De la Vallée Poussin type approximation methods

Theses of the PhD Dissertation

Author: 
Zsolt Németh

Supervisor: 
László Szili CSc
associate professor

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Graduate School: ELTE PhD School of Computer Science
Head of School: András Benczúr DSc, professor
PhD programme: Numeric and symbolic calculus
Head of Programme: Antal Járai DSc, professor emeritus
Introduction

The focus of our work is the uniform convergence of different de la Vallée Poussin type summations. We encounter this topic in theories of classical and multivariate trigonometric Fourier series, discrete Fourier series and trigonometric interpolation, and finally algebraic interpolation. There are many similarities but also some differences in our methods when dealing with these problems.

First we discuss the historical background of our study, establish the most important notations and definitions and recall some fundamental results on which our results are based upon.

Let $C_{2\pi}$ denote the linear space of complex valued $2\pi$-periodic continuous functions defined on the real numbers $\mathbb{R}$. It is well known that $C_{2\pi}$ endowed with the maximum norm

$$
\|f\|_{\infty} := \max_{x \in \mathbb{R}} |f(x)| \quad (f \in C_{2\pi})
$$

is a complete normed space, i.e. Banach space.

We denote by

$$
\varepsilon_j(x) := e^{ijx} \quad (x \in \mathbb{R}, \ j \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}).
$$

the complex trigonometric system, and by $\mathcal{T}_n \ (n \in \mathbb{N} := \{0, 1, \ldots\})$ the linear space of all complex valued trigonometric polynomials of degree not exceeding $n$.

For a function $f \in C_{2\pi}$ denote the trigonometric Fourier coefficients by

$$
\hat{f}(j) := \langle f, \varepsilon_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ijt}dt \quad (j \in \mathbb{Z}).
$$

The trigonometric Fourier series of $f$ is given by

$$
S[f] := \sum_{j \in \mathbb{Z}} \hat{f}(j)\varepsilon_j.
$$
Denote the $n$-th partial sum of this series by

$$(S_n f)(x) := \sum_{j=-n}^{n} \hat{f}(j) \varepsilon_j(x) \quad (x \in \mathbb{R}).$$

In the study of convergence, the concept of the operator norm plays an important role. In this case, for a map $T : (C_{2\pi}, \| \cdot \|_\infty) \to (C_{2\pi}, \| \cdot \|_\infty)$, the norm of operator $T$ is defined by

$$\|T\| = \max_{f \in C_{2\pi}} \frac{\|Tf\|_\infty}{\|f\|_\infty} \leq 1.$$ 

Usually we don’t evaluate the exact value of this expression, only give estimations.

It is known that the sequence of the partial sums of Fourier series of $f$ is not uniformly convergent for all $f \in C_{2\pi}$, since the operators in question are not uniformly bounded. This problem is usually avoided by replacing $S_n f \ (n \in \mathbb{N})$ with a bounded linear operator obtained by applying summation over the Fourier series (see e.g. Fejér summation).

We investigate a generalization of this idea called $\varphi$-summation, i.e. a summation generated by a function $\varphi \in \Phi$, where $\Phi$ is defined in the dissertation.

The $n$-th $\varphi$-sum of the trigonometric series of $f \in C_{2\pi}$ is defined as $(S_n^\varphi f)(x) := (S_n f)(x)$ for $n = 0$, and otherwise by

$$(S_n^\varphi f)(x) := \sum_{j \in \mathbb{Z}} \varphi \left( \frac{j}{n} \right) \hat{f}(j) e^{ijx} = \sum_{j=-n}^{n} \varphi \left( \frac{j}{n} \right) \hat{f}(j) e^{ijx}, \quad (x \in \mathbb{R}, \ 1 \leq n \in \mathbb{N}).$$ \hspace{1cm} (4)

Next we recall a fundamental result, the so-called Natanson–Zuk theorem. (see [2, p. 168]).

Denote by $L^1(\mathbb{R})$ the usual linear space (over $\mathbb{R}$) of measurable functions $g : \mathbb{R} \to \mathbb{R}$ for which the Lebesgue integral

$$\int_{-\infty}^{+\infty} |g(x)| \, dx$$
is finite.

The Fourier transform of the function \( g \in L^1(\mathbb{R}) \) is defined by

\[
\hat{g}(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t)e^{-ixt} dt \quad (x \in \mathbb{R}).
\]  \(5\)

It follows immediately from the definition that the Fourier transform of every function \( g \in L^1(\mathbb{R}) \) exists for all \( x \in \mathbb{R} \).

**Theorem A.** Suppose that \( \varphi \in \Phi \). Then \((S_n^\varphi f, n \in \mathbb{N})\) uniformly converges to \( f \) \( (i.e. \| (S_n^\varphi f - f) \|_\infty \to 0 \text{ as } n \to +\infty) \) on \( \mathbb{R} \) for every function \( f \in C_{2\pi} \) if and only if the Fourier transform of \( \varphi \) is (Lebesgue) integrable on \( \mathbb{R} \).

The de la Vallée Poussin sums provide a solution to the problem of uniform convergence, and can be defined as arithmetic means of partial sums, namely for two parameters \( n, m \in \mathbb{N} \), we may take the average of the partial sums \( S_n, S_{n+1}, \ldots, S_{n+m} \) and define the operator \( G_{n,m} \) as

\[
G_{n,m} := \frac{1}{m+1} \sum_{j=n}^{n+m} S_j \quad (n, m \in \mathbb{N}).
\]

For the norm of \( G_{n,m} \), \((n, m \in \mathbb{N})\), we state the following known results (see [1, Vol. II,Ch. 2/1]) and [12]).

**Proposition.** Suppose that \((n, m \in \mathbb{N})\). We have

\[
\| G_{n,m} \| = \frac{4}{\pi^2} \log \frac{n + m + 1}{m + 1} + O(1),
\]

i.e. there exist positive constants \( c_1, c_2 \) independent of \( n, m \) such that

\[
\frac{4}{\pi^2} \log \frac{n + m + 1}{m + 1} + c_1 \leq \| G_{n,m} \| \leq \frac{4}{\pi^2} \log \frac{n + m + 1}{m + 1} + c_2.
\]
It is also clear that $G_{n,m} : C_{2\pi} \to \mathcal{T}_{n+m}$ and $(G_{n,m}g)(x) = g(x)$ for any $g \in \mathcal{T}_n, x \in \mathbb{R}$, so the operator has some kind of projection property. In fact, we have an analogue of the Faber–Marczinkiewicz–Berman theorem due to Nikolaev [7].

**Theorem B.** Fix $n, m \in \mathbb{N}$, $n \geq 1$ and let $T_n : C_{2\pi} \to \mathcal{T}_{n+m}$ denote a de la Vallée Poussin type trigonometric projection operator, i.e. suppose that $(T_n g)(x) = g(x)$, $(g \in \mathcal{T}_n, x \in \mathbb{R})$. Now there exist a positive constant $c \in \mathbb{R}$ independent of $n, m$ such that

$$
\|T_{n,m}\| \geq c \log \frac{n + m}{m + 1}.
$$

The operator $G_{n,m}$ can be expressed as a specific $\varphi$-sum as well.

**Definition.** For $\alpha \in [0, 1)$ let $\varphi_\alpha$ be the unique function which is 1 on the interval $[-\alpha, \alpha]$, 0 on $\mathbb{R}\setminus[-1, 1]$, and is linear on the nonempty intervals $[-1, -\alpha]$ and $[\alpha, 1]$, i.e. if $\alpha \neq 1$ then on $x \in [0, 1]$ we have

$$
\varphi_\alpha(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq \alpha \\
\frac{1 - x}{1 - \alpha}, & \text{if } \alpha < x \leq 1.
\end{cases}
$$

Now it is clear that if $n,m \in \mathbb{N}$, $n + m \geq 1$ and the relation $\alpha = \frac{n}{n + m + 1}$ holds, then by (4) we have $G_{n,m} = S^{\varphi_{\alpha}}_{n+m+1}$.

The discrete version of the Fourier series (3) can be defined as follows (cf. e.g. [12], Vol. II, Chapter X). Let us fix a natural number $M \in \mathbb{N}^+ := \{1, 2, \ldots\}$ and consider the equidistant point system

$$
X_M := \{x_{k,M} := k \frac{2\pi}{M} : k = 0, 1, \ldots, M - 1\}.
$$
The discrete trigonometric Fourier coefficients with respect to the point system (6) of \( f \in C_{2\pi} \) are defined by

\[
\hat{f}_M(j) := (f, \varepsilon_j)_M = \frac{1}{M} \sum_{k=0}^{M-1} f(x_{k,M}) e^{-ijx_{k,M}} \quad (j \in \mathbb{Z}).
\] (7)

The discrete Fourier series with respect to the point system (6) of the function \( f \in C_{2\pi} \) is defined by

\[
S_M[f] := \sum_{j \in \mathbb{Z}} \hat{f}_M(j) \varepsilon_j.
\] (8)

Now we introduce the discrete version of (4). Fix the summation function \( \varphi \in \Phi \) and the number \( M \in \mathbb{N}^+ \). The \( n \)-th discrete \( \varphi \)-sums with respect to the point system (6) of the function \( f \in C_{2\pi} \) are defined by

\[
(S^n_{\varphi,M} f)(x) := \sum_{j \in \mathbb{Z}} \varphi\left(\frac{j}{n}\right) \hat{f}_M(j) \varepsilon_j(x) = \sum_{j=-n}^{n} \varphi\left(\frac{j}{n}\right) \hat{f}_M(j) \varepsilon_j(x)
\] (9)

\((x \in \mathbb{R}, f \in C_{2\pi}, m \in \mathbb{N})\).

Thus for every function \( \varphi \in \Phi \) we have a two-parameter operator family.

Conditions of uniform convergence and other properties for these general operators are investigated in [9]. There the authors give a discrete version of the Natanson–Zuk theorem.

**Thesis 1. Multivariate de la Vallée Poussin type projection operators**

This part of our work concerns the multivariate extensions of the de la Vallée Poussin means. We define the de la Vallée Poussin means of the triangular partial sums of multivariate Fourier series. We determine the exact order of the corresponding operator norms. The lower estimation of these norms were extended to a class of projection operators having similar projection properties as the de la Vallée Poussin mean.
Theorem. Fix $d \geq 1$ and suppose that $n, m \in \mathbb{N}, n \geq 1$. For any de la Vallée Poussin type projection operator $T_{n,m,d}$, we have

$$\|T_{n,m,d}\| \geq c \left( \log \frac{n + m}{m + 1} \right)^d,$$

where $c > 0$ is a positive constant independent of $n, m$.

Further, for the operator $G_{n,m,d}$ we have

$$\|G_{n,m,d}\| \leq c \left\{ \left( \log \frac{n + m}{m + 1} \right)^d + 1 \right\}.$$

The first inequality is a multivariate extension of Nikolaev’s mentioned result Theorem B for de la Vallée Poussin type projection operators in one dimension. The second inequality is a weaker $d$-dimensional variant of the above Proposition.

The details are worked out in Chapter 2 of the dissertation, based on paper [3].

Thesis 2. De la Vallée Poussin sums in trigonometric interpolation

We discuss some classic methods of trigonometric interpolation, mainly the well known (trigonometric) Lagrange and Hermite–Fejér interpolations, using the tools of discrete $\varphi$-summations (9). We define the de la Vallée Poussin sums by the operator $S_{n,M}$ (see Definition), which were used as a tool to describe the transition between these two classic methods. We give some general properties, the precise operator norm and (uniform) convergence order for these cases, based on our following result.

Theorem. 

$$\|S_{n,M}\| \sim \frac{2}{\pi} \log N + O(1),$$
i.e. to any interpolatory operator $S_{n,M}^{\alpha}$ there exist positive constants $c_1, c_2$ independent of $n$ and $M$ such that the following inequalities hold

$$\frac{2}{\pi} \log N + c_1 \leq \|S_{n,M}^{\alpha}\| \leq \frac{2}{\pi} \log N + c_2,$$

for every above numbers $n, M$, where $N := \frac{M}{2n-M}$.

The details are worked out in Chapter 3 of the dissertation, based on paper [4].

**Thesis 3. Weighted interpolation on the roots of Chebyshev polynomials**

Finally, we are establishing a connection between the trigonometric interpolations, obtained as sums of discrete (trigonometric) Fourier series (9), and algebraic interpolations on the closed interval $[-1,1]$. It is known that the results concerning the former transfer naturally to the case of algebraic interpolation on the roots of first kind of Chebyshev polynomials. Now we are considering a more general approach, using the roots of all four kinds of Chebyshev polynomials for the point systems.

Achieving uniform convergence on the whole interval $[-1,1]$ is problematic in these cases because of unpleasant behaviour near the endpoints, but two slightly different approaches are known to deal with this. The first is to supplement the problematic point systems with suitable endpoints (see e.g. [8] and our own work [5]). The other technique is multiplying the functions by suitable weight functions before dealing with the problem, thus considering the convergence in weighted spaces of continuous functions (see e.g. [10] and [11]). In our dissertation we follow this latter method.

Starting from discrete (algebraic) Fourier series we construct discrete interpolation processes on the roots of four kinds of Chebyshev polynomials generated by
suitable summation functions \( \varphi \in \Phi_+ \). These functions slightly differ from the previous summation functions \( \varphi \in \Phi \). We prove the following general result, similar to the Natanson–Zuk theorem, stating that if the cosine transform of \( \varphi \) is integrable then these processes are uniformly convergent on the whole interval \([-1, 1]\) in some weighted spaces of continuous functions.

**Theorem.** Let \( |\alpha| = |\beta| = \frac{1}{2} \) and \((\gamma, \delta)\) the corresponding parameters. Suppose that \( \varphi \in \Phi_+ \) and

\[
n_m \to +\infty \quad (m \to +\infty) \quad \text{and} \quad n_m \leq 2M_m.
\]

If \( \hat{\varphi}_c \in L^1(\mathbb{R}^+) \) then for any \( f \in C_{w_{\gamma, \delta}} \) we have

\[
\|f - S_{n_m, M_m}^\varphi(f, X_{M_m}(w_{\alpha, \beta}), \cdot)\|_{w_{\gamma, \delta}} \to 0 \quad (m \to +\infty).
\] (12)

The polynomials \( S_{n_m, M_m}^\varphi \) and the suitable parameters are defined in the dissertation.

We also examine necessary and sufficient conditions for the interpolation. As applications, we obtain various new results for the Lagrange interpolation and its arithmetic means; the Grünwald, the de la Vallée Poussin and the Hermite–Fejér interpolation.

The details are worked out in Chapter 4 of the dissertation, based on paper [6].
References


