Summary of Ph.D. thesis

Rank functions and Polish groups in descriptive set theory

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1 Introduction

Descriptive set theory deals with the definable subsets of a *Polish space*, a separable, completely metrizable topological space. It has a large variety of applications in ergodic theory, functional analysis, the theory of group actions and mathematical logic, among others.

One of the main areas of descriptive set theory studies the complexity of subsets and functions. The most important notion in this area is the notion of a Borel set: a subset $B$ of a Polish space $X$ is a *Borel set* if it is contained in the $\sigma$-algebra generated by the open subsets of $X$. The Borel hierarchy arranges the Borel sets in classes according to their complexity. We define the elements of the Borel hierarchy by transfinite induction:

$$\Sigma^0_1(X)$$ is the collection of open subsets of $X$,

and for $\xi < \omega_1$,

$$\Pi^0_\xi(X) = \{X \setminus G : G \in \Sigma^0_\xi(X)\},$$

and

$$\Sigma^0_\xi(X) = \left\{ \bigcup_n F_n : F_n \in \Pi^0_{\xi_n}(X) \text{ for some } \xi_n < \xi \right\}.$$

It is well-known that a subset is Borel iff it is contained in the above hierarchy.

A real-valued function defined on the Polish space is called *Borel measurable* (or *Borel*) if the inverse image of every open set is Borel. Similarly to the above hierarchy, one can define the *Baire hierarchy* of Borel measurable, real-valued functions defined on $X$ the following way. A function $f : X \to \mathbb{R}$ is of *Baire class* 0 if it is continuous. And for $\xi < \omega_1$, $f$ is of *Baire class* $\xi$, if it is the pointwise limit of functions from smaller classes. It is well-known that a function is Borel measurable iff it is of Baire class $\xi$ for some ordinal $\xi < \omega_1$, and it is of Baire class $\xi$ iff the inverse image of every open subset of $\mathbb{R}$ is in $\Sigma^0_{\xi+1}(X)$. We denote by $B_\xi(X)$ the family of Baire class $\xi$ functions with domain $X$.

A *rank* defined on a family of functions is a map assigning a countable ordinal to each element of the family, typically measuring its complexity. We could interpret the Baire hierarchy as a rank function on the Borel measurable functions: for a Borel measurable function $f$, assign the smallest ordinal $\xi$ such that $f$ is of Baire class $\xi$. In this way, a rank function defined on the family of Baire class $\xi$ functions can be interpreted as a refinement of the Baire hierarchy.

The theory of ranks of Baire class 1 functions defined on a compact metric space was developed by Kechris and Louveau [10]. In Chapter 2 we generalize most of their results to the family of Baire class $\xi$ functions defined on an arbitrary Polish space. We use these ranks to answer a question of Elekes and Laczkovich [8] concerning the solvability cardinal of a system of difference equations, an area that is connected to paradoxical geometric decompositions. We also show that, surprisingly, some of the natural generalizations of the Baire class 1 ranks fail to have nice properties. The results of this chapter are partly joint with M. Elekes and Z. Vidnyánszky [3], [4].

Another important area of descriptive set theory deals with natural $\sigma$-ideals that help measuring the size of subsets of Polish spaces. In Chapter 3 we investigate
the structure of the random element of Polish groups. Since such a group is not necessarily locally compact, there is no natural translation invariant measure on it. However, following Christensen [5], one can define the $\sigma$-ideal of Haar null sets which is a generalization of the Haar measure zero sets from the locally compact case to the general Polish case. Roughly speaking, a subset of a Polish group $G$ is Haar null, if every translate of it is of measure zero with respect to a Borel probability measure defined on $G$. Since we are primarily interested in homeomorphism and automorphism groups, and in these groups conjugate elements are considered isomorphic, we are only interested in conjugate invariant properties of the elements. We characterize the size of conjugacy classes of certain Polish groups with respect to the $\sigma$-ideal of Haar null sets. We also show that a large number of Polish groups can be partitioned into the union of a Haar null and a meager set. These results are joint with U. B. Darji, M. Elekes, K. Kalina and Z. Vidnyánszky [2], [1].

2 Ranks on Baire class $\xi$ functions

The purpose of Chapter 2 is to extend the theory of ranks defined on Baire class 1 functions to higher Baire classes. A rank defined on a family of functions is a map assigning countable ordinals to every function from the family, typically measuring their complexity. In their seminal paper [10], Kechris and Louveau systematically investigated three very important ranks on the Baire class 1 functions defined on a compact metric space. The theory has no straightforward generalization to the case of Baire class $\xi$ functions, hence the following very natural but somewhat vague question arises.

**Question 2.1.1.** Is there a natural extension of the theory of Kechris and Louveau to the case of Baire class $\xi$ functions?

In Chapter 2 we show that some quite natural generalizations of these ranks turn out to be bounded in $\omega_1$, hence degenerate. The main tool that we use to define ranks with nice properties is topology refinement. Since Kechris and Louveau only dealt with Baire class 1 functions defined on compact metric spaces we reprove most of their results to functions defined on arbitrary Polish spaces. Then we are able to use topology refinements to obtain results concerning Baire class $\xi$ functions.

Let us define one of the ranks of Kechris and Louveau, the so-called convergence rank, to present our technique. In order to do so, we first define the notion of a derivative.

**Definition.** A derivative on the closed subsets of $X$ is a map $D : \Pi^0_1(X) \to \Pi^0_1(X)$ such that $D(A) \subset A$ and $A \subset B \Rightarrow D(A) \subset D(B)$ for every $A, B \in \Pi^0_1(X)$. For a derivative $D$ we define the iterated derivatives of the closed set $F$ as follows $D^0(F) = F$, $D^{\eta+1}(F) = D(D^\eta(F))$ and $D^\eta(F) = \bigcap_{\theta < \eta} D^\theta(F)$ if $\eta$ is a limit. The rank of $D$ is the smallest ordinal $\eta$, such that $D^\eta(X) = \emptyset$, if such ordinal exists, $\omega_1$ otherwise. We denote the rank of $D$ by $\text{rk}(D)$.
**Definition.** Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of real-valued functions on \(X\). The oscillation of this sequence at a point \(x\) restricted to a closed set \(F \subseteq X\) is

\[
\omega((f_n)_{n \in \mathbb{N}}, x, F) = \inf_{U \in \text{open}} \inf_{N \in \mathbb{N}} \sup \{|f_m(y) - f_n(y)| : n, m \geq N, y \in U \cap F\}.
\]

Now let a derivative be defined as \(D_{(f_n)_{n \in \mathbb{N}}, \varepsilon}(F) = \{x \in F : \omega((f_n)_{n \in \mathbb{N}}, x, F) \geq \varepsilon\}\), and let \(\gamma((f_n)_{n \in \mathbb{N}}) = \sup_{\varepsilon > 0} \text{rk}(D_{(f_n)_{n \in \mathbb{N}}, \varepsilon})\). For a Baire class 1 function \(f\) let the convergence rank of \(f\) be defined by

\[
\gamma(f) = \min\{\gamma((f_n)_{n \in \mathbb{N}}) : \forall n \ f_n \text{ is continuous and } f_n \to f \text{ pointwise}\}.
\]

Now let \(\xi \geq 1\) be a countable ordinal. To define a rank on the Baire class \(\xi\) functions, let \(T_{\xi} = \{\tau' : \tau' \supseteq \tau\} \subseteq \Sigma_{\xi}^0(f), f \in \mathcal{B}_1((X, \tau'))\},\text{ where } \tau\text{ denotes the original topology on } X\). Then for \(f \in \mathcal{B}_\xi(X)\) let \(\gamma_\xi(f) = \min_{\tau' \in T_{\xi}} \gamma_{\tau'}(f)\), where \(\gamma_{\tau'}(f)\) is just the \(\gamma\) rank of \(f\) in the topology \(\tau'\).

We present here a result from Chapter 2 concerning the nice properties of \(\gamma_\xi\), that answers a question of Elekes and Laczkovich from [8]. In order to be able to formulate this we need some preparation. For \(\theta, \theta' < \omega\) let us define the relation \(\theta \lessdot \theta'\text{ if } \theta' \leq \omega^n \implies \theta \leq \omega^n\text{ for every } 1 \leq n < \omega_1\) (we use ordinal exponentiation here). We will also use the notation \(\theta \approx \theta'\text{ if } \theta \lessdot \theta'\text{ and } \theta' \lessdot \theta\). Then \(\approx\) is an equivalence relation. The characteristic function of a set \(H\) is denoted by \(\chi_H\). A set is called perfect if it is closed and has no isolated points. Define the translation map \(T_t : \mathbb{R} \to \mathbb{R}\) by \(T_t(x) = x + t\) for every \(x \in \mathbb{R}\).

**Corollary 2.7.1.** The rank \(\gamma_{\xi} : \mathcal{B}_{\xi}(\mathbb{R}) \to \omega_1\) satisfies the following:

- \(\gamma_{\xi}\) is unbounded in \(\omega_1\), moreover, for every non-empty perfect set \(P \subseteq \mathbb{R}\) and ordinal \(\zeta < \omega_1\) there is a function \(f \in \mathcal{B}_{\xi}(\mathbb{R})\) such that \(f\) is 0 outside of \(P\) and \(\gamma_{\xi}(f) \geq \zeta\),
- \(\gamma_{\xi}\) is translation-invariant, i.e., \(\gamma_{\xi}(f \circ T_t) = \gamma_{\xi}(f)\) for every \(f \in \mathcal{B}_{\xi}(\mathbb{R})\) and \(t \in \mathbb{R}\),
- \(\gamma_{\xi}\) is essentially linear, i.e., \(\gamma_{\xi}(cf) \approx \gamma_{\xi}(f)\) and \(\gamma_{\xi}(f + g) \lesssim \max\{\gamma_{\xi}(f), \gamma_{\xi}(g)\}\) for every \(f, g \in \mathcal{B}_{\xi}(\mathbb{R})\) and \(c \in \mathbb{R} \setminus \{0\}\),
- \(\gamma_{\xi}(f \cdot \chi_F) \lesssim \gamma_{\xi}(f)\) for every closed set \(F \subseteq \mathbb{R}\) and \(f \in \mathcal{B}_{\xi}(\mathbb{R})\).

As pointed out in [8], a rank with such properties can be used to derive a result about the solvability of certain systems of difference equations that has connections to the theory of paradoxical decompositions.

### 2.1 Bounded Baire class \(\xi\) functions

For an ordinal \(1 \leq \lambda < \omega_1\), let \(\mathcal{B}_1^\lambda(X) = \{f \in \mathcal{B}_1(X) : f\text{ is bounded, } \gamma(f) \leq \omega^\lambda\}\). Kechris and Louveau proved that one can get the class \(\mathcal{B}_1^\lambda(X)\) from \(\bigcup_{\eta < \lambda} \mathcal{B}_1^\eta(X)\) quite naturally. Their result works only for functions defined on compact metric
spaces, and turns out to be false for arbitrary Polish spaces. However, we managed to modify their construction to obtain a similar result.

If $F$ is a class of bounded Baire class 1 functions defined on $X$ then let $\Phi(F) = \{f \in B_1(X) : f$ is bounded, $\exists(f_n)_{n \in \mathbb{N}} \in F^\mathbb{N}(\gamma((f_n)_{n \in \mathbb{N}}) \leq \omega$ and $f_n \to f$ pointwise$\}$.

Now we define inductively the families $\Phi_\lambda(X)$ of functions by $\Phi_0(X) = B_1(X)$ and for $0 < \lambda < \omega_1$, $\Phi_\lambda(X) = \Phi(\bigcup_{\eta < \lambda} \Phi_\eta(X))$.

**Theorem 2.3.55.** For every ordinal $\lambda < \omega_1$, we have $\Phi_\lambda(X) = B^{\lambda+1}_1(X)$.

Using topology refinements, we also proved an analogous result for bounded Baire class $\xi$ functions.

## 3 Random elements of Polish groups

The study of generic elements of Polish groups (in the sense of Baire category) is a flourishing field with a large number of applications (see e.g. [11], [12], [9]). It is natural to ask whether there exist measure theoretic analogues of these results. Unfortunately, on non-locally compact groups there is no natural invariant $\sigma$-finite measure. However, a generalization of the ideal of measure zero sets can be defined in every Polish group as follows:

**Definition** (Christensen, [5]). Let $G$ be a Polish group and $B \subset G$ be Borel. We say that $B$ is Haar null if there exists a Borel probability measure $\mu$ on $G$ such that for every $g, h \in G$ we have $\mu(gBh) = 0$. An arbitrary set $S$ is called Haar null if $S \subset B$ for some Borel Haar null set $B$.

It is known that the collection of Haar null sets forms a $\sigma$-ideal in every Polish group and it coincides with the ideal of measure zero sets in locally compact groups with respect to every left (or equivalently right) Haar measure. Using this definition, it makes sense to talk about the properties of random elements of a Polish group. A property $P$ of elements of a Polish group $G$ is said to hold almost surely or almost every element of $G$ has property $P$ if the set $\{g \in G : g$ has property $P\}$ is co-Haar null.

Since we are primarily interested in homeomorphism and automorphism groups, and in such groups conjugate elements can be considered isomorphic, we are only interested in the conjugacy invariant properties of the elements of our Polish groups. Hence, in order to describe the random element, one must give a complete description of the size of the conjugacy classes with respect to the Haar null ideal. The investigation of this question has been started by Dougherty and Mycielski [7] in the permutation group of a countably infinite set, $S_\infty$. If $f \in S_\infty$ and $a$ is an element of the underlying set then the set $\{f^k(a) : k \in \mathbb{Z}\}$ is called the orbit of $a$ (under $f$), while the cardinality of this set is called orbit length. Thus, each $f \in S_\infty$ has a collection of orbits (associated to the elements of the underlying set). It is easy to show that two elements of $S_\infty$ are conjugate if and only if they have the same (possibly infinite) number of orbits for each possible orbit length.
Theorem (Dougherty, Mycielski, [7]). Almost every element of $S_\infty$ has infinitely many infinite orbits and only finitely many finite ones.

Therefore, almost all permutations belong to the union of a countable set of conjugacy classes.

Theorem (Dougherty, Mycielski, [7]). All of these countably many conjugacy classes are non-Haar null.

Thus, the above theorems give a complete description of the non-Haar null conjugacy classes and the (conjugacy invariant) properties of a random element. The aim of Chapter 3 is to initiate a systematic study of the size of the conjugacy classes of homeomorphism groups of compact metric spaces and automorphism groups of countable structures.

One can prove that in $S_\infty$ a generic element has no infinite orbits. This shows that the generic and random behaviors are quite different. In particular, $S_\infty$ can be decomposed into the union of a Haar null and a meager set. Of course, this is possible in every locally compact group, but the situation is not clear in the non-locally compact case. Thus, the following question of U. B. Darji arises naturally:

Question 3.1.4. Suppose that $G$ is an uncountable Polish group. Can it be written as the union of a meager and a Haar null set?

We also investigate this question for various Polish groups.

We now state our main results.

3.1 Homeomorphism groups

Let $\text{Homeo}^+([0,1])$ denote the space of increasing homeomorphisms of the unit interval $[0,1]$ with the topology of uniform convergence. For a homeomorphism $f \in \text{Homeo}^+([0,1])$ we let $\text{Fix}(f) = \{ x \in [0,1] : f(x) = x \}$. Our theorem concerning $\text{Homeo}^+([0,1])$ is the following, which we prove in Section 3.4.

Theorem 3.4.2. The conjugacy class of $f \in \text{Homeo}^+([0,1])$ is non-Haar null if and only if $\text{Fix}(f)$ has no limit point in the open interval $(0,1)$, and if $x_0 \in \text{Fix}(f) \cap (0,1)$ then $f(x) - x$ does not have a local extremum point at $x_0$. Moreover, the union of the non-Haar null conjugacy classes is co-Haar null.

Corollary. There are only countably many non-Haar null conjugacy classes in the group $\text{Homeo}^+([0,1])$.

Cohen and Kallman [6] investigated Question 3.1.4 and developed a technique to find Haar null-meager decompositions. However, their method does not work for the group $\text{Homeo}^+([0,1])$ as they have also noted. With the help of Theorem 3.4.2, we have the following corollary.

Corollary 3.10.1. The group $\text{Homeo}^+([0,1])$ can be partitioned into a Haar null and a meager set.
In Section 3.5 we turn to the group Homeo\(^+\)(S\(^1\)) of order-preserving homeomorphisms of the circle. In what follows, \(\tau(f)\) denotes the rotation number of a homeomorphism \(f \in \text{Homeo}^+(S^1)\). If \(f \in \text{Homeo}^+(S^1)\) has finitely many periodic points then we call one of its periodic points \(x_0\) of period \(q\) crossing, if for a lift \(F\) of \(f^q\) with \(x_0 \in \text{Fix}(F)\), \(x_0\) is not a local extremum point of \(F(x) - x\). For the formal definition of these notions, see Section 3.5.

**Theorem 3.5.5.** The conjugacy class of \(f \in \text{Homeo}^+(S^1)\) is non-Haar null if and only if \(\tau(f) \in \mathbb{Q}\), \(f\) has finitely many periodic points and all of them are crossing. Every such conjugacy class necessarily contains an even number of periodic orbits, and for every rational number \(0 \leq r < 1\) and positive integer \(k\) there is a unique non-Haar null conjugacy class with rotation number \(r\) containing \(2k\) periodic orbits. Moreover, the union of the non-Haar null conjugacy classes and \(\{f \in \text{Homeo}^+(S^1) : \tau(f) \notin \mathbb{Q}\}\) is co-Haar null.

**Corollary.** There are only countably many non-Haar null conjugacy classes in \(\text{Homeo}^+(S^1)\).

We could not settle the question whether the union of the non-Haar null conjugacy classes is co-Haar null, since the following question remains open.

**Question 3.3.3.** Is the set \(\{f \in \text{Homeo}^+(S^1) : \tau(f) \notin \mathbb{Q}\}\) Haar null?

Again, we could use the characterization to answer Question 3.1.4 for the group \(\text{Homeo}^+(S^1)\).

**Corollary 3.10.2.** The group \(\text{Homeo}^+(S^1)\) can be partitioned into a Haar null and a meager set.

In Section 3.6 we introduce basic results of spectral theory to prove the following theorem about the group \(U(\ell^2)\) of unitary transformations of the separable, infinite dimensional Hilbert space.

**Theorem 3.6.3.** Let \(U \in U(\ell^2)\) be given with spectral measure \(\mu_U\) and multiplicity function \(n_U\). If the conjugacy class of \(U\) is non-Haar null then \(\mu_U \simeq \lambda\) and \(n_U\) is constant \(\lambda\)-a. e., where \(\lambda\) denotes Lebesgue measure.

**Corollary 3.6.4.** In the unitary group every conjugacy class is Haar null possibly except for a countable set of classes, namely the conjugacy classes of the multishifts.

For the group \(U(\ell^2)\) the fact that it can be written as the union of a meager and a Haar null set has already been proved, see [6].

**Remark.** In Corollary 3.10.9 we show that the automorphism group of the countable atomless Boolean algebra can be partitioned into a Haar null and a meager set, but this group is well known to be isomorphic to the group of homeomorphisms of the Cantor set, hence we have another important homeomorphism group that can be partitioned into a Haar null and a meager set.
3.2 General results about countable structures.

We start with defining the crucial notion for the description of the orbits of a random element of an automorphism group. Informally, the following definition says that our structure is free enough: if we want to extend a partial automorphism defined on a finite set, there are only finitely many points for which we have only finitely many options.

**Definition 3.3.5.** Let $G$ be a closed subgroup of $S_\infty$. We say that $G$ has the finite algebraic closure property (FACP) if for every finite $S \subset \omega$ the set $\{b : |G(S)(b)| < \infty\}$ is finite, where $G(S)$ denotes the pointwise stabilizer of $S$.

Generalizing the results of Dougherty and Mycielski we show that the FACP is equivalent to some properties of the orbit structure of a random element of the group.

**Theorem 3.7.13.** Let $G \leq S_\infty$ be a closed subgroup. If $G$ has the FACP then the sets

$$
\mathcal{F} = \{ g \in G : g \text{ has finitely many finite orbits} \},
$$

$$
\mathcal{I} = \{ g \in G : g \text{ has infinitely many infinite orbits} \}
$$

are both co-Haar null. Moreover, if $\mathcal{F}$ is co-Haar null then $G$ has the FACP.

In Theorem 3.7.14 we also show an equivalent model theoretic condition of the FACP. We state three important special cases of the above theorem, namely when $G$ is one of the automorphism group of the countably infinite random graph ($\text{Aut}(\mathcal{R})$), the automorphism group of the rational numbers ($\text{Aut}(\mathbb{Q}, \prec)$) or the automorphism group of the countable atomless Boolean algebra ($\text{Aut}(\mathcal{B}_\infty)$).

**Corollary 3.3.7.** In $\text{Aut}(\mathcal{R})$, $\text{Aut}(\mathbb{Q}, \prec)$ and $\text{Aut}(\mathcal{B}_\infty)$ almost every element has finitely many finite and infinitely many infinite orbits.

Using these results we show the following.

**Corollary 3.10.9.** $\text{Aut}(\mathcal{R})$, $\text{Aut}(\mathbb{Q}, \prec)$ and $\text{Aut}(\mathcal{B}_\infty)$ can be decomposed into the union of an (even conjugacy invariant) Haar null and a meager set.

Unfortunately, these results are typically far from the full description of the behavior of the random elements. We continue with the detailed study of two special cases, $\text{Aut}(\mathbb{Q}, \prec)$ and $\text{Aut}(\mathcal{R})$.

3.3 $\text{Aut}(\mathbb{Q}, \prec)$ and $\text{Aut}(\mathcal{R})$

In our characterization of the non-Haar null conjugacy classes of $\text{Aut}(\mathbb{Q}, \prec)$ and $\text{Aut}(\mathcal{R})$, we also prove that each non-Haar null conjugacy class is actually compact biter, that is, it contains a translate of a non-empty portion of every (non-empty) compact set, and each fixed-point free non-Haar null class is compact catcher, that is, it contains a translate of every compact set.
In our following result concerning the automorphism group of the rational numbers, for an automorphism \( f \in \text{Aut}(\mathbb{Q},<) \), \( \mathcal{O}_f^* \) denotes the set of orbitals of \( f \), the collection of the convex hulls (relative to \( \mathbb{Q} \)) of the orbits of the rational numbers, that is

\[
\mathcal{O}_f^* = \{ \text{conv} \{ f^n(r) : n \in \mathbb{Z} \} : r \in \mathbb{Q} \}.
\]

For an orbital \( O \in \mathcal{O}_f^* \), \( s_f(O) \) denotes the sign of \( O \), that is, by picking an arbitrary \( x \in O \), \( s_f(O) = 0 \) \( \Leftrightarrow \) \( f(x) = x \), \( s_f(O) = 1 \) \( \Leftrightarrow \) \( f(x) > x \) and \( s_f(O) = -1 \) \( \Leftrightarrow \) \( f(x) < x \).

For distinct orbitals \( O_1, O_2 \in \mathcal{O}_f^* \) we use the notation \( O_1 < O_2 \) if for any \( (\text{or equivalently, for all}) \) \( x_1 \in O_1, x_2 \in O_2 \) we have \( x_1 < x_2 \).

**Theorem 3.8.4.** For almost every element \( f \) of \( \text{Aut}(\mathbb{Q},<) \)

1. for distinct orbitals \( O_1, O_2 \in \mathcal{O}_f^* \) with \( O_1 < O_2 \) such that \( s_f(O_1) = s_f(O_2) = 1 \) or \( s_f(O_1) = s_f(O_2) = -1 \), there exists an orbital \( O_3 \in \mathcal{O}_f^* \) with \( O_1 < O_3 < O_2 \) and \( s_f(O_3) \neq s_f(O_1) \),

2. \( f \) has only finitely many fixed points.

These properties characterize the non-Haar null conjugacy classes, i.e., a conjugacy class is non-Haar null if and only if one (or equivalently every) of its elements has properties (1) and (2).

Moreover, every non-Haar null conjugacy class is compact biter and those non-Haar null classes in which the elements have no rational fixed points are compact catchers.

In what follows, \( V \) is the vertex set of the random graph, \( R \) is the edge relation, and for a set \( U \subset V \), \( \mathcal{O}^f(U) \) denotes the union of the orbits of \( U \) with respect to \( f \), that is, \( \mathcal{O}^f(U) = \{ v \in V : \exists u \in U \exists k \in \mathbb{Z} (f^k(u) = v) \} \).

**Theorem 3.9.29.** For almost every element \( f \) of \( \text{Aut}(R) \)

1. for every pair of finite disjoint sets, \( A, B \subset V \) there exists \( v \in V \) such that \((\forall x \in A)((x,v) \in R)\) and \((\forall y \in B)((y,v) \not\in R)\) and \( v \not\in \mathcal{O}^f(A \cup B) \),

2. \( f \) has only finitely many finite orbits.

These properties characterize the non-Haar null conjugacy classes, i.e., a conjugacy class is non-Haar null if and only if one (or equivalently every) of its elements has properties (1) and (2).

Moreover, every non-Haar null conjugacy class is compact biter and those non-Haar null classes in which the elements have no fixed points are compact catchers.

This yields the following surprising corollary:

**Corollary.** There are continuum many non-Haar null conjugacy classes in the groups \( \text{Aut}(\mathbb{Q},<) \) and \( \text{Aut}(R) \), and their union is co-Haar null.
The Ph.D. thesis is based on the following papers


References


