \((A, \prec_X)\). The existence of such an element follows immediately from the following Proposition.

1.4.9 Proposition. Set \(A\) is finite.

Proof: Assume, to seeking for contradiction, that \(A\) is infinite. Pick a nested sequence \((A_n)\) of nonempty subsets of \(A\) with \(\bigcap_{n=0}^{\infty} A_n = \emptyset\). On the one hand, \(b(A_n) \to 0\) by property 3. On the other hand, every \(A_n\) separates an \(x \in A\) from \(t\) and hence

\[ b(A_n) \geq \lambda_b(x, t) = \lambda_b(s, t) > 0, \]

which is a contradiction. ■

For the second part of Lemma 1.4.7 let \(s'\) be a \(\prec_X\)-minimal element of \(X\). Then by Proposition 1.4.8 \(s' \in A\) thus \(\lambda_b(s', t) = \lambda_b(s, t)\) by the definition of \(A\). ■

1.4.10 Claim. For any \(s \in V\) the family \(C_s := \{X_{u,s} : u \in V \setminus \{s\}\}\) of optimal cuts is laminar.

Proof: Suppose, to the contrary, that \(\{X_{u,s}, X_{v,s}\} \subseteq C_s\) is not laminar. If \(u \in X_{v,s}\), then \(X_{u,s} \cap X_{v,s}\) is an \(u - s\) cut and \(X_{u,s} \cup X_{v,s}\) is a \(v - s\) cut. By submodularity \(X_{u,s} \cap X_{v,s}\) is an optimal \(u - s\) cut (and \(X_{u,s} \cup X_{v,s}\) is an optimal \(v - s\) cut) which contradicts to the definition of \(X_{u,s}\). For \(v \in X_{u,s}\) the argument is the same. Finally if \(u \in X_{u,s} \setminus X_{v,s}\) and \(v \in X_{v,s} \setminus X_{u,s}\), then \(X_{u,s} \setminus X_{v,s}\) is an \(u - s\) cut and \(X_{v,s} \setminus X_{u,s}\) is an \(v - s\) cut thus by applying Remark 1.2.4 follows that they are also optimal, contradicting to the definitions of \(X_{v,s}\) and \(X_{u,s}\). ■

Let \(\prec_V\) be the trivial partial ordering on \(V\) (i.e. under which there are no comparable elements).

1.4.11 Lemma. Let \(X\) be either an optimal cut or \(V\). Pick an \(\prec_X\)-minimal element \(s\) of \(X\) (see Lemma 1.4.7). Then the \(\subseteq\)-maximal elements of the laminar system \(C_{s,X} := \{X_{u,s} : u \in X \setminus \{s\}\}\) forms a partition of \(X \setminus \{s\}\).
Proof: By the choice of $s$ we know that $\bigcup C_{s,X} \subseteq X \setminus \{s\}$. It is enough to show that for any $u^* \in X \setminus \{s\}$ the laminar system $C_{s,X}$ has a maximal element that contains $u^*$. Assume, seeking for contradiction, that it is false and $(X_{u^*,s})$ is a strictly increasing sequence that shows this. On the one hand, this sequence is essential because $X_{u^*,s}$ may not be an optimal $u_n - s$ cut for $m < n$ since $X_{u^*,s}$ is the $\subseteq$-smallest such a cut. On the other hand, $\lim X_{u^*,s} \subseteq V \setminus \{s\}$ which contradicts to Lemma 1.4.3.

We build the desired abstract Gomory-Hu tree for $b$ by using Lemma 1.4.11 repeatedly. Pick an arbitrary $r \in V$ for root. It makes possible to define a unique fundamental cut for each edge $e$ of the tree, namely the vertex set of the component after deletion of $e$ that does not contain $r$. Let $\{X_i\}_{i \in I_0}$ consists of the maximal elements of the laminar system $C_r$. Let $x_i$ be a $\prec_{X_i}$-minimal element of $X_i$ and draw the tree-edges $rx_i$ for $i \in I_0$. Note that Lemma 1.4.7 ensures that the fundamental cut corresponds to $rx_i$ will separate optimally $r$ and $x_i$ assuming that $X_i$ will be the vertex set of the subtree rooted at $x_i$. Take now for each $i \in I_0$ the $\subseteq$-maximal elements $\{X_{i,j}\}_{j \in I_1}$ of $C_{x_i,X_i}$ and choose a $\prec_{X_{i,j}}$-minimal element $x_{i,j}$ of $X_{i,j}$. Draw the tree-edges $x_ix_{i,j}$ for all $i \in I_0$ and $j \in I_1$. By continuing recursively the process we claim that every $v \in V$ has to appear in the tree. Indeed, if some $v$ does not, then we would obtain a nested essential sequence of optimal cuts such that its limit contains $v$ which contradicts to Lemma 1.4.3.
Chapter 2

Highly connected infinite digraphs without edge-disjoint back and forth paths between a certain vertex pair

2.1 Background and Motivation

R. Aharoni and C. Thomassen proved by a construction the following theorem that shows that several edge-connectivity related theorems of finite graphs and digraphs become “very” false in the infinite case.

2.1.1 Theorem (R. Aharoni, C. Thomassen [3]). For all $k < \omega$ there is an infinite graph $G = (V, E)$ and $s, t \in V$ such that $E$ has a $k$-edge-connected orientation but for each path $P$ between $s$ and $t$ the graph $G = (V, E \setminus E(P))$ is not connected.

If $D$ is a $k$-edge-connected finite digraph, then for all $s_1, t_1, \ldots, s_k, t_k \in V(D)$ there are pairwise edge-disjoint paths $P_1, \ldots, P_k$ such that $P_i$ is an $s_i \rightarrow t_i$ path. This fact is implied by the following Theorem of W. Mader as well as the (strong form of) Edmonds' Branching theorem (see [11] p. 349 Theorem 10.2.1).

2.1.2 Theorem (W. Mader, [24]). Let $D = (V, A)$ be a $k+1$-edge-connected finite digraph and $s, t \in V$. Then there is an $s \rightarrow t$ path $P$ such that $(V, A \setminus A(P))$ is $k$-edge-connected.
We will show that in the infinite case there is no \( k < \omega \) such that \( k \)-edge-connectivity guarantees for infinite digraphs even the existence of edge-disjoint \( s_1 \to t_1 \) and \( s_2 \to t_2 \) paths for all choice of the vertices \( s_1, t_1, s_2, t_2 \). Not even in the special case where the two ordered vertex pairs are the converse of each other.

### 2.2 Main result

#### 2.2.1 Theorem (A. Joó, [18]). For all \( k < \omega \) there exists a \( k \)-edge-connected digraph without back and forth edge-disjoint paths between a certain vertex pair.

**Proof.** Let \( k \geq 2 \) be fixed and let \( I := \{0, \ldots, 2k - 1\} \), \( I_e := \{i \in I : i \text{ is even }\} \), \( I_o := I \setminus I_e \). Denote by \( I^* \) the set of the finite sequences from \( I \). Let the vertex set \( V \) of the digraph is the union of the disjoint sets \( \{s_\mu : \mu \in I^*\} \) (we mean \( s_\mu = s_\nu \) iff \( \mu = \nu \)) and \( \{t_\mu : \mu \in I^*\} \) (\( t_\mu = t_\nu \) iff \( \mu = \nu \)). If \( \mu \) is the empty sequence we write simply \( s, t \) and we denote the concatenation of sequences by writing them successively. For \( \nu \in I^* \) we define the set \( V_\nu := \{r_{\nu \mu} : r \in \{s, t\}, \mu \in I^*\} \subseteq V \). The edge-set \( A \) of the digraph consists of the following edges (see Figure 2.1). For all \( \mu \in I^* \) there are \( k \) edges in both directions between the two elements of the following pairs:

\[
\{s_\mu, t_\mu\}, \{s_{\mu i}, t_{\mu(i+2)}\} (i = 0, \ldots, 2k - 3), \{s_{\mu(2k-2)}, t_\mu\}.
\]

The following are single directed edges for any \( \mu \in I^* \):

\[
(s_\mu, t_{\mu 0}), (t_{\mu i}, s_{\mu(i+1)})_{i \in I_e}, (s_{\mu i}, t_{\mu(i+1)})_{i \in I_o \setminus \{2k-1\}}, (s_{\mu(2k-1)}, t_\mu).
\]

Finally let \( D := (V, A) \).

#### 2.2.2 Remark. One can avoid using parallel edges (without losing the desired properties of the digraph) by dividing each of these edges with one new vertex and drawing between them \( k(k - 1) \)-many new directed edges, one for each ordered pair. One can also achieve \( k \)-connectivity instead of \( k \)-edge-connectivity by using some similarly easy modification.

#### 2.2.3 Proposition. For \( \nu \in I^* \) the function \( f_\nu : V \to V_\nu \), \( f_\nu(r_\mu) := r_{\nu \mu} (r \in \{s, t\}) \) is an isomorphism between \( D \) and \( D[V_\nu] \).
Figure 2.1: The digraph $D$ in the case $k = 3$. Thick, two-headed arrows stand for $k$ parallel edges in both directions. The subdigraphs of the form $D[V_i]$ are isomorphic to the whole $D$ by Proposition 2.2.3.

Proof: It is a direct consequence of the definition of the edges since the number of edges from $r_\mu$ to $r'_\mu'$ are the same as from $r_\nu\mu$ to $r'_\nu\mu'$ for all $r, r', \nu, \mu, \mu' \in I^*$. □

2.2.4 Proposition. Denote by $D_v$ the digraph that we obtain from $D$ by contracting for all $i \in I$ the set $V_i$ to a vertex $v_i$. Then $D_v$ is $k$-edge-connected.

Proof: In the vertex-sequence $s, v_1, v_3, \ldots, v_{2k-1}$ there are $k$ edges in both directions between the neighboring vertices such as in the sequence $v_0, v_2, \ldots, v_{2k-2}, t$. Finally between the vertex sets of the sequences above there are in both directions at least $k$ edges. □

Let $\lambda\{u, v\} := \min\{\lambda(u, v), \lambda(v, u)\}$.

2.2.5 Proposition. $D$ is connected.

Proof: We will show that $\lambda\{s, r_\mu\} \geq 1$ for all $r \in \{s, t\}$, $\mu \in I^*$. We will use induction on length of $\mu$ (which is denoted by $|\mu|$). Consider first the $|\mu| = 0, 1$ cases directly.

The path $s, t_0, s_1, t_2, s_3, \ldots, t_{2k-2}, s_{2k-1}, t$ shows that $\lambda(s, t) \geq 1$. Using the isomorphism $f_i$ (see Proposition 2.2.3) we may fix for all $i \in I$ an $s_i \to t_i$ path $P_{s_i, t_i}$
in $D[V_i]$. (It will be sometimes convenient to identify paths with the corresponding vertex sequences.) The path
\[ t, P_{s_{k-2},t_{k-2}}, \ldots, P_{s_{2j},t_{2j}}, \ldots, P_{s_0,t_0}, P_{s_1,t_1}, s \]
justifies that $\lambda(t,s) \geq 1$ (thus $\lambda\{s,t\} \geq 1$). Then we may fix a $t_i \rightarrow s_i$ path $P_{t_i,s_i}$ in $D[V_i]$ ($i \in I$). The paths
\[ s, P_{t_1,s_1}, P_{t_3,s_3}, \ldots, P_{t_{2j+1},s_{2j+1}}, \ldots, P_{t_{2k-1},s_{2k-1}} \]
\[ P_{s_{2k-1},t_{2k-1}}, P_{s_{2k-3},t_{2k-3}}, \ldots, P_{s_{2k-1-2j},t_{2k-1-2j}}, \ldots, P_{s_1,t_1}, s \]
certify that $\lambda\{s,r_i\} \geq 1$ if $r \in \{s,t\}$, $i \in I_e$. The paths
\[ t, P_{s_{k-2},t_{k-2}}, P_{s_{k-4},t_{k-4}}, \ldots, P_{s_{2k-2-2j},t_{2k-2-2j}}, \ldots, P_{s_0,t_0} \]
\[ P_{t_0,s_0}, P_{t_2,s_2}, \ldots, P_{t_{2j},s_{2j}}, \ldots, P_{t_{2k-2},s_{2k-2}}, t \]
certify that $\lambda\{t,r_i\} \geq 1$ if $r \in \{s,t\}$, $i \in I_e$ and thus (by $\lambda\{s,t\} \geq 1$ and by transitivity) $\lambda\{s,r_i\} \geq 1$ if $r \in \{s,t\}$, $i \in I_e$. Hence the cases $\mu \in I^*$ with $|\mu| \leq 1$ are settled.

Let be $l \geq 1$ and suppose $\lambda\{s,r_{\mu}\} \geq 1$ if $r \in \{s,t\}$, $\mu \in I^*$, $|\mu| \leq l$. Let $\nu = \mu i$, where $i \in I$ and $|\nu| = l$. By the induction hypothesis we have $\lambda\{s,s_{\mu}\} \geq 1$. By the induction hypothesis for $l = 1$ we have $\lambda\{s,r_i\} \geq 1$ and hence $\lambda\{s_{\mu},r_{\mu i}\} \geq 1$ by the isomorphism $f_{\mu i}$. Combining these, we get $\lambda\{s,r_{\mu i}\} \geq 1$. 

2.2.6 Lemma. $D$ is $k$-edge-connected.

Proof: Let $k > l \geq 1$.

2.2.7 Proposition. Let $\mu \in I^*$ be arbitrary. If we delete at most $l$ edges of the digraph $D[V_\mu]$ in such a way that its subdigraphs $D[V_\mu_i]$ ($i \in I$) remain connected after the deletion, then $D[V_\mu]$ also remains connected after the deletion.

Proof: Because the isomorphism $f_\mu$ it is enough to deal with the case where $\mu$ is the empty sequence. Denote by $D'$ the digraph that we have after the deletion. Let $D'_v$ be the digraph that we get from $D'$ by contracting the sets $V_i$ to a vertex $v_i$ for
all $i \in I$. The digraphs $D'[V_i] \ (i \in I)$ are connected by assumption, thus $D'$ is connected if $D'_v$ is connected. The digraph $D'_v$ arises by deleting at most $l < k$ edges of the $k$-edge-connected digraph $D_v$ (see Proposition 2.2.4) hence it is connected. ■

We will prove that if $D$ is $l$-edge-connected, then it is also $l+1$-edge-connected. This is enough since we have already proved the connectivity (i.e. 1-connectivity) of $D$ in Proposition 2.2.5. Assume that $D$ is $l$-edge-connected. Let $C \subseteq A$, $|C| = l$ arbitrary and let $D' = (V, A \setminus C)$. By the definition of $l+1$-edge-connectivity we need to show that $D'$ is connected. Suppose, seeking for contradiction, that it is not. Since the connectivity of the subdigraphs $D'[V_i] \ (i \in I)$ implies the connectivity of $D'$ (by Proposition 2.2.7) there is an $i_0 \in I$ such that $D'[V_{i_0}]$ is not connected. Since the connectivity of the subdigraphs $D'[V_{i_0}, l]$ ($i \in I$) implies the connectivity of $D'[V_{i_0}]$ there is an $i_1 \in I$ such that $D'[V_{i_0}, i_1]$ is not connected. By recursion, we obtain an infinite sequence $(i_n)$ such that the digraphs $D'[V_{i_0}, i_n] \ (n < \omega)$ are all disconnected. Note that the digraphs $D[V_{i_0}, i_n] \ (n < \omega)$ are $l$-connected because $D$ is $l$-connected by assumption and they are isomorphic to it, hence necessarily $C \subseteq \text{span}(V_{i_0}, i_n)$ for all $n < \omega$. But then

$$C \subseteq \bigcap_{n=0}^{\infty} \text{span}(V_{i_0}, i_n) = \text{span} \left( \bigcap_{n=0}^{\infty} V_{i_0}, i_n \right) = \text{span}(\emptyset) = \emptyset$$

which is a contradiction since $|C| = l \geq 1$. ■

### 2.2.8 Lemma

There are no edge-disjoint back and forth paths between $s$ and $t$ in $D$.

**Proof:** Suppose, to the contrary, that there are. Let $P_{s,t}$ be an $s \to t$ path and $P_{t,s}$ be a $t \to s$ path such that they are edge-disjoint and have a minimal sum of lengths among these path pairs. For $u, v \in V$ call a set $U \subseteq V$ a $u - v$ cut if $u \in U$ and $v \notin U$. The set $\{t\} \cup \bigcup \{V_i : i \in I_t\}$ is a $t - s$ cut and its outgoing edges are $\{(t_i, s_{i+1})\}_{i \in I_t}$. Let $i_0 \in I_t$ be the maximal index such that $P_{t,s}$ uses the edge $(t_{i_0}, s_{i_0+1})$. Then an initial segment of $P_{t,s}$ is necessarily of the form $t, P_{s_{i_2}, t_{i_2}}, P_{s_{i_3}, t_{i_3}}, \ldots, P_{s_{i_0}, t_{i_0}}, s_{i_0+1}$ where $P_{s_{i_k}, t_{i_k}}$ is an $s_i \to t_i$ path in $D[V_i]$. The set $T := \{t\} \cup \bigcup \{V_i : i_0 \leq i \in I\}$ is also a $t - s$ cut and all the tails of its
outgoing edges are in \{t_{i_0}, t_{i_0+1}\}. \( P_{t,s} \) has already used the edge \((t_{i_0}, s_{i_0+1})\) so it may not use another edge with tail \( t_{i_0} \) hence \( P_{t,s} \) leave \( T \) using an edge with tail \( t_{i_0+1} \). Then \( P_{t,s} \) contains an \( s_{i_0+1} \to t_{i_0+1} \) subpath \( P_{s_{i_0+1}, t_{i_0+1}} \) in \( D[V_{i_0+1}] \).

\[ S := \{s\} \cup \bigcup \{V_i : i_0 + 1 \geq i \in I\} \]

is an \( s-t \) cut and all the tails of its outgoing edges are in \( \{s_{i_0}, s_{i_0+1}\} \). Therefore \( P_{s,t} \) has an initial segment in \( D[S] \) that terminates in this set. We know that \( P_{s,t} \) does not use the edge \((t_{i_0}, s_{i_0+1})\) because \( P_{t,s} \) has already used it. Therefore there is an \( m \in \{i_0, i_0 + 1\} \) such that \( P_{s,t} \) has a \( t_m \to s_m \) subpath \( P_{t_m,s_m} \) in \( D[V_m] \). But then the paths \( P_{t_m,s_m} \) and \( P_{s_m,t_m} \) are proper subpaths of \( P_{s,t} \) and \( P_{t,s} \), respectively. By Proposition 2.2.3 \( f_m \) is an isomorphism between \( D \) and \( D[V_m] \) and thus the inverse-images of the paths \( P_{t_m,s_m} \) and \( P_{s_m,t_m} \) are edge-disjoint back and forth paths between \( s \) and \( t \) with a sum of lengths strictly less than the sum of the lengths of the paths \( P_{s,t} \) and \( P_{t,s} \); this contradicts to the choice of \( P_{s,t} \) and \( P_{t,s} \).
Chapter 3

On partitioning of edges of infinite digraphs into directed cycles

Nash-Williams proved the following theorem in [25] (see page 235 Theorem 3).

3.0.9 Theorem (Nash-Williams). If $G$ is an undirected graph, then $E(G)$ can be partitioned into cycles if and only if every cut has either even or infinite number of edges.

L. Soukup gave a new shorter proof to the theorem above (Theorem 5.1 of [27]) and worked out a general method based on elementary submodel techniques to handle similar problems (Theorem 5.4 in [27]). Nash-Williams was aware of the following directed analogue of his theorem as well (he mentioned it in [25], page 237 Theorem 3’ but we could not find any written proof of it.)

3.0.10 Theorem (Nash-Williams). If $D = (V, A)$ is a directed graph, then $A$ can be partitioned into directed cycles if and only if for all $X \subseteq V$ the cardinalities of the ingoing and the outgoing edges of $X$ are equal.

Without knowing this we conjectured and proved Theorem 3.0.10. The main difficulty in contrast to the undirected case using elementary submodel approach is that in the undirected case one can find a finite witness for the violation of the condition but in the directed case we do not necessarily have a small witness. To handle this we apply a modified version of the general framework of L. Soukup.
3.1 Preparations

We call $X \subseteq V$ overloaded (with respect to $D = (V, A)$) if $|\text{out}_D(X)| < |\text{in}_D(X)|$ and we call $D$ balanced if there is no such an $X$. If $M$ is an arbitrary set, then let $D(M) := (V \cap M, A \cap M)$ and let $D \setminus M := (V, A \setminus M)$.

An observation about overloaded sets

We need the following basic observation to find overloaded sets in an unbalanced digraph in a special form.

3.1.1 Lemma. If $D = (V, A)$ is an unbalanced digraph, then it has a weakly connected component with vertex set $Z$ and an $X \cup Y$ partition of $Z$ such that $D[X]$ and $D[Y]$ are weakly connected and $X$ is overloaded in $D$.

Proof: Let $X' \subseteq V$ be overloaded and let $X_i (i \in I)$ be the vertex sets of the weakly connected components of $D[X']$. Then

$$\sum_{i \in I} |\text{out}_D(X_i)| = |\text{out}_D(X')| < |\text{in}_D(X')| = \sum_{i \in I} |\text{in}_D(X_i)|$$

therefore there is an $i_0 \in I$ such that $|\text{out}_D(X_{i_0})| < |\text{in}_D(X_{i_0})|$. Let $Y_j (j \in J)$ be the vertex sets of the weakly connected components of $D[V \setminus X_{i_0}]$ then

$$\sum_{j \in J} |\text{in}_D(Y_j)| = |\text{out}_D(X_{i_0})| < |\text{in}_D(X_{i_0})| = \sum_{j \in J} |\text{out}_D(Y_j)|$$

thus there is an $j_0 \in J$ such that $|\text{in}_D(Y_{j_0})| < |\text{out}_D(Y_{j_0})|$. Denote by $Z$ the vertex set of the weakly connected component of $D$ that contains $Y_{j_0}$ then $X := Z \setminus Y_{j_0}$ is appropriate and $Y_{j_0}$ will be the desired $Y$. ■

Elementary submodels

We give here a quick survey about elementary submodel techniques that we use to prove the main result of this chapter. One can find a more detailed survey with many combinatorial applications in [27].
All the formulas and models in this chapter are in the first order language of set theory and the models are ∈-models i.e. the “element of” relation in them is the real “∈”. A model $M_0$ is an elementary submodel of $M_1$ if $M_0 \subseteq M_1$ and for each formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M_0 : M_0 \models \varphi(a_1, \ldots, a_n)$ if and only if $M_1 \models \varphi(a_1, \ldots, a_n)$. Let $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ be a finite set of formulas where the free variables of $\varphi_i$ are $x_{i,1}, \ldots, x_{i,n_i}$. We call a set $M$ a $\Sigma$-elementary submodel if the formulas in $\Sigma$ are absolute between $M$ and the universe i.e.

$$\bigwedge_{i=1}^n \forall a_1, \ldots, a_n \in M [ (M \models \varphi_i(a_1, \ldots, a_n)) \iff \varphi_i(a_1, \ldots, a_n) ].$$

The common practice by elementary submodel techniques is to fix a large enough finite set $\Sigma$ of formulas at the beginning and do not say explicitly what it is. After that, during the proof the author refers finitely many times that this and that formula is in $\Sigma$. If it is not satisfactory for someone, then he or she may consider $\Sigma$ as the set of those formulas that have length at most $10^{10}$ and contains at most the variables: $v_1, \ldots, v_{10^{10}}$. Anyway, from now on $\Sigma$ is a fixed, large enough set of formulas.

Our next goal is to create $\Sigma$-elementary submodels. We will use the following two well-known theorems. One can find them in [22] as well as in other textbooks in the topic.

3.1.2 Theorem (Levy’s Reflection Theorem). For any ordinal $\alpha$ there is an ordinal $\beta \geq \alpha$ such that $V_\beta$ is a $\Sigma$-elementary submodel.

3.1.3 Theorem (Downward Löwenheim–Skolem-Tarski Theorem). Let $\mathfrak{A}$ be a first order structure for language $\mathcal{L}$ with basic set $A$. Denote the set of $\mathcal{L}$-formulas by $\text{Form}(\mathcal{L})$. Assume that $|\text{Form}(\mathcal{L})| \leq |A|$. Then for all $B \subseteq A$ there exists an elementary submodel $\mathfrak{C}$ of $\mathfrak{A}$ with basic set $C$ such that $B \subseteq C$ and $|C| = |\text{Form}(\mathcal{L})| + |B|$.

3.1.4 Remark. In the case of set theory $|\text{Form}(\mathcal{L})| = \aleph_0$ so if $B$ is infinite, then we may write $|C| = |B|$ in Theorem 3.1.3.

Now we can prove a fundamental fact about $\Sigma$-elementary submodels.
3.1.5 Proposition. For all infinite set $B$ there is a $\Sigma$-elementary submodel $M$ such that $B \subseteq M$ and $|M| = |B|$.

Proof: By Theorem 3.1.2 there is a $\beta \geq \text{rank}(B)$ such that $V_\beta$ is a $\Sigma$-elementary submodel. Then $B \subseteq V_\beta$ since $\beta \geq \text{rank}(B)$. Thus by using Theorem 3.1.3 with $\mathfrak{A} = V_\beta$ and with $B$ we get an elementary submodel $M$ of $V_\beta$ such that $|M| = |B|$ and $B \subseteq M$. Finally $M$ is a $\Sigma$-elementary submodel because it is an elementary submodel of a $\Sigma$-elementary submodel. ■

3.2 Main result

Proof of Theorem 3.0.10. A directed cycle has the same number of ingoing and outgoing edges for an $X \subseteq V$ thus if $A$ can be partitioned into directed cycles, then $D$ must be balanced. Next we deal with the nontrivial direction of the equivalence.

Observe that the weakly connected components of a balanced digraph are strongly connected thus each of their edges are in some directed cycle. Furthermore, a balanced digraph remains balanced after the deletion of the edges of a directed cycle. If a balanced digraph is at most countable and its edges are: $e_1, e_2, \ldots$, then we can create a desired partition by the following recursion: in the $n$-th step delete the edges of a directed cycle which contains $e_n$ from the remaining digraph if it still contains $e_n$, otherwise do nothing.

In the uncountable case the naïve recursive method above does not work because in a transfinite recursion one can not ensure that after the first limit step the remaining digraph is still balanced.

3.2.1 Lemma. For all infinite set $B$ there is a $\Sigma$-elementary submodel $M$ such that $B \subseteq M$, $|M| = |B|$, and for any balanced digraph $D \in M$ the edge-set $A(D) \cap M$ can be partitioned into directed cycles.

Theorem 3.0.10 follows directly from Lemma 3.2.1: let $D$ be an arbitrary balanced digraph and use Lemma 3.2.1 with $B = \{D\} \cup V(D) \cup A(D)$. Then
$D = D(M)$ and hence we get a desired partition.

**Proof:** We prove Lemma 3.2.1 by transfinite induction on $|B|$. Consider first the case $|B| = \aleph_0$ and let $M$ be a $\Sigma$-elementary submodel such that $B \subseteq M$ and $|M| = \aleph_0$. It exists by Proposition 3.1.5. Assume that $D \in M$ is a digraph such that $A(D) \cap M$ can not be partitioned into directed cycles. We have to show that $D$ is unbalanced. We know that $D(M)$ must be unbalanced because it is countable and we have already proved Theorem 3.0.10 for countable digraphs. Let $X \subseteq V \cap M$ be an overloaded set in $D(M)$. Then $|\text{out}_{D(M)}(X)|$ is finite because $|\text{out}_{D(M)}(X)| < |\text{in}_{D(M)}(X)| \leq |M| = \aleph_0$. Let $S$ be the set whose elements are the tails of the edges in $\text{out}_{D(M)}(X)$ and the heads of $|\text{out}_{D(M)}(X)| + 1$ many edges of $\text{in}_{D(M)}(X)$. Consider the set $X'$ of vertices that are reachable from $S$ in $D(M)$ without using the edges in $\text{out}_{D(M)}(X)$. Note that $X'$ is definable in $M$ as a certain subset of $V$ using finitely many parameters from $A \cap M$. We may assume that $\Sigma$ contains the appropriate instances of the subset axiom of ZFC hence $X' \in M$. Furthermore $\text{out}_{D(M)}(X) = \text{out}_{D(M)}(X')$ and $X'$ has at least $|\text{in}_{D(M)}(X)| + 1$ ingoing edges hence it is true in the model $M$ that $X'$ is an overloaded set in $D$. We also assume that the formula $\varphi(x)$ that says: “$x$ is an unbalanced digraph” is in $\Sigma$ thus from $M \models \varphi(D)$ we may conclude $D$ is really unbalanced.

Let $\lambda > \aleph_0$ be a cardinal and assume that Lemma 3.2.1 is true for sets with size lesser than $\lambda$. Let $B = \{b_\alpha : \alpha < \lambda\}$ be arbitrary and let $B_\alpha = \{b_\gamma : \gamma < \alpha\}$. We define a chain of $\Sigma$-elementary submodels $\langle M_\alpha : \omega \leq \alpha < \lambda \rangle$ by transfinite recursion such that for all $\omega \leq \alpha < \lambda$:

1. $\alpha, B_\alpha \subseteq M_\alpha$,
2. $|M_\alpha| = |\alpha|$, 
3. $M_\gamma \in M_\alpha$ and $M_\gamma \subseteq M_\alpha$ if $\gamma < \alpha$,
4. if $D \in M_{\alpha+1}$ is a balanced digraph, then the edge-set of $D(M_{\alpha+1})$ (i.e. $A \cap M_{\alpha+1}$) can be partitioned into directed cycles,
5. $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ if $\alpha$ is a limit ordinal.
$M_\omega$ can be an arbitrary countable $\Sigma$-elementary submodel with $B_\omega \subseteq M_\omega$.

Suppose that $M_\gamma$ is already defined if $\omega \leq \gamma < \alpha$ for some $\omega < \alpha < \lambda$ and satisfies the conditions above. If $\alpha$ is a limit ordinal, then let $M_\alpha = \bigcup \{M_\gamma : \gamma < \alpha\}$. If $\alpha = \delta + 1$, then do the following. Let $S_\alpha = \alpha \cup B_\alpha \cup M_\delta \cup \{M_\delta\}$ thus $|S_\alpha| \leq |\alpha| + |\alpha| + |\alpha| = |\alpha| < \lambda$. By the induction hypothesis there is a $\Sigma$-elementary submodel $M_\alpha$ such that $S_\alpha \subseteq M_\alpha$, $|M_\alpha| = |S_\alpha| = |\alpha|$, and for all balanced digraph $D \in M_\alpha$ the edge-set $A(D) \cap M$ can be partitioned into directed cycles. The recursion is done.

Let $M = \bigcup \{M_\alpha : \omega \leq \alpha < \lambda\}$. Then $B \subseteq M$ and $|M| = \lambda = |B|$. Clearly $M$ is $\Sigma$-elementary submodel since $M$ is the union of an increasing chain of $\Sigma$-elementary submodels. Let $D \in M$ be balanced and let $\beta + 1 < \lambda$ be the smallest ordinal such that $D \in M_\beta + 1$. Let $D_\beta = D(M_\beta + 1)$ and for $\beta < \alpha < \lambda$ let $D_\alpha = (D \setminus M_\alpha)(M_\alpha + 1)$. These are edge-disjoint subdigraphs of $D(M)$ moreover $A(D_\alpha)$ ($\beta < \alpha < \lambda$) is a partition of $A(D) \cap M$. Since $D, M_\alpha \in M_\alpha + 1$ we get (by using $\Sigma$-elementarity with an appropriate formula) $(D \setminus M_\alpha) \in M_\alpha + 1$.

3.2.2 Claim. If $M$ is a $\Sigma$-elementary submodel with $|M| \subseteq M$ and $D \in M$ is a balanced digraph, then $D \setminus M$ is also balanced.

If we prove Claim 3.2.2, then we are done with the proof of Lemma 3.2.1 as well. Indeed, by Claim 3.2.2, the digraphs $D \setminus M_\alpha$ are balanced and therefore by using the 4th property of the recursion with $D \setminus M_\alpha$ and with $M_\alpha + 1$ we can partition $A(D_\alpha)$ into directed cycles for all $\beta \leq \alpha < \lambda$ thus we get a desired partition of $A \cap M$ by uniting the partitions of the edge sets $A(D_\alpha)$.

Before the proof of Claim 3.2.2 we need some preparations.

3.2.3 Proposition. Let $G$ be an undirected graph an let $M$ be a $\Sigma$-elementary submodel such that $G \in M$ and $|M| \subseteq M$. Assume that $\lambda_{G\setminus M}(u, v) > 0$ for some $u \neq v \in V(G) \cap M$. Then $\lambda_G(u, v) > |M|$.

Proof: We assume that $\Sigma$ contains the formulas that expressing the followings:

1. $\lambda_G(u, v) = \kappa$, $(\forall G)(\forall u \neq v \in V(G))\exists \kappa(\lambda_G(u, v) = \kappa)$

2. $E' \subseteq E(G)$ separates the vertices $u$ and $v$ in graph $G$, 32
3. $f$ is a bijection between the sets $X$ and $Y$.

Let $u \neq v \in V(G) \cap M$ arbitrary and suppose that $\lambda_G(u, v) = \kappa \leq |M|$. We have to show that $\lambda_{G \setminus M}(u, v) = 0$. Since $G, u, v \in M$ and $\kappa$ is definable from them by a formula in $\Sigma$ (see 1 above) we know that $\kappa \in M$ and $M \models \lambda_G(u, v) = \kappa$. Then there is some $E'$ such that $M \models "E' \subseteq E(G)"$ separates the vertices $u$ and $v$ in graph $G$ and $f$ is a bijection between $\kappa$ and $E'$. Formulas 2 and 3 ensures that $E' \subseteq E(G)$ separates the vertices $u$ and $v$ in graph $G$ and $f$ is a bijection between $\kappa$ and $E'$. Since $f \in M$ and $\kappa \leq |M| \subseteq M$ the range of $f$ is a subset of $M$ i.e. $E' \subseteq M$ therefore $\lambda_{G \setminus M}(u, v) = 0$ since $E'$ separates $v$ and $u$. ■

We need the following result of L. Soukup (see [27] Lemma 5.3 on p. 16):

**3.2.4 Proposition.** Let $G$ be an undirected graph and let $M$ be a $\Sigma$-elementary submodel such that $G \in M$ and $|M| \subseteq M$. Assume that $x \neq y \in V(G)$ are in the same component of $G \setminus M$ and $F \subseteq E(G \setminus M)$ separates them where $|F| \leq |M|$. Then $F$ separates $x$ and $y$ in the whole $G$.

**Proof:** Assume (reductio ad absurdum) that it is false and $G, F, x, y, M$ witness it. Take a path $P$ between $x$ and $y$ in $G \setminus F$. Denote by $x'$ and by $y'$ the first and the last intersection of $P$ with $V \cap M$ with respect to some direction of $P$. The vertices $x'$ and $y'$ are well-defined and distinct since $P$ necessarily uses some edge from $E(G) \cap M$. Fix also a path $Q$ between $x$ and $y$ in $G \setminus M$. The paths $P[x', x]$, $Q$, $P[y, y']$ shows that $x'$ and $y'$ are in the same component of $G \setminus M$. Thus by Proposition 3.2.3 $\lambda_G(x', y') > |M|$. We may fix a path $R$ between $x'$ and $y'$ in $G \setminus F$ since $\lambda_G(x', y') > |M| \geq |F|$. But then $P[x, x']$, $R$, $P[y', y]$ shows that $F$ does not separate $x$ and $y$ in $G \setminus M$ which is a contradiction. ■

Now we turn to the proof of Claim 3.2.2. Assume, seeking for contradiction, that $D \setminus M$ is unbalanced. Then by Lemma 3.1.1 there is a weakly connected component of $D \setminus M$ with vertex set $Z$ and an $X \cup Y$ partition of $Z$ such that $(D \setminus M)[X]$ and $(D \setminus M)[Y]$ are weakly connected and $X$ is overloaded in $D \setminus M$. Let $F = \text{cut}_{D \setminus M}(X)$. We want to show that $|F| \leq |M|$. We may suppose that $F$
is infinite and thus \( \text{cut}_D(X) \) as well since \( F \subseteq \text{cut}_D(X) \). Thus \( \aleph_0 \leq |\text{out}_D(X)| = |\text{in}_D(X)| \). The inequality \(|\text{out}_{D \setminus M}(X)| < |\text{out}_D(X)|\) holds because otherwise

\[
|\text{out}_{D \setminus M}(X)| = |\text{out}_D(X)| = |\text{in}_D(X)| \geq |\text{in}_{D \setminus M}(X)|
\]

which contradicts to the choice of \( X \). Hence \( M \) contains \(|\text{out}_D(X)|\) elements of \( \text{out}_D(X) \) and thus \(|\text{out}_D(X)| \leq |M|\). Then

\[
|F| = |\text{in}_{D \setminus M}(X)| + |\text{out}_{D \setminus M}(X)| \leq |\text{in}_D(X)| + |\text{out}_D(X)| = |\text{out}_D(X)| \leq |M|.
\]

By using Proposition 3.2.4 to the undirected underlying graph of \( D \) with \( F \) and with arbitrary \( x \in X \) and \( y \in Y \) vertices we conclude that \( X \) and \( Y \) belongs to distinct weakly connected components of \( D \setminus F \). Let us denote by \( X' \) and \( Y' \) the vertex set of these components. We claim that \( \text{cut}_{D \setminus M}(X) = \text{cut}_D(X') \). Indeed, \( \text{cut}_D(X') \) might not have element that not in \( F \) by the definition of \( X' \) and the elements of \( F \) goes between \( X \) and \( Y \) and therefore between \( X' \) and \( Y' \). But then

\[
|\text{out}_{D \setminus M}(X)| = |\text{out}_D(X')| \quad \text{and} \quad |\text{in}_{D \setminus M}(X)| = |\text{in}_D(X')| \quad \text{thus}
\]

\[
|\text{out}_D(X')| = |\text{out}_{D \setminus M}(X)| < |\text{in}_{D \setminus M}(X)| = |\text{in}_D(X')|
\]

therefore \( X' \) is overloaded in \( D \) which is a contradiction. ■■
Chapter 4

T-joins in infinite graphs

4.1 Introduction

The 2-edge-connected components of a graph are its maximal 2-edge-connected subgraphs (a graph consists of a single vertex is considered 2-edge-connected). A T-join in a graph $G$, where $T \subseteq V(G)$, is a system $\mathcal{P}$ of edge-disjoint paths in $G$ such that the end vertices of the paths in $\mathcal{P}$ create a partition of $T$ into two-element sets. In other words we match by edge-disjoint paths the vertices in $T$. In the finite case the existence of an $F \subseteq E(G)$ for which $d_F(v)$ is odd if and only if $v \in T$ is equivalent with the existence of a T-join. Indeed, the united edge sets of the paths in $\mathcal{P}$ forms such an $F$, and such an $F$ can be decomposed into a T-join and some cycles by the greedy method. Sometimes $F$ itself is called a T-join. The two possible definitions are no more closely related in the infinite case. Take for example a one-way infinite path where $T$ contains only its end vertex. Then there is no T-join according to the path-system based definition (which we will use) but the whole edge set forms a T-join with respect to the second definition.

T-join is a common tool in combinatorial optimization problems for a detailed survey we refer to [10]. For finite connected graphs the necessary and sufficient condition for the existence of a T-join is quite simple, $|T|$ must be even. Indeed, the necessity of the condition is trivial. For the sufficiency let $|T| = 2k$ and we apply induction on $k$. The case $k = 0$ is clear. If $|T| = 2k + 2$, then remove two
vertices, \( u \) and \( v \) say, of \( T \) to obtain \( T' \). By induction we have a \( T' \)-join. Take the symmetric difference of the edge set of a \( T' \)-join and the edges of an arbitrary path between \( u \) and \( v \). By the greedy method we can partition the resulting edge set into a \( T \)-join and some cycles. If \( |T| \) is even but \( G \) is infinite, then the same proof works. In this chapter we investigate questions related to the existence of \( T \)-joins where \( T \) is infinite. For an infinite \( T \) one can not guarantee in general the existence of a \( T \)-join in a connected graph. Consider for example an infinite star and subdivide all of its edges by a new vertex (see figure 4.1) and let \( T \) consist of the whole vertex set.

![Figure 4.1: A subdivided infinite star. It has no T-join if T is the whole vertex set.](image)

Our main result in this topic is that essentially Figure 4.1 is the only counterexample.

4.1.1 Theorem. The connected graph \( G \) does not contain a \( T \)-join for some infinite \( T \subseteq V(G) \) if and only if one can obtain the graph in Figure 4.1 from \( G \) by contracting edges and deleting the resulting loops.

The following reformulation of the theorem will be more convenient:

4.1.2 Theorem. A connected infinite graph \( G \) contains a \( T \)-join for every infinite \( T \subseteq V(G) \) if and only if there is no \( U \subseteq V(G) \) for which every connected component of \( G - U \) connects to \( U \) in \( G \) by a single edge and infinitely many of them are nontrivial (i.e. not consist of a single vertex).
The “if” direction is straightforward since if such an $U$ exists, then one can choose one vertex from $U$ and two from each nontrivial connected component of $G - U$ to obtain a $T$ for which there is no $T$-join in $G$.

Our other result describes the effect of finite modifications of $T$ on the existence of a $T$-join. For finite $T$ if $G$ does not contain a $T$-join, then it contains a $T'$-join whenever $|T \triangle T'|$ is odd. (It is obvious, since in this case the existence of a $T$-join depends just on the parity of $|T|$.) Surprisingly this property remains true for infinite $T$ as well. We have the following result about this.

**4.1.3 Theorem.** The class $\{(G,T) : G$ is a connected graph and $T \subseteq V(T)\}$ can be partitioned into three nonempty subclasses defined by the following three properties:

(A) $G$ contains a $T'$-join whenever $|T' \triangle T| < \infty$,

(B) $G$ contains $T'$-join if $|T' \triangle T|$ is finite and even, but it does not when it is odd,

(C) $G$ contains $T'$-join if $|T' \triangle T|$ is finite and odd, but it does not when it is even.

To make the descriptions of the $T$-join-constructing processes more reader-friendly we introduce the following single player game terminology. There is an abstract set of tokens and every token is on some vertex of a graph $G$. (At the beginning typically we have exactly one token on each element of a prescribed vertex set $T$ and none on the other vertices.) If two tokens are on the same vertex, then we may remove them (we say that we match them to each other). If we have a token $t$ on the vertex $u$ and $uv$ is an edge of $G$, then we may move $t$ from $u$ to $v$, but then we have to delete $uv$ from $G$. A gameplay is a transfinite sequence of the steps above in which we move every token just finitely many times. Limit steps are defined by the earlier steps in a natural way. Indeed, we just delete all the edges that have been deleted earlier, remove the tokens that have been removed before, and put all the remaining tokens to their stabilized positions. We call a gameplay winning if we remove all the tokens eventually. To make talking about the relevant
part of the graph easier we also allow as a feasible step deleting vertices without tokens on them and deleting edges. Clearly if there is a $T$-join in $G$, then there is a winning gameplay for the game on $G$ where the initial token distribution is defined by $T$. On the other hand, from a winning gameplay we can get a $T$-join.

4.2 The 2-edge-connected case

A subgraph of $G$ is called $t$-infinite if it contains infinitely many tokens. We define the notions $t$-finite, $t$-empty, $t$-odd and $t$-even similarly.

4.2.1 Lemma. If $G$ is 2-edge-connected and $t$-infinite, then there is a winning gameplay.

4.2.2 Claim. Assume that $G$ is 2-edge-connected and contains even number of tokens, but at least four including $s \neq t$. Then there is a winning gameplay in which $s$ and $t$ are not matched with each other and $t$ is not moved.

Proof: We may assume that $G$ is finite otherwise we may take a finite 2-edge-connected subgraph that contains all the tokens. Take two edge-disjoint paths between the vertices that contain $s$ and $t$ and let $H$ be the Eulerian subgraph of $G$ consists of these paths. We can build up $G$ from $H$ by adding ears (as in the ear decomposition). We apply induction on the number of ears. If there is no ear i.e. $G = H$, then we take an Eulerian Cycle $O$ in $G$. Fix an Eulerian orientation of $O$. Either this orientation or the reverse of it induces a desired gameplay.

Otherwise let $Q$ be the last ear. If the number of tokens on the new vertices given by $Q$ is odd (even), then match inside $Q$ all but one (two) of these tokens and move the exception(s) to the part of the $G$ before the addition of $Q$. Delete the remaining part of $Q$. We are done by applying the induction hypothesis. ■

By contracting the 2-edge-connected components of a connected graph, we obtain a tree. If $R$ is a 2-edge-connected component of a connected graph $G$, then we denote by $\text{tree}(G; R)$ the tree of the 2-edge-connected components of $G$ rooted at $R$. We usually pick such a root $R$ arbitrarily without mentioning it explicitly.
We write $\text{tree}(G)$ if it is not rooted. We do not distinguish strictly the subtrees of $\text{tree}(G; R)$ and the corresponding subgraphs of $G$.

4.2.3 Claim. Let $G$ be a connected graph with infinitely many tokens on it. Assume that $G$ has only finitely many 2-edge-connected components. Then there is a finite gameplay after which all the components of the resulting $G'$ are 2-edge-connected and contain infinitely many tokens.

Proof: By the pigeon hole principle there is a $t$-infinite 2-edge-connected component $R$ of $G$. We use induction on the number $k$ of the 2-edge-connected components. For $k = 1$ we do nothing. If $k > 1$ we take a leaf $C$ of $\text{tree}(G; R)$. If $C$ is $t$-infinite we remove the unique outgoing edge of $V(C)$ in $G$ and we use induction to the arising component other than $C$. If $C$ is $t$-even, then we match the tokens on $C$ with each other inside $C$ and we delete the remaining part of $C$ and its unique outgoing edge and use induction. In the $t$-odd case we match all but one tokens of $C$ inside $C$ and move one to the parent component. It is doable by adding another “phantom-token” $t$ on the vertex of $C$ incident with the cut edge to the parent component and applying Claim 4.2.2 with this $t$ and an arbitrary $s$. We delete the remaining part of $C$ again and use induction. ■

Now we turn to the proof of Lemma 4.2.1. If $t$ is a token and $H$ is a subgraph of $G$, then we use the abbreviation $t \in H$ to express the fact that $t$ is on some vertex of $H$. Assume first that $T$ is countable. For technical reasons we assume the following weaker condition instead of 2-edge-connectedness.

All the connected components are 2-edge-connected and $t$-infinite. \hfill(4.1)

Let $t_0$ be arbitrary. It is enough to show, that there is a finite gameplay such that we remove $t_0$ and the resulting system still satisfies the condition 4.1. Pick two edge-disjoint paths $P_1, P_2$ between $t_0$ and any other token $t^*$ that lies in the same component as $t_0$. For $i \in \{1, 2\}$ let $K_i$ be the connected component of $G - E(P_i)$ that contains $t^*$. We claim that either $K_1$ or $K_2$ is $t$-infinite. Suppose that $K_1$ is
not. Then there is a \( t \)-infinite component \( K_{\text{inf}} \) of \( G - E(P_1) \) which does not contain \( t^* \). Note that \( P_2 \) necessarily lies in \( K_1 \). Hence \( K_{\text{inf}} \) is a subgraph of \( G - E(P_2) \) and \( P_1 \) ensures that \( K_{\text{inf}} \) belongs to the same component of \( G - E(P_2) \) as \( t^* \). Thus \( K_{\text{inf}} \) ensures that this component is \( t \)-infinite. We will need the following basic observation.

4.2.4 Observation. If each component of a graph \( G \) has finitely many 2-edge-connected components, then so has \( G - f \) for every \( f \in E(G) \).

By symmetry we may assume that \( K_1 \) is \( t \)-infinite. Move \( t_0 \) along the edges of \( P_1 \) one by one. If the following edge \( e \) is a bridge, and moving \( t_0 \) along \( e \) would create a \( t \)-odd component \( K_{\text{odd}} \), then \( t^* \notin K_{\text{odd}} \) because the component of \( t^* \) is \( t \)-infinite. In this case we delete \( e \) without moving \( t_0 \) and obtain a \( t \)-infinite component and a \( t \)-even component \( K_{\text{even}} \) that contains \( t_0 \). We match the tokens on \( K_{\text{even}} \) and erase the remaining part of it and the first phase of the process is done. If this case does not occur, then we move \( t_0 \) to \( t^* \) along \( P_1 \) and remove both.

We need to fix the condition 4.1. Each component of the resulting graph is either \( t \)-even or \( t \)-infinite. We match the tokens on \( t \)-even components and erase the remaining part of them. Each \( t \)-infinite component has finitely many 2-edge-connected components by Observation 4.2.4 thus we are done by applying Claim 4.2.3.

Consider now the general case where \( T \) can be arbitrary large. Pick a new vertex \( z \) and draw all the \( zv \ (v \in T) \) edges to obtain \( H \). Finding a \( T \)-join in \( G \) is clearly equivalent with covering in \( H \) all the edges incident with \( z \) by edge-disjoint cycles. Let us introduce the following condition for graphs.

4.2.5 Condition. \( z \in V(H') \) and the connected components of \( H' - z \) are 2-edge-connected and connect to \( z \) in \( H' \) by either zero or infinitely many edges.

Clearly \( H \) satisfies this condition. To reduce the problem to the countable case it is enough to prove the following claim.

4.2.6 Claim. There is a partition of \( E(H) \) into countable sets \( E_i \ (i \in I) \) in such a way, that for all \( i \in I \) the graphs \( H_i := (V(H), E_i) \) satisfy condition 4.2.5.
Proof: Our proof will be a basic application of the elementary submodel technique that we used intensively in the previous chapter (see our survey about the method from page 28). We apply transfinite induction on $\lambda = |E(H)|$. For $\lambda = \aleph_0$ the claim is trivial hence suppose that $\lambda > \aleph_0$ and we know the claim for graphs with less than $\lambda$ edges. Let $\langle M_\alpha : \omega \leq \alpha < \lambda \rangle$ be a chain of $\Sigma$-elementary submodels such that for all $\omega \leq \alpha < \lambda$:

1. $H, z \in M_\omega$,
2. $\alpha \subseteq M_\alpha$,
3. $|M_\alpha| = |\alpha|$, 
4. $M_\gamma \in M_\alpha$ and $M_\gamma \subseteq M_\alpha$ if $\gamma < \alpha$,
5. $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ if $\alpha$ is a limit ordinal.

Applying the notations of the previous chapter (see from page 28) take

$$H_\alpha := (H \upharpoonright M_\alpha)(M_{\alpha+1})$$

for $\omega \leq \alpha < \lambda$ and $H_\lambda := (V(H)), E(H(M_\omega))$. It is enough to check that they satisfy Condition 4.2.5 because they are edge-disjoint subgraphs of $H$ with less than $\lambda$ edges and their edge sets form a partition of $E(H)$. Hence by using the induction hypothesis to the graphs $\{H_\alpha\}_{\omega \leq \alpha \leq \lambda}$ we will be done.

4.2.7 Proposition. If $M$ is a $\Sigma$-elementary submodel with $|M| \subseteq M$ and $H', z \in M$ and $H'$ is a graph that satisfies Condition 4.2.5, then the graphs $H'(M)$ and $H' \upharpoonright M$ also satisfy Condition 4.2.5.

Proof: If there would be a bridge edge $e$ in $H'(M) - z = (H' - z)(M)$, then by $\Sigma$-elementarity it would be a bridge edge in $H' - z$ as well which is impossible. Similarly if the set $F$ of edges between $z$ and some component of $(H' - z)(M)$ is finite and nonempty, then $F$ is a violating cut in $H' - z$ as well. If there would be a cut $F$ in $(H' \upharpoonright M) - z = (H' - z) \upharpoonright M$ that violates Condition 4.2.5 and
\( xy \in F \), then by Proposition 3.2.4 (applying with \( x, y \) and \( F \)) we obtain that \( F \) would violate Condition ?? in \( H' \).

Thus we know that \( H(M_\omega) \) satisfies Condition 4.2.5 and hence \( H_\lambda \) as well since Condition 4.2.5 is invariant under adding isolated vertices. For \( \omega \leq \alpha < \lambda \) we obtained that the graph \( H \setminus M_\alpha \) satisfies Condition 4.2.5. Since \( H, M_\alpha \in M_{\alpha + 1} \) we get (by applying \( \Sigma \)-elementarity with an appropriate formula) \( (H \setminus M_\alpha) \in M_{\alpha + 1} \). Therefore by applying Proposition 4.2.7 again with \( M_{\alpha + 1} \) and with \( (H \setminus M_\alpha) \) we obtain that \( H_\alpha := (H \setminus M_\alpha)(M_{\alpha + 1}) \) satisfies Condition 4.2.5 which completes the proof of Claim 4.2.6 and hence the proof of Lemma 4.2.1 as well.

4.3 The simplification process

Lemma 4.2.1 makes it possible to decide the existence of a \( T \)-join by just investigating the structure of the 2-edge-connected components and the quantity of tokens on them. Let \( G \) be a connected graph, \( T \subseteq V(G) \) and let \( R \) be a 2-edge-connected component of it. We define a graph \( H = H(G, R, T) \) with a token-distribution on it in the following way. To obtain \( H \) we apply a gameplay that we call simplification process. We denote by \( \text{subt}(C; G, R) \) the subtree of the descendants of \( C \) rooted at \( C \) in \( \text{tree}(G; R) \). Delete all those 2-edge-connected components \( C \) for which \( \text{subt}(C; G, R) \) does not contain any token. Then consider the \( t \)-finite leafs of the reminder of \( \text{tree}(G; R) \) (we do not consider the root as a leaf.) Match the tokens on any \( t \)-even leaf \( C \) inside \( C \) and for \( t \)-odd leafs \( C \) move one token to the parent and match the others inside. In both cases delete the remaining part of \( C \). Iterate the steps above as long as possible and denote by \( H \) the resulting graph. Clearly either \( H = R \) with some tokens on it or if \( \text{tree}(H) \) is nontrivial, then \( \text{subt}(C; H, R) \) must be \( t \)-infinite for all 2-edge-connected components \( C \) of \( H \).

4.3.1 Claim. There is a winning gameplay for the original system if \( H \) is \( t \)-even or \( t \)-infinite.

Proof: The \( t \)-even case it is obvious since \( H \) is connected. Assume that \( H \) is \( t \)-infinite. We may suppose that \( H \) is not 2-edge-connected because otherwise we
are done by applying Lemma 4.2.1. Then \( \text{subt}(C; H, R) \) is \( t \)-infinite for all 2-edge-connected components \( C \) of \( H \). For each 2-edge-connected components of \( H \) we fix a path \( P_C \) in the tree \( \text{subt}(C; H, R) \) that starts at \( C \) and either terminates at some leaf of \( \text{tree}(H; R) \) or it is one-way infinite and meets some 2-edge-connected components of \( H \) other than \( C \) which is not \( t \)-empty.

After these preparations we do the following. If the root \( R \) is \( t \)-even or \( t \)-infinite, then we match all the tokens of it inside \( R \) (use Lemma 4.2.1 in the \( t \)-infinite case) and delete the remaining part of \( R \). If it is \( t \)-odd, then we move one of its tokens, say it will turn to be \( t^* \), to some child of \( R \) determined by the path \( P_R \) and we define \( P_{t^*} := P_R \). We match the other tokens inside \( R \) and then delete the remaining part of \( R \). At the next step we deal with the \( \text{subt}(C; H, R) \) trees where \( C \) is a child of \( R \). In the cases where \( C \) is \( t \)-infinite or \( t \)-even we do the same as earlier. Assume that \( C \) is \( t \)-odd. If there is no token on \( C \) that comes from \( R \), then we do the same as earlier. Suppose that there is, say \( t_0 \). If there is a token on \( C \) other than \( t_0 \), then we match here \( t_0 \) and send forward some other token \( t_1 \) in the direction defined by \( P_C \) (apply Claim 4.2.2 and a phantom-token) and we let \( P_{t_1} := P_C \). If \( t_0 \) is the only token of \( C_0 \), then we move \( t_0 \) in the direction \( P_{t_0} \). We iterate the process recursively. The only not entirely trivial thing that we need to justify is that we do not move a token infinitely many times. If we moved some token \( t \) at the previous step, then we match it at the current step unless it is the only token at the corresponding 2-edge-connected component. On the other hand, when we move \( t \) for the first time we define the path \( P_t \). The later movements of \( t \) are led by \( P_t \) which ensure that eventually \( t \) will meet some other token. ■

### 4.4 Proof of the theorems

Now we are able to prove the nontrivial direction of Theorem 4.1.2.

**Proof.** Let \( G \) be an infinite connected graph such that there is no \( U \subseteq V(G) \) for which the connected components of \( G - U \) connect to \( U \) in \( G \) by a single edge and infinitely many of them are nontrivial. Let \( T \subseteq V(G) \) be infinite. Consider the vertices \( v \) of degree one that are in \( T \) i.e. there is a token on them. Move these
tokens to the only possible direction and then delete all the vertices of degree one or zero. We denote the resulting graph by \( G' \) and we fix a 2-edge-connected component \( R \) of it. If \( G' \) has a \( t \)-infinite 2-edge-connected component, then it cannot vanish during the simplification process, thus the resulting \( H \) will be \( t \)-infinite and we are done by Claim 4.3.1.

Assume there is no such component. The degree of \( C_0 := R \) in \( \text{tree}(G') \) must be finite otherwise \( U := V(C_0) \) would violate the assumption about \( G \). Since \( C_0 \) is \( t \)-finite by the pigeonhole principle there is a child \( C_1 \) of \( C_0 \) such that \( \text{subt}(C_1; G', R) \) contains infinitely many tokens. By recursion we obtain a one-way infinite path of \( \text{tree}(G') \) with vertices \( C_n \) \((n \in \mathbb{N})\) such that for every \( n \) the tree \( \text{subt}(C_n; G', R) \) contains infinitely many tokens. The set \( U := \bigcup_{n=0}^{\infty} V(C_n) \) may not have infinitely many outgoing edges in \( G' \) otherwise \( U \) violates the condition about \( G \). Thus for large enough \( n \) the tree \( \text{subt}(C_n; G', R) \) is just a terminal segment of the one-way infinite path we constructed. This implies that infinitely many of the \( C_n \)'s contain at least one token. Since such a path cannot vanish during the simplification process, it terminates with a \( t \)-infinite \( H \) again. □

We turn to the proof of Theorem 4.1.3.

**Proof.** Let a connected \( G \) and a 2-edge-connected component \( R \) of it be fixed. Let \( T \subseteq V(G) \). We will show that case (A) of Theorem 4.1.3 occurs if and only if the simplification process terminates with infinitely many tokens and (B)/(C) occurs if and only if it terminates with an even/odd number of tokens, respectively.

Assume first that the result \( H \) of the simplification process initialized by \( T \) (\( T \)-process from now on) is \( t \)-infinite. Let \( T' \subseteq V(G) \) such that \(|T' \triangle T| < \infty\). Call \( T' \)-process the simplification process with the initial tokens given by \( T' \) and denote by \( H' \) the result of it. If \( G \) has a 2-edge-connected \( t \)-infinite component \( C \) with respect to \( T \), then \( C \) is \( t \)-infinite with respect to \( T' \) as well. Observe that such a \( C \) remains untouched during the simplification process. Thus \( H' \) is \( t \)-infinite and therefore there is a \( T' \)-join in \( G \) by Claim 4.3.1.

We may suppose that there is no 2-edge-connected, \( t \)-infinite component in \( G \). If such a component \( C \) arises during the \( T \)-process, then \( C \) receives a token.
from infinitely many children of it. Since $|T \triangle T'| < \infty$ we have $|V(D) \cap T| = |V(D) \cap T'|$ for all but finitely many 2-edge-connected component $D$. Hence the token-structure of $\text{sub}(D;G, R)$ is the same for all but finitely many child $D$ of $C$ at the case of $T'$. Thus $C$ will receive infinitely many tokens during the $T'$-process as well.

Finally we suppose that such a component does not arise, i.e. $H$ has no $t$-infinite components. Then $\text{tree}(H)$ must be an infinite tree since $H$ is $t$-infinite. Furthermore, $\text{sub}(C; H, R)$ must contain at least one token for all 2-edge-connected component $C$ of $H$ otherwise we may erase $\text{sub}(C; H, R)$ to continue the simplification process. Fix a one-way infinite path $P$ in $\text{tree}(H)$ with vertices $C_n$ ($n \in \mathbb{N}$), where $C_0 = R$ and infinitely many $C_n$ contain at least one token. For a large enough $n_0$, the token-distribution of $\text{sub}(C_{n_0}; G, R)$ is the same at the $T$ and the $T'$ cases. Hence the $T$-process and $T'$-process runs identical on the subgraph of $G$ corresponding to $\text{sub}(C_{n_0}; G, R)$. Thus $\text{sub}(C_{n_0}; H, R) = \text{sub}(C_{n_0}; H', R)$ holds and the tokens on them are the same. But then the terminal segment of path $P$ in $\text{tree}(H')$ shows that $H'$ is $t$-infinite as well.

4.4.1 Claim. There is no $T$-join in $G$ if the simplification process terminates with an odd number of tokens.

Proof: Remember that no $t$-infinite 2-edge-connected component may arise during the simplification process in this case. Assume, to the contrary, that there is a $T$-join $P$ in $G$. We play a winning gameplay induced by $P$ i.e. every step we move some token along the appropriate $P \in P$ towards its partner. If for a 2-edge-connected component $C$ the subgraph $\text{sub}(C; G, R)$ does not contain any vertex from $T$, then clearly no $P \in P$ comes here. Hence we may delete these parts of the graph. If $C$ is a leaf of (the remaining part of) $\text{tree}(G; R)$ with $|T \cap V(C)|$ even, then the corresponding paths are inside $C$. In the odd case exactly one $P \in P$ uses the unique outgoing edge of $C$ and some other paths match inside $C$ the other $T$-vertices of $C$. Thus along $P$ we may move one token to the parent component and match the others along the other paths in $C$. Therefore after we do these steps the quantity of the tokens on the 2-edge-connected components will be the same as after the first step of the simplification process. Similar arguments show that
it remains true after successor steps of the simplification process as well. Since the first difference may not arise at a limit step for all steps of the simplification process we have a corresponding position of the gameplay induced by \( \mathcal{P} \) where the token quantities on the 2-edge-connected components are the same. On the other hand, it cannot be true for the terminating position since the play induced by \( \mathcal{P} \) is a winning gameplay and hence it cannot arise a system with an odd number of tokens during the play which is a contradiction. ■

4.4.2 Claim. If the simplification process for \( T \) terminates with an even (odd) number of tokens and \(|T \Delta T'| = 1|\), then the simplification process for \( T' \) terminates with an odd (even) number of tokens.

Proof: By symmetry we may let \( T' = T \cup \{v\} \) where \( v \notin T \). If \( v \in V(R) \), then the simplification process for \( T' \) runs in the same way as for \( T \) except that at the end we have the extra token on \( v \) which changes the parity of the remaining tokens as we claimed. If \( v \) is not in the root \( R \), then it is in \( \text{subt}(C; G, R) \) for some child \( C \) of \( R \). This \( C \) is closer in \( \text{tree}(G) \) to the 2-edge-connected component that contains \( v \) than \( R \). On the one hand, we know by induction that the parity of the number of tokens on \( C \) will be different when \( C \) will become a leaf in the case of the \( T' \)-process. On the other hand, for the other children of \( R \) this parity will be clearly the same, thus the parity of the number of tokens on \( H \) and \( H' \) are different again. ■

The remaining part of Theorem 4.1.3 follows from Claim 4.4.1 and from the repeated application of Claim 4.4.2. \( \square \)
Chapter 5

Menger’s theorem for countable digraphs with finitary matroid constraints on the ingoing edges

In this chapter we generalize the countable version of Menger’s theorem of Aharoni [1] by applying the results of Lawler and Martel about polymatroidal flows (see [23]).

Let us recall Menger’s theorem (directed, edge version).

5.0.3 Theorem (Menger). Let $D = (V, A)$ be a finite digraph with $s \neq t \in V$. Then the maximum number of the pairwise edge-disjoint $s \to t$ paths is equal to the minimal number of edges that cover all the $s \to t$ paths.

Erdős observed during his school years that the theorem above remains true for infinite digraphs (by saying cardinalities instead of numbers). He felt that this is not the “right” infinite generalization of the finite theorem and he conjectured the “right” generalization which was known as the Erdős-Menger conjecture. It is based on the observation that in Theorem 5.0.3 an optimal cover consists of one edge from each path of an optimal path-system. The Erdős-Menger conjecture states that for arbitrary large digraphs there is a path-system and a cover that satisfy these complementarity conditions. After a long sequence of partial results the countable case has been settled affirmatively by R. Aharoni:
5.0.4 Theorem (R. Aharoni, [1]). Let $D = (V, A)$ be a countable digraph with $s \neq t \in V$. Then there is a system $P$ of edge-disjoint $s \to t$ paths such that there is a edge set $C$ that covers all the $s \to t$ paths in $D$ and $C$ consists of choosing one edge from each $P \in P$.

It is worth to mention that R. Aharoni and E. Berger proved the Erdős-Menger conjecture in its full generality in 2009 (see [2]) which was one of the greatest achievements in the theory of infinite graphs. We present the following strengthening of the countable Menger’s theorem above.

5.0.5 Theorem (A. Joó, [17]). Let $D = (V, A)$ be a countable digraph with $s \neq t \in V$. Assume that there is a finitary matroid $M_v$ on the ingoing edges of $v$ for any $v \in V$. Let $M$ be the direct sum of the $M_v$ matroids. Then there is a system of edge-disjoint $s \to t$ paths $P$ such that the united edge set of the paths is $M$-independent, and there is an edge set $C$ consists of one edge from each element of $P$ for which $\text{span}_M(C)$ covers all the $s \to t$ paths in $D$.

Instead of dealing with covers directly, we are focusing on $t - s$ cuts ($X \subseteq V$ is a $t - s$ cut $t \in X \subseteq V \setminus \{s\}$). Let us call a path-system $P$ independent if $A(P)$ is independent in $M$. Suppose that an independent system $P$ of edge-disjoint $s \to t$ paths and a $t - s$ cut $X$ satisfy the complementarity conditions:

5.0.6 Condition.

1. $A(P) \cap \text{out}_D(X) = \emptyset$,

2. $A(P) \cap \text{in}_D(X)$ spans $\text{in}_D(X)$ in $M$.

Then clearly $P$ and $C := A(P) \cap \text{in}_D(X)$ satisfy the demands of Theorem 5.0.5. Therefore it is enough to prove the following reformulation of the theorem.

5.0.7 Theorem. Let $D = (V, A)$ be a countable digraph with $s \neq t \in V$ and suppose that there is a finitary matroid $M_v$ on $\text{in}_D(v)$ for each $v \in V$ and let $M = \bigoplus_{v \in V} M_v$. Then there is a system $P$ of edge-disjoint $s \to t$ paths where $A(P)$ is independent in $M$ and a $t - s$ cut $X$ such that $P$ and $X$ satisfy the complementarity conditions (Condition 5.0.6).
Proof. Without loss of generality we may assume that $\mathcal{M}$ does not contain loops. A pair $(\mathcal{W}, X)$ is called a wave if $X$ is a $t - s$ cut and $\mathcal{W}$ is an independent system of edge-disjoint strict $s \rightarrow X$ paths such that the second complementarity condition holds for $\mathcal{W}$ and $X$ (i.e. $A_{\text{last}}(\mathcal{W})$ spans in $D(X)$ in $\mathcal{M}$).

5.0.8 Remark. By picking an arbitrary base $B$ of $\text{out}(s)$ and taking $\mathcal{W} := B$ as a set of single-edge paths and $X := V \setminus \{s\}$ we obtain a wave $(\mathcal{W}, X)$ thus always exists some wave.

We say that the wave $(\mathcal{W}_1, X_1)$ extends the wave $(\mathcal{W}_0, X_0)$ and write $(\mathcal{W}_0, X_0) \leq (\mathcal{W}_1, X_1)$ if

1. $X_1 \subseteq X_0$,
2. $\mathcal{W}_1$ consists of the forward continuations of some of the paths in $\mathcal{W}_0$ such that the continuations lie in $X_0$,
3. $\mathcal{W}_1$ contains all of those paths of $\mathcal{W}_0$ that meet $X_1$.

If in addition $\mathcal{W}_1$ contains a forward-continuation of all the elements of $\mathcal{W}_0$, then the extension is called complete. Note that $\leq$ is a partial ordering on the waves and if $(\mathcal{W}_0, X_0) \leq (\mathcal{W}_1, X_1)$ holds, then the extension is proper (i.e. $(\mathcal{W}_0, X_0) < (\mathcal{W}_1, X_1)$) iff $X_1 \subsetneq X_0$.

5.0.9 Observation. If $(\mathcal{W}_1, X_1)$ is an incomplete extension of $(\mathcal{W}_0, X_0)$, then it is a proper extension thus $X_1 \subseteq X_0$. Furthermore, $\mathcal{W}_1$ and $X_0$ do not satisfy the second complementarity condition (Condition 5.0.6/2).

5.0.10 Lemma. If a nonempty set $\mathcal{X}$ of waves is linearly ordered by $\leq$, then $\mathcal{X}$ has a unique smallest upper bound $\text{sup}(\mathcal{X})$.

Proof: We may suppose that $\mathcal{X}$ has no maximal element. Let $\langle (\mathcal{W}_\xi, X_\xi) : \xi < \kappa \rangle$ be a cofinal sequence of $(\mathcal{X}, \leq)$. We define $X := \bigcap_{\xi < \kappa} X_\xi$ and

$$\mathcal{W} := \bigcup_{\xi < \kappa} \bigcap_{\xi < \xi} \mathcal{W}_\xi.$$ 

For $P \in \mathcal{W}$ we have $V(P) \cap X_\xi = \{\text{end}(P)\}$ for all large enough $\xi < \kappa$ hence $V(P) \cap X = \{\text{end}(P)\}$. The paths in $\mathcal{W}$ are pairwise edge-disjoint since $P_1, P_2 \in \mathcal{W}$
implies that \( P_1, P_2 \in W_\xi \) for all large enough \( \xi \). Since the matroid \( \mathcal{M} \) is finitary the same argument shows that \( \mathcal{W} \) is independent.

Suppose that \( e \in \text{in}_D(X) \setminus A(\mathcal{W}) \). For a large enough \( \xi < \kappa \) we have \( e \in \text{in}_D(X_\xi) \). Then the last edges of those elements of \( W_\xi \) that terminate in \( \text{head}(e) \) spans \( e \) in \( \mathcal{M} \). These paths have to be elements of all the further waves of the sequence (because of the definition of \( \leq \)) and thus of \( \mathcal{W} \) as well. Therefore \((W, X)\) is a wave and clearly an upper bound.

Suppose that \((Q, Y)\) is another upper bound for \( \mathcal{X} \). Then \( X_\xi \supseteq Y \) for all \( \xi < \kappa \) and hence \( X \supseteq Y \). Let \( Q \in \mathcal{Q} \) be arbitrary. We know that \( W_\xi \) contains an initial segment \( Q_\xi \) of \( Q \) for all \( \xi < \kappa \) because \((Q, Y)\) is an upper bound (see the definition of \( \leq \)). For \( \xi < \zeta < \kappa \) the path \( Q_\xi \) is a (not necessarily proper) forward continuation of \( Q_\xi \). From some index the sequence \( \langle Q_\xi : \xi < \kappa \rangle \) need to be constant, say \( Q^* \), since \( Q \) is a finite path. But then \( Q^* \in \mathcal{W} \). Thus any \( Q \in \mathcal{Q} \) is a forward continuation of a path in \( \mathcal{W} \). Finally assume that some \( P \in \mathcal{W} \) meets \( Y \). Pick a \( \xi < \kappa \) for which \( P \in W_\xi \). Then \((W_\xi, X_\xi) \leq (Q, Y)\) guarantees \( P \in \mathcal{Q} \). Therefore \((W, X) \leq (Q, Y)\).

The Remark 5.0.8 and Lemma 5.0.10 imply via Zorn's Lemma the following.

**5.0.11 Corollary.** There exists a maximal wave. Furthermore, there is a maximal wave which is greater or equal to an arbitrary prescribed wave.

Let \((W, X)\) be a maximal wave. To prove Theorem 5.0.7 it is enough to show that there is an independent system of edge-disjoint \( s \rightarrow t \) paths \( \mathcal{P} \) that consists of the forward-continuation of all the paths in \( \mathcal{W} \). Indeed, condition \( A(\mathcal{P}) \cap \text{out}_D(X) = \emptyset \) will be true automatically (otherwise \( \mathcal{P} \) would violate independence, when the violating path “comes back” to \( X \)) and hence \( \mathcal{P} \) and \( X \) will satisfy the complementarity conditions.

We need a method developed by Lawler and Martel in [23] for the augmentation of polymatroidal flows in finite networks which works in the infinite case as well.

**5.0.12 Lemma.** Let \( \mathcal{P} \) be an independent system of edge-disjoint \( s \rightarrow t \) paths. Then there is either an independent system of edge-disjoint \( s \rightarrow t \) paths \( \mathcal{P}' \) with
\( \text{span}_{M_t}(A_{\text{last}}(P)) \nsubseteq \text{span}_{M_t}(A_{\text{last}}(P')) \) or there is a \( t - s \) cut \( X \) such that the complementarity conditions (Condition 5.0.6) hold for \( P \) and \( X \).

**Proof:** Call \( W \) an augmenting walk if

1. \( W \) is a directed walk with respect to the digraph that we obtain from \( D \) by changing the direction of edges in \( A(P) \),
2. \( \text{start}(W) = s \) and \( W \) meets no more \( s \),
3. \( A(W) \triangle A(P) \) is independent,
4. if for some initial segment \( W' \) of \( W \) the set \( A(W') \triangle A(P) \) is not independent, then for the one edge longer initial segment \( W'' = W'e \) the set \( A(W'') \triangle A(P) \) is independent again.

If there is an augmenting walk terminating in \( t \), then let \( W \) be a shortest such a walk. Build \( P' \) from the edges \( A(W) \triangle A(P) \) in the following way. Keep untouched those \( P \in P \) for which \( A(W) \cap A(P) = \emptyset \) and replace the remaining finitely many paths, say \( Q \subseteq P \) where \( |Q| = k \), by \( k + 1 \) new \( s \to t \) paths constructed from the edges \( A(W) \triangle A(Q) \) by the greedy method. Obviously \( P' \) is an independent system of edge-disjoint \( s \to t \) paths. We need to show that

\[ \text{span}_{M_t}(A_{\text{last}}(P)) \nsubseteq \text{span}_{M_t}(A_{\text{last}}(P')). \]

If only the last vertex of \( W \) is \( t \), then it is clear. Let \( f_1, e_1, \ldots, f_n, e_n, f_{n+1} \) be the edges of \( W \) incident with \( t \) enumerated with respect to the direction of \( W \). The initial segments of \( W \) up to the inner appearances of \( t \) may not be augmenting walks (since \( W \) is a shortest that terminates in \( t \)) hence by condition 4 the one edge longer and the one edge shorter segments are. It follows that for any \( 1 \leq i \leq n \) there is a \( M_t \)-circuit \( C_i \) in

\[ A_i := A(P) \cap \text{in}_D(t) + f_1 - e_1 + f_2 - e_2 + \cdots + f_i. \]

Furthermore, \( f_i \notin A(P) \) and \( e_i \in C_i \cap A(P) \). It implies by induction that \( A_i \setminus \{e_i\} \) spans the same set in \( M_t \) as \( A(P) \cap \text{in}_D(t) \) whenever \( 1 \leq i \leq n \) and hence
\( A_n \cup \{f_{n+1}\} \) spans a strictly larger.

Suppose now that none of the augmenting walks terminate in \( t \). Let us denote the set of the last vertices of the augmenting walks by \( Y \). We show that \( \mathcal{P} \) and \( X := V \setminus Y \) satisfy the complementarity conditions. Obviously \( X \) is a \( t-s \) cut. Suppose, to the contrary, that \( e \in A(\mathcal{P}) \cap \text{out}_D(X) \). Pick an augmenting walk \( W \) terminating in \( \text{head}(e) \). Necessarily \( e \in A(W) \), otherwise \( W \) would be an augmenting walk contradicting to the definition of \( X \). Consider the initial segment \( W' \) of \( W \) for which the following edge is \( e \). Then \( W'e \) is an augmenting walk (if \( W' \) itself is not, then it is because of condition 4) which leads to the same contradiction.

To show the second complementarity condition assume that \( f \in \text{in}_D(X) \setminus A(\mathcal{P}) \). Choose an augmenting walk \( W \) that terminates in \( \text{tail}(f) \). We may suppose that \( f \notin A(W) \) otherwise we consider the initial segment \( W' \) of \( W \) for which the following edge is \( f \) (it is an augmenting walk, otherwise \( W'f \) would be by applying condition 4). The initial segments of \( Wf \) that terminate in \( \text{head}(f) \) may not be augmenting walks. Let \( f_1, e_1, \ldots, f_n, e_n \) be the ingoing-outgoing edge pairs of \( \text{head}(f) \) in \( W \) with respect to the direction of \( W \) (enumerating with respect to the direction of \( W \)) and let \( f_{n+1} := f \). Then for any \( 1 \leq i \leq n+1 \) there is a unique \( \mathcal{M} \)-circuit \( C_i \) in

\[
A(\mathcal{P}) \cap \text{in}_D(\text{head}(f)) + f_1 - e_1 + f_2 - e_2 + \cdots + f_i.
\]

It follows by using condition 4 and the definition of \( X \) that for \( 1 \leq i \leq n \)

1. \( f_i \notin A(\mathcal{P}) \) and \( e_i \in C_i \cap A(\mathcal{P}) \),

2. \( \text{tail}(e_i), \text{tail}(f_i) \in Y \) (tail with respect to the original direction),

3. \( C_i \subseteq \text{in}_D(X) \).

Assume that we already know for some \( 1 \leq i \leq n \) that \( f_j \) is spanned by \( F := A(\mathcal{P}) \cap \text{in}_D(X) \) in \( \mathcal{M} \) whenever \( j < i \). Any element of \( C_i \setminus \{f_i\} \) which is not in \( F \) has a form \( f_j \) for some \( j < i \) thus by the induction hypothesis it is spanned by
and hence we obtain that \( f_i \in \text{span}_\mathcal{M}(F) \) as well. By induction it is true for \( i = n + 1 \).

5.0.13 Proposition. Assume that \((W, X)\) and \((Q, Y)\) are waves where \( Y \subseteq X \) and \( Q \) consists of the forward-continuation of some of the paths in \( W \) where the new terminal segments lie in \( X \). Let \( W_Y := \{ P \in W : \text{end}(P) \in Y \} \). Then for an appropriate \( Q' \subseteq Q \) the pair \((W_Y \cup Q', Y)\) is a wave with \((W, X) \preceq (W_Y \cup Q', Y)\).

Proof: The path-system \( W_Y \cup Q \) (not necessarily disjoint union) is edge-disjoint since the edges in \( A(Q) \setminus A(W) \) lie in \( X \). For the same reason it may violate independence only at the vertices \( \{ \text{end}(P) : P \in W_Y \} \subseteq Y \). Pick a base \( B \) of \( \text{in}_D(Y) \) for which \( A_{\text{last}}(W_Y) \subseteq B \subseteq A_{\text{last}}(W_Y) \cup A_{\text{last}}(Q) \).

It is routine to check that the choice \( Q' = \{ P \in Q : A(P) \cap B \neq \emptyset \} \) is suitable.

For \( A_0 \subseteq A \) let us denote \((D - \text{span}_\mathcal{M}(A_0), \mathcal{M}/\text{span}_\mathcal{M}(A_0))\) by \( \mathcal{D}(A_0) \). Note that for any \( A_0 \) the matroid corresponding to \( \mathcal{D}(A_0) \) has no loops and \((D, \mathcal{M}) = \mathcal{D}(\emptyset) =: \mathcal{D} \).

5.0.14 Observation. If \((W, X)\) is a wave and for some \( A_0 \subseteq A \setminus A(W) \) the set \( A_0 \cup A(W) \) is independent, then \((W, X)\) is a \( \mathcal{D}(A_0) \)-wave as well.

5.0.15 Lemma. If \((W, X)\) is a maximal \( \mathcal{D} \)-wave and \( e \in A \setminus A(W) \) for which \( A(W) \cup \{ e \} \) is independent, then all the extensions of the \( \mathcal{D}(e) \)-wave \((W, X)\) in \( \mathcal{D} \) are complete.

Proof: Seeking a contradiction, assume that we have an incomplete extension \((Q, Y)\) of \((W, X)\) with respect to \( \mathcal{D}(e) \). Observe that necessarily \( e \in \text{in}_D(Y) \) and \( r_{\mathcal{M}}(\text{in}_D(Y)/A_{\text{last}}(Q)) = 1 \). Furthermore, \( Y \subseteq X \) by Observation 5.0.9.

We show that \((W, X)\) has a proper extension with respect to \( \mathcal{D} \) as well contradicting with its maximality. Without loss of generality we may assume that \( \text{in}_D(X) = A_{\text{last}}(W) \). Indeed, otherwise we delete the edges \( \text{in}_D(X) \setminus A(W) \) from \( D \) and from \( \mathcal{M} \). It is routine to check that after the deletion \((W, X)\) is still a wave and a proper extension of it remains a proper extension after putting back these edges.
Contract $V \setminus X$ to $s$ and contract $Y$ to $t$ in $D$ and keep $\mathcal{M}$ unchanged. Apply the augmenting walk method (Lemma 5.0.12) in the resulting system with the strict $V \setminus X \to Y$ terminal segments of the paths in $\mathcal{Q}$. If the augmentation is possible, then the assumption $\text{in}_D(X) = A_{\text{last}}(\mathcal{W})$ ensures that the first edge of any element of the resulting path-system $\mathcal{R}$ is a last edge of some path in $\mathcal{W}$. By uniting the elements of $\mathcal{R}$ with the corresponding paths from $\mathcal{W}$ we can get a new independent system of edge-disjoint strict $s \to Y$ paths $\mathcal{Q}'$ (with respect to $\mathcal{Q}$). Furthermore, $r_{\mathcal{M}}(\text{in}_D(Y)/A_{\text{last}}(\mathcal{Q})) = 1$ guarantees that $A_{\text{last}}(\mathcal{Q}')$ spans $\text{in}_D(Y)$ in $\mathcal{M}$ and hence $(\mathcal{Q}', Y)$ is a wave. Thus by Proposition 5.0.13 we get an extension of $(\mathcal{W}, X)$ and it is proper because $Y \subseteq X$ which is impossible.

Thus the augmentation must be unsuccessful which implies by Lemma 5.0.12 that there is some $Z$ with $Y \subseteq Z \subseteq X$ such that $Z$ and $Q$ satisfy the complementarity conditions. By Proposition 5.0.9 we know that $Z \subseteq X$. For the initial segments $Q_Z$ of the paths in $\mathcal{Q}$ up to $Z$ the pair $(Q_Z, Z)$ forms a wave. Thus by applying Proposition 5.0.13 with $(\mathcal{W}, X)$ and $(Q_Z, Z)$ we obtain an extension of $(\mathcal{W}, X)$ which is proper because $Z \subseteq X$ contradicting to the maximality of $(\mathcal{W}, X)$.

5.0.16 Proposition. If $(\mathcal{W}, X)$ is a maximal wave and $v \in X$, then there is a $v \to t$ path $Q$ in $D[X]$ such that $\text{A}(\mathcal{W}) \cup \text{A}(Q)$ is independent.

Proof: It is equivalent to show that there exists a $v \to t$ path $Q$ in $D - \text{span}_M(\text{A}(\mathcal{W}))$ (path $Q$ will necessarily lie in $D[X]$ because $D - \text{span}_M(\text{A}(\mathcal{W}))$ does not contain any edge entering into $X$.) Suppose, to the contrary, that it is not the case. Let $X' \subseteq X$ be the set of those vertices in $X$ that are unreachable from $v$ in $D - \text{span}_M(\text{A}(\mathcal{W}))$ (note that $v \notin X'$ but $t \in X'$ by the indirect assumption). Let $\mathcal{W}'$ be consist of the paths in $\mathcal{W}$ that meet $X'$. If we prove that $(\mathcal{W}', X')$ is a wave, then we are done since it would be a proper extension of the maximal wave $(\mathcal{W}, X)$. Assume that $f \in \text{in}_D(X') \setminus \text{A}(\mathcal{W}')$. Then by the definition of $X'$ we have tail$(f) \in V \setminus X$ thus $f \in \text{in}_D(X)$. Hence $f$ is spanned by the last edges of the paths in $\mathcal{W}$ terminating in head$(f)$ and all these paths are in $\mathcal{W}'$ as well. Therefore $(\mathcal{W}', X')$ is a wave.
5.0.17 Lemma. Let \((W, X_0)\) be a maximal wave and assume that \(P \in W\) and let \(W_0 = W \setminus \{P\}\). Then there is an \(s\)-arborescence \(A\) such that

1. \(A(P) \subseteq A(A)\),
2. \(A(A) \cap A(W_0) = \emptyset\),
3. \(A(A) \cup A(W_0)\) is independent,
4. \(t \in V(A)\),
5. there is a maximal wave with respect to \(\mathcal{D}(A(A))\) which is a complete extension of the \(\mathcal{D}(A(A))\)-wave \((W_0, X_0)\).

Proof:

5.0.18 Proposition. The pair \((W_0, X_0) = (W \setminus \{P\}, X_0)\) is a maximal wave with respect to \(\mathcal{D}(A(P))\).

Proof: It is clearly a wave thus we show just the maximality. Suppose that \((Q, Y)\) is a proper extension of \((W \setminus \{P\}, X_0)\) with respect to \(\mathcal{D}(A(P))\). Necessarily \(\text{end}(P) \in Y\) otherwise it would be a wave with respect to \(\mathcal{D}\) which properly extends \((W, X_0)\). Let \(e\) be the last edge of \(P\). We know that \(A_{\text{last}}(Q)\) spans \(\text{in}_D(Y)\) in \(\mathcal{M}/e\). Since \(A(Q)\) is \([\mathcal{M}/\text{span}_{\mathcal{M}}(A(P))]\)-independent it follows that \((Q \cup \{P\}, Y)\) is a \(\mathcal{D}\)-wave. But then it properly extends \((W, X_0)\) which is a contradiction. \(\blacksquare\)

Fix a well-ordering of \(A\) with order type \(|A| \leq \omega\). We build the arborescence \(A\) by recursion. Let \(A_0 := P\). Assume that \(A_m, W_m\) and \(X_m\) has already defined for \(m \leq n\) in such a way that

1. \(A(A_m) \cap A(W_m) = \emptyset\),
2. \(A(A_m) \cup A(W_m)\) is independent,
3. \((W_m, X_m)\) is a maximal wave with respect to \(\mathcal{D}_m := \mathcal{D}(A(A_m))\) and a complete extension of the \(\mathcal{D}_m\)-wave \((W_k, X_k)\) whenever \(k < m\),
4. for \(0 \leq k < n\) we have \(A_{k+1} = A_k + e_k\) for some \(e_k \in \text{out}_D(V(A_k))\).
If \( t \in V(A_n) \), then \( A_n \) satisfies the requirements of Lemma 5.0.17 thus we are done. Hence we may assume that \( t \notin V(A_n) \).

**5.0.19 Proposition.** \( \text{out}_{D-\text{span}_M(A(W_n))}(V(A_n)) \neq \emptyset \).

**Proof:** We claim that the \( D \)-wave \( (W_n, X_n) \) is not a \( D \)-wave. Indeed, suppose it is, then \( \text{end}(P) \notin X_n \) (since \( A_{\text{last}}(W_n) \) does not span the last edge \( e \) of \( P \)) and therefore \( X_n \subseteq X_0 \) thus it extends \( (\mathcal{W}, X_0) \) properly with respect to \( D \) contradicting to the maximality of \( (\mathcal{W}, X_0) \). Hence the \( s \)-arborescence \( A_n \) need to have an edge \( e \in \text{in}_D(X) \). Let \( Q \) be a path that we obtain by applying Proposition 5.0.16 with \( (W_n, X_n) \) and \( \text{head}(e) \) in the system \( \mathcal{D}_n \). Consider the last vertex \( v \) of \( Q \) which is in \( V(A_n) \). Since \( v \neq t \) there is an outgoing edge \( f \) of \( v \) in \( Q \) and hence \( f \in \text{out}_{D-\text{span}_M(A(W_n))}(V(A_n)) \). \( \blacksquare \)

Pick the smallest element \( e_n \) of \( \text{out}_{D-\text{span}_M(A(W_n))}(V(A_n)) \) and let \( A_{n+1} := A_n + e_n \). Let \( (W_{n+1}, X_{n+1}) \) be a maximal wave with respect to \( \mathcal{D}_{n+1} \) which extends \( (W_n, X_n) \) (exists by Corollary 5.0.11). Lemma 5.0.15 ensures that it is a complete extension.

Suppose, to the contrary, that the recursion does not stop after finitely many steps. Let

\[
A_\infty := \left( \bigcup_{n=0}^{\infty} V(A_n), \bigcup_{n=0}^{\infty} A(A_n) \right).
\]

Note that \( A(A_\infty) \) is independent and \( \{ (W_n, X_n) : n < \omega \} \) is an \( \leq \)-increasing sequence of \( \mathcal{D}(A(A_\infty)) \)-waves. Let \( (W_\infty, X_\infty) \) be a maximal \( \mathcal{D}(A(A_\infty)) \)-wave which extends \( \sup_n (W_n, X_n) \) (see Lemma 5.0.10).

It may not be a wave with respect to \( \mathcal{D} \). Hence the \( s \)-arborescence \( A_\infty \) contains an edge \( e \in \text{in}_D(X_\infty) \). Apply Proposition 5.0.16 with \( (W_\infty, X_\infty) \) and \( \text{head}(e) \) in the system \( \mathcal{D}(A(A_\infty)) \). Consider the last vertex \( v \) of the resulting \( Q \) which is in \( V(A_\infty) \). Since \( v \neq t \) by assumption there is an outgoing edge \( f \) of \( v \) in \( Q \). Then \( f \in \text{out}_{D-\text{span}_M(A(W_\infty))}(V(A_\infty)) \) which implies that for some \( n_0 < \omega \) we have \( f \in \text{out}_{D-\text{span}_M(A(W_n))}(V(A_n)) \) whenever \( n > n_0 \). But then the infinitely many pairwise distinct edges \( \{ e_n : n_0 < n < \omega \} \) are all smaller than \( f \) in our fixed well-ordering on \( A \) which contradicts to the fact that the type of this well-ordering
is at most $\omega$. \hfill \blacksquare

The Theorem follows easily from Lemma 5.0.17. Indeed, pick a maximal wave $(W_0, X_0)$ with respect to $D_0 := D$ where $W_0 = \{P_n\}_{n<\omega}$. Apply Lemma 5.0.17 with $P_0 \in W_0$. The resulting arborescence $A_0$ contain a unique $s \rightarrow t$ path $P_0^*$ which is necessarily a forward-continuation of $P_0$ (usage of a new edge from $in_D(X_0)$ would lead to dependence). Then by Lemma 5.0.17 we have a maximal wave $(W_1, X_1)$ (where $X_1 \subseteq X_0$) with respect to $D_1 := D_0(A(A_0))$ such that $W_1 = \{P_1^1\}_{1 \leq n < \omega}$ where $P_1^1$ is a forward continuation of $P_n$. Then we apply Lemma 5.0.17 with the $D_1$-wave $(W_1, X_1)$ and $P_1^1 \in W_1$ and continue the process recursively. By the construction $\bigcup_{n<m} A(P_n^*)$ is independent for each $m < \omega$. Since $M$ is finitary $\bigcup_{n<\omega} A(P_n^*)$ is independent as well thus $P := \{P_n^*\}_{n<\omega}$ is a desired paths-system that satisfies the complementarity conditions with $X_0$.

\section*{5.1 Related open problems}

We suspect that one can omit the countability condition for $D$ in Theorem 5.0.7 by analysing the famous infinite Menger's theorem [2] of Aharoni and Berger. We also think that it is possible to put matroid constraints on the outgoing edges of each vertex as well but this generalization contains the Matroid intersection conjecture for finitary matroids as a special case, the latter problem is hard enough itself. The finitariness of the matroids are used several times in the proof; we do not know yet if one can omit this condition.
Chapter 6

King-serf duo by monochromatic paths in edge-coloured tournaments

6.1 Introduction

A tournament $T = (V(T), A(T))$ is a directed graph obtained by orienting the edge set of a (possibly infinite) complete undirected graph. For a finite or infinite cardinal $\kappa$ let $\exp_0(\kappa) = \kappa$ and let $\exp_{k+1}(\kappa) = 2^{\exp_k(\kappa)}$. A $\kappa$-edge-colouring of a tournament $T$ is a function $c : A(T) \to \kappa$. A monochromatic path is a directed path with edges having the same colour. We call a directed cycle quasi-monochromatic if all but at most one of its edges have the same colour.

Our investigation was motivated by the following conjecture of Erdős [26, p. 274].

6.1.1 Conjecture (Erdős). For every positive integer $k$ there is a (least) positive integer $f(k)$ so that every $k$-edge-coloured finite tournament admits a subset $S \subseteq V(T)$ of size at most $f(k)$ such that $S$ is reachable from every vertex by a monochromatic path.

It is known that $f(1) = f(2) = 1$, and there is an example showing that $f(3) \geq 3$ (see [26]). However, there is no known constant upper bound for $f(3)$, although it is conjectured to be 3 by Erdős. As a weakening of the original conjecture, we
consider source-sink pairs instead of one sink set $S$. However, we may add bounds on the length of the monochromatic paths. More precisely, a king-serf duo by monochromatic paths consists of disjoint vertex sets $K, S \subseteq V(D)$ so that every vertex $v$ has a monochromatic path of length at most two from $K$ to $v$ or from $v$ to $S$. The size of the duo is defined as $|K| + |S|$. An edge $uv$ of an edge-coloured tournament $T$ is called forbidding if there is no monochromatic path of length at most two from $v$ to $u$. Note that if $T'$ is a subtournament of $T$ containing a forbidding edge $uv$, then $uv$ is forbidding edge with respect to $T'$ as well.

The main result of the chapter is the following.

6.1.2 Theorem. For every (finite or infinite) cardinal $\kappa > 0$ there is a cardinal $\lambda_{\kappa} \leq \exp_{10}(\kappa)$ such that in every $\kappa$-edge-coloured tournament there exists a king-serf duo by monochromatic paths of size at most $\lambda_{\kappa}$. For finite $\kappa$ one can guarantee $\lambda_{\kappa} \leq \kappa^{62500\kappa}$.

Proof. The proof relies on the following theorem of A. Hajnal that we gain by putting together the earlier finite version of it [9] and his later work where he extends it to infinitely many colours in [13].

6.1.3 Theorem (A. Hajnal). For every finite simple graph $H$ and cardinal $\kappa > 0$ there is a simple graph $G$ of size at most $\exp_{|V(H)|+5}(\kappa)$ (at most $\kappa^{500|V(H)|^3\kappa}$ in the finite case) such that in any $\kappa$-edge-colouring of $G$ one can find a monochromatic induced subgraph isomorphic to $H$.

With the help of Theorem 6.1.3, first we prove the following.

6.1.4 Lemma. For every cardinal $\kappa > 0$ there exists a tournament $T_{\kappa}$ of size at most $\exp_{10}(\kappa)$ (at most $\kappa^{62500\kappa}$ in the finite case) such that in any $\kappa$-edge-colouring of $T_{\kappa}$ there exists a quasi-monochromatic directed cycle of length three.

Proof: Pick a graph $G$ ensured by Theorem 6.1.3 for $\kappa$ and $H := C_5$, that is, a cycle of length 5. Fix a well-ordering of $V(G)$. Let $T_{\kappa}$ denote the tournament obtained by orienting the edges of $G$ forward according to the ordering, and by adding all missing edges as backward edges. We claim that $T_{\kappa}$ satisfies the conditions of the lemma.
Take an arbitrary $\kappa$-edge-colouring of $T_\kappa$. The choice of $G$ implies that there is a monochromatic (not necessarily directed) cycle $C$ of length 5 in the graph such that $A(C)$ consists of forward edges, and all the other edges induced by $V(C)$ in $T_\kappa$ are backward edges.

No matter how the edges of $C$ are oriented, we can always find a directed path of length two in $A(C)$. Take such a path, say $uv$ and $vw$. These edges together with $wu$ form a quasi-monochromatic directed cycle, which completes the proof of the lemma.

We claim that $\lambda_\kappa := |V(T_\kappa)|$ satisfies the conditions of the theorem. Suppose, to the contrary, that there exists a $\kappa$-edge-coloured tournament $T$ not containing a king-serf duo by monochromatic paths of size at most $\lambda_\kappa$. Let $T_\kappa$ be a tournament that we obtain by applying Lemma 6.1.4.

6.1.5 Lemma. $T$ has a subtournament isomorphic to $T_\kappa$ consisting of forbidding edges.

Proof: We build up the desired subtournament by transfinite recursion. Let $V(T_\kappa) = \{u_\gamma\}_{\gamma < \lambda_\kappa}$. Assume that for some $\alpha < \lambda_\kappa$ we have already found an $\subseteq$-increasing chain $\langle f_\beta : \beta < \alpha \rangle$ of $T_\kappa \to T$ embeddings where $\text{dom}(f_\beta) = \{u_\gamma\}_{\gamma < \beta}$ and the images of the edges of $T_\kappa$ are forbidding edges of $T$. If $\beta$ is a limit ordinal, we may simply take $f_\beta := \bigcup_{\gamma < \beta} f_\gamma$ to keep the conditions. Assume that $\beta = \delta + 1$. Let $O = \{\gamma < \delta : u_\delta u_\gamma \in A(T_\kappa)\}$. As $T$ is a counterexample, the sets $K := \{f_\delta(u_\gamma)\}_{\gamma \in O}$ and $S := \{f_\delta(u_\gamma)\}_{\gamma \in \delta \setminus O}$ cannot form a king-serf duo by monochromatic paths. Therefore there is a vertex $v \in V(T)$ such that there is a forbidding edge from $v$ to every element of $K$, and there is a forbidding edge from every element of $S$ to $v$. But then $f_{\delta + 1} := f_\delta \cup \{(u_\delta, v)\}$ maintains the conditions. Finally, the image of $f := \bigcup_{\gamma < \lambda_\kappa} f_\gamma$ gives the desired copy of $T_\kappa$.

The $\kappa$-edge-colouring of $T$ defines a $\kappa$-edge-colouring of its $T_\kappa$ subdigraph as well. Therefore, by the choice of $T_\kappa$, there is a quasi-monochromatic directed cycle $C$ of length three in $T_\kappa$. Let $uv$ denote the edge of $C$ with different colour than the others if $C$ contains two colours, and let $uv$ be an arbitrary edge of $C$ if it is monochromatic. Then $C - uv$ is a monochromatic path of length two from $v$ to $u$, contradicting $uv$ being a forbidding edge of $T$. \qed
Chapter 7

Independent and maximal branching packing in infinite matroid-rooted digraphs

7.1 Introduction

The investigation of packing branchings started with a result of Edmonds.

7.1.1 Theorem (Edmonds’ branching theorem (strong form), [8]). Let \( D = (V, A) \) be a finite digraph and \( R_i \subseteq V \) nonempty for \( i = 1, \ldots, k \). Suppose that for each nonempty \( X \subseteq V \) has at least as many ingoing edges as many \( R_i \) are disjoint from it. Then there exists a system \( \{B_i\}_{1 \leq i \leq k} \) of pairwise edge-disjoint spanning branchings in \( D \) where the root set of \( B_i \) is \( R_i \).

The theorem above of Edmonds fails for infinite digraphs. To see this choose \( D \) as in Theorem 2.2.1 for \( k = 2 \) and let \( R_1 = \{s\}, R_2 = \{t\} \). Even so it is possible to give infinite generalizations under reasonable restrictions. Our first result in this topic is the following.

7.1.2 Theorem (A. Joó, [15]). In Edmonds’ branching theorem (Theorem 7.1.1) instead of finiteness of \( D \) it is enough to assume that any forward-infinite path of \( D \) meets all the sets \( R_i \).
We investigated later the possibility of packing infinitely many branchings with prescribed nonempty root sets \( \{ R_i \}_{i < \omega} \). The literal generalization of the condition of Edmonds does not even imply that any vertex \( v \) is simultaneously reachable by edge-disjoint paths from the root sets \( R_i \) (which would be obviously necessary). We proved the following theorem.

**7.1.3 Theorem** (A. Joó, [16]). Let \( D = (V, A) \) be a digraph and let \( R_i \subseteq V \) be nonempty for \( i < \omega \). Suppose that for any \( v \in V \) there is a system of pairwise edge-disjoint directed paths \( \{ P^v_i \}_{i < \omega} \) in \( D \) such that \( P^v_i \) goes from \( R_i \) to \( v \). Assume that any backward-infinite path in \( D \) meets all the sets \( R_i \). Then there exists a system \( \{ B_i \}_{i < \omega} \) of pairwise edge-disjoint spanning branchings in \( D \) where the root set of \( B_i \) is \( R_i \).

The branching packing theorem of Edmonds has been generalized in several different directions in the finite case. One of these directions is a matroid involved, reachability-based branching packing of Cs. Király [20]. Our most up to date result in this topic is a common generalization of our two theorems above (Theorem 7.1.2 and 7.1.3) and [20] (which itself is a common generalization of the reachability based branching packing theorem [19] and a matroid-based branching packing result [7]). We need some preparation to state it.

### 7.2 Matroid-rooted digraphs

We call a triple \( \mathfrak{R} = (D_{\mathfrak{R}}, M_{\mathfrak{R}}, \pi_{\mathfrak{R}}) \) a matroid-rooted digraph if \( D_{\mathfrak{R}} = (V, A) \) is a digraph, \( M_{\mathfrak{R}} = (S, I) \) is a matroid and \( \pi_{\mathfrak{R}} : S \to P(V) \setminus \{ \emptyset \} \). We will omit the subscripts whenever they are clear from the context. For an \( I \in \mathcal{I} \) and \( T \subseteq V \) an \( (I, T) \)-linkage is a system of edge-disjoint paths \( \{ P_i \}_{i \in I} \) such that \( P_i \) goes from \( \pi(i) \) to \( T \). In a strict linkage \( P_i \) goes strictly from \( \pi(i) \) to \( T \). We say that \( I \) is \( T \)-linkable if there exists an \( (I, T) \)-linkage. A branching packing \( \mathcal{B} \) with respect to \( \mathfrak{R} \) is a system of edge-disjoint branchings \( \mathcal{B} = \{ B_i \}_{i \in S} \) in \( D \) where the root set of \( B_i \) is \( \pi(i) \). A branching packing is trivial if none of the branchings in it have any edges. For \( X \subseteq V \) let \( \mathcal{S}(X) = \{ i \in S : \pi(i) \cap X \neq \emptyset \} \). The matroid-rooted digraph is called independent if \( \mathcal{S}(v) \in \mathcal{I} \) for all \( v \in V \). A branching
packing is called independent if the matroid-rooted digraph $B := (D, M, \pi_B)$ is independent where $\pi_B(i) = V(B_i)$. Let us denote $\text{span}(S(\text{top}_D(X)))$ by $\mathcal{N}(X)$ (the need of $X$). Clearly $\mathcal{N}(X) = \text{span}(\bigcup_{v \in X} \mathcal{N}(v))$. The branching packing $B$ is maximal if for all $v \in V$ the set $S_B(v) := \{i \in S : \pi_B(i) \cap X \neq \emptyset\}$ spans $\mathcal{N}(v)$. Hence a branching packing $B$ is independent and maximal if $S_B(v)$ is a base of $\mathcal{N}(v)$ for all $v \in V$.

7.3 Preparations

The linkage condition

For the existence of a maximal independent branching packing the independence of $\mathcal{R}$ is obviously necessary since $S(v) \subseteq S_B(v)$ holds for any branching packing $B$. The maximality criteria leads to the following necessary condition.

7.3.1 Condition (linkage condition). For all $v \in V$ there exists a $(B, v)$-linkage in $D$ where $B$ is a base of $\mathcal{N}(v)$.

Indeed, since a maximal branching packing contains such a linkage for any $v$.

If we suppose that $\mathcal{M}$ and $D$ are finite, then independence and linkage condition are enough to ensure the existence of a maximal independent branching packing as shown by Cs. Király in [20]. In fact, instead of Condition 7.3.1 he used the condition “$r(S(X)) + |\text{in}_D(X)| \geq r(\mathcal{N}(X))$ holds for all nonempty $X \subseteq V$”. Simple examples show that the literal infinite generalization of this inequality with cardinals fails to be sufficient in the infinite case. In fact it does not even imply our Condition 7.3.1 although they are equivalent in the finite case.

We need a formally stronger (but in fact equivalent) version of Condition 7.3.1 which is more similar with the condition of Cs. Király.

7.3.2 Condition. For all nonempty $X \subseteq V$ there exists a $(B, X)$-linkage in $D$ where $B$ is a base of $\mathcal{N}(X)$.

A linkage above is called a linkage for $X$ if it is strict and $B$ contains a base of $S(X)$. Clearly, one can always ensure these extra regularity conditions by
taking the appropriate segments of the paths and replace some of them with trivial paths. Sometimes we will not want to deal with these trivial paths. Throwing them away, the indices of the remaining paths form a base of \( \mathcal{N}(X)/\mathcal{S}(X) \). A **reduced linkage for \( X \)** is a strict \((B, X)\)-linkage where \( B \) is a base of \( \mathcal{N}(X)/\mathcal{S}(X) \).

### 7.3.3 Observation.
Condition 7.3.2 is equivalent to demanding the existence of a (reduced) linkage for all nonempty \( X \subseteq V \).

### 7.3.4 Proposition.
Condition 7.3.1 and 7.3.2 are equivalent.

**Proof:** Condition 7.3.1 is just the restriction of Condition 7.3.2 to the singleton sets \( X = \{v\} \) \((v \in V)\). We give a proof sketch for the nontrivial direction. Well-order \( X \) and pick a linkage \( \{P_i\}_{i \in G_0} \) for the smallest element \( x_0 \) of \( X \). Take a linkage \( \{P'_i\}_{i \in B} \) for the following element \( x_1 \). Let \( B' = \{i \in B : i \not\in \text{span}(G_0)\} \). We claim that for \( j \in B' \) the path \( P'_j \) may not have a common edge (or even common vertex) with any path in \( \{P_i\}_{i \in G_0} \). Indeed, if it would, then \( j \in \mathcal{N}(x_0) \) and therefore \( j \in \text{span}(G_0) \) (since \( G_0 \) spans \( \mathcal{N}(x_0) \)), contradicting to the choice \( j \in B' \). But then \( G_1 := G_0 \cup B' \) spans \( \mathcal{N}([x_0, x_1]) \) and the path-system \( \{P_i\}_{i \in G_0} \cup \{P'_i\}_{i \in B'} \) is edge-disjoint. One can finish the proof by transfinite recursion taking union of the path-systems at limit steps and trim the final system to be independent at the end.

---

**The statement of the main result and feasible extensions**

We propose the following two possible relaxations of the finiteness of \( D \) and \( M \).

### 7.3.5 Condition.
The matroid \( M \) has finite rank, and for any forward-infinite path \( P \) the set \( \mathcal{S}(V(P)) \) spans \( \mathcal{N}(V(P)) \).

### 7.3.6 Condition.
The matroid \( M \) has at most countably many components all, of which has finite rank. Furthermore, for any backward-infinite path \( P \) the set \( \mathcal{S}(V(P)) \) spans \( \mathcal{N}(V(P)) \).

Now we state our main result.
7.3.7 Theorem. If the matroid-rooted digraph \( R = (D, \mathcal{M}, \pi) \) satisfies independence, the linkage condition, and either Condition 7.3.5 or Condition 7.3.6, then there is an independent maximal branching packing for \( R \).

Instead of dealing with the branchings directly, we introduce the notion of feasible extension of an \( R \). Let \( i_0 \in S \) and \( e_0 \in A \) such that \( e_0 \in \text{out}_D(\pi(i_0)) \) and \( S(\text{head}(e_0)) \cup \{i_0\} \) is independent. The matroid-rooted digraph obtained by \((i_0, e_0)\)-extension from \( R = (D, \pi, \mathcal{M}) \) is \( R_1 := (D - e_0, \mathcal{M}, \pi_1) \) where

\[
\pi_1(i) = \begin{cases} 
\pi(i) & \text{if } i \neq i_0 \\
\pi(i) \cup \{\text{head}(e_0)\} & \text{if } i = i_0.
\end{cases}
\]

This extension is an imitation of giving edge \( e_0 \) to the branching \( B_{i_0} \). A matroid-rooted digraph \( R' \) is an extension of \( R \) if there is a transfinite sequence (build-sequence) of matroid-rooted digraphs \( \langle R_\xi : \xi \leq \alpha \rangle \) (where \( R_\xi = (D_\xi, \mathcal{M}, \pi_\xi) \) and \( D_\xi = (V, A_\xi) \)) with the following properties.

1. \( R_0 = R \), \( R_\alpha = R' \),
2. \( R_{\beta+1} \) is an \((i_\beta, e_\beta)\)-extension of \( R_\beta \) for some \( i_\beta, e_\beta \),
3. for a limit \( \beta \) we have \( \pi_\beta(i) = \bigcup_{\gamma < \beta} \pi_\gamma(i) \) and \( D_\beta := (V, \bigcap_{\gamma < \beta} A_\gamma) \).

For an \( R' \) extension of \( R \), the sequence above is not necessarily unique but the order \(|\alpha|\) of the extension is \(|\alpha| = |A(D_R) \setminus A(D_{R'})|\). Let \( S' \subseteq S \). If \( \pi_R(i) = \pi_{R'}(i) \) whenever \( i \notin S' \), then we say that \( R' \) is an \( S' \)-extension of \( R \). We define the limit of a transfinite sequence of consecutive extensions in the same way as we defined the limit of transfinite sequence of \((i, e)\)-extensions at the limit steps.

It is routine to check that for any \( v \in V \) and any extension \( R' \) of \( R \), we have \( N_{R'}(v) \subseteq N(v) \). We call \( R' \) a feasible extension (with respect to \( R \)) if it satisfies the following condition.

7.3.8 Condition. \( R' \) is independent and satisfies the linkage condition; furthermore, \( N_{R'}(v) = N(v) \) for all \( v \in V \).
In longer terms: $R'$ is independent, and for all $v \in V$ there is a $(B,v)$-linkage where $B$ is a base of $\mathcal{N}(v)$. It is easy to see that finding a branching packing $\{B_i\}_{i \in S}$ for $R$ is equivalent to finding a feasible extension $R'$ of $R$ such that $S_{R'}(v)$ is a base of $\mathcal{N}(v)$ for all $v \in V$. Here $A(B_i)$ will consist of those edges $e$ for which we had an $(i,e)$-extension in some fixed build-sequence of the extension $R'$.

Our plan is to construct a build-sequence of such an $R'$ extension. Any extension of an infeasible extension of $R$ is an infeasible extension of $R$, thus every member of the build-sequence needs to be feasible. On the one hand, a feasible extension of a feasible extension of $R$ is clearly a feasible extension of $R$. On the other hand, the limit of feasible extensions is not necessary feasible, therefore it is not enough to ensure the existence of a single feasible $(i,e)$-extension. (In the finite case of course it is enough since after at most $|A|$-many $(i,e)$-extensions we are done. Furthermore, in this case, for any independent $R$ that satisfies the linkage condition there exists a feasible $(i,e)$-extension for some $i$ and $e$ unless the trivial branching packing is already maximal.)

**Counterexamples**

As we have already mentioned independence and the linkage condition are not enough to ensure the existence of an independent maximal branching packing. We show this fact by an example (Figure 7.1) where we do not even have a feasible $(i,e)$-extension although for any vertex $v$ the set $S(v)$ is not a base of $\mathcal{N}(v)$. Let $V = \{u_n\}_{n < \omega} \cup \{v_n\}_{n < \omega}$. The edges are $u_1v_0$, $v_1u_0$ furthermore for $n < \omega$

$$u_n u_{n+1}, \quad v_n v_{n+1}, \quad u_{2n+3} u_{2n+1}, \quad v_{2n+3} v_{2n+1}.$$ 

Finally take the free matroid on $\{0,1\}$, let $\pi(0) = \{u_{2n}\}_{n < \omega}$, and let $\pi(1) = \{v_{2n}\}_{n < \omega}$. It is routine to check (by using Figure 7.1) that the linkage condition holds, i.e. every vertex is reachable simultaneously by edge-disjoint paths from the sets $\pi(0)$ and $\pi(1)$. To justify that there is no feasible $(i,e)$-extension, we give for any $e \in \operatorname{out}_D(\pi(0))$ a vertex set $X_e$ such that for the $(0,e)$-extension $R_1$ we have

$$\mathcal{N}_{R_1}(X_e) = \{0\} \subseteq \{0,1\} = \mathcal{N}(X_e),$$
which shows the infeasibility. We also do the same for any \( e \in \text{out}_D(\pi(1)) \). For \( n < \omega \) let \( X_{u_n, u_{n+1}} = \{u_k\}_{n < k < \omega} \) and let \( X_{v_nv_{n+1}} = \{v_k\}_{n < k < \omega} \).

\[
\begin{array}{ccccccc}
  u_0 & \rightarrow & u_1 & \rightarrow & u_2 & \rightarrow & u_3 \\
  \downarrow & & \downarrow & & \downarrow & & \downarrow \\
  v_0 & \rightarrow & v_1 & \rightarrow & v_2 & \rightarrow & v_3 \\
\end{array}
\]

Figure 7.1: An independent matroid-rooted digraph that satisfies linkage condition but has no feasible \((i,e)\)-extension although \(S(v)\) is not a base of \(N(v)\) for any \(v\). Elements of \(\pi(0)\) are circled and elements of \(\pi(1)\) are in rectangle in the figure.

In the example above the structure of the matroid was as simple as possible but the one-way infinite paths violate Condition 7.3.5 and Condition 7.3.6. Let us give another counterexample (Figure 7.2) in which, beyond the independence and the linkage condition, there is no infinite path at all (not even undirected) and the matroid is just a little bit more complicated than what Condition 7.3.6 allows. Let \(V = \{u_n\}_{n<\omega} \cup \{v_n\}_{n<\omega} \cup \{w\}\) and let \(A = \{u_nv_n\}_{n<\omega} \cup \{v_nw\}_{n<\omega}\). The matroid will be a countable subset of the vectorspace \(\mathbb{R}^\omega\) with the linear independence. We define

\[
S(u_n) := \{(0, \ldots, 0, 1, 0, \ldots)\},
\]
\[
S(v_n) := \{(0, \ldots, 0, 1, 0, \ldots), (0, \ldots, 0, 1, -1, 0, \ldots)\},
S(w) := \emptyset.
\]

The resulting matroid-rooted digraph is clearly independent. The unique elements of the sets \(S(u_n)\) form a base of \(N(w)\), and paths \(u_nv_n, w \ (n < \omega)\) form a linkage for \(w\). Considering the other vertices, \(S(u_n)\) and \(S(v_n)\) already spans \(N(u_n)\) and \(N(v_n)\), thus the linkage condition holds. On the one hand, a hypothetical independent and maximal branching packing may not use any edge in the from \(u_nv_n\) otherwise it would violate independence at \(v_n\). On the other hand, we claim that one cannot obtain a base for \(N(w)\) by taking at most one element from each \(S(v_n)\).
Indeed, a nontrivial linear combination of such vectors must have a nonzero component other than the 0th which ensures that they cannot span $(1, 0, \ldots)$. Hence there is no independent and maximal branching packing.

$$w$$

$v_0(1, \pm 1, 0, \ldots)$ $v_3(0, 1, \pm 1, 0, \ldots)$ $v_2(0, 0, 1, \pm 1, 0, \ldots)$ $v_3(0, 0, 0, 1, \pm 1, 0, \ldots)$ $\ldots$

$u_0(1, 0, \ldots)$ $u_1(0, 1, 0, \ldots)$ $u_2(0, 0, 1, 0, \ldots)$ $u_3(0, 0, 0, 1, 0, \ldots)$

Figure 7.2: An independent matroid-rooted digraph $\mathcal{R} = (D, \mathcal{M}, \pi)$ that satisfies linkage condition. Furthermore $D$ does not contain even undirected infinite paths and $\mathcal{M}$ is countable and finitary but there is no independent, maximal branching packing. For every vertex $v$ we listed the elements of $S(v)$ next to $v$.

There is an asymmetry in the matroid restriction part of Condition 7.3.5 and Condition 7.3.6. In our last example we show that one cannot replace the “$\mathcal{M}$ has a finite rank” part of Condition 7.3.5 by the condition that $\mathcal{M}$ has countably many components all of which have finite rank. Let $V = \{t\} \cup \{(m, n) < \omega \times \omega : m \leq n\}$. The set $A$ consists of the following edges (see Figure 7.3). For all $m, n < \omega$ for which it makes sense

1. infinitely many parallel edges from $(m, n + 1)$ to $(m, n)$,

2. edge from $(m, n)$ to $(m + 1, n)$,

3. edge from $(2m + 2, n)$ to $(2m, n)$,

4. edge from $(m, m)$ to $t$,

5. edge from $t$ to $(2m + 1, n)$ (not in the figure).
Figure 7.3: An illustration that in Condition 7.3.5 one cannot replace the restriction of the matroid by the weaker restriction of Condition 7.3.6. The outgoing edges of \( t \) (a single edge to each vertex in an odd row) are not on the figure. The thick horizontal edges stand for infinitely many parallel edges.

Observe that after the deletion of \( t \) just finitely many vertices are reachable from each vertex, which shows that there is no forward-infinite path in \( D := (V, A) \). Let \( \mathcal{M} \) be the free matroid on \( \omega \) and \( \pi(n) = \{(0, n)\} \). It is easy to check (using Figure 7.3) that \( \mathcal{N}(v) = \omega \) for all \( v \in V \) and the linkage condition holds. We have to show that there are no edge-disjoint spanning branchings with the prescribed roots. Suppose, to the contrary, that there are, and fix a a counterexample \( \mathcal{B} = \{\mathcal{B}_n\}_{n<\omega} \). For \( \mathcal{B}_0 \) the only possibility to reach \( t \) is to use the edge \(((0, 0), t)\). Suppose that we already know for some \( 0 < N \) that \( \mathcal{B}_n \) contains the path \((0, n), (1, n), \ldots, (n, n), t\) whenever \( n < N \). By using just the remaining edges \( t \) is no longer reachable from columns up to \( N - 1 \) hence for \( \mathcal{B}_N \) the path \((0, N), (1, N), \ldots, (N, N), t\) is the only possible option to reach \( t \). On the other hand, after the deletion of the edges of these paths for all \( n \) the vertices \( \{(0, n) : 1 \leq n < \omega\} \) are no longer reachable from \( \{(0, 0), t\} \). This prevents \( \mathcal{B}_0 \) from being a spanning branching rooted at \((0, 0)\) which is a contradiction.
7.4 Duality and the characterisation of the infeasible \((i, e)\)-extensions

Assume that linkage condition and independence hold for \(\mathcal{R}\) and let us focus first just on a single \((i_0, e_0)\)-extension \(\mathcal{R}_1\) of \(\mathcal{R}\). We cannot ruin independence in this extension, as it is built into the definition of the \((i, e)\)-extension. If for some nonempty \(X \subseteq V\) any linkage for \(X\) necessarily uses all the ingoing edges of \(X\), then we call \(X\) tight (with respect to \(\mathcal{R}\)). If \(X\) is tight and \(i_0 \in \text{span}(\mathcal{S}(X))\), then \(X\) is called \(i_0\)-dangerous. We claim that if \(e_0\) is an ingoing edge of an \(i_0\)-dangerous set \(X\), then the \((i_0, e_0)\)-extension is infeasible. On the one hand, \(i_0 \in \text{span}(\mathcal{S}(X))\) implies that \(\text{span}(\mathcal{S}_{\mathcal{R}_1}(X)) = \text{span}(\mathcal{S}(X))\) and hence \(\mathcal{N}(X) / \mathcal{S}_{\mathcal{R}_1}(X) = \mathcal{N}(X) / \mathcal{S}(X)\). On the other hand, by the tightness of \(X\) (with respect to \(\mathcal{R}\)) any \((B, X)\)-linkage where \(B\) is a base of \(\mathcal{N}(X) / \mathcal{S}(X) = \mathcal{N}(X) / \mathcal{S}_{\mathcal{R}_1}(X)\) uses all the incoming edges of \(X\) including \(e_0\) thus there is no longer desired linkage for \(X\) with respect to \(\mathcal{R}_1\). It will turn out that surprisingly this is the only possible reason for the infeasibility of an \((i_0, e_0)\)-extension.

In the finite case one can justify this easily in the following way. We use without proof that if \(\mathcal{M}\) has finite rank, then the consequence

\[ \mathcal{R}' \text{ is independent and } r(\mathcal{S}_{\mathcal{R}'_1}(X)) + |\text{in}_{\mathcal{D}_{\mathcal{R}'_1}}(X)| \geq r(\mathcal{N}(X)) \text{ for all nonempty } X \subseteq V \]

of Condition 7.3.8 is actually equivalent with it. Furthermore, tightness of \(X\) is equivalent with the fact that there is equality for \(X\) in the inequality above. (Of course in the finite case we do not need to know this equivalence or anything about our Condition 7.3.8 at all. One can simply define tightness based on the inequality.)

If the \((i_0, e_0)\)-extension is infeasible in the finite case and \(X^*\) is a violating set with respect to the resulting \(\mathcal{R}_1\), then the extension necessarily reduces the number of ingoing edges of \(X^*\) (i.e. \(|\text{in}_D(X^*)| = |\text{in}_D - e_0(X^*)| + 1\) and hence \(e_0 \in \text{in}_D(X^*)\)) but does not increase the rank of the submatroid corresponding to \(X^*\) (i.e. \(r(\mathcal{S}_{\mathcal{R}_1}(X^*)) = r(\mathcal{S}(X^*))\)) thus \(i_0 \in \text{span}(\mathcal{S}(X^*))\), furthermore there must
be originally equality for \( X^* \). Summarizing these we obtain that \( e_0 \) is an incoming edge of the \( i_0 \)-dangerous set \( X^* \).

As we mentioned the same characterisation of infeasible extensions remains true in the general case, although we need to use more complex arguments to prove it. We devote the rest of the section this proof and the corresponding preparations.

A set \( X \subseteq V \) is called \( t \)-good for some \( t \in V \) if there is a system of edge-disjoint paths \( \{ P_b \}_{b \in B} \cup \{ P_e \}_{e \in \text{in}_D(X)} \) in \( D[X] \) such that \( B \) is a base of \( S(X) \) and \( \{ P_b \}_{b \in B} \) is a \((B,t)\)-linkage and \( P_e \) goes from head \((e)\) to \( t \).

### 7.4.1 Definition (complementarity conditions)

The complementarity conditions for an \((I,t)\)-linkage \( \{P_i\}_{i \in I} \) and a vertex set \( X \ni t \) are the following.

1. \( I_{in} := I \cap S(X) \) is a base of \( S(X) \),
2. paths \( \{P_i\}_{i \in I_{in}} \) lie in \( D[X] \),
3. for \( i \in I \setminus S(X) =: I_{out} \) we have \( |A(P_i) \cap \text{in}_D(X)| = 1 \),
4. \( \bigcup_{i \in I_{out}} A(P_i) \supseteq \text{in}_D(X) \).

For \( i \in I_{out} \) let us denote by \( e_i \) the first edge of \( P_i \) that enters \( X \). Note that if the complementarity conditions hold for \( \mathcal{P} \) and \( X \), then \( X \) is \( t \)-good, as shown by the paths

\[
\{P_i\}_{i \in I_{in}} \cup \{P_i[\text{head}(e_i), t]\}_{i \in I_{out}}.
\]

One can replace the conditions 2,3,4 in Definition 7.4.1 by the single condition

\[
A_{last}(\{P_i[\text{start}(P_i), \text{head}(e_i)]\}_{i \in I_{out}}) = \text{in}_D(X).
\]

### 7.4.2 Claim

There exists a \( \subseteq \)-largest \( t \)-good set.

**Proof:** First of all, we always have a smallest \( t \)-good set, namely \( \{t\} \).

### 7.4.3 Proposition

For any \( \subseteq \)-increasing nonempty chain \( \langle X_\beta : \beta < \alpha \rangle \) of \( t \)-good sets \( \bigcup_{\beta < \alpha} X_\beta \) is \( t \)-good.
Proof: Note that the definition of \(t\)-goodness is equivalent if we demand a generator system \(G\) (a set that contains a base) instead of a base \(B\) of \(S(X)\). We define for all \(\beta \leq \alpha\) a path-system \(P_\beta\) that shows the \(t\)-goodness of \(X_\beta\). Let \(P_0\) be an arbitrary system that witnesses the \(t\)-goodness of \(X_0\). If some \(P_\beta = \{P_g\}_{g \in G_\beta} \cup \{P_e : e \in \text{in}_D(X_\beta)\}\) has been defined, then we obtain \(P_\beta+1\) in the following way. Let \(P = \{P'_g\}_{g \in G'} \cup \{P'_e : e \in \text{in}_D(X_{\beta+1})\}\) be an arbitrary path-system that shows the \(t\)-goodness of \(X_{\beta+1}\). Throw away the elements of \(G'\) that are spanned by \(G_\beta\) and take the union of \(G_\beta\) and the reminder of \(G'\) to obtain \(G_{\beta+1}\). For \(g \in G_\beta\) we keep the path \(P_g\) unchanged. Observe that for \(g \in G_{\beta+1} \setminus G_\beta\), the path \(P'_g \in P\) may not start inside \(X_\beta\), because then \(G_\beta\) would span \(g\) since \(G_\beta\) is a generator for \(S(X_\beta)\). For \(g \in G_{\beta+1} \setminus G_\beta\), let \(e_g\) be the first edge of \(P'_g\) that enters \(X_\beta\). We obtain \(P_g\) as a concatenation of paths \(P'_g[\text{start}(P'_g), \text{head}(e_g)]\) and \(P_{e_g} \in P_\beta\). We do the same terminal segment replacement process with all the paths \(\{P'_e : e \in \text{in}_D(X_{\beta+1})\}\) as well for getting \(\{P_e : e \in \text{in}_D(X_{\beta+1})\}\). Note that the resulting system \(P_{\beta+1}\) is really edge-disjoint.

Let \(\beta \leq \alpha\) be a limit ordinal. Observe that \(G_\beta := \bigcup_{\gamma < \beta} G_\gamma\) is a generator system for \(S(X_\beta)\). Indeed, if \(i \in S(X_\beta)\), then \(i \in S(X_\gamma)\) for some \(\gamma < \beta\), hence \(i \in \text{span}(G_\gamma) \subseteq \text{span}(G_\beta)\). If \(e \in \text{in}_D(X_\beta)\), then \(e \in \text{in}_D(X_\gamma)\) for some \(\gamma < \beta\), thus \(P_e\) has already been defined, as well as the paths \(\{P_b\}_{b \in G_\beta}\). Furthermore, the path-system \(P_\beta := \{P_e\}_{e \in \text{in}_D(X_\gamma)} \cup \{P_b\}_{b \in G_\beta}\) is obviously edge-disjoint since any two elements of \(P_\gamma\) for some \(\gamma < \beta\).

7.4.4 Proposition. If \(X\) and \(Y\) are \(t\)-good sets, then \(X \cup Y\) is a \(t\)-good set as well.

Proof: Let \(P = \{P_b\}_{b \in B_X} \cup \{P_e\}_{e \in \text{in}_D(X)}\) and \(Q = \{Q_b\}_{b \in B_Y} \cup \{Q_e\}_{e \in \text{in}_D(Y)}\) be path-systems that show the \(t\)-goodness of \(X\) and \(Y\), respectively. Note that all the common edges of the two path-systems are lie inside \(X \cap Y\). Let us define \(B'_Y = \{b \in B_Y : b \notin \text{span}(B_X)\}\). For \(s \in B'_Y \cup [\text{in}_D(Y) \cap \text{in}_D(X \cup Y)] \setminus \text{in}_D(X)\) let \(R_s\) be the path that we obtain by taking the terminal segment of \(Q_s\) up to the first vertex in \(X\) and concatenate it with \(P_e\) where \(e\) is the last edge of this
terminal segment. The path-system

\[ \{P_s : s \in B_X \cup (\text{in}_D(X) \cap \text{in}_D(X \cup Y)) \} \cup \{R_s : s \in B'_X \cup (\text{in}_D(Y) \cap \text{in}_D(X \cup Y)) \} \backslash \text{in}_D(X) \]

shows that \( X \cup Y \) is \( t \)-good. ■

Proposition 7.4.3 and 7.4.4 imply that the union of arbitrary many \( t \)-good sets is \( t \)-good thus the union of all of them (it is not an empty union because \{t\} is in it) as well. ■ ■

Our main tool to characterize the infeasible \((i,e)\)-extensions is the following theorem.

7.4.5 Theorem. If the \((I,t)\)-linkage \( \mathcal{P} = \{P_i\}_{i \in I} \) does not satisfy the complementarity conditions with the largest \( t \)-good set \( T \), then there is a \( t \)-linkable \( I' \) for which \( \text{span}(I') \supseteq \text{span}(I) \).

Proof. Assume that \( \mathcal{P} \) and \( T \) do not satisfy the complementarity conditions. We need to find the desired \( I' \). For \( i \in I \setminus S(T) =: I_{\text{out}} \) let the first edge of \( P_i \) that enters into \( T \) be \( e_i \). First we show that we may suppose without loss of generality that there is a path-system \( \{P_i\}_{i \in B} \cup \{P_e\}_{e \in \text{in}_D(T)} \) such that

1. \( \{P_i\}_{i \in B} \cup \{P_e\}_{e \in \text{in}_D(T)} \) shows the \( t \)-goodness of \( T \),
2. \( B \subseteq I \) and \( \{P_i\}_{i \in B} \subseteq \mathcal{P} \),
3. for \( i \in I_{\text{out}} \) we have \( A(P_i) \cap \text{in}_D(T) = \{e_i\} \), and \( P_i[\text{head}(e_i), t] = P_{e_i} \).

Indeed, otherwise let \( J \) be a maximal \( I/S(T) \)-independent subset of \( I_{\text{out}} \) and for \( j \in J \) take the segments \( \{P_j[\text{start}(P_j), \text{head}(e_j)]\}_{j \in J} \) from \( \mathcal{P} \) and extend it to an \((J \cup B,t)\)-linkage \( \mathcal{Q} \) by using the \( t \)-goodness of \( T \). Clearly \( I \subseteq \text{span}(J \cup B) \).

We may assume that \( \text{span}(I) = \text{span}(J \cup B) \) otherwise we are done with the proof of Theorem 7.4.5. We check that \( A(\mathcal{Q}) \cap \text{in}_D(T) = \{e_j\}_{j \in J} \subseteq \text{in}_D(T) \) by applying the fact that \( \mathcal{P} \) and \( T \) do not satisfy the complementarity conditions. Assume that the first complementarity condition fails for \( \mathcal{P} \) and \( T \). We know \( S(X) \subseteq \text{span}(I) \) because of \( \text{span}(I) = \text{span}(J \cup B) \). Thus \( \{e_j\}_{j \in J} = \text{in}_D(T) \) would
mean that complementarity conditions hold for $P$ and $T$ which is not the case. Finally the edges $\{e_j\}_{j \in J} \setminus \text{in}_D(T)$ are unused by $Q$, hence $Q$ and $T$ do no satisfy the complementarity conditions either. If $P$ and $T$ satisfy the first complementarity condition, then by using the alternative formulation of complementarity conditions 2,3,4 we obtain that
\[
\text{in}_D(X) \setminus A_{\text{last}}(\{P_i[\text{start}(P_i), \text{head}(e_i)]\}_{i \in I_{\text{out}}}) \neq \emptyset.
\]
These edges will be unused by $Q$.

Let us denote $\{i \in S : I + i \in I\} = I \cup (S \setminus \text{span}(I))$ by $I^*$. We build an auxiliary digraph by extending $D$. Pick the new vertices $\{u_i\}_{i \in I^*}, \{w_i\}_{i \in S}$ and $s$ and draw the following additional edges

1. $\{su_i : i \in I^*\}$,
2. $\{u_iw_i : i \in I^*\}$,
3. $\{w_iv : i \in S \land v \in \pi(i)\}$.

We denote the resulting digraph by $D_0^+ = (V^+, A_0)$. For $i \in I$ we extend the path $P_i$ with the new initial vertices $s, u_i, w_i$ to obtain the $s \to t$ path $P_i^+$ in $D_0^+$. Let $P^+ = \{P_i^+\}_{i \in I}$. Finally, change the direction of the edges in $A(P^+)$ to obtain $D_0^*$. We called these redirected edges backward edges and the others forward edges. Let $U_0^+$ be the set of vertices of $D_0^*$ that are unreachable from $s$ and let $U_0 = U_0^+ \cap V$.

Assume first that $t \notin U_0$ i.e. there is an $s \to t$ path $P^+$ in $D_0^+$. Let its first edge be $su_{i_0}$. Note that $i_0 \in I^* \setminus I$. Use the standard augmentation path (augmentation walk) technique (see Theorem 5.0.12 with free matroids) to obtain a system of edge-disjoint $s \to t$ paths $\{Q_i^+\}_{i \in I+{i_0}}$ in $D_0^+$ where the first edge of $Q_i^+$ is $su_i$. By cutting off the three initial vertices of the paths $Q_i^+$ we obtain a system of edge-disjoint paths $Q = \{Q_i\}_{i \in I+\{i_0\}}$ in $D$ such that for any $i \in I \cup \{i_0\}$, path $Q_i$ goes from $\pi(i)$ to $t$, i.e. we get an $(I + i_0, t)$-linkage.

Suppose that $t \in U_0$. Clearly the paths in $P^+$ use all the edges in $\text{in}_{D_0^+}(U_0^+)$ and none of the edges in $\text{out}_{D_0^+}(U_0^+)$. Therefore the same holds for $P$ with respect to
We claim that $T \subseteq U_0$. Assume, to the contrary, that we have a strict $s \to T$ path $P^+$ in $D_0^+$ with last edge $f$. It follows from our additional assumptions in the first paragraph that $\mathcal{P}$ does not use any edge from $\text{out}_D(T)$, thus $f$ cannot be a backward edge. If $f \in \text{in}_D(T) \setminus A(\mathcal{P})$, then path $P_f \in \{P_v \mid v \in \text{in}_D(T)\}$ would show the reachability of $t$ from $s$ in $D_0^+$ contradicting $t \in U_0$. Finally, suppose that $f = w_iv$ for some $v \in T$. Then $i \in S(T)$ and therefore $i \in \text{span}(B)$. For $j \in S \setminus I^*$ the vertex $w_j$ has no ingoing edge in $D_0^+$ hence we know that $i \in I^*$. Thus $I + i$ is independent hence $B + i$ as well. It follows that necessarily $i \in B$. But then the unique ingoing edge of $w_i$ in $D_0^+$ comes from $\text{start}(P_i) \in T$ and it contradicts the strictness of the $s \to T$ path $P^+$.

Let $I_{\text{in},0} = I \cap S(U_0)$ and let $I_{\text{out},0} = I \setminus S(U_0)$. We claim that for $i \in I_{\text{in},0}$ we have $\text{start}(P_i) \in U_0$. Indeed, otherwise the backward edge $\text{start}(P_i)w_i$ and any forward edge $w_iv$ with $v \in \pi(i) \cap U_0$ would lead to a contradiction with the definition of $U_0^+$. Since $A(\mathcal{P}) \cap \text{out}_D(U_0) = \emptyset$ we obtain that the path-system $\{P_i\}_{i \in I_{\text{in},0}}$ lies in $D[U_0]$. Then clearly the paths $\{P_i\}_{i \in I_{\text{out},0}}$ have to use all the edges in $\text{in}_D(U_0)$ because $\text{in}_D(U_0) \subseteq A(\mathcal{P})$. It follows that each of them uses exactly one such an edge, thus $\mathcal{P}$ and $U_0$ satisfy all but possibly the first complementary conditions.

Let us define $F_0 := S(U_0) \setminus \text{span}(I_{\text{in},0})$. Observe that $F_0 \neq \emptyset$, otherwise the first complementarity condition would hold for $\mathcal{P}$ and $U_0$ and hence $U_0$ would be a $t$-good set with $U_0 \supseteq T$ (clearly $U_0 \neq T$, since $\mathcal{P}$ and $T$ do not satisfy the complementarity conditions by assumption) which contradicts the choice of $T$.

We know that $S(U_0) \subseteq \text{span}(I)$, since for $i \in S \setminus \text{span}(I)$ the path $s, u_i, w_i, v$ where $v \in \pi(i)$ shows that $\pi(i) \cap U_0 = \emptyset$ i.e. $i \notin S(U_0)$. Fix a well-ordering of $I_{\text{out},0}$. For $i \in F_0$ let $s_0(i)$ be the smallest element of $I_{\text{out},0} \cap C(i, I)$. Extend $D_0^+$ with the new edges $\{u_{s_0(i)}w_i : i \in F_0\}$ to obtain $D_1^+ = (V^+, A_1)$. We get $D_1^+$ by changing the direction of edges in $A(\mathcal{P})$ in $D_0^+$. Assume first that there is some $s \to t$ path $P^+$ in $D_1^+$. For the first edge $su_{i_0}$ of $P^+$ we have $i_0 \in I^+ \setminus I$. Consider $S_{\text{rep}} := \{i \in F_0 : u_{s_0(i)}w_i \in A(P^+)\}$ and take the smallest element $s_0(i)$ of $s_0[S_{\text{rep}}]$. The set $(I + i_0) - s_0(i) + i$ is independent, spans $I + i_0$, and the remaining elements of $s_0[S_{\text{rep}}]$ have the same fundamental circuit on it as on $I$. We can do recursively.
in increasing order the other replacements thus $I' := (I + i_0) + S_{\text{rep}} - s_0[S_{\text{rep}}]$ is independent and spans $I + i_0$. Applying $P^+$ in the augmentation path method results in a desired $(I', t)$-linkage.

Assume that such a $P^+$ does not exist. Let $U_1^+ \ni t$ be the set of the vertices of $D_1^+$ that are not reachable from $s$ and let $U_1 = U_1^+ \cap V$. Because of the new edges, $U_1^+ \subseteq U_0^+$ holds. Observe that the vertices $\{w_i\}_{i \in S(T)}$ did not get any new ingoing edge ($B$ ensures $S(T) \cap F_0 = \emptyset$) hence $T \subseteq U_1$ follows in the same way as we proved $T \subseteq U_0$. Let us define $I_{\text{in},1} = I \cap S(U_1)$ and $I_{\text{out},1} = I \setminus S(U_1)$. The complementarity conditions hold for $P$ and $U_1$ except the first which may not, and $S(U_1) \subseteq \text{span}(I)$ holds. The proof of these facts are the same as for $U_0$. Note that the new edges ensure that $F_0 \cap S(U_1) = \emptyset$ hence for $F_1 := S(U_1) \setminus \text{span}(I_{\text{in},1})$ we have $F_0 \cap F_1 = \emptyset$. Let us extend the well-ordering of $I_{\text{out},0}$ to a well ordering of $I_{\text{out},1}$ in such a way that $I_{\text{out},1} \setminus I_{\text{out},0}$ is a terminal segment in it. This choice ensures that for an edge $u_{s_0(i)} w_i$ the element $s_0(i)$ be the is the smallest in $I_{\text{out},1} \cap C(i, I)$, not just in $I_{\text{out},0} \cap C(i, I)$. For $i \in F_1$ let $s_1(i)$ the smallest element of $I_{\text{out},1} \cap C(i, I)$. We obtain $D_2^*$ from $D_1^+$ by adding the new edges $\{u_{s_1(i)} w_i : i \in F_1\}$.

We define the corresponding notions $D_2^*, U_2^+, U_2, I_{\text{in},2}, I_{\text{out},2}, F_2$ and continue the process recursively. Suppose, to the contrary, that we do not find a desired $I'$. Let us define $D_\omega^+ = (V^+, \bigcup_{n<\omega} A_n)$ and the corresponding notions as earlier. Note that $U_\omega = \bigcap_{n<\omega} U_n \supseteq T$ and it satisfies all but the first complementarity conditions (thus $F_\omega \neq \emptyset$) with $P$ because of the usual reasons. Obviously $I_{\text{in},\omega} \subseteq \bigcap_{n<\omega} I_{\text{in},n}$ but in fact $I_{\text{in},\omega} = \bigcap_{n<\omega} I_{\text{in},n}$ holds. Indeed, $i \in \bigcap_{n<\omega} I_{\text{in},n}$ implies that $P_i$ lies in $\bigcap_{n<\omega} U_n = U_\omega$ and start($P_i$) $\in \pi(i)$ shows $i \in I_{\text{in},n}$. Let $i \in F_\omega = S(U_\omega) \setminus \text{span}(I_{\text{in},\omega})$ be arbitrary. Then $i \notin F_n$ for all $n < \omega$ otherwise by the new edges we would have $i \notin S(U_{n+1}) \supseteq S(U_\omega)$. On the other hand, $i \in S(U_\omega) \subseteq S(U_n)$ and by putting these together we obtain $i \in \text{span}(I_{\text{in},n})$ for all $n < \omega$. We can not have $i \in I_{\text{in},n}$ for all $n < \omega$ since then $i \in I_{\text{in},\omega}$ would follow. Suppose that $i \notin I_{\text{in},n}$ if $n > n_0$. Thus for any $n > n_0$ we have $(C(i, I) - i) \subseteq I_{\text{in},n}$ but then

$$(C(i, I) - i) \subseteq \bigcap_{n < n_0} I_{\text{in},n} = \bigcap_{n < \omega} I_{\text{in},n} = I_{\text{in},\omega}$$

witnesses $i \in \text{span}(I_{\text{in},\omega})$, hence $i \notin F_\omega$ and thus $F_\omega = \emptyset$ which is a contradiction.
Now we are able to prove the characterization of the infeasible \((i, e)\)-extensions.

**7.4.6 Lemma.** The \((i_0, e_0)\)-extension is infeasible if and only if \(e_0\) enters into some \(i_0\)-dangerous set.

**Proof:** We have already checked the “if” part so now we prove the other direction. Assume that vertex \(t\) witnesses the failure of the linkage condition in the \((i_0, e_0)\)-extension \(R_1\) of \(\mathcal{R}\). We claim that the largest \(t\)-good set \(X\) with respect to \(R_1\) is a desired \(i_0\)-dangerous set (with respect to \(\mathcal{R}\)). Let \(P = \{P_b\}_{b \in B_{\text{out}}}\) be a reduced linkage for \(X\) with respect to \(\mathcal{R}\), i.e. a strict \((B_{\text{out}}, X)\)-linkage where \(B_{\text{out}}\) is a base of \(\mathcal{N}(t)/S(X)\). Note that \(B_{\text{out}}\) contains a base \(B'_{\text{out}}\) of \(\mathcal{N}(t)/S_{R_1}(X)\). Let \(P' = \{P_b\}_{b \in B'_{\text{out}}}\). Clearly \(e_0 \in \mathcal{A}(P')\), otherwise \(P'\) is a \((B'_{\text{out}}, X)\)-linkage with respect to \(R_1\) as well from which one can get a \((B, t)\)-linkage with respect to \(\mathcal{R}\) where \(B\) is a base of \(\mathcal{N}(t)\) by using the \(t\)-goodness of \(X\). Suppose that \(e_0 \in \mathcal{A}(P_{b_1})\) for a \(b_1 \in B'_{\text{out}}\). Then we are able to construct a strict \((B - b_1, t)\)-linkage \(L\) with respect to \(\mathcal{R}_1\) from \(P' \setminus \{P_{b_1}\}\) via \(t\)-goodness as above. By Fact 0.3.1, an augmentation of this linkage in the sense of Theorem 7.4.5 would lead to a linkage for \(t\) with respect to \(\mathcal{R}_1\), which is impossible; therefore, by Theorem 7.4.5, the linkage \(L\) satisfies the complementarity conditions with \(X\), hence \(P' \setminus \{P_{b_1}\}\) needs to use all the edges in \(\text{in}_{D - e_0}(X)\). The only way for this to be true is if \(e_0\) is the last edge of \(P_{b_1}\). \(A_{\text{last}}(P') = \text{in}_{D}(X)\) and \(P' = P\). Thus \(e_0 \in \text{in}_{D}(X)\) and \(X\) is tight with respect to \(\mathcal{R}\). Furthermore, \(P' = P\) implies \(S(X) = S_{R_1}(X)\), hence \(i_0 \in \text{span}(S(X))\). Therefore, \(X\) is \(i_0\)-dangerous. □

### 7.5 New matroid-rooted digraphs from tight sets

In finite combinatorics, it is a common proof technique to subdivide the problem into smaller sub-problems by using an appropriate notion of tightness and then solve the smaller sub-problems by induction from which one can obtain a solution for the original problem. Unfortunately, in infinite combinatorics usually the resulting sub-problems are no longer “smaller” in any sense that would make
possible such an induction. Even though they do not lead to such an immediate success, the investigation of them could be fruitful, as happened in this topic.

Through this chapter we have some fixed matroid-rooted digraph $\mathcal{R}$ that satisfies independence and linkage condition.

**7.5.1 Claim.** If $X$ is a tight set and $\{P_b\}_{b \in B_0}$ is a linkage for $Z$ where $\emptyset \neq Z \subseteq X$, then

1. the set $B^* := B_0 \cap S(X)$ is a base of $N(Z) \cap \text{span}(S(X))$,
2. $P_b$ is a path of $D[X]$ for $b \in B^*$,
3. for $b \in B_0 \setminus B^*$ we have $|A(P_b) \cap \text{in}_D(X)| = 1$,
4. all the edges $\{e \in \text{in}_D(X) : \text{head}(e) \in \text{to}_D(Z)\}$ are used by the paths $\{P_b\}_{b \in (B_0 \setminus B^*)}$.

**Proof:** Pick a linkage $\{Q_b\}_{b \in B_1}$ for $X$ and let $B_0 \subseteq B \subseteq (B_0 \cup B_1)$ a base of $N(X)$. Note that if $A(P_{b_0}) \cap A(Q_{b_1}) \neq \emptyset$ for some $b_0 \in B_0$ and $b_1 \in B_1 \setminus B_0$, then $b_1 \in N(Z)$ and therefore $b_1 \in \text{span}(B_0)$. For $b \in B \setminus B_0$ let $P_b = Q_b$, then $\mathcal{P}' := \{P_b\}_{b \in B}$ is a $(B, X)$-linkage. Suppose, seeking for contradiction, that $B^*$ is not a base of $N(Z) \cap \text{span}(S(X)) \subseteq \text{span}(B_0)$.

Pick an $i \in (N(Z) \cap \text{span}(S(X))) \setminus \text{span}(B^*)$ and some

$$j \in C(i, B_0) \setminus B^* = C(i, B) \setminus B^*.$$

$B - j + i$ is a base of $N(X)$ and since $i \in \text{span}(S(X))$ for a suitable $k \in S(X)$ the set $B - j + k$ as well (by Corollary 0.3.9 with $I := B - j + i$ and $J := S(X)$). Note that $\text{start}(P_j) \notin X$ because $j \in B_0 \setminus B^*$ and therefore $A(Q_i) \cap \text{in}_D(X) \neq \emptyset$. But then we may replace $P_j$ by a trivial path $P_k$ (consisting of a single vertex from $\pi(k) \cap X$ in $\mathcal{P}'$ and the new linkage does not use the edges $A(P_j) \cap \text{in}_D(X) \neq \emptyset$ which contradicts the tightness of $X$.

If for some $b \in B^*$ path $P_b$ is not entirely in $X$, then it uses some element of $\text{in}_D(X)$. Replace $P_b$ by a trivial path consisting of an element of $\pi(b) \cap X$ to get a contradiction as above.
Assume that for some $b \in B_0$ path $P_b$ uses more than one ingoing edge of $X$, then we may replace it in $Q$ by its own initial segment up to the head of its first edge in $\text{in}_D(X)$ and get contradiction.

If the linkage $\mathcal{P} = \{P_b\}_{b \in B_0}$ does not use all the edges $\{e \in \text{in}_D(X) : \text{head}(e) \in \text{to}_D(Z)\}$, then the linkage $\mathcal{P}' = \{P_b\}_{b \in B}$ does not use these edges either (since for $b \in B \setminus B_0$ their heads may not even be reachable from $\pi(b)$) which contradicts the tightness of $X$.

7.5.2 Corollary. Under Condition 7.3.5 (Condition 7.3.6), for a tight set $X$, a forward-infinite (backward-infinite) path $P$ of $D[X]$ may not be reachable in $D$ from outside $X$ (equivalently from $\{\text{head}(e) : e \in \text{in}_D(X)\}$).

Proof: Since

$$\mathcal{N}(V(P)) = \text{span}(S(V(P))) \subseteq \mathcal{N}(V(P)) \cap \text{span}(S(X)),$$

by applying the first statement of Claim 7.5.1 with $Z := V(P)$ we obtain $B^* = B_0$ thus $B_0 \setminus B^* = \emptyset$. Hence the Corollary follows from the fourth statement of Claim 7.5.1.

For a tight set $X$ let $\mathfrak{R}[X]$ be the matroid-rooted digraph with $D_{\mathfrak{R}[X]} = D[X]$, $\mathcal{M}_{\mathfrak{R}[X]} = S(X) \oplus \{i_e : e \in \text{in}_D(X)\}$ where $i_e$ are some new elements, distinct from the elements of $S$, and we consider $\{i_e : e \in \text{in}_D(X)\}$ as a free matroid. Finally, let $\pi_{\mathfrak{R}[X]}(i) = \pi(i) \cap X$ for $i \in S(X)$ and let $\pi_{\mathfrak{R}[X]}(i_e) = \{\text{head}(e)\}$ for $e \in \text{in}_D(X)$.

7.5.3 Observation. For $U \cup \{i\} \subseteq S(X)$ we have $i \in \text{span}(S(U))$ iff

$$i \in \text{span}_{\mathcal{M}_{\mathfrak{R}[X]}}(S_{\mathfrak{R}[X]}(U)).$$

Applying Claim 7.5.1 we prove some basic facts related to $\mathfrak{R}[X]$.

7.5.4 Proposition.

1. $\mathfrak{R}[X]$ satisfies linkage condition and independence,

2. $\text{span}(\mathcal{N}(Z) \cap S(X)) = \mathcal{N}(Z) \cap \text{span}(S(X))$ ($Z \subseteq X$),
3. $\mathcal{N}_{\mathcal{R}[X]}(Z) = \mathcal{N}(Z) \cap \mathcal{S}(X) \cup \{i_e : e \in \text{in}_D(X) \land \text{head}(e) \in \text{to}_D(Z)\}$ ($Z \subseteq X$).

Proof: Let $v \in X$ be arbitrary and pick a linkage $\{P_b\}_{b \in B}$ for $v$. Take the terminal segments of paths $P_b$ from their first vertex in $X$. Claim 7.5.1 and the definition of $\mathcal{R}[X]$ ensure that the result is a linkage for $v$ with respect to $\mathcal{R}[X]$. The independence preserving part follows from the fact that the circuits of $\mathcal{M}_{\mathcal{R}[X]}$ are exactly those circuits of $\mathcal{M}$ that lie in $\mathcal{S}(X)$ and for $Z \subseteq X$ we have $\mathcal{S}(Z) = \mathcal{S}_{\mathcal{R}[X]}(Z) \cap \mathcal{S}(X)$. Thus if we have an $\mathcal{M}_{\mathcal{R}[X]}$-circuit $C \subseteq \mathcal{S}_{\mathcal{R}[X]}(v)$ for some $v \in X$, then $C \subseteq \mathcal{S}(v)$ and $C$ would be an $\mathcal{M}$-circuit as well.

Assume that $i \in \text{span}(\mathcal{N}(Z) \cap \mathcal{S}(X))$. Then by monotonicity $i \in \text{span}(\mathcal{S}(X))$ and $i \in \text{span}(\mathcal{N}(Z)) = \mathcal{N}(Z)$, i.e. $i \in \mathcal{N}(Z) \cap \text{span}(\mathcal{S}(X))$. Suppose now $i \in \mathcal{N}(Z) \cap \text{span}(\mathcal{S}(X))$. By the first statement of Claim 7.5.1 we know that there is a base $B^* \subseteq \mathcal{S}(X) \cap \mathcal{N}(Z)$ of $\mathcal{N}(Z) \cap \text{span}(\mathcal{S}(X))$, hence

$$i \in \text{span}(B^*) \subseteq \text{span}(\mathcal{N}(Z) \cap \mathcal{S}(X)).$$

At the third statement of this Proposition, the inclusion “$\subseteq$” is straightforward. The linkage $\{P_b\}_{b \in B^*}$ from the first two statement of Claim 7.5.1 ensures $\mathcal{N}_{\mathcal{R}[X]}(Z) \supseteq \mathcal{N}(Z) \cap \mathcal{S}(X)$ and the last two statements of Claim 7.5.1 show

$$\mathcal{N}_{\mathcal{R}[X]}(Z) \supseteq \{i_e : e \in \text{in}_D(X) \land \text{head}(e) \in \text{to}_D(Z)\}. \quad \blacksquare$$

7.5.5 Proposition. If $X$ is a tight set and $Z \subseteq X$, then $Z$ is tight with respect to $\mathcal{R}$ iff $Z$ is tight with respect to $\mathcal{R}[X]$.

Proof: Suppose that $Z \neq \emptyset$ is not tight with respect to $\mathcal{R}$ and let $\mathcal{P} = \{P_b\}_{b \in B_0}$ be a linkage for $Z$ such that for some $f \in \text{in}_D(Z)$ we have $f \notin A(\mathcal{P})$. Let $q_b$ be the first vertex of $P_b$ in $X$. We show that the paths $\{P_b[q_b, \text{end}(P_b)]\}_{b \in B_0}$ witnesses that $Z$ is not tight with respect to $\mathcal{R}[X]$. By Claim 7.5.1 we know that these are really paths in $D[X]$. Let $B^* = \{b \in B_0 : \text{start}(P_b) = q_b\}$. According to Claim 7.5.1 $B^*$ is a base of $\mathcal{N}(Z) \cap \text{span}(\mathcal{S}(X))$. By the forth statement of Claim 7.5.1 the set of the last edges of the paths $\{P_b[\text{start}(P_b), q_b]\}_{b \in B_0 \setminus B^*}$ is $\{e \in \text{in}_D(X) : \text{head}(e) \in \text{to}_D(Z)\} =: A_0$. For $e \in A_0$ let $P_e = P_b[q_b, \text{end}(P_b)]$ where $e$ is the last
edge of $P_b[\text{start}(P_b), q_b]$. The third statement of Proposition 7.5.4 ensures that the linkage

$$\{P_b\}_{b \in B^*} \cup \{P_i\}_{i \in A_0}$$

corresponds to a base of $\mathcal{N}_R[X](Z)$. This linkage clearly does not use $f \in \text{in}_D(Z)$ but we are not done yet since we need to show $f \in \text{in}_D(X) \cap \text{in}_D(Z)$. Suppose, to the contrary, that $f \notin \text{in}_D(X)(Z)$, then necessarily $f \in \text{in}_D(X) \cap \text{in}_D(Z)$. But then by the last statement of Claim 7.5.1 we obtain $f \in A(P)$ contradicting to the choice of $f$. This completes the proof of the “only if” part of the statement.

The proof of the other direction is very similar hence let us give just a sketch. Take a linkage for $Z$ which witnesses the untightness of $Z$ with respect to $\mathcal{R}[X]$. Then give a backward continuation for its paths in the form $P_i$ by using an arbitrary linkage for $Z$ with respect to $\mathcal{R}$. The resulting linkage for $Z$ with respect to $\mathcal{R}$ shows untightness of $Z$ with respect to $\mathcal{R}$. ■

Observation 7.5.3 leads to the following consequence of the Proposition above.

**7.5.6 Corollary.** If $X$ is a tight set, $Z \subseteq X$ and $i \in S$, then $Z$ is $i$-dangerous with respect to $\mathcal{R}$ iff $Z$ is $i$-dangerous with respect to $\mathcal{R}[X]$.

**7.5.7 Claim.** If $X$ and $Y$ are tight sets with $X \cap Y \neq \emptyset$, then $X \cap Y$ is tight as well. Furthermore if $X$ and $Y$ are $i$-dangerous and $i \in \mathcal{N}(X \cap Y)$, then $X \cap Y$ is $i$-dangerous.

**Proof:** Let $\{P_b\}_{b \in B_0}$ be a linkage for $X \cap Y =: Z$. If some edge enters to $Z$, then it enters into $X$ or enters into $Y$ thus by applying the last statement of Claim 7.5.1 to $X$ with $Z$ and then to $Y$ with $Z$ we obtain that the paths $\{P_b\}_{b \in B_0}$ use all the edges in $\text{in}_D(Z)$.

Let us turn to the dangerousness part of the claim. By the first statements of Claim 7.5.1 $B^*_X := B_0 \cap S(X)$ is a base of $\mathcal{N}(Z) \cap \text{span}(S(X))$ and $B^*_Y := B_0 \cap S(Y)$ is a base of $\mathcal{N}(Z) \cap \text{span}(S(Y))$. By definition both $B^*_X$ and $B^*_Y$ need to contain a base of $S(Z)$. This two bases of $S(Z)$ must be the same since $B^*_X \cup B^*_Y$ is independent. Therefore on the one hand, $B^*_X \cap B^*_Y$ contains a base of $S(Z)$. On
the other hand, by the second statement of Claim 7.5.1 for \( b \in B^*_X \cap B^*_Y \) we have
\[
\text{start}(P_b) \in X \cap Y = Z \quad \text{and hence} \quad b \in S(Z) \quad \text{thus} \quad B^*_X \cap B^*_Y \subseteq S(Z).
\]
It follows that \( B^*_X \cap B^*_Y \) is a base of \( S(Z) \).

Assume now that \( i \in \text{span}(S(X)) \cap \text{span}(S(Y)) \) and \( i \in N(Z) \) (hence \( i \in N(X) \cap N(Y) \)). Then \( i \in \text{span}(B^*_X) \cap \text{span}(B^*_Y) \). If \( i \in B^*_X \cap B^*_Y \), then \( i \in S(Z) \) and we are done. If exactly one element of \( \{B^*_X, B^*_Y\} \) contains \( i \), then \( B^*_X \cup B^*_Y \) would contain a circuit through \( i \) which is impossible. Finally if \( i \) has a fundamental circuit on \( B^*_X \) and on \( B^*_Y \), then by the independence of \( B^*_X \cup B^*_Y \) and the weak circuit elimination (Fact 0.3.6) these two circuits must be the same and therefore lie in \( B^*_X \cap B^*_Y + i \). Hence
\[
i \in \text{span}(B^*_X \cap B^*_Y) = \text{span}(S(Z)).
\]

**7.5.8 Claim.** If the \((i,e)\)-extension of \( \mathcal{R}[X] \) is feasible, where \( i \in S \), then the \((i,e)\)-extension of \( \mathcal{R} \) is feasible as well.

**Proof:** Suppose, to the contrary, that it is not. Then by Lemma 7.4.6 \( e \) in an ingoing edge of some \( i \)-dangerous set \( Y \). Using the fact that \( e \) lies in \( X \) we have
\[
in \text{span}(S(X)) \cap \text{span}(S(Y))
\]
hence \( i \) is \( i \)-dangerous too. Edge \( e \) witnesses that \( i \in N(X \cap Y) \), thus by Claim 7.5.7 \( Z := X \cap Y \) is \( i \)-dangerous with respect to \( \mathcal{R} \) thus by Claim 7.5.6 it is \( i \)-dangerous with respect to \( \mathcal{R}[X] \) as well. But then the \((i,e)\)-extension of \( \mathcal{R}[X] \) is infeasible since \( e \in \text{in}_{D[X]}(Z) \) and \( Z \) is \( i \)-dangerous which is a contradiction. 

**7.5.9 Corollary.** Let \( S' \subseteq S \). Then a feasible \( S' \)-extension \( \mathcal{R}[X]^* \) of \( \mathcal{R}[X] \) of order \( n \) determines a unique feasible \( S' \)-extension \( \mathcal{R}^* \) of \( \mathcal{R} \) of order \( n \) characterized by the property \( \mathcal{R}^*[X] = \mathcal{R}[X]^* \).

**7.6 Augmentations at a prescribed vertex**

In this section we prove a Lemma that allows us a kind of local augmentation. The Lemma will imply immediately Theorem 7.3.7 in the case of countable \( D \) and
one can derive from it Theorem 7.3.7 itself as well without too much effort as we
will do it in the last section.

7.6.1 Lemma. Assume that \( R = (D, M, \pi) \) is independent and satisfies the linkage
condition and either Condition 7.3.5 or Condition 7.3.6. Then for any \( v \in V \) and for any \( W \subseteq S \) which is the union of finitely many components of \( M \) there
is a finite-order feasible \( W \)-extension \( R^* \) of \( R \) such that \( S_{R^*}(v) \cap W \) is a base of \( N(v) \cap W \).

Proof. Assume, to the contrary, that the lemma is false and choose an arbitrary
counterexample triple \( R = (D, M, \pi), v_0, W \). We may assume (by replacing \( R \) by
some feasible finite-order \( W \)-extension of itself) that we are not able to augmenting
at \( v_0 \) even by one. More precisely for any feasible finite-order \( W \)-extensions \( R' \)
of \( R \) we have \( S_{R'}(v_0) = S_R(v_0) \). Similarly we may suppose that \( R \) minimize the
following expression among the feasible finite-order \( W \)-extensions \( R' \) of \( R \).

\[
\min \{|A(P_i)| : \{P_i\}_{i \in B} \text{ is a reduced linkage for } v_0 \text{ with respect to } R', i_0 \in B \cap W\} 
\]

(7.1)

Let the minimum for \( R \) be taken on \( P_{i_0} \in \{P_i\}_{i \in B} \). Consider the first edge \( e_0 \)
of \( P_{i_0} \).

7.6.2 Proposition. The \((i_0, e_0)\)-extension of \( R \) is defined but not feasible.

Proof: Suppose, to the contrary, that it is undefined i.e. \( i_0 \in \text{span}(\text{head}(e_0)) \).
Then by Corollary 0.3.9 there is some \( i'_0 \in S(\text{head}(e_0)) \cap W \subseteq N(v_0) \) such that \( B - i_0 + i'_0 \) is a base of \( N(v_0) \) (which implies \( \text{head}(e_0) \neq v_0 \)). But then we may replace \( i_0 \) by \( i'_0 \) and \( P_{i_0} \) by \( P_{i'_0} := P_{i_0}[\text{head}(e_0), v_0] \) to get a contradiction with the fact that the minimum at (7.1) for \( R \) is \(|A(P_{i_0})|\). On the other hand, the
\((i_0, e_0)\)-extension cannot be feasible since otherwise the resulting extension would
have a smaller minimum showed by the linkage that we would obtain from \( \{P_i\}_{i \in B} \)
by replacing \( P_{i_0} \) with \( P_{i_0}[\text{head}(e_0), v_0] \). \( \blacksquare \)

It follows by Lemma 7.4.6 that \( e_0 \) enters into some \( i_0 \)-dangerous set \( X \).
7.6.3 Proposition. The set $X$ does not contain $v_0$.

Proof: Suppose, seeking for contradiction, that $v_0 \in X$. Then all the paths in $\{P_b\}_{b \in B}$ meet $X$. Pick a reduced linkage $\{P'_b\}_{b \in B'_X}$ for $X$ and note that if some $P'_b$ have a common edge (or just a common vertex) with a path in $\{P_b\}_{b \in B}$, then $b_0 \in \text{span}(B)$. Let $B'_X = \{b \in B_X : b \notin \text{span}(B)\}$. The set $(B \cup B'_X) \setminus \{i_0\}$ contains a base of $\mathcal{N}(X)/\mathcal{S}(X)$ since $B \cup B'_X$ clearly does and $i_0 \in \text{span}(\mathcal{S}(X))$ implies that $i_0$ is a loop in $\mathcal{N}(X)/\mathcal{S}(X)$. The path-system $(\{P_b\}_{b \in B} \setminus \{P_{i_0}\}) \cup \{P'_b\}_{b \in B'_X}$ is edge-disjoint and shows that $X$ is not tight, since the edge $e_0 \in \text{in}_D(X)$ is unused, which is a contradiction. ■

By the first statement of Proposition 7.5.4 $\mathfrak{R}[X]$ is independent and satisfies the linkage condition.

7.6.4 Claim. $\mathfrak{R}[X]$ satisfies Condition 7.3.5 (Condition 7.3.6) if $\mathfrak{R}$ does.

Proof: Under Condition 7.3.5 the bases of $\mathcal{M}$ are finite and hence a tight set may have just finitely many ingoing edges. Thus the bases of $\mathcal{M}_{\mathfrak{R}[X]}$ are finite as well and therefore $\mathfrak{R}[X]$ satisfies the first part of Condition 7.3.5.

In the case of Condition 7.3.6 observe that the independence of $\mathfrak{R}$ implies that there are no loops in $\mathcal{M}$. Thus from a component $C$ of $\mathcal{M}$ we get at most $r(C \cap \mathcal{S}(X)) \leq r(C) < \infty$ components of $\mathcal{M}_{\mathfrak{R}[X]}$. Since under Condition 7.3.6 the bases of $\mathcal{M}$ are countable, a tight set may have just countably many ingoing edges. Hence the set of the further single-element components $\{\{i_e\} : e \in \text{in}_D(X)\}$ is countable. Thus $\mathfrak{R}[X]$ satisfies the first part of Condition 7.3.6.

To show the second part of Condition 7.3.5 (Condition 7.3.6) for $\mathfrak{R}[X]$ take a forward-infinite (backward-infinite) path $P$ of $D[X]$ and let $i \in \mathcal{N}_{\mathfrak{R}[X]}(V(P))$ be arbitrary. By Corollary 7.5.2 and by the third statement of Proposition 7.5.4 (with $Z := V(P)$) we obtain

$$i \in \mathcal{N}_{\mathfrak{R}[X]}(V(P)) = \mathcal{N}(V(P)) \cap \mathcal{S}(X).$$

Since Condition 7.3.5 (Condition 7.3.6) holds for $\mathfrak{R}$ and $i \in \mathcal{N}(V(P))$ and Corollary 7.5.2 ensures

$$\mathcal{S}(V(P)) = \mathcal{S}_{\mathfrak{R}[X]}(V(P)) \subseteq \mathcal{S}(X)$$
we have
\[ i \in \text{span}(\mathcal{S}(V(P))) = \text{span}(\mathcal{S}_{\mathbb{R}[X]}(V(P))). \]
Hence by Observation 7.5.3 \( i \in \text{span}_{\mathcal{M}_{\mathbb{R}[X]}}(\mathcal{S}_{\mathbb{R}[X]}(V(P))). \)

The proof of the following sublemma based on the same idea as the analogous step in [16] but the more general circumstances (matroids and reachability based packing) makes it more technical and much longer. After proving of the sublemma we continue the proof of Lemma 7.6.1 at page 90.

7.6.5 Sublemma. Let \( v_1 \) be the last vertex of \( P_{i_0} \) in \( X \). Then \( \mathcal{R}[X], v_1, W \cap \mathcal{S}(X) = W^* \) is a counterexample for Lemma 7.6.1.

Proof: Suppose, to the contrary, that it is not, and choose a feasible finite-order \( W^* \)-extension \( \mathcal{R}[X]^* \) of \( \mathcal{R}[X] \) such that \( \mathcal{S}_{\mathbb{R}[X]}^*(v_1) \cap W^* \) is a base of \( \mathcal{N}_{\mathbb{R}[X]}(v_1) \cap W^* \). Then Corollary 7.5.9 gives a feasible finite-order \( W^* \)-extension \( \mathcal{R}^* \) of \( \mathcal{R} \) such that \( \mathcal{R}^*[X] = \mathcal{R}[X]^* \).

7.6.6 Proposition. There is some \( i_1 \in \mathcal{S}_{\mathbb{R}^*}(v_1) \cap W^* \) for which \( B' := B - i_0 + i_1 \) is a base of \( \mathcal{N}(v_0)/\mathcal{S}(v_0) \).

Proof: By the \( i_0 \)-dangerousness of \( X \) we have \( i_0 \in \text{span}(\mathcal{S}(X)) \) and path \( P_{i_0} \) shows \( i_0 \in \mathcal{N}(v_1) \). Thus by applying Proposition 7.5.4 with \( Z := \{v_1\} \) we obtain
\[
  i_0 \in \mathcal{N}(v_1) \cap \text{span}(\mathcal{S}(X))
  = \text{span}(\mathcal{N}(v_1) \cap \mathcal{S}(X))
  = \text{span}(\mathcal{N}_{\mathbb{R}[X]}(v_1) \cap \mathcal{S}(X)).
\]
Since \( W \ni i_0 \) and any circuit through \( i_0 \) lies in \( W \) it implies
\[
i_0 \in \text{span}(\mathcal{N}_{\mathbb{R}[X]}(v_1) \cap \mathcal{S}(X) \cap W) = \text{span}(\mathcal{N}_{\mathbb{R}[X]}(v_1) \cap W^*) = \text{span}(\mathcal{S}_{\mathbb{R}[X]}^*(v_1) \cap W^*).
\]
Finally apply Corollary 0.3.9 with \( i := i_0, J := B, \) and \( J := \mathcal{S}_{\mathbb{R}^*}(v_1) \cap W^* \) and let \( i_1 \) be the resulting \( j \) of Corollary 0.3.9 .
7.6.7 Claim. There is a \((B^*, v_0)\)-linkage \(P^*\) with respect to \(\mathcal{R}^*\) such that \(i_1 \in B^*\) and \(P^* \ni P_{i_1}^* := P_{i_0}[v_1, v_0]\). Hence (7.1) is smaller for \(\mathcal{R}^*\) than for \(\mathcal{R}\) (which is a contradiction).

Let \(B' = B - i_0 + i_1\) (see Proposition 7.6.6). If the paths \(\{P_i\}_{i \in B' - i_1}\) have no edge in \(A(D) \setminus A(D_{\mathcal{R}^*}) =: A_{\text{lost}}\), then \(\{P_{i_1}^*\} \cup \{P_i\}_{i \in B' - i_1}\) shows that a desired linkage exists and we are done. We may assume that it is not the case. Remember that \(A_{\text{lost}} \subseteq A(D[X])\).

Consider the indices of those paths from \(\{P_i\}_{i \in B' - i_1}\) that meet \(X\) i.e. \(B_{\text{ess}} := \{i \in B' - i_1 : V(P_i) \cap X \neq \emptyset\}\) (the essential paths). Let us define the nonessential paths \(B_{\text{non}} := B' \setminus B_{\text{ess}}\) as well. For \(i \in B_{\text{ess}}\) we denote by \(q_i\) and \(z_i\) the first and the last vertex of \(P_i\) in \(X\) respectively. Whenever for some \(i \in B_{\text{ess}}\) the path \(P_i[q_i, z_i]\) use an edge \(e \in \text{out}_D(X)\), then there is an edge \(h_e \in \text{in}_D(X)\) of \(P_i[q_i, z_i]\) which corresponds to the first “come back” to \(X\) after \(e\) (see Figure 7.4). For all such an \(e\) we extend \(D_{\mathcal{R}^*}[X]\) with a new edge \(g(i, e)\) that goes from \(\text{tail}(e)\) to \(\text{head}(h_e)\).

Furthermore, pick a new vertex \(t\) and for all \(i \in B_{\text{ess}}\) draw an edge \(f_i\) from \(z_i\) to \(t\) to obtain \(H\). Let \(B_{\text{in}} = \{i \in B_{\text{ess}} : \text{start}(P_i) \in X\}\) and let \(B_{\text{out}} = B_{\text{ess}} \setminus B_{\text{in}}\).

Figure 7.4: The construction of the digraph \(H\). We have \(i \in B_{\text{out}}\) and \(j \in B_{\text{in}}\).

We claim that one can justify Claim 7.6.7 by proving the following Claim.

7.6.8 Claim. There is a \(B'_{\text{in}} \subseteq S(X)\) such that the set \((B' \setminus B_{\text{in}}) \cup B'_{\text{in}} = B_{\text{non}} \cup B_{\text{out}} \cup B'_{\text{in}}\) is a base of \(\mathcal{N}(v_0)/S_{\mathcal{R}}(v_0)\) and there is a system of edge-disjoint paths \(\{Q_i\}_{i \in B_{\text{out}} \cup B'_{\text{in}}\} \subset H\) such that for \(i \in B_{\text{out}}\) path \(Q_i\) goes from \(q_i\) to \(t\) and for \(i \in B'_{\text{in}}\)
it goes from \( \pi_{\mathcal{R}[X]^*}(i) \) to \( t \).

Indeed, for \( i \in B_{\text{non}} \) let \( P_i^* = P_i \) if \( i \neq i_1 \) and let \( P_{i_1}^* = P_{i_0}[v_0, v_1] \). For \( i \in B_{\text{out}} \) replace first the edges in the form \( g(j, e) \) of \( Q_i \) with the corresponding path segments \( P_j[\text{tail}(e), \text{head}(h_e)] \). Then simplify the resulting walk to a path and delete its last edge, say \( f_k \). Denote the result by \( \tilde{Q}_i \). Concatenate \( P_i[\text{start}(P_i), q_i] \) with \( \tilde{Q}_i \) and \( P_k[z_k, v_0] \) to obtain \( P_i^* \). In the case \( i \in B_{\text{in}} \) we do the same, except we need to concatenate just \( \tilde{Q}_i \) and \( P_k[z_k, v_0] \) to get \( P_i^* \). Finally \( \{P_i^*\}_{i \in B_{\text{non}} \cup B_{\text{out}} \cup B_{\text{in}}} \) is a desired linkage.

Let us define a matroid-rooted digraph that makes possible a reformulation of Claim 7.6.8. For \( j \in B_{\text{out}} \) let \( F(j) = i_e \in S_{\mathcal{M}[X]} \) where \( e \) is the unique in-going edge of \( q_j \) in \( P_j \) and let \( F \) be the identity on \( S(X) \). We define \( \mathcal{M}_\mathcal{Q} := \{S(X)/(B_{\text{non}} \cup B_{\text{out}} \cup S(v_0))\} \bigoplus F[B_{\text{out}}] \). Note that \( S_{\mathcal{M}_\mathcal{Q}} \subseteq S_{\mathcal{M}_{\mathcal{R}[X]^*}} \). Finally let \( \mathcal{Q} := (H, \mathcal{M}_\mathcal{Q}, \pi_{\mathcal{R}[X]^*}|_{S_{\mathcal{M}_\mathcal{Q}}}) \).

7.6.9 Observation. The \( \mathcal{M}_\mathcal{Q} \)-independent sets are \( \mathcal{M}_{\mathcal{R}[X]^*} \)-independent and for \( S' \subseteq S_{\mathcal{M}_{\mathcal{R}[X]^*}} \)

\[
\text{span}_{\mathcal{M}_{\mathcal{R}[X]^*}}(S') \cap S_{\mathcal{M}_{\mathcal{Q}}} \subseteq \text{span}_{\mathcal{M}_\mathcal{Q}}(S' \cap S_{\mathcal{M}_\mathcal{Q}}). \tag{7.2}
\]

For any \( T \subseteq X \) we have \( S_\mathcal{Q}(T) = \mathcal{R}[X]^*(T) \cap S_{\mathcal{M}_{\mathcal{Q}}} \) which implies by using (7.2) with \( S' := S_{\mathcal{R}[X]^*}(T) \)

\[
\text{span}_{\mathcal{M}_{\mathcal{R}[X]^*}}(S_{\mathcal{R}[X]^*}(T)) \cap S_{\mathcal{M}_{\mathcal{Q}}} \subseteq \text{span}_{\mathcal{M}_\mathcal{Q}}(S_\mathcal{Q}(T)). \tag{7.3}
\]

7.6.10 Proposition. For \( T \subseteq X \) we have \( \mathcal{N}_{\mathcal{R}[X]}(T) \cap S_{\mathcal{M}_{\mathcal{Q}}} \subseteq \mathcal{N}_{\mathcal{Q}}(T) \).

Proof: We know that \( \mathcal{N}_{\mathcal{R}[X]} = \mathcal{N}_{\mathcal{R}[X]^*} \) since \( \mathcal{R}[X]^* \) is a feasible extension of \( \mathcal{R}[X] \). Obviously \( \text{to}_{D_{\mathcal{R}[X]^*}}(T) \subseteq \text{to}_H(T) \) because \( D_{\mathcal{R}[X]^*} \) is a subdigraph of \( H \). Then by applying (7.3) with \( T := \text{to}_{D_{\mathcal{R}[X]^*}}(T) \)

\[
\mathcal{N}_{\mathcal{R}[X]}(T) \cap S_{\mathcal{M}_\mathcal{Q}} = \mathcal{R}[X]^*(T) \cap S_{\mathcal{M}_\mathcal{Q}} \subseteq \text{span}_{\mathcal{M}_{\mathcal{R}[X]^*}}[S_{\mathcal{R}[X]^*}(\text{to}_{D_{\mathcal{R}[X]^*}}(T))] \cap S_{\mathcal{M}_\mathcal{Q}} \subseteq \text{span}_{\mathcal{M}_\mathcal{Q}}[S_\mathcal{Q}(\text{to}_H(T))] = \mathcal{N}_\mathcal{Q}(T). \]

Clearly \( B_0 := F[B_{\text{ess}}] \subseteq \mathcal{N}_\mathcal{Q}(t) \) is \( \mathcal{M}_\mathcal{Q} \)-independent. In fact it is a base of \( \mathcal{N}_\mathcal{Q}(t) \). Indeed, if there is an \( \mathcal{M}_\mathcal{Q} \)-independent \( I \) with \( B_0 \not\subseteq I \subseteq \mathcal{N}_\mathcal{Q}(t) \), then we would
obtain

\[ B' \subseteq B_{non} \cup B_{out} \cup I \subseteq \mathcal{N}(v_0)/\mathcal{S}(v_0) \]

where \( B_{non} \cup B_{out} \cup I \) is \( \mathcal{N}(v_0)/\mathcal{S}(v_0) \)-independent which is impossible since \( B' \) is a base of \( \mathcal{N}(v_0)/\mathcal{S}(v_0) \). Thus an equivalent formulation of Claim 7.6.8, that we will actually prove, is the following.

**7.6.11 Claim.** There is a \((\hat{B}, t)\)-linkage with respect to \( \Omega \) where \( \hat{B} \) is a base of \( \mathcal{N}_\Omega(t) \).

**Proof:** Fix a build sequence of \( \mathcal{R}[X]^* \) from \( \mathcal{R}[X] \) and let the corresponding sequence of edges be \( \{h_m : m < M\} \). Note that \( \{h_m : m < M\} = A_{lost} \). For \( n \leq M \) we denote the extension of \( H \) with the edges \( \{h_m : n \leq m < M\} \) by \( H_n \) and we define \( \Omega_n = (H_n, \mathcal{M}_\Omega, \pi_\Omega) \). Note that \( H_M = H \) and hence \( \Omega_M = \Omega \).

**7.6.12 Observation.** If \( H_n \) contains an \( u \rightarrow v \) path and \( v \neq t \), then \( D \) as well since we can just replace the edges in form \( g(j, e) \) by the corresponding paths of \( D \).

**7.6.13 Proposition.** For all \( n \leq M \) we have \( \mathcal{N}_{\Omega_n}(t) = \mathcal{N}_\Omega(t) \).

**Proof:** Obviously \( \mathcal{N}_{\Omega_n}(t) \supseteq \mathcal{N}_\Omega(t) = \text{span}_{\mathcal{M}_\Omega}(B_0) \). Suppose, to the contrary, that \( \mathcal{N}_{\Omega_n}(t) \setminus \mathcal{N}_\Omega(t) \neq \emptyset \). Then there is some \( i \in \mathcal{N}_{\Omega_n}(t) \setminus \text{span}_{\mathcal{M}_\Omega}(B_0) \) such that \( t \) (and hence \( \{z_i\}_{i \in \mathcal{B}_{ess}} \)) is reachable from \( \pi_\Omega(i) \) in \( H_n \). Necessarily \( i \in \mathcal{S}(X) \) because \( \mathcal{S}_{\mathcal{M}_\Omega} \setminus \mathcal{S}(X) = \mathcal{F}[B_{out}] \subseteq B_0 \). Then by Observation 7.6.12 \( \{z_i\}_{i \in \mathcal{B}_{ess}} \) is reachable from \( \pi_\Omega(i) = \pi_{\mathcal{R}^{-1}}(i) \cap \mathcal{X} \) in \( D \). But then from \( \pi(i) \) as well, since all the new vertices that get \( \pi(i) \) in an extension were originally reachable from \( \pi(i) \). It follows that \( i \in \mathcal{N}(\{z_i\}_{i \in \mathcal{B}_{ess}}) \subseteq \mathcal{N}(v_0) \) because \( v_0 \) is reachable in \( D \) from any element of \( \{z_i\}_{i \in \mathcal{B}_{ess}} \). But then \( B' \cup \{i\} \subseteq \mathcal{N}(v_0)/\mathcal{S}(v_0) \) would be independent which is a contradiction since \( B' \) is a base of \( \mathcal{N}(v_0)/\mathcal{S}(v_0) \) and clearly \( i \notin B' \) since \( i \in S_{\mathcal{M}_\Omega} \) and

\[ B' \cap S_{\mathcal{M}_\Omega} = B_{in} \subseteq B_0 \subseteq \text{span}_{\mathcal{M}_\Omega}(B_0) \neq i. \]

We prove by induction that for all \( n \leq M \) there is a \((B_n, t)\)-linkage with respect to \( \Omega_n \) where \( B_n \) is a base of \( \mathcal{N}_\Omega(t) \). For \( n = M \) we will obtain a desired linkage for Claim 7.6.11.
Let us start with the case \( n = 0 \). For \( i \in B_{\text{ess}} \) consider \( P_i[q_i, z_i] \) and for any \( e \in \text{out}_D(X) \cap A(P_i[q_i, z_i]) \) replace the segment \( P_i[tail(e), head(h_e)] \) by the edge \( g(i, e) \) to obtain a path \( P^0_F(i) \) in \( H_0 \). The linkage \( P_0 = \{ P^0_i \}_{i \in B_0} \) is suitable.

Suppose that there is a \((B_n, t)\)-linkage \( P_n = \{ P^n_i \}_{i \in B_n} \) with respect to \( \Omega_n \) such that \( n < M \) and \( B_n \) is a base of \( N_{\Omega}(t) \). We need to give a desired linkage with respect to \( \Omega_{n+1} \). We may assume that for some \( j_0 \in B_n \) we have \( h_n \in A(P^n_{j_0}) \) otherwise \( \Omega_{n+1} \) satisfy the complementarity conditions since otherwise Theorem 7.4.5 provides us a desired linkage. For the last edge \( f_{i(j_0)} \) of \( P^n_{j_0} \) clearly \( z_{i(j_0)} \in T^+ \) otherwise \( f_{i(j_0)} \in \text{in}_{H_{n+1}}(T^+) \setminus A(P^n_{j_0}) \) contradicting to the complementarity conditions.

We build \( P_{n+1} \) in three steps. First let \( B^n_{\text{in}} = (B_n - j_0) \cap S_{\Omega}(T^+) \) and for \( i \in B^n_{\text{in}} \) let \( P^{n+1}_i = P^n_i \). By the first complementarity condition \( B^n_{\text{in}} \) is a \( \mathcal{M}_{\Omega} \)-base of \( S_{\Omega}(T^+) \) and these paths lie inside \( T^+ \). Let \( T = T^+ - t \) and we define \( B^n_{\text{un}} = B_n \setminus N_{\mathfrak{N}[X]}(T) \).

**7.6.14 Proposition.** \( j_0 \notin B^n_{\text{un}} \).

**Proof:** Since \( z_{i(j_0)} \in T \) the path \( P^n_{j_0} \) shows by applying Observation 7.6.12 that \( T \) is reachable from \( \pi_{\Omega}(j_0) = \pi_{\mathfrak{N}[X]}(j_0) \) in \( D \) and hence from \( \pi_{\mathfrak{N}[X]}(j_0) \) as well thus \( j_0 \in N_{\mathfrak{N}[X]}(T) \). ■

In the second step we define \( P^{n+1}_i := P^n_i \) for \( i \in B^n_{\text{un}} \). Proposition above ensures that these paths are in \( H_{n+1} \). To construct the third part take a reduced linkage \( \mathcal{R} = \{ R_i \}_{i \in B_r} \) for \( T \) with respect to \( \mathfrak{N}[X]^* \). Because of \( D_{\mathfrak{N}[X]} \) is a subdigraph of \( H_{n+1} \) the path-system \( \mathcal{R} \) is in \( H_{n+1} \). Since \( \mathfrak{N}[X]^* \) is a feasible extension of \( \mathfrak{N}[X] \) the set \( B_r \) is a base of \( N_{\mathfrak{N}[X]}(T)/S_{\mathfrak{N}[X]^*}(T) \).

**7.6.15 Proposition.** \( B_n \setminus (B^n_{\text{in}} \cup B^n_{\text{un}}) \subseteq \text{span}_{\mathcal{M}_{\Omega}}(B^n_{\text{in}} \cup (B_r \cap S_{\mathcal{M}_{\Omega}})) \).
Proof: Let \( i \in B_n \setminus (B_n^m \cup B_n^u) \) be arbitrary. Then \( i \notin \text{span}_{\mathcal{M}_Q}(B_n^m) = \text{span}_{\mathcal{M}_Q}(\mathcal{S}_\Omega(T)) \) hence by (7.3) of Observation 7.6.9 \( i \notin \text{span}_{\mathcal{M}_{R[t]}}(\mathcal{S}_\Omega(T)). \) On the other hand, \( i \in \mathcal{N}_{\mathcal{R}[t]}(T) \) because \( i \notin B_n^u \). It shows that \( i \in \mathcal{N}_{\mathcal{R}[t]}(T)/\mathcal{S}_\Omega(T) \).

Then \( i \in \text{span}_{\mathcal{M}_R[X]}(B_n^m \cup \mathcal{S}_R[X]^*(T)) \) because the choice of \( B_n \). Hence by Observation 7.6.9 \( i \in \text{span}_{\mathcal{M}_Q}(B_n^m) \). We may take a \( B_r' \subseteq B_r \cap \mathcal{S}_\Omega \) for which \( B_n^m \cup B_n^u \cup B_r' \) is a maximal \( \mathcal{M}_Q \)-independent subset of \( B_n^m \cup B_n^u \cup (B_r \cap \mathcal{S}_\Omega) \). For \( i \in B_r' \) concatenate \( R_i \) with the terminal segment of the path \( P^*_j \) that corresponds to the last edge of \( R_i \) to obtain \( P^*_j \). These paths also witnesses that \( B_r' \subseteq \mathcal{N}_\Omega(t) \) and therefore \( B_n^m \cup B_n^u \cup B_r' \) is a base of \( \mathcal{N}_\Omega(t) \) since it is independent and spans such a base namely \( B_n \).

We need to check that the paths \( \{P^*_j\}_{i \in B_r'} \) have no common edges with the paths \( \{P^n_i\}_{i \in B_n^m \cup B_n^u} \). The path-system \( \{P^n_i\}_{i \in B_n^m} \) lies in \( T^+ \) and the terminal segments of the paths \( \{P^*_i\}_{i \in B_r'} \) from the first (and only) entering to \( T^+ \) are some other elements of \( P_n \) which itself is an edge-disjoint system. Hence the path-system \( \{P^*_i\}_{i \in B_r'} \cup \{P^n_i\}_{i \in B_n^m} \) is edge-disjoint. From the definition of \( B_n^u \) it follows that

\[
\text{to}_{\mathcal{R}[t]^*}(T) \cap \bigcup_{i \in B_n^u} V(P^n_i) = \emptyset.
\]

On the other hand,

\[
\text{to}_{\mathcal{R}[t]^*}(T) \supseteq \bigcup_{i \in B_r} V(P^*_i) \setminus \{t\},
\]

thus the two paths-systems may not even have common vertex other than \( t \). Now the proof of Claim 7.6.11 is complete and hence the proof of Claim 7.6.7 and the proof of Sublemma 7.6.5 as well. \( \blacksquare \)

We continue the proof of Lemma 7.6.1. We obtained by Sublemma 7.6.5 and by Proposition 7.6.3 that if \( \mathcal{R}_0, v_0, W_0 \) is a counterexample triple, then there is a feasible finite-order \( W_0 \)-extension \( \mathcal{R}_1 \) of \( \mathcal{R}_0 \) such that there is a vertex set \( X =: \)
$X_1 \neq v_0$ which is tight with respect to $R_1$ and for a suitable $v_1 \in X_1$ the triple $R_1[X], v_1, S_{R_1}(X) \cap W_0 =: W_1$ is a counterexample again. Furthermore, we know that there is an $e_1 \in \text{in}_{D_{R_1}}(X_1)$ and there is a path, namely $P_{i_1}^*$ see Figure 7.4, that goes strictly from $X_1$ to $v_0$ and starts at $v_1$. The path $P_{i_0}$ shows that $v_1$ is reachable outside $X_1$ in $D_{R_1}$. We may apply these observations with the new counterexample triple and iterate the process recursively to get an infinite sequence of counterexample triples $\langle (R_n[X_n], W_n, v_n) : n < \omega \rangle$ with $X_0 := V$.

Here $R_{n+1}$ is a finite-order feasible $W_n$-extension of $R_n$ where the extension use edges only from $D[R_n[X_n]]$, $\langle X_n : n < \omega \rangle$ is a nested sequence of vertex sets such that $X_n$ is tight with respect to $R_n$ and $v_n \in X_n$ but $v_n \notin X_{n+1}$. We also have a path $P_n$ in $D_{R_n}$ from $X_{n+1}$ to $v_n$ with start($P_n$) = $v_{n+1}$ and some edge $e_{n+1} \in \text{in}_{D_{R_{n+1}[X_n]}(X_{n+1})}$.

If $R_0$ satisfies Condition 7.3.6, then we build a backward-infinite path $P$ by concatenating the paths $P_n$ for $n = 1, 2, \ldots$. Then $P$ lies in the $R_1$-tight $X_1$ and $V(P)$ is reachable in $D_{R_1}$ from outside $X_1$ in $D_{R_1}$ (showed by $P_n$) contradicting to Corollary 7.5.2. It proves Lemma 7.6.1 in the case when $R_0$ satisfies Condition 7.3.6.

Suppose now that $R_0$ satisfies Condition 7.3.5. The sequence $\langle \mathcal{N}(X_n) : n < \omega \rangle$ is $\subseteq$-decreasing thus $\langle r(\mathcal{N}(X_n)) : n < \omega \rangle$ is a decreasing sequence of natural numbers therefore by throwing away some initial elements we may assume that $r(\mathcal{N}(X_n))$ does not depend on $n$. On the other hand, the $(n+1)$-th extension uses edges only from $D_{R_n[X_n]}$ thus we have

$$S_{R_n}(X_n) = S_{R_{n+1}}(X_n) \supseteq S_{R_{n+1}}(X_{n+1})$$

and therefore

$$r(S_{R_n}(X_n)) \geq r(S_{R_{n+1}}(X_{n+1})).$$

But then

$$r(\mathcal{N}(X_n)/S_{R_n}(X_n)) = r(\mathcal{N}(X_n)) - r(S_{R_n}(X_n))$$

is an increasing function of $n$ (bounded by $r(M) < \infty$) hence similarly we may suppose that it is constant, say $m_0$. Pick a reduced linkage $P_1$ for $X_1$ with respect to
It consists of $m_0$ paths and these paths use all the elements of $\text{in}_{\mathcal{R}_1}(X_1) \ni e_1$ because of the tightness of $X_1$. Then pick a reduced linkage $Q$ for $X_2$ in $\mathcal{R}_2$. Observe that these paths also use all the elements of $\text{in}_{\mathcal{R}_1}(X_1) = \text{in}_{\mathcal{R}_2}(X_1)$. Take the set of the terminal segments of the elements of $Q$ from the first vertex in $X_1$ and denote it by $Q'$. From $\mathcal{P}_1$ obtain via concatenation with elements in $Q'$ a reduced linkage $\mathcal{P}_2$ for $X_2$ with respect to $\mathcal{R}_2$. Iterate the process recursively. In a general step we have a reduced linkage $\mathcal{P}_n$ for $X_n$ with respect to $\mathcal{R}_n$ and we find forward-continuations for the elements of $\mathcal{P}_n$ to obtain a reduced linkage for $X_{n+1}$ with respect to $\mathcal{R}_{n+1}$. By tightness of $X_{n+1}$ with respect to $\mathcal{R}_{n+1}$ necessarily $e_{n+1} \in A(\mathcal{P}_{n+1})$. Eventually we obtain an edge disjoint path-system $\mathcal{P}$ with $m_0$ members. Since the edges $\{e_n\}_{1 \leq n < \omega} \subseteq A(\mathcal{P})$ are pairwise distinct there is a $P \in \mathcal{P}$ that contains infinitely many of them. A terminal segment of the forward-infinite path $P$ lies inside $X_1$ and reachable from outside $X_1$ in $\mathcal{D}_{\mathcal{R}_1}$ (shown by $P$ itself) which contradicts to Corollary 7.5.2. Now the proof of Lemma 7.6.1 is complete.

7.7 Iteration of local augmentations

Now we are able to prove our main result Theorem 7.3.7. Suppose first that $V$ is countable and $V = \{v_n\}_{n < \omega}$ and the components of $\mathcal{M}$ are $\{C_n\}_{n < \omega}$. Let $\omega \times \omega = \{p_n : n < \omega\}$. We build recursively a sequence $\langle \mathcal{R}_n : n \leq \omega \rangle$ such that $\mathcal{R}_0 = \mathcal{R}$ and if $p_n = \langle m, k \rangle$, then we obtain $\mathcal{R}_{n+1}$ by applying Lemma 7.6.1 to $\mathcal{R}_n = (D_n, \mathcal{M}, \pi_n)$ with $v_m$ and $C_k$. Finally let $\mathcal{R}_\omega = (D_\omega, \mathcal{M}, \pi_\omega)$ where $D_\omega = (V, \bigcap_{n < \omega} A(D_n))$ and $\pi_\omega(i) = \bigcup_{n < \omega} \pi_n(i)$. By the construction for any $v \in V$ and any component $C$ of $\mathcal{M}$ the set $S_\omega(v) \cap C$ is a base of $\mathcal{N}(v) \cap C$ thus for all $v \in V$ the set $S_\omega(v)$ is a base of $\mathcal{N}(v)$.

In a general case we should organize the recursion more warily to ensure that after limit steps Condition 7.3.8 holds. Let $V = \{v_\xi : \xi < \kappa\}$. To obtain $\mathcal{R}_{\xi+1}$ from $\mathcal{R}_\xi$ we consider $v_\xi$ and all the finitely many vertices that lost some ingoing edge since the last limit step. We apply to these vertices $v$ one by one in an arbitrary order Lemma 7.6.1 with the smallest $n$ for which $S_\xi(v) \cap C_n$ is not a base
of $\mathcal{N}(v) \cap C_n$ (if such an $n$ does not exists for some $v$, then do nothing with that $v$). Observe that it ensures that after a limit step $\alpha$ a $v \in V$ either keeps all of its ingoing edges or $S_\alpha(v)$ is a base of $\mathcal{N}(v)$. We need to justify that at limit steps we obtain feasible extensions in the process above.

7.7.1 Proposition. Let $\alpha < \kappa$ be a limit ordinal and suppose that $\langle R_\beta : \beta < \alpha \rangle$ has been already defined during the process and the appearing extensions of $R_0 = \mathcal{R}$ are feasible. Then the limit $R_\alpha$ of the sequence is also a feasible extension of $\mathcal{R}$.

Proof: Let $v \in V$ arbitrary and pick a linkage $\{P_b\}_{b \in B}$ for $v$ with respect to $\mathcal{R}$. If some $P_b$ is not a path in $D_\alpha$ then replace it by the terminal segment $Q_b$ of itself that starts at the head $u_b$ of the last deleted edge of $P_b$ otherwise let $Q_b = P_b$ and let $u_b$ be the first vertex of $P_b$. Note that $b \in \text{span}(S_\alpha(u_b))$. It is enough to show that there is a transversal for $\{S_\alpha(u_b)\}_{b \in B}$ which is a base of $\mathcal{N}(v)$. To do so we prove that for any component $C$ of $\mathcal{M}$ there is a transversal for $\{S_\alpha(u_b)\}_{b \in B \cap C}$ which is a base of $\mathcal{N}(v) \cap C$.

Let $C$ be fixed and let $B^*_C := B \cap C = \{b_1, \ldots, b_\ell_0\}$. Pick a base $B'_C = \{b'_1, \ldots, b'_\ell_0\}$ of $\mathcal{N}(v) \cap C$ for which $b'_\ell \in \text{span}(S_\alpha(u_{b_\ell}))$ holds for all $1 \leq \ell \leq \ell_0$ and $b'_\ell \in S_\alpha(u_{b_\ell})$ for as many $\ell$ as possible. Assume, to the contrary, that $b'_{\ell_1} \notin S_\alpha(u_{b_{\ell_1}})$ for some $1 \leq \ell_1 \leq \ell_0$. The fact $b''_{\ell_1} \in \text{span}(S_\alpha(u_{b_{\ell_1}})) \setminus S_\alpha(u_{b_{\ell_1}})$ implies that there is a circuit $C \ni b'_{\ell_1}$ such that $(C \setminus \{b'_{\ell_1}\}) \subseteq S_\alpha(u_{b_{\ell_1}})$. Note that $C \subseteq C$ because $b'_{\ell_1} \in C$. Since $(C \setminus \{b'_{\ell_1}\}) \notin \text{span}(B'_C - b''_{\ell_1})$ (otherwise $b''_{\ell_1} \in \text{span}(B'_C - b'_1)$ there is some $b''_{\ell_1} \in (C \setminus \{b'_{\ell_1}\})$ for which $B'_C - b''_{\ell_1}$ is still a base of $\mathcal{N}(v) \cap C$ contradicting to the choice of $B'_C$. ■
Summary

Infinite graph theory has a great tradition in Hungary. It was one of the favourite topics of Pál Erdős. In the dissertation we are focusing on infinite generalizations of theorems in the theory of finite graphs where the proofs need new ideas well beyond the proof of the finite counterpart. In each of the seven chapters of the dissertation we present one of our results in the theory of infinite graphs.

In the first chapter we prove that any edge-weighted graph with finite total weight admits a Gomory-Hu tree. We show by an example that one can not omit the condition that the sum of the weights is finite. In the second chapter we construct for any $k \in \mathbb{N}$ an infinite $k$-edge-connected digraph in which there is no edge-disjoint back and forth directed path between some vertex pair. Note that such a finite digraph does not exist even for $k = 2$. In the third chapter we give a simple proof to a theorem of Nash-Williams. We prove that the edge set of a digraph can be partitioned into directed cycles iff for each vertex set the cardinality of the ingoing and the outgoing edges are the same. In the fourth chapter we investigate questions about the existence of $T$-joins in infinite graphs. In the fifth chapter we generalize the countable Menger’s theorem of R. Aharoni by assuming finitary matroid constraints on the ingoing edges of each vertex. In the sixth chapter we consider problems related to reachability by monochromatic paths in edge-coloured tournaments. In the last chapter we are considering branching packings in infinite digraphs. Even the finite branching packing theorems fails for infinite digraphs it turned out, that the infinite generalization is possible by adding some restrictions about the forward-infinite or backward-infinite paths.
Összefoglaló

A végtelen gráfok vizsgálatának nagy tradíciója van Magyarországon. Ez volt az egyik kedvenc területe Erdős Pálnak. Disszertációinkban főként véges gráfelméleti tételek végtelen általánosításait mutatjuk be, ahol a végtelen eset tárgyalása a véges esethez képest lényeges új gondolatokat igényel. A disszertáció hét fejezetének mindegyikében egy-egy végtelen gráfelméleti eredményünk kerül bemutatásra.

Bibliography


ADATLAP

a doktori értekezés nyilvánosságra hozatalához

I. A doktori értekezés adatai
A szerző neve: Joó Attila
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A témavezető munkahelye: Budapesti Műszaki és Gazdaságtudományi Egyetem, Rényi Alfréd Matematikai Kutatóintézet

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1. A doktori értekezés szerzőjeként hozzájárulok, hogy a doktori fokozat megszerzését követően a doktori értekezésem és a tézisek nyilvánosságra kerüljenek az ELTE Digitális Intézményi Tudástárban. Felhatalmazom a Természettudományi kar Dékáni Hivatal Doktori, Habilitációs és Nemzetközi Ügyek Csoportjának ügyintézőjét, hogy az értekezést és a téziseket feltöltse az ELTE Digitális Intézményi Tudástárba, és ennek során kitöltsze a feltöltéshez szükséges nyilatkozatokat.

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3. A doktori értekezés szerzőjeként hozzájárulok a doktori értekezés és a tézisek szövegének plágiumkereső adatbázisba helyezéséhez és plágiumellenőrző vizsgálatok lefuttatásához.


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a doktori értekezés szerzőjének aláírása