Finite size matrix elements and thermal correlators in integrable quantum field theories

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Ph.D. Thesis

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Chapter 1

Introduction
Our world is believed to be inherently 4 space-time dimensional, but in certain circumstances the effective dimension of a system can be considered smaller. For example, the dominant extension of a nanowire is 1+1 dimensional (one space and one time dimensions), and an atomic layer, like graphene, is a 2+1 dimensional material. Low-dimensional quantum systems have been in the focus of theoretical and experimental physics for many decades. The interest was many times fuelled by new experimental discoveries in condensed matter physics, such as the quantum Hall effect or high temperature superconductivity. These interesting phenomena arise as a collective behaviour of the quantum many-body system when the correlation in the system is strong. When the correlation length is much longer than the microscopic distance scale, quantum field theory (QFT) provides a natural framework for theoretical description. Condensed matter systems are usually non-relativistic, but in many cases the resulting QFT is of a relativistic form, where the role of the speed of light is played by some other quantities, like the Fermi velocity for the Luttinger liquids or the speed of sound for acoustic phonons. Another important consequence of strong correlation in the systems is the limited applicability of the of the usual perturbative approach of QFTs, that calls for a deeper understanding and treatment of QFTs, and for the development of non-perturbative methods.

One of the revolutionary ideas to better understand QFTs was the Wilsonian renormalisation group (RG) method [Wil75]. It led to the understanding of criticality, phase transitions of quantum systems, and their universal behaviour around the critical points. The RG relates the description of a system on different energy scales, schematically speaking its flow in the space of theories while changing the energy scale. The flows either end in a critical point, where the correlation length diverges, or go to infinity. At the critical points, where second order phase transitions happen, the correlation length in the system goes to infinity leading to scale invariance. Scale invariance typically manifests in power law behaviour of thermodynamic quantities around the critical point. For systems with short range interaction, scale invariance also leads to conformal symmetry, and hence critical points can be described by conformal field theories (CFT). In the CFT framework, quantum theories are mainly not described by their Lagrangians, which is not known for many cases, rather by the properties of the operators in the theory and their algebraic structure. For two dimensional theories, the conformal symmetry is so restrictive that many models can eventually be solved exactly. For example, two dimensional CFTs with central charge smaller than one are completely classified [BPZ84]. Quantum theories around critical points can be described as perturbation of the CFT, and their physical properties, like critical exponents and universal ratios, are determined by the CFT. Since many models flow to the same CFT in the UV or IR, they
have the same critical exponents and universal ratios that explains the universality observed in experiments.

In two dimensions, there is also another interesting class of QFTs that drew great attention in theoretical physics, the so-called integrable QFTs (IQFT). A common notion of integrability is the existence of infinitely many non-trivial conserved quantities in the theory that restricts the scattering of the particles. The above mentioned two dimensional CFTs can be also considered as integrable theories [BLZ96, BLZ97a, BLZ99].

In integrable theories, there is no particle creation in physical scattering processes, and every scattering process factorises into products of two-particle scatterings. The two-particle scattering matrix satisfies certain relations, like unitarity, crossing-relation or the consistency of the factorisation, the so-called the Yang-Baxter equations. These properties together give rise to the scattering matrix bootstrap program, that was pioneered, amongst others, by the Zamolodchikov brothers [ZZ79]. In the bootstrap program, the goal is to find solutions to the equations describing the properties of the S-matrices\(^1\). The solutions of the S-matrix bootstrap describe the full non-perturbative scattering process in IQFTs. Similar to the S-matrix bootstrap, in integrable theories, there is a form factor bootstrap program that aims to construct the matrix elements of operators as solutions to certain equations [KW78, Smi92]. With the help of the form factor solutions, it is possible to express correlation functions that are fundamental to QFT calculations.

The existence of infinitely many non-trivial conserved charges in physical systems is quite a subtle situation, however there are many examples of condensed matter systems that can be described by IQFTs. One of the first example came from the topic of conducting polymers, the relation of the polyacetylene polymers to the Gross-Neveu models [BK81, CB82]. IQFTs appear in the solution of the Kondo problem [WT84], in the description of inelastic neutron scattering on copper nitrate [TNL+09], and the famous $E_8$ spectrum of Zamolodchikov [Zam89] for the critical Ising-model with magnetic field was also observed in experiments [CTW+10]. The advancement of cold atom experiments makes it also possible to construct integrable models in laboratory and investigate their non-equilibrium behaviour. A famous examples is the quantum Newton’s cradle [KWW06]. For a nice review about application of integrable techniques, especially the use of form factors, to condensed matter systems we refer the reader to [EK05].

IQFTs can play also a less direct, but useful role in understanding quantum systems. For example, many universality classes contain integrable perturbation of the critical CFT. This

---

\(^1\)S-matrix is a shorthand notation for the scattering matrix.
gives a great toolbox to calculate universal properties of the universality class with the help of IQFT techniques, e.g. the calculation of universal ratios [DC98].

One more important application of integrability and integrable techniques lies in the understanding of the AdS/CFT correspondence [Mal97]. In [MZ03], it was shown that in the planar limit of the $\mathcal{N} = 4$ super Yang-Mills theory, the CFT side of the correspondence, the scaling dimension of the operators can be mapped to the energy levels of a $SO(6)$ integrable spin-chain. On the AdS (string theory) side of the correspondence, classical integrable structure was found in [BPR04]. These observations motivated extensive study of integrability in the AdS/CFT context; for a summary we refer the interested reader to [BAA+12].

This thesis focuses on thermal correlation functions and finite size matrix elements in a subset of IQFTs, namely relativistic models with massive particles and diagonal scattering. We are interested in theories defined on a cylinder with flat Euclidean metric. This setting, on one hand, is the imaginary time formalism of Matsubara [Mat55] introduced to describe finite temperature QFTs. On the other hand, it is a finite size QFT at zero temperature. In two dimensional relativistic theories, the two descriptions are related, and there is a correspondence between quantities in the complementary pictures, for example the finite volume ground state energy is related to the free energy density of the finite temperature system.

An efficient evaluation of thermal correlators is important to understand phase transitions of a system and to relate theoretical predictions to experimental results. Form factor spectral expansion at zero temperature turned out to be a successful way to calculate correlators [YZ91]. For finite temperature correlators, LeClair and Mussardo conjectured a series expansion for the one- and two-point functions in [LM99]. Saleur proved the LeClair-Mussardo (LM) conjecture for one-point functions for operators that are the density of conserved currents in [Sal00], but he also questioned the validity of the conjecture for two-point functions. With finite size regularisation and finite size form factors, Pozsgay and Takacs proved the LM conjecture up to third order exponential corrections in the volume in [PT08b]. Later, Pozsgay showed the equivalence of the LM conjecture and the diagonal finite volume form factor formula to all orders in [Poz11]. Pozsgay and Takacs also applied the finite volume regularisation to calculate thermal two-point functions with form factor spectral expansion in [PT10]. The formalism is well-defined, albeit the calculation is tedious, and the result does not seem to resemble the conjecture by LeClair and Mussardo.

As mentioned in the previous paragraph, finite size regularisation and finite size matrix elements can play an important role in calculation of quantities, like thermal one-point functions that are ill-defined in infinite volume due to the presence of singularities. Finite
volume regularisation leads to a well-defined finite result. Finite size matrix elements and the understanding of their volume dependence play also an important role in calculations, where the natural setting is that of a finite size system, such as e.g. lattice QCD. Finite volume form factors also appeared recently in the context of AdS/CFT correspondence and the calculation of the three-point functions, related to the OPE coefficients [BJW14, BJ15, BKV15]. In [Poz13], Pozsgay used the finite size vacuum expectation value interpretation of the LM-conjecture, and generalised it to general diagonal finite size matrix elements containing all exponential corrections in the volume parameter.

As we see, thermal and finite size correlators play important role in performing calculations, and their mutual interplay helps to understand the structure of correlators better. This thesis aims to verify the LM conjecture and Pozsgay’s generalisation using both numerical and analytical approaches, since there is no proof for their validity from first principles yet. The convergence and applicability of the series are also an open question, and numerical evaluation can give an intuition for these issues. Moreover, we also aim to further develop the finite size regularisation of the two-point functions and crosscheck the expressions with properties expected to hold for them, like the clustering in case of large spacial separation of the operators.

The thesis is organised as follows. In Chapter 2, we summarise the basic properties of the S-matrices and form factors in IQFTs, the bootstrap programs. In Chapter 3, we show the effect of finite size setup on the spectrum, energies, and matrix elements of an IQFT.

In Chapter 4, we present the LM conjecture, its properties, and the numerical verification of the expression. In Chapter 5, we show Pozsgay’s generalisation of the LM conjecture, we prove it for the operator of the trace of the stress-energy tensor. We also provide numerical verification of the conjecture for another operator. We also comment on the convergence of the series, and possible critical volumes where it can break down.

In Chapter 6, we turn to the evaluation of the thermal two-point functions, correct the result presented in [PT10] for the $D_{22}$ contribution, and show the validity of the clustering property up to third order.

In Chapter 7, we conclude the thesis and discuss further interesting research directions.
Chapter 2

Integrable models and bootstrap
In this chapter, we briefly summarise the consequence of integrability on scattering processes and matrix elements. The constraints coming form integrability can be restrictive enough to fix the scattering matrix and matrix elements totally, this is the topic of the scattering matrix and form factor bootstrap. For more detailed review of integrability and the scattering matrix bootstrap we refer the reader to the reviews [Dor97, Mus92].

2.1 Integrable quantum field theories

The notion of integrability in classical mechanics is well-defined, however, for quantum theories the definition is not straightforward [CM11]. Here, we will consider a relativistic QFTs integrable if there exist infinite many commuting, local, conserved quantities with Lorenz spin higher than zero\(^1\), and we assume that the conserved quantities are complete in some appropriate sense\(^2\). For massive quantum theories in dimension higher than 1 + 1, the existence of any higher Lorenz spin conserved charge leads to a trivial theory according to the Coleman-Mandula theorem [CM67]. This theorem does not hold for 1 + 1 dimensional theories, however, the existence of these charges have nontrivial consequences. To state these consequences, let us introduce the asymptotic states of the theory and focus on the scattering processes.

We only consider theories with massive particles with non-degenerate masses. We denote \(n\)-particle states like

\[
|\theta_1, \ldots, \theta_n\rangle_{\alpha_1, \ldots, \alpha_n}
\]

where \(\theta_i\) and \(\alpha_i\) are the rapidities and types of the one-particle states building up the multi-particle state. The normalisation of one-particle states are

\[
\beta\langle \theta' | \theta \rangle_\alpha = 2\pi \delta_{\beta \alpha} \delta(\theta' - \theta).
\]

Energy and momentum of the multiparticle state is the sum of the one-particle states\(^3\)

\[
E\{\theta_i\}_{\alpha_i} = \sum_i m_{\alpha_i} \cosh (\theta_i), \quad (2.1.1)
\]

\[
P\{\theta_i\} = \sum_i m_{\alpha_i} \sinh (\theta_i). \quad (2.1.2)
\]

\(^1\)We also impose energy and momentum conservation that are Lorenz spin zero quantities.

\(^2\)There is no part of the dynamics that is decoupled from the rest and is not constrained by these charges.

\(^3\)The energy of the vacuum is set to zero.
The asymptotic *in* states are the ones with rapidities ordered as

\[ \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n, \]

while the *out* states are ordered as

\[ \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n. \]

Both the *in* and *out* states form a basis in the Hilbert space of the theory. Scattering is nothing but the connection of the two bases, and the S-matrix is

\[ S_{n \to m} = \langle \theta_1', \ldots, \theta_m' | \theta_1, \ldots, \theta_n \rangle^{\text{out}}_{\beta_1, \ldots, \beta_n} \langle \theta_1, \ldots, \theta_n | \theta_1', \ldots, \theta_m' \rangle^{\text{in}}_{\alpha_1, \ldots, \alpha_m}. \]

The consequence of integrability on the scattering processes is the lack of particle creation, i.e. there exist only \( n \to n \) scatterings\(^4\). Moreover, the set of rapidities and particle types are equal for the initial and final states participating in the scattering, i.e. the scattering is elastic. Integrability also ensures the factorisation of the scattering processes, meaning every scattering can be described as a product of only \( 2 \to 2 \) scatterings. The consistency of the factorisation property are called the Yang-Baxter equations.

As mentioned before, we only consider theories with non-degenerate mass spectrum; this kind of scattering is called diagonal. Yang-Baxter equations are trivially satisfied for diagonal scattering, but other constraints for the two-particle scattering matrix can fix its form almost totally. In the following Section we give a brief overview of this so-called S-matrix bootstrap program.

### 2.2 Scattering matrix bootstrap

For diagonal elastic scattering we can express the action of two-particle S-matrix on two-particle states as

\[ |\theta_1, \theta_2\rangle^{\text{in}}_{\alpha_1, \alpha_2} = S_{\alpha_1, \alpha} (\theta_1 - \theta_2) |\theta_2, \theta_1\rangle^{\text{out}}_{\alpha_2, \alpha_1}, \]

\(^4\)The only exception is for special rapidity distribution with complex values that is related to bound states and will give us important constraints for the two-particle scattering matrix.
where due to Lorenz invariance the amplitude only depends on difference of the rapidities. Moreover, for parity invariant theories the S-matrix is symmetric
\[ S_{\alpha_1 \alpha_2} (\theta) = S_{\alpha_2 \alpha_1} (\theta) . \]

General field theoretical considerations lead to the following properties of the S-matrix:

- **Hermitian analyticity:** The scattering matrix and its Hermitian conjugate are the opposite boundary value of the same analytic function. For time reversal symmetric theories this simplifies to the real analyticity condition [Mir99] that reads in rapidity variables as
  \[ S_{\alpha \beta} (\theta) = S_{\alpha \beta} (-\theta^*)^* , \]
  meaning the S-matrix is real for purely imaginary rapidities.

- **Unitarity:** Unitarity of the S-matrix means \( SS^\dagger = 1 \), that can be translated with the real analyticity condition to the following relation in rapidity space
  \[ S_{\alpha \beta} (\theta) S_{\alpha \beta} (-\theta) = 1 . \quad (2.2.1) \]

- **Crossing:**
  \[ S_{\alpha \beta} (\theta) = S_{\alpha \bar{\beta}} (i\pi - \theta) , \quad (2.2.2) \]
  where \( \bar{\beta} \) means the antiparticle of \( \beta \).

- **Maximal analyticity:** The S-matrix is a meromorphic function of the rapidity on the physical strip \((0 < \Im (\theta) < i\pi)\) with poles only on the imaginary axis\(^5\).

Single poles of the S-matrix are bound states, and the higher order poles have field theoretical explanation, e.g. the so-called Coleman-Thun diagrams. If the scattering of a type-\( \alpha \) and a type-\( \beta \) particles creates a bound state of type-\( \gamma \) at rapidity \( i\mu_{\alpha \beta} \), the S-matrix around that pole takes the form
\[ S_{\alpha \beta} (\theta) \sim \frac{i \left( \Gamma_{\gamma_{\alpha \beta}} \right)^2}{\theta - i\mu_{\alpha \beta}^\gamma} , \quad (2.2.3) \]

\(^5\)In some theories the scattering matrix can have resonance poles that are not on the imaginary axis.
where $\Gamma_{\alpha\beta}^\gamma$ is the three point coupling. The conservation of energy and momentum in the scattering implies a relation between the masses of the particles and the pole position

$$m_{\gamma}^2 = m_{\alpha}^2 + m_{\beta}^2 + 2m_{\alpha}m_{\beta}\cos\left(u_{\alpha\beta}^\gamma\right).$$

The existence of bound states implies poles in alternative channels, i.e. $S_{\alpha\gamma}^\beta \left(iu_{\alpha\gamma}^\beta\right)$ and $S_{\beta\gamma}^\alpha \left(iu_{\beta\gamma}^\alpha\right)$ are also singular. The mass relations imply the masses of the particles in the bound state formation (fusion) form the so-called mass triangle. The $u$ parameters are called fusion angles and hence they satisfy

$$u_{\alpha\beta}^\gamma + u_{\beta\gamma}^\alpha + u_{\alpha\gamma}^\beta = 2\pi. \quad (2.2.4)$$

The consistency conditions for the S-matrix amplitudes following from the existence of bound states are

$$S_{\delta\gamma}^\theta = S_{\alpha\delta}^\left(\theta + i\bar{u}_{\gamma\alpha}^\beta\right) S_{\beta\delta}^\left(\theta - i\bar{u}_{\gamma\beta}^\alpha\right), \quad (2.2.5)$$

where $\bar{u} = \pi - u$, and $\delta$ is the type of the “spectator” particle. Equation (2.2.5) is called the bootstrap principle, also known as the nuclear democracy principle by Geoffrey Chew [Che66].

The above properties of the S-matrix, i.e. unitarity, crossing, ... form the basis of the S-matrix bootstrap. Considering these equations we can search for solutions of them, and fortunately in many cases they are restrictive enough to fix the S-matrix absolutely [ZZ79]. This procedure is called the S-matrix bootstrap.

The solutions of the bootstrap equations can have some ambiguity, i.e we can multiply the solutions with functions, that satisfy the unitarity and crossing equations, and does not include new poles to the physical strip. This is called the CDD-ambiguity [CDD56], and we need other means to identify the corresponding field theory to an S-matrix solution, that is a nontrivial task. Perturbation theory can help to match up Lagrangians to the S-matrices, however, not every theory can be described by a Lagrangian. An alternative way is to use of the TBA equations (3.2.13) to extract the central charge of the ultraviolet limit of the theory and possibly the scaling dimension of the perturbing operator.

Examples for factorised scattering theories and they S-matrices are given in Appendix A for the sinh-Gordon theory\(^6\) and for the $T_2$ model.

---

\(^6\)The scattering matrix of the sinh-Gordon theory is not obtained through the bootstrap approach, but from the analytic continuation of the sine-Gordon S-matrix and perturbation theoretic considerations. Strictly speaking it is a CDD factor.
2.3 Form factor bootstrap

The matrix elements of a local operator $\mathcal{O}(x, t)$ between asymptotic states are

$$
\begin{align*}
&\langle\theta'_{1}, \ldots, \theta'_{m} | \mathcal{O}(x, t) | \theta_{1}, \ldots, \theta_{n}\rangle_{\alpha_{1}, \ldots, \alpha_{m}} \\
&= e^{it\Delta E - i\Delta P} F_{\beta_{1}, \ldots, \beta_{m}; \alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta'_{1}, \ldots, \theta'_{m} | \theta_{1}, \ldots, \theta_{n}) ,
\end{align*}
$$

(2.3.1)

where $\Delta E = \sum_{i} m_{\alpha_{i}} \cosh (\theta_{i}) - \sum_{i} m_{\beta_{i}} \cosh (\theta'_{i})$, $\Delta P = \sum_{i} m_{\alpha_{i}} \sinh (\theta_{i}) - \sum_{i} m_{\beta_{i}} \sinh (\theta'_{i})$, and the $F_{\mathcal{O}}$ function is called form factor. With the help of crossing relations

$$
\begin{align*}
&F_{\beta_{1}, \ldots, \beta_{m}; \alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta'_{1}, \ldots, \theta'_{m} | \theta_{1}, \ldots, \theta_{n}) \\
&= \sum_{k=1}^{n} 2\pi \delta_{\beta_{m}\alpha_{k}} \delta (\theta'_{m} - \theta_{k}) \prod_{l=1}^{m} S_{\alpha_{l}\alpha_{l}} (\theta_{l} - \theta_{k}) \\
&\times F_{\beta_{1}, \ldots, \beta_{m-1}; \beta_{m+1}, \alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta'_{1}, \ldots, \theta'_{m-1} | \theta_{1}, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots, \theta_{n}) ,
\end{align*}
$$

(2.3.2)

all form factors can be expressed in terms of the elementary form factors

$$
F_{\alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta_{1}, \ldots, \theta_{n}) = \langle 0 | \mathcal{O}(0, 0) | \theta_{1}, \ldots, \theta_{n}\rangle_{\alpha_{1}, \ldots, \alpha_{m}}^{in}.
$$

Similarly to the S-matrix bootstrap, the form factors also satisfy certain equations [KW78, KS87, Smi92]

Lorentz symmetry:

$$
F_{\alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta_{1} + \Lambda, \theta_{2} + \Lambda, \ldots, \theta_{n} + \Lambda) = \exp (s_{\mathcal{O}} \Lambda) F_{\alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta_{1}, \theta_{2}, \ldots, \theta_{n}) ,
$$

(2.3.3)

Exchange:

$$
F_{\alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta_{1}, \ldots, \theta_{k}, \theta_{k+1}, \ldots, \theta_{n}) = S_{\alpha_{k}\alpha_{k+1}} (\theta_{k} - \theta_{k+1}) F_{\alpha_{1}, \ldots, \alpha_{k+1}, \alpha_{k}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta_{1}, \ldots, \theta_{k+1}, \theta_{k}, \ldots, \theta_{n}) ,
$$

(2.3.4)

Cyclic property:

$$
F_{\alpha_{1}, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta_{1} + 2i\pi, \theta_{2}, \ldots, \theta_{n})
$$

(2.3.5)

$$
= F_{\alpha_{2}, \ldots, \alpha_{n}, \alpha_{1}}^{\mathcal{O}} (\theta_{2}, \ldots, \theta_{n}, \theta_{1}) ,
$$

Kinematical poles:

$$
- i \text{Res}_{\theta' = \theta} F_{\alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta + i\pi, \theta', \theta_{1}, \ldots, \theta_{n})
$$

(2.3.6)

$$
= \left( 1 - \prod_{k=1}^{n} S_{\alpha_{k}} (\theta' - \theta_{k}) \right) F_{\alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta_{1}, \ldots, \theta_{n}) ,
$$

Dynamical poles:

$$
- i \text{Res}_{\theta' = \theta} F_{\alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta' + i\bar{\alpha}_{\gamma}^{2}, \theta - i\bar{\gamma}_{\beta}^{2}, \theta_{1}, \ldots, \theta_{n})
$$

(2.3.7)

$$
= \Gamma_{\alpha_{\beta}}^{\gamma} F_{\gamma, \alpha_{1}, \ldots, \alpha_{n}}^{\mathcal{O}} (\theta, \theta_{1}, \ldots, \theta_{n}) ,
$$

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where \( s_\mathcal{O} \) denotes the Lorentz spin of the operator \( \mathcal{O} \).

Finding solutions to the form factor equations is called the form factor bootstrap program. Similarly to the S-matrix bootstrap, it is also a nontrivial task to match the solutions of the form factor bootstrap to specific operators of the theory considered. Examples for solutions of the form factor bootstrap are given in Appendix A for the sinh-Gordon theory and the \( T_2 \) model.

### 2.3.1 Diagonal form factors

It follows from the crossing relation (2.3.2) that the diagonal evaluation of a form factor

\[
F_{\alpha_n, \ldots, \alpha_1, \ldots, \alpha_n}^{\mathcal{O}}(\theta_n, \ldots, \theta_1|\theta_1, \ldots, \theta_n),
\]

is singular. By shifting one set of the rapidities infinitesimally, we tune away the form factor from its singularity, and it is possible to express it by an elementary form factor

\[
F_{\alpha_n, \ldots, \alpha_1, \ldots, \alpha_n}^{\mathcal{O}}(\theta_n + i\pi + i\varepsilon_n, \ldots, \theta_1 + i\pi + i\varepsilon_1, \theta_1, \ldots, \theta_n).
\]

There is no unique way to assign a finite physical value to the diagonal matrix elements, however, there are two evaluations that play important role in form factor calculations. The first, is the so-called symmetric evaluation, when we set every \( \varepsilon_i \) to the same value \( \varepsilon \) and take the \( \varepsilon \to 0 \) limit simultaneously

\[
F_{\alpha_n, \ldots, \alpha_1, \ldots, \alpha_n}^{\mathcal{O}, s}(\theta_1, \ldots, \theta_n) = \lim_{\varepsilon \to 0} F_{\alpha_n, \ldots, \alpha_1, \ldots, \alpha_n}^{\mathcal{O}}(\theta_n + i\pi + i\varepsilon_n, \ldots, \theta_1 + i\pi + i\varepsilon_1, \theta_1, \ldots, \theta_n).
\]

The second, is the connected evaluation of the diagonal form factor

\[
F_{\alpha_n, \ldots, \alpha_1, \ldots, \alpha_n}^{\mathcal{O}, c}(\theta_1, \ldots, \theta_n) = F.P. F_{\alpha_n, \ldots, \alpha_1, \ldots, \alpha_n}^{\mathcal{O}}(\theta_n + i\pi + i\varepsilon_n, \ldots, \theta_1 + i\pi + i\varepsilon_1, \theta_1, \ldots, \theta_n),
\]

where \( F.P. \) stand for the finite, i.e. the \( \varepsilon \), independent part of the form factor.

A consequence of the exchange axiom (2.3.4) for the symmetric and connected diagonal form factors is that they are completely symmetric in their rapidity arguments. Therefore,
we can simplify the notation for the diagonal form factors as follows:

\[ F_{2n_1,\ldots,2n_k,s/c}^O (\theta_1, \ldots, \theta_{\tilde{N}}) \]

where \( k \) is the number of particle species, \( n_i \) are the number of particles of type-\( i \), \( \tilde{N} = \sum n_i \) is the total number of particles. The \( j \)th particle with rapidity \( \theta_j \) is type-\( \alpha_j \) particle, that is not implicit in the new notation.
Chapter 3

Integrable models in finite volume
In this chapter, we review the volume dependence of energy levels and matrix elements of local operators in an IQFT when the model is considered on a space-time cylinder.

3.1 Finite volume and its relation to finite temperature

We consider our theory defined on a cylinder with flat Euclidean metric (see Figure 3.1.1). We further impose periodic boundary conditions in the compactified direction. By choosing the space direction along the circumference of the cylinder, we have the QFT defined in finite volume. Alternatively, we can define the time direction along the circumference which corresponds to a finite temperature QFT with the Matsubara imaginary time formalism [Mat55]. In the latter case, the circumference of the cylinder $L$ is just the inverse temperature $1/T$.

For relativistic theories, the assignment of time and space can be swapped without changing the theory, which is manifest in a path integral formulation. Therefore the partition functions and correlators in finite volume and at finite temperature are related to each other, entailing relations between other physical quantities as well. For example, as we will see in Section 3.2.2, the ground state energy of the finite volume system is related to the free energy density of the finite temperature system. The relation of the two descriptions will also play a crucial role in Chapters 4 and 5.

![Figure 3.1.1. The cylinder with circumference $L$, where we define our theories.](image)
3.2 Finite volume energy

In this section, we present the Bethe ansatz approach to calculate the energy of the vacuum and excited states in finite volume. First, we introduce the Bethe-Yang equations that allow the calculation of excited state gaps up to corrections exponentially suppressed for large volume. Supplementing these with the idea of swapping time with space, as discussed above, Zamolodchikov was able to construct an exact expression, the so-called thermodynamic Bethe ansatz, for the ground state energy, which was later generalised to describe the exact excited state levels as a function of the volume.

3.2.1 Bethe-Yang equations

We consider a theory with multiple kinds of massive particles and diagonal scattering. In finite volume, the rapidities in the multiparticle state \(|\theta_1, \ldots, \theta_n\rangle_{\alpha_1, \ldots, \alpha_n}\) become quantised due to the periodic boundary condition. The quantisation condition for large volume \(L\) is a system of coupled nonlinear equations, the Bethe-Yang equations

\[
e^{im_\alpha_i \sinh(\theta_i)} \prod_{j \neq i} S_{\alpha_i \alpha_j} (\theta_i - \theta_j) = 1,
\]

(3.2.1)

where \(m_\alpha_i\) is the mass of the type-\(\alpha_i\) particle, and \(S_{\alpha_i \alpha_j}\) is the two-particle scattering matrix. \(m_\alpha_i \sinh(\theta_i)\) is the momentum of a single particle. The \(i\)th equation in (3.2.1) characterises the monodromy of the wave function in the variable \(\theta_i\) obtained in the procedure moving the \(i\)th particle around the circumference of the cylinder. (3.2.1) is the analogy of the box quantisation in quantum mechanics for integrable systems, and it is valid up to exponential correction in the volume \(L\). The above consideration can be extended for the two-particle states in higher dimensional theories under the inelastic threshold [Lüs86b, Lüs91]. The reason, (3.2.1) holds for general state in integrable QFTs, is the factorisation of the scattering and lack of particle production.

With the solution of the Bethe-Yang equations (3.2.1) the energy of the multi-particle state is expressed as

\[
E (\{\theta_i\}_{\alpha_i}) = E_0 (L) + \sum_i m_\alpha_i \cosh (\theta_i) + \mathcal{O} (e^{-\mu L}),
\]

(3.2.2)

where \(E_0 (L)\) is the finite volume ground state energy, the Casimir energy, and \(\mu\) is some characteristic scale.

The finite volume vacuum energy depends on the normalisation. From the infinite volume (IR limit) prospective it is natural to choose the vacuum energy to be zero up to exponential
corrections in the volume. Alternatively, we choose a different normalisation that is motivated by the ultraviolet limit, i.e. the CFT description in the $L = 0$ limit, and it is

$$E_0 (L) = BL + \mathcal{O} \left( e^{-\mu L} \right), \quad (3.2.3)$$

where $B$ is the bulk energy density depending on the theory considered.

The exponential corrections come from virtual scattering processes that are the consequence of the compactness of the cylinder, i.e. virtual particles winding around the cylinder [Lüs86a, KM91]. Figure 3.2.1a shows the so-called $\mu$-term corrections, when a particle decays into a virtual particle pair that recombine after winding around the cylinder into the original particle. Figure 3.2.1b shows the so-called F-term corrections, when a particle emits a virtual particle that winds around the cylinder and gets absorbed by the original particle. These processes are exponentially suppressed for large volumes.

The logarithm of quantisation condition (3.2.1) is

$$Q_i = m_\alpha L \sinh (\theta_i) + \sum_{j \neq i} \delta_{\alpha_i \alpha_j} (\theta_i - \theta_j) = 2\pi I_i, \quad (3.2.4)$$

where

$$\delta_{ij} (x) = -i \log \nu_{ij} S_{ij} (x), \quad (3.2.5)$$

is the phase shift of the scattering matrices and $\nu_{ij} = S_{ij} (0)$. The quantum numbers $\{I_i\}$

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characterise the finite volume multi-particle states, and hence we introduce the notation

$$|\{I_1, \ldots, I_n\}\rangle_{\alpha_1, \ldots, \alpha_n, L}.$$ \hspace{1cm} (3.2.6)

We include $\nu_{ij}$ in the definition of the phase shift to make $\delta_{ij}$ a continuous function that makes the numerical solution of (3.2.4) stable. As a consequence of this phase shift convention, the set of $\{I_i\}$ contains integer or half-integer numbers depending on the particle content and the number of negative $\nu$ factors.

We define the density of states in rapidity space as the Jacobian of the mapping between rapidity and quantum number spaces (3.2.4)

$$\rho(\theta_1, \ldots, \theta_k) = \det K_{ij},$$ \hspace{1cm} (3.2.7)

with

$$K_{ij} = \frac{\partial Q_i}{\partial \theta_j}.\hspace{1cm} (3.2.8)$$

We introduce the restricted density belonging to any bipartite partition of the rapidities $\{\theta_1, \ldots, \theta_n\} = \{\theta_+\} \cup \{\theta_-\}$ as

$$\rho(\{\theta_+\} | \{\theta_-\}) = \det K_+,\hspace{1cm} (3.2.9)$$

where $K_+$ is the submatrix corresponding to the subset $\{\theta_+\}$.

### 3.2.2 Thermodynamic Bethe ansatz

In [Zam90], Zamolodchikov considered an integrable QFT defined on a torus with the two circumferences being $L$ and $R$, where the second one is large\(^1\) (see Figure 3.2.2).

The partition function of the theory can be expressed in two different Hamiltonian pictures by choosing the time direction in line with one of the circumferences of the torus. In the large $R$ limit, this results in two expressions for the partition function. On the one hand, it can be expressed by the ground state energy of a finite volume system

$$Z(R, L) \sim e^{-RE(L)},$$ \hspace{1cm} (3.2.10)

and the other hand, by the free energy density of a finite temperature system

$$Z(R, L) \sim e^{-LRf(L)}.$$ \hspace{1cm} (3.2.11)

\(^1\)Here we exchanged the usual notation of the two directions.
Zamolodchikov calculated the free energy density taking the thermodynamic limit of the Bethe-Yang equations (3.2.1), and hence the name of the method thermodynamic Bethe ansatz (TBA).

The result for the finite volume ground state energy is

\[
E_{\text{TBA}}(L) = E_0(L) - BL = -\sum_\beta \int \frac{d\theta'}{2\pi} m_\beta \cosh(\theta) \log \left( 1 + e^{-\varepsilon_\beta(\theta)} \right),
\]

(3.2.12)

where \(\varepsilon_\beta\) are the so-called pseudoenergy functions satisfying the TBA integral equations

\[
\varepsilon_\alpha(\theta) = m_\alpha L \cosh(\theta) - \sum_\beta \int \frac{d\theta'}{2\pi} \varphi_{\alpha\beta}(\theta - \theta') \log \left( 1 + e^{-\varepsilon_\beta(\theta')} \right),
\]

(3.2.13)

with the kernels given by the logarithmic derivatives of the two-particle scattering phases

\[
\varphi_{\alpha\beta}(\theta) = -i \frac{\partial}{\partial \theta} \log S_{\alpha\beta}(\theta).
\]

(3.2.14)

The TBA energy is calculated with the normalisation in regards to the infrared limit and therefore it does not contain the bulk energy density

\[
E_{\text{TBA}}(L) = E_0(L) - BL.
\]

(3.2.15)

The result contains every exponential correction in the volume, all the F- and \(\mu\)-terms.

The vacuum expectation value of \(\Theta\), the trace of the stress-energy tensor\(^2\), can also be

\(^2\)The stress-energy tensor is normalised following the conventions of Zamolodchikov’s paper [Zam90].
expressed by the following derivative of $E_{TBA}(L)$ [Zam90]

$$
\langle \Theta \rangle_L = \langle \Theta \rangle_\infty + 2\pi \frac{1}{L} \frac{d}{dL} [LE_{TBA}(L)] = 2\pi \left[ \frac{E_{TBA}(L)}{L} + \frac{dE_{TBA}(L)}{dL} \right]
$$

$$
= \langle \Theta \rangle_\infty + \sum_a m_a \int_{-\infty}^{\infty} d\theta \frac{1}{1 + e^{\varepsilon_a(\theta)}} \left[ \cosh \theta \partial_L \varepsilon_a(\theta) - \frac{1}{L} \sinh \theta \partial_\theta \varepsilon_a(\theta) \right]
$$

(3.2.16)

where the derivatives of the TBA pseudoenergy solve the following linear equations

$$
\partial_L \varepsilon_\alpha(\theta) = m_a \cosh \theta + \sum_b \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi_{\alpha\beta}(\theta - \theta') \frac{1}{1 + e^{\varepsilon_\beta(\theta')}} \partial_L \varepsilon_\beta(\theta') ,
$$

$$
\partial_\theta \varepsilon_\alpha(\theta) = m_a L \sinh \theta + \sum_b \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi_{\alpha\beta}(\theta - \theta') \frac{1}{1 + e^{\varepsilon_\beta(\theta')}} \partial_\theta \varepsilon_\beta(\theta') .
$$

(3.2.17)

### 3.2.3 Excited state thermodynamic Bethe ansatz

From the work by Dorey and Tateo [DT96, DT98], it is known that starting from the TBA equation of the ground state one can reach the excited states by analytic continuation in the volume parameter. The same equations were also obtained in [BLZ97b] using a different approach. When performing the analytic continuation, singularities of the log $(1 + e^{-\varepsilon_\beta})$ terms, corresponding to locations where $Y_\beta = e^{\varepsilon_\beta} = -1$, cross the integration contour modifying the TBA equations as

$$
\varepsilon_\alpha(\theta) = m_a L \cosh (\theta) - \sum_{j=1}^{N} \eta_j \log S_{\alpha\alpha_j}(\theta - \tilde{\theta}_j) - \sum_\beta \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi_{\alpha\beta}(\theta - \theta') \log \left(1 + e^{-\varepsilon_\beta(\theta')} \right) ,
$$

$$
E_{TBA}(L) = \sum_{j=1}^{N} i m_{\alpha_j} \eta_j \sinh (\tilde{\theta}_j) - \sum_\beta \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m_\beta \cosh (\theta) \log \left(1 + e^{-\varepsilon_\beta(\theta)} \right) ,
$$

(3.2.18)

where the $\tilde{\theta}_i$ are the positions of the singularities of the pseudoenergy of species $\alpha_i$, which crossed the integration contour and satisfy the quantisation conditions

$$
\varepsilon_{\alpha_j}(\tilde{\theta}_j) = i\pi \left(2\tilde{I}_j + 1 \right) , \quad \tilde{I}_j \in \mathbb{Z}
$$

(3.2.19)

and the $\tilde{I}_j$ can be viewed as quantum numbers specifying the excited state. Such singularities are called *active*; their contribution further depends on the orientation $\eta_i$ of the integration.
contour around the singularity, which take the values $\pm 1$ if the singularity crossed the real axis from above/below respectively. The number and type of the active singularities depend on the excited state, and the positions of these singularities at a fixed volume $L$ fully specify the excited state\textsuperscript{3}. Therefore, the corresponding finite volume state can be denoted as

$$|\tilde{\theta}_1, \ldots, \tilde{\theta}_N\rangle_{\alpha_1, \ldots, \alpha_N, L}.$$ (3.2.20)

The term $\log (1 + e^{-\varepsilon\beta})$ also has singularities where the $Y_\beta = e^{\varepsilon\beta} = 0$. In [DT96, DT98], it was shown that whenever a singularity that corresponds to a zero of a $Y$ function crosses the integration contour, it does not generate new source terms to the TBA equations, but only rearranges the active singularities already present. The only exception is when such a singularity pinches the integration contour; for more details about dealing with this situation see Section 5.3.1.2.

In asymptotically large volume (IR limit), the integral term is negligible in (3.2.18), and the $\cosh$ term goes to infinity with the volume, while the value of the pseudoenergy is finite. This forces the imaginary part of the active singularities to specific values such that the part of the source terms containing $S$-matrices compensate the singularity, i.e. they are determined by the poles of the scattering matrix. The quantisation condition (3.2.19) for the real part of the active singularities, typically after using scattering matrix fusion properties (2.2.5), will reproduce the Bethe-Yang quantisation conditions (3.2.9). The relation between the quantum numbers $\{\tilde{I}_i\}$, the number and type of the active singularities, and the Bethe-Yang quantum numbers $\{I_i\}$ with the particle content of the Bethe-Yang equations (3.2.1), depends on the specific model considered.

The TBA system (3.2.18) can be recast in a universal functional form called the $Y$-system [Zam91, RTV93]

$$Y_\alpha\left(\theta - \frac{i\pi}{h}\right) Y_\alpha\left(\theta + \frac{i\pi}{h}\right) = \prod_{\beta=1}^{k} (1 + Y_\beta(\theta))^{I_{\alpha\beta}},$$ (3.2.21)

where $Y_\alpha(\theta) = e^{\varepsilon_\alpha(\theta)}$, $h$ is the Coxeter number, and $I_{\alpha\beta}$ is the incidence matrix of some diagram. In Section 5.3.1.2 we shall use the fact that $Y$-system relates the positions of the two types of logarithmic singularities of the TBA equations.

\textsuperscript{3} The modification of the original integration contour can alter the active singularity description of an excited state.
3.3 Form factors in finite volume

In finite volume, the matrix elements of operators are modified, but can be related to the infinite volume form factors. A formalism that gives the exact finite volume form factors to all orders in $L^{-1}$ was introduced in [PT08a, PT08b]. The finite volume behaviour of local matrix elements$^{4}$ can be given as [PT08a]

$$L\langle \{I'_1, \ldots, I'_m\}|O(0,0)|\{I_1, \ldots, I_n\}\rangle_L = \frac{F_{m+n}^O(\theta'_m + i\pi, \ldots, \theta'_1 + i\pi, \theta_1, \ldots, \theta_n)}{\sqrt{\rho(\theta_1, \ldots, \theta_n)\rho(\theta'_1, \ldots, \theta'_m)}} + O(e^{-\mu L}),$$

(3.3.1)

where $\{\theta_1, \ldots, \theta_n\}$ and $\{\theta'_1, \ldots, \theta'_m\}$ are solutions of the Bethe-Yang equations (3.2.1) with quantum numbers $\{I_1, \ldots, I_n\}$ and $\{I'_1, \ldots, I'_m\}$ respectively. $\rho$ functions are the density of states as introduced before in (3.2.7). The above relation is valid, provided that there are no disconnected terms i.e. the left and the right states do not contain particles with the same rapidity.

In the presence of nontrivial scattering, there are only two cases when exact equality of (at least some of) the rapidities can occur [PT08b]:

1. The two states are identical, i.e. $n = m$ and

$$\{I'_1, \ldots, I'_m\} = \{I_1, \ldots, I_n\},$$

(3.3.2)

in which case the corresponding diagonal matrix element can be written as a sum over all bipartite divisions of the rapidities $\{\theta\} = \{\theta_+\} \cup \{\theta_-\}$ (including the trivial ones when $\{\theta_+\}$ is the empty set or the complete set)

$$L\langle \{I_1 \ldots I_n\}|O|\{I_1 \ldots I_n\}\rangle_L = \frac{1}{\rho(\theta_1, \ldots, \theta_n)} \sum_{\{\theta_+\} \cup \{\theta_-\}} F_{2k,c}^O(\{\theta_+\}) \rho(\{\theta_-\}) + O(e^{-\mu L}),$$

(3.3.3)

where $k = |\{\theta_+\}|$ and the connected form factor $F_{2k,c}^O(\theta_1, \ldots, \theta_k)$ is defined as in (2.3.9). The density and the restricted densities are defined as in (3.2.7) and in (3.2.9). An alternative expression for the finite volume diagonal matrix element with the symmetric

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$^4$In this section we suppress the notation of particle types in the form factors, and subscripts the form factors denote the number of particle in the form factor.
evaluation of form factors (2.3.8) is

\[
L\langle \{I_1 \ldots I_n\}|O|\{I_1 \ldots I_n\}\rangle_L = \frac{1}{\rho(\theta_1, \ldots, \theta_n)} \sum_{\{\theta_+\} \cup \{\theta_-\}} F_{2k,s}^O(\{\theta_+\}) \rho(\{\theta_-\}) + O(e^{-\mu L}),
\]

(3.3.4)

where \(\rho(\{\theta_-\})\) is not the restricted density, but the normal density (3.2.7) for \(|\{\theta_-\}|\) particles with the according quantum numbers.

2. Both states are parity symmetric states in the spin zero sector, i.e.

\[
\{I_1, \ldots, I_n\} \equiv \{-I_n, \ldots, -I_1\},
\]

\[
\{I'_1, \ldots, I'_m\} \equiv \{-I'_m, \ldots, -I'_1\}.
\]

Furthermore, both states must contain one (or possibly more, in a theory with more than one species) particle of zero quantum number. Writing \(m = 2k + 1\) and \(n = 2l + 1\) and defining

\[
F_{k,l}(\theta'_1, \ldots, \theta'_k|\theta_1, \ldots, \theta_l) = \lim_{\epsilon \to 0} F_{2k+2l+2}(i\pi + \theta'_1 + \epsilon, \ldots, i\pi + \theta'_k + \epsilon, i\pi - \theta'_l + \epsilon, \ldots, i\pi - \theta'_1 + \epsilon, i\pi + \epsilon, 0, \theta_1, \ldots, \theta_l, -\theta_l, \ldots, -\theta_1),
\]

(3.3.5)

the formula for the finite-volume matrix element takes the form

\[
L\langle \{I'_1, \ldots, I'_k, 0, -I'_k, \ldots, -I'_1\}|O|\{I_1, \ldots, I_l, 0, -I_l, \ldots, -I_1\}\rangle_L (3.3.6)
\]

\[
= (\rho_{2k+1}(\theta'_1, \ldots, \theta'_k, 0, -\theta'_k, \ldots, -\theta'_1)\rho_{2l+1}(\theta_1, \ldots, \theta_l, 0, -\theta_l, \ldots, -\theta_1))^{-1/2} \\
\times \left[ F_{k,l}(\theta'_1, \ldots, \theta'_k|\theta_1, \ldots, \theta_l) \\
+ mL F_{2k+2}(i\pi + \theta'_1, \ldots, i\pi + \theta'_k, i\pi - \theta'_k, \ldots, i\pi - \theta'_1, \theta_1, \ldots, \theta_l, -\theta_l, \ldots, -\theta_1) \right] + O(e^{-\mu L}).
\]

The expressions (3.3.3) and (3.3.6) for the finite volume form factors with disconnected pieces are conjectures. They are well-supported both by analytical arguments [Sal00, PT08b] and numerical data [PT08b], but are not yet proven from first principles.

A prescription to construct the \(\mu\)-term corrections to form factors was introduced in [Poz08], however, the procedure is ambiguous in the general case, although it works for
general finite volume form factors. On the other hand, the exact finite volume dependence of diagonal matrix elements was recently conjectured in [Poz13], and Chapter 5 is devoted to generalising this conjecture and finding supporting evidence, both analytic and numerical.
Chapter 4

One-point functions in finite temperature
In this chapter, we focus on the calculation of finite temperature one-point functions with the help of form factors. First, we introduce the finite temperature one-point functions, then we review the proposal by LeClair and Mussardo to express it as an integral series containing form factors and TBA pseudoenergy. Finally, we present our numerical verification of the conjecture.

4.1 Finite temperature one-point functions

Using the Matsubara imaginary time formalism, we can express the finite temperature expectation value of an operator like

\[ \langle O(x,t) \rangle_L = \frac{1}{Z} \text{Tr} e^{-LH} O(x,t) , \]

where \( L \) is the circumference of the compactified time direction, i.e. the inverse temperature; \( H \) is the Hamiltonian of the system, \( Z = \text{Tr} e^{-LH} \) the partition function, and \( O \) is the given local operator. In the basis of multiparticle states (in a theory consisting only one type of particle with mass \( M \)), the trace reads as

\[ \langle O(x,t) \rangle_L = \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{d\theta_j}{2\pi} e^{-(\theta_j)L} \langle \theta_n, \ldots, \theta_1 | O(0,0) | \theta_1, \ldots, \theta_n \rangle , \]

where \( e(\theta) = M \cosh(\theta) \) is the energy of the particles parametrised by the rapidity. Due to translational invariance, the expression is independent of the coordinates \( x \) and \( t \).

However, the above expression is formal due to singularities of the diagonal matrix elements, and also of the partition function and therefore it is necessary to find a way to define it properly. To get insight into the nature of the singularities, let us consider the free fermionic theory, where the scattering matrix is \( S = -1 \). The form factor between the states denoted by the ordered sets \( A = \{\theta_1, \ldots, \theta_n\} \) and \( B = \{\theta'_1, \ldots, \theta'_m\} \) is described by the sum over the ways of breaking up the sets into subsets [Smi92] (a mathematically precise proof is presented in [BC15])

\[ \langle A | O | B \rangle = \sum_{A = A_1 \cup A_2, B = B_1 \cup B_2} S_{A,A_1} S_{B,B_1} \langle A_2 | B_2 \rangle \langle A_1 | O | B_1 \rangle_c , \]
where $S_{A,A_1}$ and $S_{B,B_1}$ are the product of S-matrices rearranging the rapidities like

$$
\langle A | = S_{A,A_1} \langle A_2, A_1 | , \quad |B\rangle = S_{B,B_1} |B_1, B_2\rangle ,
$$

(4.1.4)

and the connected form factors of the operator are defined as in (2.3.9). The inner product $\langle A_2|B_2\rangle$ can be calculated introducing the free creation-annihilation operators $|\theta_1,\ldots,\theta_n\rangle = A^\dagger(\theta_1)\ldots A^\dagger(\theta_n)|0\rangle$ with the anticommutator $\{A(\theta), A^\dagger(\theta')\} = 2\pi\delta(\theta - \theta')$.

Using (4.1.3) in (4.1.2), one finds that the $\delta$-functions have two effects. On the one hand, they reproduce the partition function in the numerator, and on the other hand, they modify the integration measure. As a consequence, the expression for the finite temperature one-point function in the free theory becomes [LLSS96]

$$
\langle O(x,t)\rangle_L = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{d\theta_j}{2\pi} \frac{e^{-\epsilon(\theta_j)L}}{1 + e^{-\epsilon(\theta_j)L}} \langle \theta_n,\ldots,\theta_1|O(0,0)\rangle |\theta_1,\ldots,\theta_n\rangle_c .
$$

(4.15)

LeClair and Mussardo [LM99] used this result to conjecture the finite temperature one-point function in an interactive integrable theory. This conjecture is the topic of the next section.

4.2 The Leclair-Mussardo conjecture

4.2.1 The conjecture

From the derivation of the TBA equation [Zam90], we know that the torus partition function, when the dimension $R$ is large\(^1\), can be expressed as

$$
Z(R,L) = \exp \left[ MR \int \frac{d\theta}{2\pi} \cosh(\theta) \log \left( 1 + e^{-\epsilon(\theta)} \right) \right] .
$$

(4.2.1)

In [LM99], LeClair and Mussardo showed that similarly to the non-relativistic case [YY69], the partition function can be described by free fermionic particles with dispersion relation governed by the TBA pseudoenergy

$$
Z(R,L) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{d\theta_j}{2\pi} e^{-\epsilon(\theta_j)} \langle \theta_n,\ldots,\theta_1|\theta_1,\ldots,\theta_n\rangle_c .
$$

(4.2.2)

This realisation led them to conjecture the finite temperature one-point function for the inter-

\(^{1}\)We would like to note again that our notation of $R$ and $L$ is exchanged compared to the usual notation.
acting case by replacing the dispersion relation in (4.1.5) from $e(\theta)L$ to the TBA pseudoenergy $\varepsilon(\theta)$. The conjecture for $k$ species of massive particles takes the form [LM99]

$$\langle O \rangle_L = \sum_{n_1,\ldots,n_k=0}^{\infty} \left( \prod_{j=1}^{k} n_j! \right)^{-1} \int_{-\infty}^{\infty} \prod_{j=1}^{\tilde{N}} \frac{d\theta_j}{2\pi} \left[ 1 + e^{\varepsilon_j(\theta_j)} \right] F_{2n_1,\ldots,2n_k,c}^{O}(\theta_1,\ldots,\theta_{\tilde{N}}) \tag{4.2.3}$$

where $F_{2n_1,\ldots,2n_k,c}^{O}$ are the connected diagonal form factors of the operator $O$ (2.3.9), $n_i$ are the number of particles of species $i$, $\tilde{N} = \sum n_i$ is the total number of particles, and the $j$th particle has rapidity $\theta_j$ and species $\beta_j$.

The duality between the finite volume and finite temperature description, mentioned in the Section 3.2.2, means that the LeClair-Mussardo conjecture can be understood, not only as an expression for the finite temperature one-point function, but also as the expectation value of the operator in the finite volume vacuum state

$$L \langle 0 | O | 0 \rangle_L .$$

### 4.2.2 Evidence for the LeClair-Mussardo series: TBA and finite volume form factors

LeClair and Mussardo showed that for the trace of stress-energy tensor their series coincides with the result from the TBA equations (3.2.16), derived by Zamolodchikov in [Zam90]. Saleur extended the argument and proved the equivalence of the TBA and the LM series for one-point functions of conserved charge densities [Sal00].

For general operators, where a TBA construction is not known, the LM series was demonstrated to be true up to (and including) three particle terms in [PT08b], and a proof to all orders was presented in [Poz11]. We remark that these results were proven using the finite volume form factor formalism introduced in [PT08a, PT08b], and the proofs relies on the expression of finite volume diagonal matrix elements (3.3.3), which is still only a conjecture, albeit well-supported as mentioned in Section 3.3.

Since the LM series is not proven from first principles, we present a numerical verification in the next section.

### 4.3 Numerical check of the LeClair-Mussardo series

To validate the LM series numerically, we chose the $T_2$ model for the calculation. There are several advantages in this choice. First, all the data necessary to calculate the series...
(4.3.1): the scattering theory, the form factors (of primary fields) and the TBA equations, are known for this model. Second, the $T_2$ model contains operators with known form factors that are not related to the trace of the stress-energy tensor. As mentioned before, the trace of the stress-energy tensor is not interesting from the numerical verification point of view, since the conjecture is proven for this operator due to the TBA equations. Moreover, for the $T_2$ model there exists an independent numerical method to calculate the finite volume expectation value of the operators, the Truncated Conformal Space Approach (TCSA). The details of the $T_2$ model are summarised in Appendix A, and the TCSA method with its RG-extrapolated extension is summarised in Appendix B.

The $T_2$ model contains two massive particles, and the LM conjecture (4.2.3) takes the following form in it:

$$
\langle O \rangle_L = \sum_{n_1,n_2=0}^{\infty} \frac{1}{n_1!n_2!} \int_{-\infty}^{\infty} \prod_{j=1}^{N} \frac{d\theta_j}{2\pi} \left[ 1 + e^{\varepsilon_{\alpha_j}(\theta_j)} \right] \times F_{2n_1,2n_2,c}^{O}(\theta_1,\ldots,\theta_{\tilde{N}}),
$$

(4.3.1)

where $n_1$ and $n_2$ are the number of particles with mass $m_1$ and $m_2$ respectively. $\varepsilon_{\alpha}$ are the pseudoenergy functions that satisfy the TBA integral equation

$$
\varepsilon_{\alpha}(\theta) = m_\alpha L \cosh \theta - \sum_{b=1}^{2} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi_{ab}(\theta - \theta') \log \left( 1 + e^{-\varepsilon_{\beta}(\theta')} \right),
$$

(4.3.2)

with $\varphi_{ab}$ the derivatives of the phase shift of the two-particle S-matrices defined in (3.2.14), and $F_{2n_1,2n_2,c}^{O}$ is the connected form factor defined as (2.3.9).

In large volume, a rough estimate for the magnitude of the terms in the series comes from the behaviour of the filling fractions

$$
\prod_i \frac{1}{1 + e^{\varepsilon_{\alpha_j}(\theta_j)}} \lesssim e^{-(\sum_i m_i)L},
$$

where $m_i$ are the masses of particles contained in the given state. Using this estimate, we can identify the terms of the series that give the dominant contribution. However, with decreasing volume the ordering of terms can change depending on the behaviour of the pseudoenergy functions; in addition, to maintain the accuracy it is necessary to add progressively more terms. As a result, the form factor series (4.2.3) is effectively an infrared (low energy/large
Table 4.3.1. The terms incorporated from the form factor series into the numerical calculation.

<table>
<thead>
<tr>
<th>label</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>11</th>
<th>12</th>
<th>111</th>
<th>22</th>
<th>112</th>
</tr>
</thead>
<tbody>
<tr>
<td># of type-1 integrals</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td># of type-2 integrals</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>magnitude/L</td>
<td>0</td>
<td>$m_1$</td>
<td>$m_2$</td>
<td>$2m_1$</td>
<td>$m_1 + m_2$</td>
<td>$3m_1$</td>
<td>$2m_2$</td>
<td>$2m_1 + m_2$</td>
</tr>
</tbody>
</table>

Numerical evaluation of connected diagonal form factors of the $T_2$ model is rather non-trivial to perform in a sufficiently fast and numerically stable way for a large number of particles. We describe the required tricks in Appendix A.3. The procedure can be straightforwardly generalised to connected diagonal form factors in other integrable models. For our numerical calculations, we implemented the terms with less than 4 integrals, since for higher terms the number of integrals and the size of the form factors makes the numerical integration too time-consuming; in addition, the terms incorporated already show an excellent agreement with the conjecture. Table 4.3.1 shows the terms calculated for numerics. The label 0 indicates the contribution from the infinite volume vacuum expectation value of the operator. For the numerical integration we used the Cuba library routines [Hah06], called from inside Wolfram Mathematica.²

$T_2$ is the perturbation of $\mathcal{M}_{2,7}$ minimal model by its $\Phi_{1,3}$ operator, and the trace of the stress-energy tensor is given by the relation

$$\Theta = T_{\mu}^{\mu} = 2\pi\lambda(2\Delta_{1,3} - 2)\Phi_{1,3},$$ (4.3.3)

where $\lambda$ is the coupling constant and $\Delta_{1,3}$ the conformal weight of the operator. Because of (4.3.3), the LM series of the $\Phi_{1,3}$ operator can be evaluated from TBA, and it does not give us an independent validation. However, the error in the numerical solution of the TBA equations (4.3.2) can be made very small (in our case it was of the order $10^{-14}$), and therefore this can be used to check our procedures for evaluation of the LM series; we found excellent agreement. Another application is to estimate the accuracy of the RG-extrapolated TCSA. Table 4.3.2 shows the difference of the expectation value for the $\Phi_{1,3}$ operator between the TBA prediction and RG-extrapolated TCSA evaluation in units of $m_1$ for several values of the dimensionless volume parameter, $l = m_1 L$. The results also shows the efficiency of the RG-extrapolated TCSA by comparing the RG-extrapolated TCSA value to those evaluated

at the highest value of the cutoff. As exemplified by the data at $l = 15$, the extrapolation results in an improvement of accuracy of almost two orders of magnitude for volumes $l \gtrsim 10$.

Table 4.3.2. The difference $\delta f_{13} = m_1^{6/7} (\langle \Phi_{1,3} \rangle_{\text{TBA}} - \langle \Phi_{1,3} \rangle_{\text{TCSA}})$ between the RG-extrapolated TCSA evaluation and the TBA prediction for the VEV of $\Phi_{1,3}$. The second row shows the difference between the raw TCSA value at the cut-off value $e_{\text{cut}} = 30$ and the TBA.

Now we turn to the evaluation of the LM series for the $\Phi_{1,2}$ operator. At infinite volume, only the infinite volume vacuum expectation value (A.2.4) contributes to the LM series, while for small volumes almost all terms are relevant. Table 4.3.3 shows the value of the RG-extrapolated TCSA evaluation and the LM series with the mentioned contributions for $\Phi_{1,2}$, while Table 4.3.4 shows the difference of the LM series from the RG-extrapolated TCSA data while adding more and more contributions.

For $l > 1$, there is a steady improvement as more and more terms are added, and it is clear from the table that contributions from higher states are switched on at progressively lower volumes. Comparing the columns labeled +111 and +12, it can be seen that adding the 111 contribution makes the agreement of the LM series with TCSA worse; the reason is that the term 22 is of the same order of magnitude as the contribution from 111, so consistency requires adding them to the series together. Indeed, the deviation reported in column +22 is smaller than the one in column +12.

Table 4.3.3. The values for $im_1^{4/7} \langle \Phi_{1,2} \rangle$ from the RG-extrapolated TCSA evaluation and the Leclair-Mussardo series.

For $l \lesssim 1$, the series does not converge very well; e.g. for $l = 0.5$, there is no sign of any convergence. However, for such small values of the volume higher terms of the LM series would still be significant. Indeed, the leading corrections following the 112 term are the
Table 4.3.4. The difference $\delta f_{12} = \frac{1}{4^7} \left( \langle \Phi_{1,2} \rangle_{\text{LM}} - \langle \Phi_{1,2} \rangle_{\text{TCSA}} \right)$ between the RG-extrapolated TCSA evaluation and the Leclaire-Mussardo series, depending on the multi-particle contributions included in the latter.

contributions 1111 and 122, which can be estimated to be of order

$$e^{-4m_1L} \quad \text{and} \quad e^{-(m_1+2m_2)L}$$

(4.3.4)

which at $l = 0.5$ give approximately 14% and 12%, respectively. This agrees well with the magnitude of the deviation at $l = 0.5$, as can also be seen from Table 4.3.3.

For volumes $l < 3.5$, the estimated TCSA error is smaller than $10^{-6}$, and so the deviation between TCSA and the LM series is dominated by the higher corrections to the LM series. However, the calculation of higher contributions becomes progressively slower as the number of particles to include increases (122 can be evaluated using a 10-particle form factor, cf. the remark at the end of Section A.2). Similarly to the case of 111 and 22, contributions 1111 and 122 are roughly of the same order and so they must be added to the series together. Calculating the 4 dimensional integration numerically proved to be rather difficult, so we have not evaluated the contributions 1111 and 122.

However, there is a way to verify the consistency of the above considerations further. We fitted the deviation in the last column of Table 4.3.4, the difference from the TCSA, by the ansatz

$$a e^{-4m_1L} + b e^{-(m_1+2m_2)L},$$

(4.3.5)
dictated by the particle content of the states $1111$ and $122$. As shown in Figure 4.3.1, the fit was very successful. In addition, the parameters $a$ and $b$ turned out to be of the same order of magnitude, as expected from the above considerations.

For larger volumes the agreement cannot be improved by including other contributions, since it is dominated by the residual truncation error of the RG-extrapolated TCSA. We remark, that a similar evaluation of the LM series for the case of $\Phi_{1,3}$ operator, the agreement of the LM series and TBA calculations gave an agreement of precision $10^{-13}$ with the TBA for volumes $l > 7$. Figure 4.3.2 illustrates the improvement of the LM series by adding more terms, as compared to the RG-extrapolated TCSA data.

### 4.4 Summary

In this chapter, we presented the LeClair-Mussardo conjecture, a series using form factors and TBA pseudoenergies to describe the finite temperature expectation value of a local operator in integrable theories with diagonal scattering. The series can also be seen as an expression for the finite volume expectation value of the operator due to the relativistic
Figure 4.3.2. Leclair-Mussardo series against RG-extrapolated TCSA evaluation for the VEV of $\Phi_{1,2}$ by taking account more and more contributions, in units of $m_1$

invariance of the two dimensional QFT considered. This alternative viewpoint is going to be the starting point of the next chapter, where we consider the generalisation of the LM series to express finite volume expectation values in excited states.

We summarised arguments and literature supporting the correctness of the LM conjecture, however, to only known proof of it depends on the validity of the finite volume diagonal form factor formula, for which there is no proof from first principles yet. We presented our numerical verification of the LM series in the $T_2$ model and found precise agreement in the parameter space expected. As argued in Section 4.3, the $T_2$ model is a perfect framework for the validation, since all the ingredients of the LM series are known in this model, the model contains operators for which the LM series is not proven already, and we also have an independent numerical method to crosscheck the results. Therefore we shall continue to use the $T_2$ model as a laboratory for numerical testing of theoretical ideas on the finite volume dependence of matrix elements.
Chapter 5

Finite size diagonal matrix elements
In this chapter, we present the extension of the LeClair-Mussardo series for finite volume diagonal matrix elements, conjectured by Pozsgay in [Poz13]. We prove the conjecture for the trace of the stress-energy tensor and present numerical evidence for its validity in the $T_2$ model.

5.1 Finite volume expectation values in excited states: the conjecture

The TBA equations (3.2.18) describe the finite volume ground state energy of the system. Dorey and Tateo showed in [DT96] how to arrive to the finite volume energy of the excited state by analytic continuation of the TBA equations in the volume parameter. Intuitively, analytical continuation is expected to connect not only the energy of the eigenstates (3.2.18), but also other quantities such as expectation values of local operators. As mentioned in Section 4.2.1, the LeClair-Mussardo series can be interpreted as expectation value of operators in the finite volume vacuum state, therefore, the analytic continuation of it is expected to describe the expectation value of operators in excited states, i.e. the exact diagonal finite volume matrix elements of operators. Analytic continuation on the LM series modifies the integration contours, due to the movement of TBA active singularities. It was shown in [Poz13] how to perform the residue integrals over the modified contours and resum the terms into a compact form, arriving to an analytically continued Leclair-Mussardo conjecture. That calculation was carried out in the sinh-Gordon theory, where the excited TBA system is still a conjecture [Tes08], but the result passes several consistency checks. Namely, the first $e^{-mL}$ corrections in the infrared limit agree with theoretical expectations and the result also agrees with the TBA results for the trace of the stress-energy tensor.

Let us now state the conjecture for the general form of the finite volume expectation values in the excited state described by the set of active singularities $\{\bar{\theta}_1, \ldots, \bar{\theta}_N\}$ according to the excited state TBA equations (3.2.18). It contains two kind of quantities, the “dressed version” of diagonal form factors and the densities of active singularities.
**Definition 1.** The dressed diagonal form factors of the local operator $\mathcal{O}$ are

$$
\mathcal{D}_\varepsilon^\mathcal{O} (\bar{\theta}_1, \ldots, \bar{\theta}_l) := \sum_{n_1, \ldots, n_k = 0}^\infty \prod_j n_j! \int_{-\infty}^{\infty} \prod_{j=1}^{\tilde{N}} \frac{d\theta_j}{2\pi} \left[ 1 + e^{\varepsilon \rho_j (\theta_j)} \right] 
\times F_{2l, 2n_1, \ldots, 2n_k, c}^\mathcal{O} (\bar{\theta}_1, \ldots, \bar{\theta}_l, \theta_1, \ldots, \theta_{\tilde{N}}),
$$

where $\bar{\theta}_i$ rapidities belong to a subset of active singularities, with the $i$th one corresponding to species $\alpha_i$, and $\tilde{N} = \sum_{j=1}^k n_j$ is the total number of integration.

To obtain the densities of active singularities, consider the derivative matrix with respect to the singularity positions

$$
K_{jl} = \frac{\partial Q_j}{\partial \theta_l}, \quad (5.1.2)
$$

of the quantisation conditions (3.2.19)

$$
Q_j = i\eta_j \varepsilon \alpha_j (\bar{\theta}_j | \bar{\theta}_1, \ldots, \bar{\theta}_{\tilde{N}}), \quad (5.1.3)
$$

satisfied by the positions of the active singularities.

**Definition 2.** The density of active singularities (in rapidity space) is the determinant of the derivative matrix

$$
\rho (\bar{\theta}_1, \ldots, \bar{\theta}_N) = \det K_{jl}. \quad (5.1.4)
$$

**Definition 3.** For any bipartite partition $\{ \bar{\theta}_1, \ldots, \bar{\theta}_N \} = \{ \bar{\theta}_+ \} \cup \{ \bar{\theta}_- \}$ of the active singularities, the restricted density of active singularities in the subset $\{ \bar{\theta}_+ \}$ relative to $\{ \bar{\theta}_- \}$ is defined by

$$
\rho (\{ \bar{\theta}_+ \} \mid \{ \bar{\theta}_- \}) = \det K_+, \quad (5.1.5)
$$

where $K_+$ is the sub-matrix corresponding to the subset of active singularities $\{ \bar{\theta}_+ \}$.
Using the above definitions, the main result can be stated as follows:

**Conjecture 4.** The exact finite volume expectation values of an operator $O$ in any finite volume state can be written as

$$L \langle \bar{\theta}_1, \ldots, \bar{\theta}_N | O | \bar{\theta}_1, \ldots, \bar{\theta}_N \rangle_L = \frac{1}{\rho (\bar{\theta}_1, \ldots, \bar{\theta}_N)} \sum_{\{\bar{\theta}_+\} \cup \{\bar{\theta}_-\}} D^O_\varepsilon (\{\bar{\theta}_+\}) \rho (\{\bar{\theta}_-\} | \{\bar{\theta}_+\}). \quad (5.1.6)$$

This conjecture was verified by explicit calculation for the case of one and two active singularity in the sinh-Gordon theory [Poz13].

For the trace of the stress-energy tensor the conjecture is equivalent to the excited state TBA equations for any state, similarly to the Leclair-Mussardo series (4.2.3) which for the trace of the stress-energy tensor is equivalent to the ground state TBA (3.2.16) [LM99]. The proof of this equivalence is given in Section 5.2.

The infrared limit of the formula reproduces previously known results for finite volume diagonal form factors (3.3.3). As mentioned in Section 3.2.3, in large volume the imaginary parts of the active singularities tend to fixed values, which are determined by the poles of the scattering matrix [DT96, DT98], and the real parts $\{\vartheta_j\} = \{\text{Re} \bar{\theta}_i\}$ of the singularity positions can be interpreted as rapidities of on-shell particles, where usually one particle is described by more than one singularity positions, which all have the same real parts. The quantisation conditions reduce to the logarithm Bethe-Yang equations

$$m_{\alpha_j} L \sinh \vartheta_j - i \sum_{k \neq j} \log S_{\alpha_j \alpha_k} (\vartheta_j - \vartheta_k) = 2 \pi I_j \quad (5.1.7)$$

where the momentum quantum numbers $\bar{I}_j$ are related to the TBA quantum numbers $I_j$ in (3.2.19). The density of active singularities specified in Definition 2 reduces to the usual density of states in rapidity space (3.2.7), while the restricted density in Definition 3 turns into the restricted density (3.2.9) used in the diagonal form factor formula (3.3.3).

In the same limit, the “dressed” diagonal form factors reduce to connected diagonal form factors (2.3.9), and for theories where particles are represented by a single active singularity, formula (5.1.6) reduces to finite volume diagonal matrix elements (3.3.3), which are valid up to exponential corrections in the volume.
In theories, where a particle is represented by several active singularities, the particle can be considered as a bound state of the active singularities. In infinite volume, this does not make any difference due to bootstrap equations satisfied by the scattering matrix, but in finite volume the composite nature of the particles gives exponential corrections, which are exactly the $\mu$-term corrections to the form factor described in [Poz08]. However, while in [Poz08] the description of the particles as composite objects was still ambiguous, the excited state TBA equation gives a clear prescription valid for every value of the volume. In line with the usual terminology of finite volume corrections [Lüs86a, KM91, BJ09], the terms in (5.1.6) containing rapidity integrations, originating from either the quantisation conditions (3.2.19) or the dressed form factors (5.1.1), give the F-term corrections, which describe virtual particle loops winding around the finite volume cylinder.

5.2 Equivalence of the form factor series and the TBA for the trace of the stress-energy tensor

In this section, we present the equivalence of the conjectured form factor series for excited states (5.1.6) and the TBA equations for $\Theta$, the trace of the stress-energy tensor. We proceed in three steps. First, we explicitly evaluate the TBA prediction for $\langle \Theta \rangle$, then recast it in a form which can be matched with the dependence of (5.1.6) on the densities, and then prove that the rest of the formula matches the dressed form factors of $\Theta$.

5.2.1 $\langle \Theta \rangle$ from TBA

As described in [Zam90], the expectation value of the trace of the stress-energy tensor can be expressed in the following way (cf. (3.2.16))

$$
\langle \Theta \rangle_L = \langle \Theta \rangle_\infty + 2\pi \left[ \frac{E_{\text{TBA}}(L)}{L} + \frac{dE_{\text{TBA}}(L)}{dL} \right].
$$

(5.2.1)

For an excited state with $N$ active singularities we obtain

$$
E_{\text{TBA}}(L) = \sum_{j=1}^{N} im_{\alpha_j} \eta_j \sinh(\bar{\theta}_j) - \sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \sinh(\theta) \frac{\partial \varepsilon_{\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} ,
$$

$$
dE_{\text{TBA}}(L) = \sum_{j=1}^{N} im_{\alpha_j} \eta_j \cosh(\bar{\theta}_j) \frac{d\bar{\theta}_j}{dL} + \sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \cosh(\theta) \frac{\partial \varepsilon_{\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} ,
$$

(5.2.2)
where we performed a partial integration in the energy expression. The derivatives of the pseudoenergy satisfy the following linear equations

\[
\begin{align*}
\partial_{\theta} \varepsilon_{\alpha} (\theta) &= m_{\alpha} L \sinh (\theta) - \sum_{j=1}^{N} \imath \eta_{j} \varphi_{\alpha \alpha_{j}} (\theta - \bar{\theta}_{j}) + \sum_{\beta} \int \frac{d\theta'}{2\pi} \varphi_{\alpha \beta} (\theta - \theta') \frac{\partial_{\theta} \varepsilon_{\beta} (\theta')}{{1 + e^{\varepsilon_{\beta} (\theta')}}}, \\
\partial_{L} \varepsilon_{\alpha} (\theta) &= m_{\alpha} \cosh (\theta) + \sum_{j=1}^{N} \imath \eta_{j} \varphi_{\alpha \alpha_{j}} (\theta - \bar{\theta}_{j}) \frac{d\bar{\theta}_{j}}{dL}, \quad (5.2.3)
\end{align*}
\]

The linearity of the above equations can be exploited by introducing new functions \(f\) satisfying the following equations

\[
\begin{align*}
f_{s,\alpha} (\theta) &= m_{\alpha} \sinh (\theta) + \sum_{\beta} \int \frac{d\theta'}{2\pi} \varphi_{\alpha \beta} (\theta - \theta') \frac{f_{s,\beta} (\theta')}{{1 + e^{\varepsilon_{\beta} (\theta')}}}, \\
f_{c,\alpha} (\theta) &= m_{\alpha} \cosh (\theta) + \sum_{\beta} \int \frac{d\theta'}{2\pi} \varphi_{\alpha \beta} (\theta - \theta') \frac{f_{c,\beta} (\theta')}{{1 + e^{\varepsilon_{\beta} (\theta')}}}, \\
f_{j,\alpha} (\theta) &= \varphi_{\alpha \alpha_{j}} (\theta - \bar{\theta}_{j}) + \sum_{\beta} \int \frac{d\theta'}{2\pi} \varphi_{\alpha \beta} (\theta - \theta') \frac{f_{j,\beta} (\theta')}{{1 + e^{\varepsilon_{\beta} (\theta')}}}, \quad j = 1, \ldots, N \quad (5.2.4)
\end{align*}
\]

which can be used to express the derivatives as

\[
\begin{align*}
\partial_{\theta} \varepsilon_{\alpha} (\theta) &= L f_{s,\alpha} (\theta) + \sum_{j=1}^{N} (-\imath \eta_{j}) f_{j,\alpha} (\theta), \\
\partial_{L} \varepsilon_{\alpha} (\theta) &= f_{c,\alpha} (\theta) + \sum_{j=1}^{N} \left( \imath \eta_{j} \frac{d\bar{\theta}_{j}}{dL} \right) f_{j,\alpha} (\theta). \quad (5.2.5)
\end{align*}
\]

Inserting these relation into (5.2.3)

\[
E_{TBA} (L) = \sum_{j=1}^{N} \imath m_{\alpha} \eta_{j} \sinh (\bar{\theta}_{j}) - \sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \sinh (\theta) \frac{L f_{s,\beta} (\theta) + \sum_{j=1}^{N} (-\imath \eta_{j}) f_{j,\beta} (\theta)}{{1 + e^{\varepsilon_{\beta} (\theta)}}}.
\]
\[
\frac{dE_{\text{TBA}}(L)}{dL} = \sum_{j=1}^{N} i m_{\alpha_j} \eta_j \cosh(\bar{\theta}_j) \frac{d\bar{\theta}_j}{dL} \\
+ \sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \cosh(\theta) \frac{f_{c,\beta}(\theta) + \sum_{j=1}^{N} \left( i \eta_j \frac{d\bar{\theta}_j}{dL} \right) f_{j,\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} ,
\] (5.2.6)

the expectation value \(\langle \Theta \rangle\) takes the form

\[
\langle \Theta \rangle = \langle \Theta \rangle_{\infty} + \sum_{j=1}^{N} \frac{i}{L} m_{\alpha_j} \eta_j \sinh(\bar{\theta}_j) - \sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \sinh(\theta) \frac{f_{s,\beta}(\theta) + \sum_{k=1}^{N} \left( \frac{-i}{L} \eta_j \right) f_{j,\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} \\
+ \sum_{j=1}^{N} i m_{\alpha_j} \eta_j \cosh(\bar{\theta}_j) \frac{d\bar{\theta}_j}{dL} + \sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \cosh(\theta) \frac{f_{c,\beta}(\theta) + \sum_{j=1}^{N} \left( i \eta_j \frac{d\bar{\theta}_j}{dL} \right) f_{k,\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} \\
= \langle \Theta \rangle_{\infty} + \sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \sinh(\theta) \frac{f_{c,\beta}(\theta) - m_{\beta} \sinh(\theta) f_{s,\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} \\
+ \sum_{j=1}^{N} \frac{i}{L} \eta_j \left[ m_{\alpha_j} \sinh(\bar{\theta}_j) + \sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \sinh(\theta) \frac{f_{j,\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} \right] \\
+ \sum_{k=1}^{N} i \eta_j \frac{d\bar{\theta}_j}{dL} \left[ m_{\alpha_j} \cosh(\bar{\theta}_j) + \sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \cosh(\theta) \frac{f_{j,\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} \right] .
\] (5.2.7)

Using that the derivatives of phase shift are even functions \(\varphi_{\alpha\beta}(\theta) = \varphi_{\beta\alpha}(-\theta)\), and the definition of \(f_i\) and \(f_{s,c}\) one can easily see that

\[
\sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \sinh(\theta) \frac{f_{j,\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} = \sum_{\beta} \int \frac{d\theta}{2\pi} \varphi_{\alpha_j\beta}(\bar{\theta}_j - \theta) \frac{f_{s,\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} ,
\]
\[
\sum_{\beta} \int \frac{d\theta}{2\pi} m_{\beta} \cosh(\theta) \frac{f_{j,\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} = \sum_{\beta} \int \frac{d\theta}{2\pi} \varphi_{\alpha_j\beta}(\bar{\theta}_j - \theta) \frac{f_{c,\beta}(\theta)}{1 + e^{\varepsilon_{\beta}(\theta)}} ,
\] (5.2.8)
and so $\langle \Theta \rangle$ simplifies to

$$
\frac{\langle \Theta \rangle}{2\pi} = \frac{\langle \Theta \rangle}{2\pi} + \sum_{\beta} \int_{\theta} \frac{m_{\beta} \cosh (\theta) f_{c,\beta} (\theta) - m_{\beta} \sinh (\theta) f_{s,\beta} (\theta)}{1 + e^{\epsilon_{\beta} (\theta)}} d\theta
$$

$$
+ \sum_{j=1}^{N} i \eta_{j} f_{s,\alpha_{j}} (\bar{\theta}_{j}) + \sum_{k=1}^{N} i \eta_{j} f_{c,\alpha_{j}} (\bar{\theta}_{j}) \frac{d\bar{\theta}_{j}}{dL}.
$$

(5.2.9)

The derivatives of the active singularity positions can be expressed using the quantisation conditions (5.1.3)

$$
\frac{dQ_{i}}{dL} = \sum_{j} \frac{\partial Q_{i}}{\partial \bar{\theta}_{j}} \frac{d\bar{\theta}_{j}}{dL} + \frac{\partial Q_{i}}{\partial L} = 0,
$$

$$
\frac{d\bar{\theta}_{i}}{dL} = - \sum_{j} K_{ij}^{-1} \frac{\partial Q_{j}}{\partial L},
$$

(5.2.10)

where

$$
K_{jl} = \frac{\partial Q_{j}}{\partial \bar{\theta}_{l}} = i \eta_{j} \frac{\partial \bar{\epsilon}_{\alpha_{j}} (\bar{\theta}_{j})}{\partial \bar{\theta}_{l}}
$$

$$
= i \eta_{j} \left\{ \begin{array}{ll}
L f_{s,\alpha_{j}} (\bar{\theta}_{j}) + \sum_{m \neq j} (-i \eta_{m}) f_{m,\alpha_{j}} (\bar{\theta}_{j}) & j = l \\
(i \eta_{l}) f_{l,\alpha_{j}} (\bar{\theta}_{j}) & j \neq l
\end{array} \right. .
$$

(5.2.11)

The notation $\bar{\epsilon}_{\alpha_{j}} (\bar{\theta}_{j}|\bar{\theta}_{1}, \ldots, \bar{\theta}_{N})$ means that we consider the pseudoenergy functions $\bar{\epsilon}_{\alpha_{j}} (\theta)$ obtained as a solution of the excited TBA system (5.2.2) for a fixed arrangement of singularity positions $\bar{\theta}_{1}, \ldots, \bar{\theta}_{N}$, evaluated at $\theta = \bar{\theta}_{j}$. This is nothing else than the function specifying the exact quantisation conditions for the singularity positions via (3.2.19), written in a more detailed form specifying its dependence on all the singularity positions explicitly.

Introducing the following combinations

$$
N_{j} = i \eta_{j} f_{s,\alpha_{j}} (\bar{\theta}_{j}) , \quad N_{\varphi,jl} = \eta_{j} \eta_{l} f_{l,\alpha_{j}} (\bar{\theta}_{j}) ,
$$

(5.2.12)

where $N_{\varphi,jl} = N_{\varphi,lj}$, $K$ can be rewritten as

$$
K_{jl} = \left\{ \begin{array}{ll}
L N_{j} + \sum_{m \neq j} N_{\varphi,jm} & j = l \\
- N_{\varphi,jl} & j \neq l
\end{array} \right. .
$$

(5.2.13)

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The explicit volume derivative of the quantisation condition is
\[
\frac{\partial Q_j}{\partial L} = i\eta_j \frac{\partial \bar{\alpha}_j(\bar{\theta}_j, \ldots, \bar{\theta}_N)}{\partial L} = i\eta_j f_{c,\alpha_j}(\bar{\theta}_j). \tag{5.2.14}
\]

Introducing
\[
\mathcal{M}_j = i\eta_j f_{c,\alpha_j}(\bar{\theta}_j), \tag{5.2.15}
\]
the derivative of the singularity position takes the following form
\[
\frac{d\bar{\theta}_j}{dL} = -\sum_{l} K_{jl}^{-1} \mathcal{M}_l, \tag{5.2.16}
\]
such as \(\langle \Theta \rangle\)
\[
\frac{\langle \Theta \rangle}{2\pi} = \frac{\langle \Theta \rangle}{2\pi} + \sum_{\beta} \int \frac{d\theta}{2\pi} \frac{m_\beta \cosh(\theta) f_{s,\beta}(\theta) - m_\beta \sinh(\theta) f_{s,\beta}(\theta)}{1 + e^{\varepsilon_\beta(\theta)}}
+ \sum_{i=1}^{N} \frac{N_i}{L} - \sum_{i,j=1}^{N} \mathcal{M}_i K_{ij}^{-1} \mathcal{M}_j. \tag{5.2.17}
\]

This is our final form for the TBA result for \(\langle \Theta \rangle\).

### 5.2.2 Isolating the singularity density terms

To see the equivalence of \(\langle \Theta \rangle\) to the form factor series (5.1.6), the terms containing \(N_i\) and \(\mathcal{M}_i\) need to be rearranged in order to match the structure of the singularity density terms in (5.1.6).

Let us start with the term
\[
\sum_{i,j=1}^{N} \mathcal{M}_i K_{ij}^{-1} \mathcal{M}_j. \tag{5.2.18}
\]
The inverse of \(\mathcal{K}\) can be expressed by its co-factor matrix \(\mathcal{C}\)
\[
K_{ij}^{-1} = \frac{C_{ji}}{\det \mathcal{K}}. \tag{5.2.19}
\]
The diagonal elements of the co-factor matrix are just the principal minors of \(\mathcal{K}\):
\[
C_{ii} = \det \mathcal{K} \{ \{i\} \}, \tag{5.2.20}
\]
where $\mathcal{K}(I)$ denotes the matrix obtained by omitting from $\mathcal{K}$ the rows and columns that are indexed by the set $I$. The non-diagonal elements of the co-factor matrix can be expressed with principal minors and sequences of the elements of $\mathcal{K}$ [MOVdDW89]

\[
C_{ji} = \sum_{n=0}^{N-2} \sum_{\{\alpha\}} (-1)^{n+1} K_{i\alpha_1} K_{\alpha_1 \alpha_2} \ldots K_{\alpha_n j} \det \mathcal{K}(\{j, i, \alpha_1, \ldots, \alpha_n\}) \\
= \sum_{n=0}^{N-2} \sum_{\{\alpha\}} N_{\varphi, i\alpha_1} N_{\varphi, \alpha_1 \alpha_2} \ldots N_{\varphi, \alpha_n j} \det \mathcal{K}(\{j, i, \alpha_1, \ldots, \alpha_n\}), \tag{5.2.21}
\]

where $\{\alpha\} \subset \{1, \ldots, N\} \setminus \{i, j\}$. With the help of (5.2.20) and (5.2.21) one can write

\[
\sum_{i,j=1}^{N} M_i K_{ij}^{-1} M_j = \sum_{i} \frac{\det \mathcal{K}(i)}{\det \mathcal{K}} M_i M_i + \sum_{i \neq j} \sum_{n=0}^{N-2} \sum_{\{\alpha\}} \frac{\det \mathcal{K}(j, i, \{\alpha\})}{\det \mathcal{K}} \\
\times M_i M_j N_{\varphi, i\alpha_1} N_{\varphi, \alpha_1 \alpha_2} \ldots N_{\varphi, \alpha_n j}. \tag{5.2.22}
\]

Now we rearrange the term

\[
\sum_{i=1}^{N} \frac{N_i}{L} \tag{5.2.23}
\]

in a similar manner. For this we need the following theorem:

**Theorem 5.** If the $N \times N$ matrix $\mathcal{K}^{(N)}$ has the form

\[
\mathcal{K}^{(N)}_{ij} = \begin{cases} 
LN_i + \sum_{k \neq i} N_{\varphi, ik} & i = j \\
-N_{\varphi, ij} & i \neq j
\end{cases}, \tag{5.2.24}
\]

its determinant can be expanded as

\[
\det \mathcal{K}^{(N)} = LN_i \det \mathcal{K}^{(N)}(\{i\}) \\
+ \sum_{n=1}^{N-1} \sum_{\{\alpha\}} N_{\varphi, i\alpha_1} N_{\varphi, \alpha_1 \alpha_2} \ldots N_{\varphi, \alpha_{n-1} \alpha_n} LN_{\alpha_n} \det \mathcal{K}^{(N)}(\{i, \alpha_1, \ldots, \alpha_n\}), \tag{5.2.25}
\]

where $i$ is any chosen row, $\{\alpha\} \subset \{1, \ldots, N\} \setminus \{i\}$ and $\mathcal{K}^{(N)}(I)$ is the submatrix of $\mathcal{K}^{(N)}$ as defined before.
Proof. Up to $N = 3$ it is easy to check the statement by direct evaluation. For $N > 3$ we proceed by induction. Let us suppose the theorem is valid for $N - 1$

$$\det K^{(N-1)} = LN_i \det K^{(N-1)}(\{i\}) + \sum_{n=1}^{N-2} \sum_{\{\alpha\}} \left\{ N_{\varphi,ia_1} N_{\varphi,ia_2} \cdots N_{\varphi,ia_{n-1}a_n} LN_{a_n} \times \det K^{(N-1)}(\{i, \alpha_1, \ldots, \alpha_n\}) \right\}. \quad (5.2.26)$$

The determinant for the matrix $K^{(N)}$ of size $N$ can be expanded by its row $j$

$$\det K^{(N)} = K^{(N)}_{jj} C^{(N)}_{jj} + \sum_{i \neq j} K^{(N)}_{ji} C^{(N)}_{ji}, \quad (5.2.27)$$

where $C^{(N)}$ is the co-factor matrix of $K^{(N)}$. Using (5.2.20) and (5.2.21) leads to

$$\det K^{(N)} = K^{(N)}_{jj} \det K^{(N)}(\{j\}) + \sum_{i \neq j} K^{(N)}_{ji} \det K^{(N)}(\{j, i, \alpha_1, \ldots, \alpha_n\}) \quad (5.2.28)$$

Now $K^{(N)}(\{j\})$ can be related to $K^{(N-1)}$ by observing that their off-diagonal elements are the same, while $K^{(N)}_{ii} = K^{(N-1)}_{ii} + N_{\varphi,ij}$. Implementing this by shifting $LN_i \rightarrow LN_i + N_{\varphi,ij}$ in (5.2.26) one obtains

$$\det K^{N}(\{j\}) = \det K^{(N-1)}|_{LN_i \rightarrow LN_i + N_{\varphi,ij}} = (LN_i + N_{\varphi,ij}) \det K^{(N)}(\{j, i\}) + \sum_{n=1}^{N-2} \sum_{\{\alpha\}} \left\{ N_{\varphi,ia_1} N_{\varphi,ia_2} \cdots N_{\varphi,ia_{n-1}a_n} \times (LN_{a_n} + N_{\varphi,a_nj}) \det K^{(N)}(\{j, i, \alpha_1, \ldots, \alpha_n\}) \right\}, \quad (5.2.29)$$

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and inserting this back to (5.2.28) gives

\[
\det K^{(N)} = L N_j \det K^{(N)} \{ \{ j \} \} + \sum_{i \neq j} N_{\varphi,ji} (LN_i + N_{\varphi,ij}) \det K^{(N)} \{ \{ j, i \} \}
\]

\[
+ \sum_{i \neq j} N_{\varphi,ji} \sum_{n=1}^{N-2} \sum_{\alpha} \left\{ N_{\varphi,i\alpha_1} N_{\varphi,\alpha_1\alpha_2} \ldots N_{\varphi,\alpha_{n-1}\alpha_n} \times (LN_{\alpha_n} + N_{\varphi,\alpha_j}) \det K^{(N)} \{ \{ j, i, \alpha_1, \ldots, \alpha_n \} \} \right\}
\]

\[
- \sum_{i \neq j} N_{\varphi,ji} \sum_{n=1}^{N-2} \sum_{\alpha} N_{\varphi,i\alpha_1} N_{\varphi,\alpha_1\alpha_2} \ldots N_{\varphi,\alpha_{n-1}\alpha_n} \det K^{(N)} \{ \{ i, j, \alpha_1, \ldots, \alpha_n \} \}
\]

\[
= LN_j \det K^{(N)} \{ \{ j \} \} + \sum_{n=1}^{N-1} \sum_{\alpha} N_{\varphi,ja_1} N_{\varphi,\alpha_1\alpha_2} \ldots N_{\varphi,\alpha_n L \alpha_n} \det K^{(N)} \{ \{ j, \alpha_1, \ldots, \alpha_n \} \}
\]

which is just the statement we wanted to prove. Q.e.d.

Using the above theorem we can rewrite

\[
\sum_i \frac{N_i}{L} = \sum_i \frac{N_i}{L} \det K = \sum_i \frac{N_i}{L} \det K = \sum_i \frac{N_i}{L} \det K^{(i)} + \sum_{i \neq j} \sum_{n=0}^{N-2} \sum_{\alpha} \frac{\det K^{(N)} \{ \{ i, j, \alpha_1, \ldots, \alpha_n \} \}}{\det K} \times N_{\varphi,ja_1} N_{\varphi,\alpha_1\alpha_2} \ldots N_{\varphi,\alpha_n} .
\]

(5.2.31)

which has the same structure as (5.2.22). Substituting the definitions of \( \mathcal{N} \) (5.2.12) and \( \mathcal{M} \) (5.2.15) into (5.2.22) and (5.2.31) we can see that every \( \eta_i \) factor appears twice and so drops out. Therefore the expression for \( \langle \Theta \rangle \) (5.2.17) simplifies to

\[
\frac{\langle \Theta \rangle}{2\pi} = \frac{\langle \Theta \rangle}{2\pi} + \sum_{\beta} \int \frac{d\theta}{2\pi} \frac{m_\beta \cosh (\theta) f_{c,\beta} (\theta) - m_\beta \sinh (\theta) f_{s,\beta} (\theta)}{1 + e^{\epsilon_\beta (\theta)}}
\]

\[
+ \sum_i \frac{\det K \{ \{ i \} \}}{\det K} \left[ f_{c,a_i} (\tilde{\theta}_i) f_{c,a_i} (\tilde{\theta}_i) - f_{s,a_i} (\tilde{\theta}_i) f_{s,a_i} (\tilde{\theta}_i) \right]
\]

\[
+ \sum_{i \neq j} \sum_{n=0}^{N-2} \sum_{\alpha} \frac{\det K \{ \{ i, j, \alpha_1, \ldots, \alpha_n \} \}}{\det K} \left[ f_{c,a_i} (\tilde{\theta}_i) f_{c,a_j} (\tilde{\theta}_j) - f_{s,a_i} (\tilde{\theta}_i) f_{s,a_j} (\tilde{\theta}_j) \right]
\]

\[
\times f_{a_1,a_1} (\theta_i) f_{a_2,a_1} (\theta_1) \ldots f_{j,a_n} (\theta_n) .
\]

(5.2.32)

The determinant ratios are exactly the density factors in (5.1.6); what remains to be shown
is that the other terms reproduce the dressed form factors of $\Theta$.

### 5.2.3 Dressed form factors of $\Theta$

**Theorem 6.** *In the absence of active singularities of the TBA equations, the dressed form factors of $\Theta$ are given by*

$$
\mathcal{D}_\varepsilon^\Theta = \sum_{n_1,\ldots,n_k=0}^{\infty} \prod_i n_i! \int_{-\infty}^{\infty} \prod_j \frac{d\theta_j}{2\pi} \left[ 1 + e^{\varepsilon \beta_j(\theta_j)} \right] F^{\Theta}_{2n_1,\ldots,2n_k,c} (\theta_1, \ldots, \theta_N),
$$

*(5.2.33)*

*which is equal to*

$$
\mathcal{D}_\varepsilon^\Theta = \langle \Theta \rangle_{\infty} + 2\pi \sum_\beta \int \frac{d\theta \ m_\beta \cosh (\theta) f_{c,\beta} (\theta) - m_\beta \sinh (\theta) f_{s,\beta} (\theta)}{1 + e^{\varepsilon \beta(\theta)}}.
$$

*(5.2.34)*

**Proof.** The connected diagonal form factors of $\Theta$ are given by [LM99]

$$
F^{\Theta}_{2n,c} (\theta_1, \ldots, \theta_n) = 2\pi \varphi_{12} \varphi_{23} \ldots \varphi_{n-1,n} m_{\beta_1} m_{\beta_n} \cosh (\theta_{1n}) + \text{permutations},
$$

where $\theta_{ij} = \theta_i - \theta_j$, $\beta_i$ denotes the species of the $i$th particle and $\varphi_{ij}$ is a short-hand for $\varphi_{\beta_i,\beta_j} (\theta_{ij})$. *(5.2.33)* is symmetric under re-ordering particles of the same species which results in a combinatorial factor $\prod_i n_i!$ cancelling the denominators in front of the integrals. To take into account the rest of the permutations we can rewrite $\mathcal{D}_\varepsilon^\Theta$ like

$$
\mathcal{D}_\varepsilon^\Theta = 2\pi \sum_{n=0}^{\infty} \sum_{\beta_1,\ldots,\beta_n} \prod_{j=1}^{n} \frac{d\theta_j}{2\pi} \left[ 1 + e^{\varepsilon \beta_j(\theta_j)} \right] \varphi_{12} \varphi_{23} \ldots \varphi_{n-1,n} m_{\beta_1} m_{\beta_n} \cosh (\theta_{1n})
$$

$$
= \sum_{n=0}^{\infty} \mathcal{D}_{\varepsilon,n}^\Theta.
$$

*(5.2.35)*

Following [LM99] every $\mathcal{D}_{\varepsilon,n}^\Theta$ can be graphically represented as seen in Figure 5.2.1a, where every node represents a particle with a given rapidity and species, including the integration

$$
\int \frac{d\theta_i}{2\pi \left[ 1 + e^{\varepsilon \beta_i(\theta_i)} \right]},
$$

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and the first and last node is multiplied by its mass. Every horizontal line represents a factor \( \varphi_{ij} \) and the dashed line represents the factor \( \cosh \theta_{1n} \). The whole graph is multiplied by \( 2\pi \) to account for the normalisation of the operator \( \Theta \), and summed over every possible type configuration for the nodes. The empty graph (with zero node) represents \( \mathcal{D}^{\Theta}_{\epsilon,0} = \langle \Theta \rangle_{\infty} \). Using hyperbolic addition formulas for the \( \cosh \) terms every graph can be represented as difference of two chains where the two end nodes instead of being connected by dashed line, are multiplied by \( \cosh \) or \( \sinh \) of the rapidity at the given node as shown in Figure 5.2.1b.

Since the functions \( f_c \) and \( f_s \) in (5.2.34) satisfy the self-consistent equations (5.2.4), it is convenient to expand them in the following way

\[
\begin{align*}
\varphi_{\beta} (\theta - \theta_1) & = \sum_{n=0}^{\infty} K_{n,\beta} (\theta), \\
\varphi_{\beta} (\theta - \theta_1) & = \sum_{n=0}^{\infty} J_{n,\beta} (\theta),
\end{align*}
\]

where

\[
\begin{align*}
K_{0,\beta} (\theta) & = m_{\beta} \cosh (\theta), \\
J_{0,\beta} (\theta) & = m_{\beta} \sinh (\theta),
\end{align*}
\]
\[ K_{n,\beta}(\theta) = \sum_{\beta_1, \ldots, \beta_n} \int \frac{d\theta_i}{2\pi} \frac{1}{1 + e^{\varepsilon_{\beta_i}(\theta)} - e^{\varepsilon_{\beta_i}(\theta)} - e^{\varepsilon_{\beta_i}(\theta)} - \varepsilon_{\beta_{n-1}\beta_n m_{\beta_n} \cosh(\theta_n)}}, \]

\[ J_{n,\beta}(\theta) = \sum_{\beta_1, \ldots, \beta_n} \int \frac{d\theta_i}{2\pi} \frac{1}{1 + e^{\varepsilon_{\beta_i}(\theta)} - e^{\varepsilon_{\beta_i}(\theta)} - e^{\varepsilon_{\beta_i}(\theta)} - \varepsilon_{\beta_{n-1}\beta_n m_{\beta_n} \sinh(\theta_n)}}, \] (5.2.37)

The graphical representation of \(K_{n,\beta}(\theta)\) and \(J_{n,\beta}(\theta)\) can be seen in Figure 5.2.1c; the dashed node indicates that the corresponding rapidity integral and the filling fraction belonging to that node is not included in the contribution. Comparing to Figure 5.2.1b it is clear that \(K_{n,\beta}(\theta)\) and \(J_{n,\beta}(\theta)\) describe the contribution of the chain between one of the end nodes and a dashed node with type \(\beta\) which is \(n\) steps away from the end node. Multiplying \(K_{n,\beta}(\theta)\) and \(J_{n,\beta}(\theta)\) with \(2\pi \sum_{\beta}^{\beta_{\text{d}}} \frac{d\theta}{2\pi} m_{\beta} \cosh(\theta)\) and \(-2\pi \sum_{\beta}^{\beta_{\text{d}}} \frac{d\theta}{2\pi} m_{\beta} \sinh(\theta)\) as in (5.2.34) closes the chains and they become identical to the ones in Figure 5.2.1b with length \(n + 1\), i.e. equal to \(D_{\Theta}^{\varepsilon,n+1}\). The sum for \(n\) in \(f_c\) and \(f_s\) then generates all the contributions in \(D_{\Theta}^{\varepsilon}\). Q.e.d.

**Theorem 7.** The dressed form factor of \(\Theta\) with one active singularity \(\bar{\theta}_i\) with type \(\alpha_i\) is

\[ D_{\varepsilon}^{\Theta} (\bar{\theta}_i) = \sum_{n_1, \ldots, n_k=0}^{\infty} \frac{1}{n_1!} \int_{-\infty}^{\infty} \cdots \frac{1}{n_k!} \int_{-\infty}^{\infty} F_{2n_1, \ldots, 2n_k, c} (\bar{\theta}_i, \theta_1, \ldots, \theta_N), (5.2.38) \]

which is equal to

\[ D_{\varepsilon}^{\Theta} (\bar{\theta}_i) = f_{c,\alpha_i} (\bar{\theta}_i) f_{c,\alpha_i} (\bar{\theta}_i) - f_{s,\alpha_i} (\bar{\theta}_i) f_{s,\alpha_i} (\bar{\theta}_i). \] (5.2.39)

**Proof.** The proof follows the ideas used in demonstrating Theorem 6. The factor \(\prod_{i} \frac{1}{n_i!}\) for the non-active singularities cancels as before, but now one must sum over all possible positions of the active singularity:

\[ D_{\varepsilon}^{\Theta} (\bar{\theta}_i) = 2\pi \sum_{n,m=0}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\prod_{j=1}^{n+m} \frac{d\theta_j}{2\pi} \left[ 1 + e^{\varepsilon_{\beta_j}(\theta)} \right]^{n+m}} F_{2n_1, \ldots, 2n_k, c} (\bar{\theta}_i, \theta_1, \ldots, \theta_N), (5.2.40) \]

\[ = \sum_{n,m=0}^{\infty} D_{\varepsilon,n,m}^{\Theta} (\bar{\theta}_i), \] (5.2.41)

where the sum runs for the number and the species of nodes between the active singularity.
and the end nodes (for \( n = 0 \), the active singularity is the end node on the left, while for \( m = 0 \) it is the end node on the right). The active singularity is marked by a black node in Figures 5.2.2a and 5.2.2b. \( K_{n,\alpha_i} (\bar{\theta}_i) \) and \( J_{n,\alpha_i} (\bar{\theta}_i) \) are represented in Figure 5.2.2c; they are equal to the contribution of the chain between one of the end nodes and the active singularity. Multiplying them as in 5.2.39 it follows that

\[
D_{\xi,n,m}^{\Theta} (\bar{\theta}_i) = K_{n,\alpha_i} (\bar{\theta}_i) K_{m,\alpha_i} (\bar{\theta}_i) - J_{n,\alpha_i} (\bar{\theta}_i) J_{m,\alpha_i} (\bar{\theta}_i),
\] (5.2.42)

and the summation in both of \( f_{c/s} \) the result exactly reproduces \( D_{\xi}^{\Theta} (\bar{\theta}_i) \). Q.e.d.

\[\square\]
Theorem 8. The dressed form factor of $\Theta$ with $N$ active singularities $\{\bar{\theta}_1, \ldots, \bar{\theta}_N\}$ is

$$D_\varepsilon^O (\bar{\theta}_1, \ldots, \bar{\theta}_N) = \sum_{n_1, \ldots, n_k = 0}^{\infty} \sum_{\Pi_i n_i!} \int_{-\infty}^{\infty} \prod_{j=1}^{\dot{N}} \frac{d\theta_j}{2\pi} \left[ 1 + e^{\varepsilon_j(\bar{\theta}_j)} \right] \times F_{2n_1, \ldots, 2n_k}^O (\bar{\theta}_1, \ldots, \bar{\theta}_N, \theta_1, \ldots, \theta_N),$$

which is equal to

$$D_\varepsilon (\bar{\theta}_1, \ldots, \bar{\theta}_N) = \sum_{i \neq j} \sum_{\{\alpha\}} \left[ f_{c,\alpha_i} (\bar{\theta}_i) f_{c,\alpha_j} (\bar{\theta}_j) - f_{s,\alpha_i} (\bar{\theta}_i) f_{s,\alpha_j} (\bar{\theta}_j) \right] \times f_{\alpha_1,\alpha_i} (\bar{\theta}_i) f_{\alpha_2,\alpha_1} (\bar{\theta}_1) \ldots f_{j,\alpha_n} (\bar{\theta}_n),$$

where $\{\alpha\} = \{1, \ldots, N\} \setminus \{i, j\}.$

Proof. As in the proofs of the previous theorems, (5.2.43) can be organised into a sum over terms corresponding to individual permutations of the active singularities. For a given permutation, the contribution is the sum of graphs represented in Figures 5.2.3a and 5.2.3b, where the number and type of nodes separating end nodes and the active singularities is varying.

The functions $f_{\alpha_1,\alpha_p} (\bar{\theta}_p)$ can be expanded as $f_{c/s}$ previously, using their definition in
\[ f_{\alpha q,\alpha p}(\bar{\theta}_p) = \sum_{n=0}^{\infty} L_{n,pq} , \quad (5.2.45) \]

where

\[ L_{0,pq} = \varphi_{pq} , \]
\[ L_{n,pq} = \sum_{\beta_1,\ldots,\beta_n} \int \prod_i \frac{d\theta_i}{2\pi} \frac{1}{1 + e^{\beta_i(\theta_i)}} \varphi_{\alpha_p \beta_1} \cdots \varphi_{\beta_n \alpha_q} . \quad (5.2.46) \]

\( L_{n,pq} \) is represented in Figure 5.2.3c; it generates all the possible contribution to Figure 5.2.3b between two active singularities. In a given permutation of the active singularities let us take \( \bar{\theta}_i \) and \( \bar{\theta}_j \) as the two active singularities closest to the left/right end nodes; then the terms

\[ f_{c,\alpha_i}(\bar{\theta}_i) f_{c,\alpha_j}(\bar{\theta}_j) - f_{s,\alpha_i}(\bar{\theta}_i) f_{s,\alpha_j}(\bar{\theta}_j) \]

generate all the contributions between the active singularities and the ends, and

\[ f_{\alpha_1,\alpha_i}(\bar{\theta}_i) f_{\alpha_2,\alpha_1}(\bar{\theta}_1) \cdots f_{j,\alpha_n}(\bar{\theta}_n) \]

generate all the contributions between the other active singularities. Summing up for all the permutations of the active singularities proves the theorem. Q.e.d.

Theorems 6, 7 and 8 prove the equivalence of the form factor series (5.1.6) and the TBA equations for \( \langle \Theta \rangle \) in any excited state described by the TBA system (3.2.18).

5.3 Finite volume expectation values in the \( T_2 \) model

For the numerical validation of the conjecture (5.1.6), we follow a similar strategy as we did for the Leclair-Mussardo conjecture in Chapter 4 ([STW13]), and use the \( T_2 \) model again for the validation. In this section we summarise the excited state TBA equations and the form of the conjecture (5.1.6) for the \( T_2 \) model.
5.3.1 Excited state TBA equations for states with one type-1 particle

5.3.1.1 Form and solution of the excited TBA equations

The simplest excited states in the excited TBA formalism for the $T_2$ model [DT98] are those with one type-1 particle\(^1\). For these excited states the TBA equations contain only two active singularities of type-2 with $\eta_1 = -1$, $\eta_2 = 1$:

\[
\varepsilon_a (\theta) = m_a L \cosh (\theta) + \log \left( \frac{S_{a2} (\theta - \bar{\theta}_1)}{S_{a2} (\theta - \bar{\theta}_2)} \right),
\]

(5.3.1)

\[
e^{-\varepsilon_2 (\bar{\theta}_1/2)} = -1,
\]

(5.3.2)

\[
E_{\text{TBA}} (L) = -i m_2 \left( \sinh (\bar{\theta}_1) - \sinh (\bar{\theta}_2) \right)
- \sum_{a} \frac{d \theta}{2 \pi} m_a \cosh (\theta) \log \left( 1 + e^{-\varepsilon_a (\theta)} \right),
\]

(5.3.3)

which are related by $\bar{\theta}_2 = \bar{\theta}_1^*$ for states with nonzero momentum, where $*$ is the complex conjugation; or $\bar{\theta}_2 = -\bar{\theta}_1$ for the zero momentum state.

In large volume, the imaginary part of the active singularity’s position, as explained in Section 3.2.3, is forced to $\frac{i \pi}{10}$, since $S_{22}$ is singular around $\bar{\theta}_1 - \bar{\theta}_2 \sim \frac{i \pi}{5}$. The position of the active singularity can be written as

\[
\bar{\theta}_1 = \tilde{\theta} + i \left( \frac{\pi}{10} + \delta \right),
\]

(5.3.4)

where $\tilde{\theta}$ and $\delta$ are real; $\delta$ is a correction to the imaginary part that decays exponentially in the dimensionless volume variable $m_1L$. Substituting this form into condition (5.3.2) and keeping only the first order corrections in $\delta$, the solution for the position of the active singularity is

\[
m_1 L \sinh \tilde{\theta} = 2 \pi s,
\]

\[
\delta = \cos (\pi s) \tan \left( \frac{3 \pi}{10} \right) \tan^2 \left( \frac{2 \pi}{5} \right) e^{-m_2 \cos (\frac{\pi}{10}) \sqrt{m_1^2 L^2 + (2 \pi s)^2}},
\]

(5.3.5)

where $s$ is an integer number giving the momentum quantum number of the state. Using this solution for TBA energy (5.3.3) and expanding the $\log (1 + e^{-\varepsilon})$ term in the integral, the

\(^1\)The identification of the particle content of a state comes from the IR behaviour of its energy.
energy takes the following form

\[ E_{\text{TBA}} (L) = E(L) - BL \]
\[ = 2 \sin \left( \frac{\pi}{10} \right) m_2 \cosh (\tilde{\theta}) + 2m_2 \cosh (\tilde{\theta}) \sin \left( \frac{\pi}{10} \right) \delta \]
\[ - \sum_a \int \frac{d\theta}{2\pi} m_a \cosh (\theta) e^{-m_a L \cosh(\theta)} \frac{S_{a2} (\theta - \tilde{\theta} + i \frac{\pi}{10})}{S_{a2} (\theta - \tilde{\theta} - i \frac{\pi}{10})} \]
\[ = \sqrt{m_1^2 + \left( \frac{2\pi s}{L} \right)^2} + 2 \sqrt{m_2^2 + \left( \frac{2\pi s m_2}{L m_1} \right)^2} \cos \left( \frac{\pi}{10} \right) \]
\[ \times \cos (\pi s) \tan \left( \frac{3\pi}{10} \right) \tan^2 \left( \frac{2\pi}{5} \right) e^{-m_2 \cos (\pi)} \sqrt{m_1^2 L^2 + (2\pi s)^2} \]
\[ - \sum_a \int \frac{d\theta}{2\pi} m_a \cosh (\theta) e^{-m_a L \cosh(\theta)} S_{a1} \left( \theta - \tilde{\theta} + i \frac{\pi}{2} \right), \quad (5.3.6) \]

where the bootstrap identity

\[ \frac{S_{a2} (\theta + i \frac{\pi}{10})}{S_{a2} (\theta - i \frac{\pi}{10})} = S_{a1} \left( \theta + i \frac{\pi}{2} \right), \quad (5.3.7) \]

was used. The first term gives \( L^{-1} \) corrections related to the kinetic energy of the particle in finite volume, while the second and third terms are the leading exponential corrections, the \( \mu \) and \( F \) terms [BJ09], which for a zero-momentum particle were first derived by Lüscher in [Lüs86a].

In the small volume (UV) limit, the energy is proportional to the effective central charge of the state

\[ E_{\text{TBA}} (L) = -\frac{6\pi}{L} c_{\text{eff}} (L), \quad (5.3.8) \]

which has the ultraviolet limit

\[ c_{\text{eff}} (0) = c - 12(\Delta + \bar{\Delta}), \quad (5.3.9) \]

where \( c = -68/7 \) is the central charge of the minimal model \( \mathcal{M}_{2,7} \) and \( \Delta, \bar{\Delta} \) are the left/right conformal weights of the state in the ultraviolet limit. Using the dilogarithm trick introduced in [Zam90], one can confirm the expected effective central charge for these states [DT98]. In the UV-limit, the state with one type-1 particle and \( s = 0 \) goes to the conformal state.
Figure 5.3.1. Effective central charge for the excited states with momentum quantum number \( s = 0, 1 \) from TBA and TCSA, with the UV (CFT value) and IR asymptotics (\( \mu \) and F term).

\(|\Delta_{1,2}, \bar{\Delta}_{1,2}\rangle\), and its effective central charge is \( c_{\text{eff}}^{s=0} = c - 12 (\Delta_{1,2} + \bar{\Delta}_{1,2}) = -\frac{20}{7} \); the state with one type-1 particle \( s = 1 \) goes to the CFT state \( L_{-1} |\Delta_{1,3}, \bar{\Delta}_{1,3}\rangle\), and its effective central charge is \( c_{\text{eff}}^{s=1} = c - 12 (\Delta_{1,3} + \bar{\Delta}_{1,3} + 1) = -\frac{802}{7} \). In Figure 5.3.1 we plot \( c_{\text{eff}} \) for the states \( s = 0, 1 \) showing that the TBA results match perfectly with the TCSA calculation and also reproduces the expected asymptotic.

The excited state TBA equations are solved numerically by simultaneously iterating equations (5.3.1) and (5.3.2) in large volume, where the asymptotic of the pole position (5.3.5) can be used as a starting point. Using this solution the volume is decreased and the equations are re-iterated, and continuing this process the solution can be tracked to small volume. For \( s \neq 0 \) the numerics is straightforward to perform up to precision of order \( 10^{-12} \), and all the ingredients to calculate the conjecture (5.1.6) can be readily constructed. For \( s = 0 \) there exists a critical volume \( r_c \) under which it is necessary to be more careful with the numerical calculation.

5.3.1.2 Zero-momentum state: desingularisation in small volume

As described in details in [BLZ97b, DT96, DT98], for states containing a zero-momentum particle it can happen that under a given critical volume \( r_c \) some singularity ends up on the integration contour. To describe this situation, we recall that the \( Y \)-system (3.2.21) gives relations between positions where the functions \( Y_\alpha = e^{\epsilon_\alpha} \) take the values 0 and -1, which are the logarithmic singular points of the TBA equations. For the \( T_2 \) model, the incidence

\(^2\text{Here, we used the usual CFT notation for primary fields and Virasoro generators, without introducing them properly. Since CFT is not the main topic of this thesis, we refer the interested reader to the pedagogical introduction by Ginsparg [Gin88] for definitions and notations.}\)
matrix and the Coxeter number is given by
\[ I^{[T_2]} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad h = 5. \] (5.3.10)

In large volume, for the excited state containing one type-1 particle with zero momentum, the active singularities, where \( Y_2 = -1 \), are on the imaginary axis at positions \( \pm \tilde{\theta}_1 \) with
\[ \tilde{\theta}_1 = i \frac{\pi}{10} + i \delta, \] (5.3.11)
and \( \delta > 0 \). From (3.2.21) it follows that \( Y_2 = -1 \) at \( \tilde{\theta}_1 - i \frac{2\pi}{5} \) and \( -\tilde{\theta}_1 + i \frac{2\pi}{5} \), and \( Y_1 = Y_2 = 0 \) at positions \( \tilde{\theta}_1 - i \frac{\pi}{5} \) and \( -\tilde{\theta}_1 + i \frac{\pi}{5} \). As the volume decreases, the value of \( \delta \) increases till at some critical value of the volume given by \( m_1L = r_c \) it reaches \( \delta_c = \frac{\pi}{10} \). At this point \( \tilde{\theta}_1 = -\tilde{\theta}_1 + i \frac{2\pi}{5} \) resulting in a coincidence of singularities, and also of zeros. Decreasing the volume to \( m_1L < r_c \) results in the “scattering” of the singularities on each other at right angle pushing them away from the imaginary axis with fixed imaginary part in the form
\[ \tilde{\theta}_1 = i \frac{\pi}{5} + \alpha. \] (5.3.12)

As a result, the zeros of \( Y_1 \) and \( Y_2 \) sit exactly on the integration contour, making the equation for the pseudoenergy (5.3.1) singular and leading to instabilities in the numerical solution of the TBA equations.

One possible way to handle the problem numerically is to shift the integration contour, while an alternative approach, called desingularisation, is to rearrange the self-consistent equations appropriately. Desingularisation relies on the relation [DT98]
\[
\varphi_{ad} (\theta) &= -\varphi_h (\theta) I^{[T_2]}_{ad} + \sum_b \int \frac{d\theta'}{2\pi} \varphi_{ab} (\theta - \theta') \varphi_h (\theta') I^{[T_2]}_{bd}, \\
\varphi_h (\theta) &= \frac{h}{2 \cosh \left( \frac{h}{2} \theta \right)} = \pm i \partial_\theta \log \left( \sigma_h \left( \theta \pm i \frac{\pi}{h} \right) \right), \\
\sigma_h (\theta) &= \tanh \left( \frac{h}{4} \theta \right), \] (5.3.13)
which allows the TBA equations to be recast as
\[
\tilde{\varepsilon}_a (\theta) = \varepsilon_a (\theta) - \log \left( \sigma \left( \theta' - \tilde{\theta}_1 \right) \sigma \left( \theta' - \tilde{\theta}_2 \right) \right),
\]
\[ \dot{e}_a(\theta) = m_a L \cosh(\theta) - \sum_{b} \int \frac{d\theta'}{2\pi} \varphi_{ab}(\theta - \theta') \log \left( \sigma \left( \theta' - \tilde{\theta}_1 \right) \sigma \left( \theta' - \tilde{\theta}_2 \right) + e^{-\dot{e}_b(\theta')} \right), \]

\[ e^{\dot{e}_a(\tilde{\theta}_i)} = i \coth \left( \frac{5}{2} \tilde{\theta}_1 + i \frac{\pi}{4} \right), \]

\[ E_{TBA}(L) = - \sum_{a} \int \frac{d\theta}{2\pi} m_a \cosh(\theta) \log \left( \sigma \left( \theta - \tilde{\theta}_1 \right) \sigma \left( \theta - \tilde{\theta}_2 \right) + e^{-\dot{e}_b(\theta)} \right), \quad (5.3.14) \]

where \( \tilde{\theta}_1 = \tilde{\theta}_1 - i \frac{\pi}{h}, \tilde{\theta}_2 = \tilde{\theta}_2 + i \frac{\pi}{h} \). These equations are regular and can be iterated in a stable way, however, the available precision using double precision numbers drops to the order of \( 10^{-10} \). Fortunately, that is still more than sufficient for our purposes.

To calculate the densities and \( \langle \Theta \rangle \) under \( r_c \), we need to desingularise \( f_s, f_c, f_i \) in (5.2.4) as well. It is easy to see that the equation for \( f_s \) and \( f_c \) is regular under \( r_c \), since \( \frac{1}{1 + e^{\tilde{e}_a}} \) at the singularity position is regular, because \( Y_{\alpha} = e^{\tilde{e}_a} = 0 \). For \( f_i \) the source term \( \varphi \) is singular at \( \tilde{\theta}_{1,2} \), hence it is necessary to desingularise it:

\[ \hat{f}_{i,a}(\theta) = f_{i,a}(\theta) + \varphi_h(\theta - \tilde{\theta}_i), \]

\[ \hat{f}_{i,a}(\theta) = \sum_{\beta} \int \frac{d\theta'}{2\pi} \varphi_{a\beta}(\theta - \theta') e^{\dot{e}_{\beta}(\theta')} \varphi_h(\theta - \tilde{\theta}_i) + \hat{f}_{1,\beta}(\theta') \frac{1}{1 + e^{\dot{e}_{\beta}(\theta')}}. \quad (5.3.15) \]

For numerical calculations the form

\[ \hat{f}_{1,a}(\theta) = \sum_{\beta} \int \frac{d\theta'}{2\pi} \varphi_{a\beta}(\theta - \theta') \frac{\partial \sigma \left( \theta' - \tilde{\theta}_1 \right) \sigma \left( \theta' - \tilde{\theta}_2 \right) + e^{-\dot{e}_b(\theta')} \hat{f}_{1,\beta}(\theta')}{\sigma \left( \theta' - \tilde{\theta}_1 \right) \sigma \left( \theta' - \tilde{\theta}_2 \right) + e^{-\dot{e}_b(\theta')}}, \]

\[ \hat{f}_{2,a}(\theta) = \sum_{\beta} \int \frac{d\theta'}{2\pi} \varphi_{a\beta}(\theta - \theta') \frac{\sigma \left( \theta' - \tilde{\theta}_1 \right) \partial \sigma \left( \theta' - \tilde{\theta}_2 \right) + e^{-\dot{e}_b(\theta')} \hat{f}_{1,\beta}(\theta')}{\sigma \left( \theta' - \tilde{\theta}_1 \right) \sigma \left( \theta' - \tilde{\theta}_2 \right) + e^{-\dot{e}_b(\theta')}}, \quad (5.3.16) \]

is more convenient.

### 5.3.2 Densities and the conjecture for states with one type-1 particle

The derivatives of the quantisation condition can be written in the form (5.2.11) with the help of the definitions in (5.2.4), (5.2.12) for states with one type-1 particle

\[ \frac{\partial (Q_1, Q_2)}{\partial \langle \tilde{\theta}_1, \tilde{\theta}_2 \rangle} = \begin{pmatrix} LN_1 + N_\varphi & -N_\varphi \\ -N_\varphi & LN_2 + N_\varphi \end{pmatrix}, \quad (5.3.17) \]
where

\[ N_1 = -i f_{s,2}(\tilde{\theta}_1) = -im_2 \sinh(\tilde{\theta}_1) - i \sum_{\beta} \int_{2\pi} d\theta' \frac{\varphi_{2\beta}(\tilde{\theta}_1 - \theta')}{2\pi} \frac{f_{s,\beta}(\theta')}{1 + e^{\varepsilon_{\beta}(\theta')}}; \]

\[ N_2 = i f_{s,2}(\tilde{\theta}_2) = im_2 \sinh(\tilde{\theta}_2) + i \sum_{\beta} \int_{2\pi} d\theta' \frac{\varphi_{2\beta}(\tilde{\theta}_2 - \theta')}{2\pi} \frac{f_{s,\beta}(\theta')}{1 + e^{\varepsilon_{\beta}(\theta')}}; \]

\[ N_\varphi = -f_{2,2}(\tilde{\theta}_1) = -\varphi_{22}(\tilde{\theta}_1 - \tilde{\theta}_2) - \sum_{\beta} \int_{2\pi} d\theta' \frac{\varphi_{2\beta}(\tilde{\theta}_1 - \theta')}{2\pi} \frac{f_{2,\beta}(\theta')}{1 + e^{\varepsilon_{\beta}(\theta')}}; \]

\[ = -f_{1,2}(\tilde{\theta}_2) = -\varphi_{22}(\tilde{\theta}_2 - \tilde{\theta}_1) - \sum_{\beta} \int_{2\pi} d\theta' \frac{\varphi_{2\beta}(\tilde{\theta}_2 - \theta')}{2\pi} \frac{f_{1,\beta}(\theta')}{1 + e^{\varepsilon_{\beta}(\theta')}}. \] (5.3.18)

For the case \( s = 0 \) and \( m_1 L < r_c \)

\[ N_\varphi = -\hat{f}_{2,2}(\tilde{\theta}_1) + \varphi_h(\tilde{\theta}_1 - \tilde{\theta}_2) = +\varphi_h(\tilde{\theta}_1 - \tilde{\theta}_2) \]

\[ - \sum_{\beta} \int_{2\pi} d\theta' \frac{\varphi_{\alpha\beta}(\tilde{\theta}_1 - \theta')}{2\pi} \frac{\sigma(\theta' - \tilde{\theta}_1)}{\sigma(\theta' - \tilde{\theta}_1)} \frac{\partial \sigma(\theta' - \tilde{\theta}_2)}{\sigma(\theta' - \tilde{\theta}_2)} + e^{-\varepsilon_{\beta}(\theta')} \frac{\hat{f}_{1,\beta}(\theta')}{1 + e^{\varepsilon_{\beta}(\theta')}} \]

\[ = -\hat{f}_{1,2}(\tilde{\theta}_2) + \varphi_h(\tilde{\theta}_2 - \tilde{\theta}_1) = +\varphi_h(\tilde{\theta}_2 - \tilde{\theta}_1) \]

\[ - \sum_{\beta} \int_{2\pi} d\theta' \frac{\varphi_{\alpha\beta}(\tilde{\theta}_2 - \theta')}{2\pi} \frac{\partial \sigma(\theta' - \tilde{\theta}_1)}{\sigma(\theta' - \tilde{\theta}_1)} \frac{\sigma(\theta' - \tilde{\theta}_2)}{\sigma(\theta' - \tilde{\theta}_2)} + e^{-\varepsilon_{\beta}(\theta')} \frac{\hat{f}_{1,\beta}(\theta')}{1 + e^{\varepsilon_{\beta}(\theta')}}. \] (5.3.19)

With the above densities the conjecture for the form factor series (5.1.6) for this state takes the form

\[ \langle \tilde{\theta}_1, \tilde{\theta}_2 | \mathcal{O} | \tilde{\theta}_1, \tilde{\theta}_2 \rangle_L = \sum_{n_1, n_2 = 0}^{\infty} \frac{1}{n_1! n_2!} \int_{-\infty}^{\infty} \prod_{j=1}^{\tilde{N}} \frac{d\theta_j}{2\pi} \left[ 1 + e^{\varepsilon_{\alpha_j}(\theta_j)} \right] \left[ F_{2n_1,2n_2,c}^O(\theta_1, \ldots, \theta_{\tilde{N}}) + \frac{1}{L^2 N_1 N_2 + \mathcal{N}_\varphi L (N_1 + N_2)} \left\{ (L N_1 + \mathcal{N}_\varphi) F_{2n_1,2n_2+2,c}^O(\tilde{\theta}_1, \theta_1, \ldots, \theta_{\tilde{N}}) + (L N_2 + \mathcal{N}_\varphi) F_{2n_1,2n_2+2,c}^O(\tilde{\theta}_2, \theta_1, \ldots, \theta_{\tilde{N}}) + F_{2n_1,2n_2+4,c}^O(\tilde{\theta}_1, \tilde{\theta}_2, \theta_1, \ldots, \theta_{\tilde{N}}) \right\} \right]. \] (5.3.20)
5.4 Numerical results

As mentioned in Section 4.3, operator $\Phi_{1,3}$ is related to the trace of the stress-energy tensor, hence its form factor series is equivalent to the TBA equation as proved in Section 5.2. The numerical calculation for $\Phi_{1,3}$ is therefore not a real further test for the general validity of the form factor series (5.1.6). However, similarly to Section 4.3, the calculation of it is still useful since with its expectation value known from TBA equations one can get an independent check of the numerical precision of TCSA, and the convergence of the form factor series. For $\Phi_{1,2}$ there is only TCSA and the form factor series, with the numerical deviation for $\Phi_{1,3}$ setting the expected precision for the agreement between them.

For the numerical integration we again used the Cuba library routines [Hah06], called from inside Wolfram Mathematica$^3$.

5.4.1 Moving one-particle state, $s = 1$

For the moving type-1 excited state with momentum quantum number $s = 1$, Figure 5.4.1 shows the expectation value $\langle \{1\} | \Phi_{1,3} | \{1\} \rangle_1$ calculated with RG-extrapolated TCSA, from TBA together with the results from the form factor series (5.3.20) obtained by adding progressively more terms. The precision of the TBA is of the order $10^{-12}$ and comparing it with the TCSA data, we find that the precision of the RG-extrapolated TCSA is of order $10^{-6} - 10^{-7}$. Table 5.4.1 shows the difference between the form factor series with different terms involved and the TCSA data. For volume $m_1L > 5$ the difference between the form factor series up to and including the 112 term, and the TCSA is in the order of the TCSA error, and including more terms make the agreement better for smaller volume as well.

For $\Phi_{1,2}$ the results for the quantity $i\langle \{1\} | \Phi_{1,2} | \{1\} \rangle_1$ are shown in Figure 5.4.2, and the difference between the form factor series and the TCSA is given in Table 5.4.2. We note that since from (A.2.4) it follows that the matrix elements of $\Phi_{1,2}$ are imaginary, here and in all subsequent figures and tables concerning $\Phi_{1,2}$ we multiply all data by $i$. The form factor series shows excellent agreement with the TCSA for volume $m_1L > 5$, and again including more terms the agreement is better for smaller volumes. As noted before, the correctness of the form factor series for this operator does not follow from TBA, hence this is a nontrivial verification of the form factor series.

---

Figure 5.4.1. $i\langle 1 | \Phi_{1,3} | 1 \rangle_1$ evaluated by the form factor series including different contributions, RG extrapolated TCSA and TBA. Here and in all subsequent plots matrix elements are given in units of $m_1$.

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Table 5.4.1. The difference between the evaluations of $i\langle 1 | \Phi_{1,3} | 1 \rangle_1$ from the RG-extrapolated TCSA and the form factor series, depending on the multi-particle contributions included in the latter.
Figure 5.4.2. $i_i\{\{1\}|\Phi_{1,2}|\{1\}\}_1$ evaluated by the form factor series including different contributions and RG-extrapolated TCSA.

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Table 5.4.2. The difference between the evaluations of $i_i\{\{1\}|\Phi_{1,2}|\{1\}\}_1$ from the RG-extrapolated TCSA and the form factor series, depending on the multi-particle contributions included in the latter.
As seen for the $s = 1$ case the form factor series reproduce the expectation value of local operators with very good precision in large volume, even by including only few terms from the series. The expectation is that for any state in small volume it is necessary to include higher contributions of the series, but for any desired accuracy a finite number of them is sufficient.

As shown below, this expectation is challenged by the nontrivial transition in the TBA equation for standing state at $r_c$. Figures 5.4.3 and 5.4.4 show the result for the expectation values of $\Phi_{1,3}$ and $\Phi_{1,2}$, while Tables 5.4.3 and 5.4.4 list the numerical deviations from RG-extrapolated TCSA.

For large volume ($m_1 L \gtrsim 6$) the agreement between the form factor series and the TCSA is again excellent. However, towards the critical volume ($r_c \sim 2.66$) the terms of the form factor series tend to diverge. This can be understood from the fact that the total density of the states

$$\rho_{\text{tot}} = \det K = L^2 N_1 N_2 + N_\phi L (N_1 + N_2) ,$$

which the denominator of the form factor series (5.3.20), is zero at $r_c$. Figure 5.4.5 shows the
Table 5.4.3. The difference between the evaluations of $\langle \{0\} | \Phi_{1,3} | \{0\} \rangle_1$ from the RG-extrapolated TCSA and the form factor series, depending on the multi-particle contributions included in the latter.

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Figure 5.4.4. $i_1 \langle \{0\} | \Phi_{1,2} | \{0\} \rangle_1$ calculated by the form factor series including different contributions and RG extrapolated TCSA.
The difference between the evaluations of $i_1 \langle \{0\} | \Phi_{1,2} | \{0\} \rangle_1$ from the RG-extrapolated TCSA and the form factor series, depending on the multi-particle contributions included in the latter.

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Table 5.4.4. The difference between the evaluations of $i_1 \langle \{0\} | \Phi_{1,2} | \{0\} \rangle_1$ from the RG-extrapolated TCSA and the form factor series, depending on the multi-particle contributions included in the latter.

The reason for $\rho_{tot}$ vanishing at $r_c$ can be understood from the excited TBA. Since the active singularities coincide at this point, the density which is the Jacobi determinant of the quantisation condition for the active singularities, is zero due to the degeneracy. Such singularities of the density were observed previously for the finite volume form factors formula in [STW13]; however in that case including the exponential corrections resolved (or at least shifted) the singularity. The present situation is different as all exponential corrections to the density are already included. To resolve the singularity it would be necessary to include every term of the form factor series, to compensate for the zero of the denominator.

This conclusion is also supported by the behaviour of the pseudoenergies and the filling fractions around $r_c$. Approaching $r_c$ the filling fractions no more suppress the higher order terms in the series and the ordering of terms by their magnitude is not valid anymore, i.e. every terms is important in the series. This is consistent with the procedure of desingularisation, whereby to describe the excited state level with the TBA equation under $r_c$ it was necessary to redefine the pseudoenergy to have a form which is finite and convergent under iterations.

For the form factor series a similar rearrangement is necessary close to and under $r_c$. Unfortunately such a rearrangement is not yet known, and this sets the practical validity of the form factor series to IR regions where no nontrivial transitions occur in the TBA.
5.5 Summary

In this chapter, we stated the generalisation of the LM series, a series for a general diagonal finite volume matrix element in integrable QFTs with diagonal scattering and massive particles. We proved the generalised conjecture for the trace of the stress-energy tensor operator. Furthermore, we checked the validity of the conjecture numerically in the $T_2$ model. We found precise agreement with the TCSA results for states with nonzero spin. However, for spin zero states, where the TBA equations can have a nontrivial transition, the convergence of the series breaks down at the critical volume. The resolution of this problem is presently unknown.
Chapter 6

Thermal two-point functions
In this chapter, we present a way to evaluate finite temperature two-point functions with spectral expansion. The calculation uses finite volume regularisation, and the infinite volume limit is taken at the end of the computation resulting in a spectral expansion that consists of well-defined terms order by order. We present the contribution to the expansion up to two-particle terms, and check the expressions numerically in the sinh-Gordon model.

For simplicity, in this chapter we only consider theories with one kind of particle species\(^1\), and we switch the notation of the compactified directions: \( R \) denotes the circumference in the time direction, while \( L \) is the circumference in the compactified space direction.

### 6.1 Thermal two-point functions with finite volume regularisation

A field theory with finite temperature \( T \) can be defined using a compact Euclidean (Matsubara) time \( t \):

\[
t = t + R, \quad \text{where} \quad R = 1/T.
\]

The two-point function can be expressed as

\[
\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0) \rangle^R = \frac{\text{Tr} \left( e^{-RH} \mathcal{O}_1(x,t)\mathcal{O}_2(0) \right)}{\text{Tr} \left( e^{-RH} \right)},
\]

where \( H \) is the Hamiltonian of the infinite volume system. A naive spectral sum leads to an ill-defined expression due to the presence of disconnected contributions (cf. e.g. the discussion in [PT10]). However, one can put the system in a finite spatial volume \( L \) with periodic boundary conditions

\[
x = x + L,
\]

where we assume, the volume is big enough, so the Bethe-Yang quantisation condition is valid (3.2.1). In finite volume, the two-point function takes the form

\[
\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0) \rangle^R_L = \frac{\text{Tr}_L \left( e^{-RH_L} \mathcal{O}_1(x,t)\mathcal{O}_2(0) \right)}{\text{Tr}_L \left( e^{-RH_L} \right)},
\]

where \( \text{Tr}_L \) denotes the trace over the finite volume states, \( H_L \) is the Hamiltonian in volume

---

\(^1\)The results can be easily generalised for diagonal scattering theories with multiple particle species.
This expression can be expanded inserting two complete sets of states

\[
\text{Tr}_L \left( e^{-R L} \mathcal{O}_1(x,t) \mathcal{O}_2(0) \right) = \sum_{m,n} e^{-R E_{n(L)}} \langle n | \mathcal{O}_1(x,t) | m \rangle _L \langle m | \mathcal{O}_2(0) | n \rangle _L ,
\]

(6.1.5)

where the matrix elements of local operators are also taken in the finite volume system. \(|n\rangle_L\) is a shorthand notation for the finite volume states (3.2.6) with \(n\) particles; the summation for the quantum numbers is suppressed in the expression. To evaluate (6.1.5), we need the expression for finite volume form factors summarised in Section 3.3.

### 6.1.1 The form factor expansion using finite volume regularisation

Using the finite volume description introduced in Section 3.3, we can write

\[
\langle \mathcal{O}_1(x,t) \mathcal{O}_2(0) \rangle _{R,L} = \frac{1}{Z} \sum_{N,M} C_{NM}
\]

(6.1.6)

where

\[
C_{NM} = \sum_{I_1 \ldots I_N, J_1 \ldots J_M} L \langle \{I_1, \ldots, I_N\} | \mathcal{O}_1(0) | \{J_1, \ldots, J_M\} \rangle _L \times
\]

\[
L \langle \{J_1, \ldots, J_M\} | \mathcal{O}_2(0) | \{I_1, \ldots, I_N\} \rangle _L e^{i(P_1 - P_2)x} e^{-E_1(R-t)} e^{-E_2t}
\]

(6.1.7)

and \(E_{1,2}\) and \(P_{1,2}\) are the total energies and momenta of the multiparticle states (2.1.1) and (2.1.2) with the rapidity solution of the Bethe-Yang equations (3.2.4). The task is to calculate the sum in finite volume and then take the limit \(L \to \infty\).

First, we classify the contributions into different multiparticle orders following the procedure in [EK09, PT10]. Introducing two auxiliary variables \(u\) and \(v\) (at the end both will be set to 1):

\[
\langle \mathcal{O}_1(x,t) \mathcal{O}_2(0) \rangle _{R,L} = \frac{1}{Z} \sum_{N,M} u^N v^M C_{NM} .
\]

(6.1.8)

Similarly for the partition function

\[
Z = \sum_N (uv)^N Z_N ,
\]

with \(Z_N\) denoting the \(N\)-particle contribution. The inverse of the partition function is expanded as

\[
Z^{-1} = \sum_N (uv)^N \tilde{Z}_N ,
\]

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where
\[ \bar{Z}_0 = 1, \quad \bar{Z}_1 = -Z_1, \quad \bar{Z}_2 = Z_1^2 - Z_2. \]

Putting this together, we can rewrite the expansion as
\[ \langle \mathcal{O}_1(x, t)\mathcal{O}_2(0) \rangle^R_L = \sum u^N v^N \tilde{D}_{NM}, \tag{6.1.9} \]
with
\[ \tilde{D}_{NM} = \sum_l C_{N-l,M-l}\bar{Z}_l. \tag{6.1.10} \]

The first few nontrivial terms are given by
\[ \tilde{D}_{1M} = C_{1M} - Z_1 C_{0,M-1}, \]
\[ \tilde{D}_{2M} = C_{2M} - Z_1 C_{1,M-1} + (Z_1^2 - Z_2) C_{0,M-2}. \tag{6.1.11} \]

In this way we produce a double series expansions in powers of the variables $e^{-mt}$ and $e^{-m(R-t)}$. Since these variables are independent, each quantity \( \tilde{D}_{NM} \) must have a well-defined \( L \to \infty \) limit which we denote as
\[ D_{NM} = \lim_{L \to \infty} \tilde{D}_{NM}, \tag{6.1.12} \]
and we obtain that
\[ \langle \mathcal{O}_1(x, t)\mathcal{O}_2(0) \rangle^R = \lim_{L \to \infty} \langle \mathcal{O}_1(x, t)\mathcal{O}_2(0) \rangle^R_L = \sum_{N,M} D_{NM}. \tag{6.1.13} \]

A similar reordering was also used for the expansion of the one-point function in powers of $e^{-mR}$ [PT08b], and for the boundary one-point function in [Tak08]. It is evident from (6.1.7) that the $D_{NM}$ with $N > M$ can be obtained from those with $N < M$ after a trivial exchange of $t$ with $R - t$, $x$ with $-x$ and $\mathcal{O}_1$ with $\mathcal{O}_2$. 

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6.2 The spectral expansion for finite temperature correlators

To evaluate the finite temperature two-point function, it is necessary to evaluate the summation over two sets of intermediate states. For a given $C_{NM}$ this involves an $N$ and an $M$ particle state. One can start with any of these; to simplify the calculations, it is best to start with the one containing the smaller number or particles, and do the other later. On the other hand, doing the calculation in the reverse order allows one to cross-check the result [PT10].

To evaluate the first summation, a systematic method was given in [PT10] based on a multidimensional residue method. Once this is done, all the singularities from the form factors are tamed, and the second summation can be performed by a simple transition from the discrete sum to an integral using the density of states. Then, after assembling $\tilde{D}_{NM}$ using the lower $C_{N'M'}$ coefficients as in (6.1.10), and taking the limit $L \rightarrow \infty$ the final formula for the contribution $D_{NM}$ can be obtained. Another quick validity check of the calculation is provided by the existence of the infinite volume limit.

6.2.1 Converting sums to contour integrals

For sums over one-particles states $|\{I\}\rangle_L$ with quantum number $I \in \mathbb{Z}$ we can substitute

$$\sum_I \rightarrow \sum_I \oint_{C_I} \frac{d\theta}{2\pi} \frac{\rho_1(\theta)}{e^{iQ_1(\theta)} - 1},$$

where $Q_1(\theta)$ is the logarithm of the Bethe-Yang equations for one particle (3.2.4), $\rho_1(\theta)$ is the density of the state (3.2.7) and $C_I$ are small closed curves surrounding the solution of

$$Q_1(\theta) = 2\pi I,$$

in the complex $\theta$ plane.

For two-particle sums over two-particle states $|\{I_1, I_2\}\rangle_L$ with quantum numbers $I_1, I_2 \in \mathbb{Z} + \frac{1}{2}$ we can use the multidimensional generalisation of the residue theorem to write

$$\sum_{I_1 > I_2} \rightarrow \sum_{I_1 > I_2} \oint_{C_{I_1, I_2}} \oint_{C_{I_1, I_2'}} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{\rho_2(\theta_1, \theta_2)}{e^{iQ_1(\theta_1, \theta_2)} + 1} \frac{\rho_2(\theta_1', \theta_2')}{e^{iQ_2(\theta_1', \theta_2')} + 1},$$

where $C_{I_1, I_2}$ is a multi-contour (a direct product of two closed curves encircling the pole.
locations in the variables $\theta_1$ and $\theta_2$) surrounding the Bethe-Yang solutions of

\[
Q_1(\theta_1, \theta_2) = mL \sinh \theta_1 + \delta(\theta_1 - \theta_2) = 2\pi I_1,
\]
\[
Q_2(\theta_1, \theta_2) = mL \sinh \theta_2 + \delta(\theta_2 - \theta_1) = 2\pi I_2,
\]

where due to the definition (3.2.5), $I_1$ and $I_2$ take half-integer values, and

\[
\rho_2(\theta_1, \theta_2) = \det \begin{pmatrix}
\frac{\partial Q_2}{\partial \theta_1} & \frac{\partial Q_1}{\partial \theta_1} \\
\frac{\partial Q_2}{\partial \theta_2} & \frac{\partial Q_1}{\partial \theta_2}
\end{pmatrix} = m^2 L^2 \cosh \theta_1 \cosh \theta_2 + mL(\cosh \theta_1 + \cosh \theta_2) \varphi(\theta_1 - \theta_2).
\]

Since form factors vanish when any two of their arguments coincide, we can extend the sum by adding the diagonal terms

\[
\sum_{I_1 > I_2} \rightarrow \frac{1}{2} \left( \sum_{I_1=I_2} - \sum_{I_1 < I_2} \right).
\]

In the next step, the contours are joined together and opened into straight lines, to a product contour whose components in each variable enclose the real axis. However, this can only be done by including other poles (apart from the ones needed for the state summations) in the interior, which come from singularities of the $Q$-dependent denominators and of the form factors. These must be classified and subtracted. This procedure was discussed in some detail in [PT10], and for one complex variable it is illustrated in fig. 6.2.1 (for more complex variable it must be performed in each variables separately). We shall only outline it for the case of the $D_{22}$ contribution, because of the corrections we make to the previous calculation.
performed in that paper.

6.2.2 The \( D_{22} \) contribution revisited

The \( D_{22} \) contribution is given by

\[
D_{22} = \lim_{L \to \infty} \left[ C_{22} - Z_1 C_{11} + (Z_1^2 - Z_2) C_{00} \right] = \lim_{L \to \infty} \left[ C_{22} - Z_1 \tilde{D}_{11} - Z_2 C_{00} \right],
\]

where

\[
C_{22} = \sum_{I_1 > I_2} \sum_{J_1 > J_2} L \langle \{I_1, I_2\} | \mathcal{O}_1(0) | \{J_1, J_2\} \rangle_L \langle \{J_1, J_2\} | \mathcal{O}_2(0) | \{I_1, I_2\} \rangle_L K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2),
\]

with the notation

\[
K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2) = e^{imx(\sinh \vartheta_1 + \sinh \vartheta_2 - \sinh \vartheta'_1 - \sinh \vartheta'_2)} e^{-m(R-t)(\cosh \vartheta_1 + \cosh \vartheta_2)} e^{-m(R-t)(\cosh \vartheta'_1 + \cosh \vartheta'_2)}, \tag{6.2.1}
\]

and where [PT10]

\[
Z_1 = mL \int \frac{d\vartheta_1}{2\pi} \cosh \vartheta_1 e^{-mR \cosh \vartheta_1},
\]

\[
Z_2 = \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \rho_2(\vartheta_1, \vartheta_2) e^{-2mR \cosh \vartheta_1} - \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \rho_1(\vartheta_1) e^{-2mR \cosh \vartheta_1},
\]

\[
C_{00} = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle,
\]

\[
D_{11} = \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F_{2s}^{\mathcal{O}_1} (\vartheta_1 + i\pi, \vartheta_2) F_{2s}^{\mathcal{O}_2} (\vartheta_2 + i\pi, \vartheta_1) e^{imx(\sinh \vartheta_1 - \sinh \vartheta_2)} e^{-m(R-t) \cosh \vartheta_1} e^{-mt \cosh \vartheta_2}
\]

\[
+ \left[ \langle \mathcal{O}_1 \rangle F_{2s}^{\mathcal{O}_1} + \langle \mathcal{O}_2 \rangle F_{2s}^{\mathcal{O}_2} \right] \int \frac{d\vartheta_1}{2\pi} e^{-mR \cosh \vartheta_1},
\]

The rapidities are quantised by the Bethe-yang equations (3.2.4), namely

\[
Q_1(\vartheta_1, \vartheta_2) = mL \sinh \vartheta_1 + \delta(\vartheta_1 - \vartheta_2) = 2\pi I_1, \tag{6.2.2}
\]

\[
Q_2(\vartheta_1, \vartheta_2) = mL \sinh \vartheta_2 + \delta(\vartheta_2 - \vartheta_1) = 2\pi I_2,
\]

and

\[
Q'_1(\vartheta'_1, \vartheta'_2) = mL \sinh \vartheta'_1 + \delta(\vartheta'_1 - \vartheta'_2) = 2\pi J_1,
\]

\[
Q'_2(\vartheta'_1, \vartheta'_2) = mL \sinh \vartheta'_2 + \delta(\vartheta'_2 - \vartheta'_1) = 2\pi J_2.
\]
We perform the sum for $J_1$ and $J_2$ first and separate it into a diagonal and an off-diagonal piece:
\[
\sum_{J_1 > J_2} = (\{J_1, J_2\} = \{I_1, I_2\} \text{ term}) + \sum_{J_1 > J_2}'
\]

because the finite volume form factor expressions are different for the two types of contributions. In the second term, the prime indicates that the diagonal contributions are excluded.

6.2.2.1 The diagonal piece

This calculation is exactly the same as in [PT10], so we only highlight the main steps. Starting from
\[
\sum_{J_1 > J_2} = \sum_{I_1 > I_2} L \langle \{I_1, I_2\} | O_1(0) | \{I_1, I_2\} \rangle L \langle \{I_1, I_2\} | O_2(0) | \{I_1, I_2\} \rangle L \langle \{I_1, I_2\} | O_2(0) | \{I_1, I_2\} \rangle L K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta_2),
\]

where we use the diagonal form factor formula with the symmetric form factors (3.3.4)
\[
L \langle \{I_1, I_2\} | O(0) | \{I_1, I_2\} \rangle L = F_{4s}^{O}(\vartheta_1, \vartheta_2) + (\rho_1(\vartheta_1) + \rho_1(\vartheta_2)) F_{2s}^{O} + \rho_2(\vartheta_1, \vartheta_2) \langle O \rangle.
\]

Writing the sum in terms of contour integrals, after opening the contours and performing the large $L$ limit the diagonal contribution becomes
\[
C_{22}^{\text{diag}} = \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \left[ F_{4s}^{O_1}(i\pi, 0) F_{2s}^{O_2}(i\pi, 0) e^{-mR(cosh \vartheta_1 + cosh \vartheta_2)} + F_{4s}^{O_2}(i\pi, 0) F_{2s}^{O_1}(i\pi, 0) e^{-mR(cosh \vartheta_1 + cosh \vartheta_2)} \right]
\]

\[
+ \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{\cosh \vartheta_1 + \cosh \vartheta_2}{\cosh \vartheta_1 \cosh \vartheta_2} F_{2s}^{O_1}(i\pi, 0) F_{2s}^{O_2}(i\pi, 0) e^{-mR(cosh \vartheta_1 + cosh \vartheta_2)}
\]

\[
- \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \left[ F_{2s}^{O_1}(i\pi, 0) \langle O_2 \rangle + F_{2s}^{O_2}(i\pi, 0) \langle O_1 \rangle \right] e^{-m2R \cosh \vartheta_1}
\]

\[
+ \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} mL \cosh \vartheta_1 \left[ F_{2s}^{O_1}(i\pi, 0) \langle O_2 \rangle + F_{2s}^{O_2}(i\pi, 0) \langle O_1 \rangle \right] e^{-mR(cosh \vartheta_1 + cosh \vartheta_2)}
\]

\[
+ Z_2 C_{00}.
\]

6.2.2.2 The non-diagonal part

In the non-diagonal part, one can use (3.3.1)

\[
L \langle I_1, I_2 | O(0) | J_1, J_2 \rangle_L = \frac{F_{4s}^{O}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1, \vartheta'_2)}{\sqrt{\rho_2(\vartheta_1, \vartheta_2) \rho_2(\vartheta'_1, \vartheta'_2)}}.
\]

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to write

$$C_{22}^{nondiag} = \sum_{I_1 > I_2} \sum_{J_1 > J_2} \left\{ \frac{F_4^{O_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1, \vartheta'_2) F_4^{O_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2, \vartheta'_1)}{\rho_2(\vartheta_1, \vartheta_2) \rho_2(\vartheta'_1, \vartheta'_2)} \right\}$$

$$\times K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2)$$

$$= \sum_{I_1 > I_2} \tilde{C}_{22} \rho_2(\vartheta_1, \vartheta_2),$$

where

$$\tilde{C}_{22} = \sum_{J_1 > J_2} \oint_{C_{j_1} \times C_{j_2}} \oint_{2\pi} d\vartheta'_1 d\vartheta'_2 \left\{ \frac{F_4^{O_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1, \vartheta'_2) F_4^{O_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2, \vartheta'_1)}{[e^{iQ'_1(\vartheta'_1, \vartheta'_2)} + 1] [e^{iQ'_2(\vartheta'_1, \vartheta'_2)} + 1]} \right\}$$

and the prime denotes the omission of the \{J_1, J_2\} = \{1, 2\} term. We can substitute

$$\sum_{J_1 > J_2} \rightarrow \frac{1}{2} \sum_{J_1, J_2}$$

since the form factors vanish when any two of their rapidity arguments are identical.

Now we open the contours to encircle the real axis in \vartheta'_1 and \vartheta'_2. However, that brings more singularities inside the contour whose contribution must then be subtracted. These can be classified as follows:

1. **Spurious QQ-poles.** There are two such terms, which come from including the poles with \(J_{1,2} = I_{1,2}\) or \(J_{1,2} = I_{2,1}\). Their contribution vanishes for \(L \rightarrow \infty\) \[PT10\], hence the term “spurious”. However, they must be included in the numerical tests, therefore we provide their form in equation (C.2.11). Note that the form factors are not singular in this case, although their limits in such points are direction dependent.

2. **QF-poles.** In this case one of the integrations has a pole from a form factor, and the
other one from a $Q$-term:

\[
\begin{align*}
\text{QFI} : & \quad \varphi'_1 = \varphi_1 \quad \& \quad Q'_2(\varphi_1, \varphi'_2) = 2\pi J_2; \\
\text{QFII} : & \quad \varphi'_1 = \varphi_2 \quad \& \quad Q'_2(\varphi_2, \varphi'_2) = 2\pi J_2; \\
\text{QFIII} : & \quad \varphi'_2 = \varphi_1 \quad \& \quad Q'_1(\varphi'_1, \varphi_1) = 2\pi J_1; \\
\text{QFIV} : & \quad \varphi'_2 = \varphi_2 \quad \& \quad Q'_1(\varphi'_1, \varphi_2) = 2\pi J_1.
\end{align*}
\]

3. **FF poles.** In this case poles in both integrals come from form factors:

\[
\begin{align*}
\text{FFI} : & \quad \varphi'_1 = \varphi'_2 = \varphi_1; \\
\text{FFII} : & \quad \varphi'_1 = \varphi'_2 = \varphi_2.
\end{align*}
\]

The poles of the form factors can be separated by introducing the regular connected part $F_{1rc}$:

\[
\begin{align*}
F^Q_4(\varphi_2 + i\pi, \varphi_1 + i\pi, \varphi'_1, \varphi'_2) & = \frac{A}{\varphi_2 - \varphi'_1} + \frac{B}{\varphi_2 - \varphi'_2} + \frac{C}{\varphi_1 - \varphi'_1} + \frac{D}{\varphi_1 - \varphi'_2} \\
& + F^Q_{1rc}(\varphi_2 + i\pi, \varphi_1 + i\pi|\varphi'_1, \varphi'_2), \\
F^Q_2(\varphi_1 + i\pi, \varphi_2 + i\pi, \varphi'_1, \varphi'_2) & = \frac{E}{\varphi_1 - \varphi'_2} + \frac{F}{\varphi_1 - \varphi'_1} + \frac{G}{\varphi_2 - \varphi'_2} + \frac{H}{\varphi_2 - \varphi'_1} \\
& + F^Q_{2rc}(\varphi_1 + i\pi, \varphi_2 + i\pi|\varphi'_1, \varphi'_2),
\end{align*}
\]

where

\[
\begin{align*}
A & = i(S(\varphi_2 - \varphi_1) - S(\varphi'_1 - \varphi'_2)) F^Q_2(\varphi_1 + i\pi, \varphi'_2), \\
B & = i(S(\varphi'_1 - \varphi'_2)S(\varphi_2 - \varphi_1) - 1) F^Q_2(\varphi_1 + i\pi, \varphi'_1), \\
C & = i((1 - S(\varphi_2 - \varphi_1)S(\varphi'_1 - \varphi'_2)) F^Q_2(\varphi_2 + i\pi, \varphi'_2), \\
D & = i(S(\varphi'_1 - \varphi'_2) - S(\varphi_2 - \varphi_1)) F^Q_2(\varphi_2 + i\pi, \varphi'_1), \\
E & = i(S(\varphi_1 - \varphi_2) - S(\varphi'_2 - \varphi'_1)) F^Q_2(\varphi_2 + i\pi, \varphi'_1), \\
F & = i(S(\varphi'_2 - \varphi'_1)S(\varphi_1 - \varphi_2) - 1) F^Q_2(\varphi_2 + i\pi, \varphi'_2), \\
G & = i((1 - S(\varphi_1 - \varphi_2)S(\varphi'_2 - \varphi'_1)) F^Q_2(\varphi_1 + i\pi, \varphi'_1), \\
H & = i(S(\varphi'_2 - \varphi'_1) - S(\varphi_1 - \varphi_2)) F^Q_2(\varphi_1 + i\pi, \varphi'_2).
\end{align*}
\]

Using the above notation, the pole terms resulting from the form factors can be obtained:
from which one can identify the terms giving $QF$ and $FF$ type singularities. For the residue calculation, the formulas of Appendix Section C.1 can be used. This results in certain differences from the result derived in [PT10], where too simplistic evaluation of residues resulted in some inaccuracies in the end result.

Once the residues are calculated, in the case of the $QF$ terms a further summation remains which must be converted into an integral. It has the general form (here written for the case $QFI$):

$$
\sum_{J_2 \neq J_2} \frac{G(\vartheta_1, \vartheta_2, \vartheta'_2)}{\vartheta'_2 - \vartheta'_1},
$$

where $\vartheta'_2$ is a solution to

$$Q'_2(\vartheta_1, \vartheta'_2) = 2\pi J_2,
$$

and the case $J_2 = I_2$ was omitted since it is a spurious $QQ$ singularity. One can convert the $J_2$ summation into integrals using the residue formula

$$
- \sum_{J_2 \neq J_2} \oint_{C_{J_2}} \frac{d\vartheta'_2}{2\pi i} \frac{G(\vartheta_1, \vartheta_2, \vartheta'_2)}{e^{iQ'_2(\vartheta_1, \vartheta'_2)} + 1}.
$$
Opening the contours and taking care to eliminate the contributions resulting from possible poles of the function $G$ lying on the real $\vartheta'_2$ axis:
where the second term corrects for the subtraction of the $J_2 = I_2$ case and $\vartheta'_2$ denotes the location of the poles of $G$. The notation $\leftrightarrow$ corresponds to the straight line contours enclosing the real axis as illustrated in Figure 6.2.1. The full results of the residue calculations are given in Appendix Section C.2.

The $J_2 = I_2$ term typically is of order $O(1/L)$, except for second order pole contributions. This results in the following contribution to the $QF$ terms:

\[
F_2^{O_1}(i\pi, 0)F_2^{O_2}(i\pi, 0)K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_2, \vartheta_1) \\
\times \left( \frac{mL \cosh \vartheta_1 - \varphi (\vartheta_2 - \vartheta_1)}{mL \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} + \frac{mL \cosh \vartheta_2 - \varphi (\vartheta_2 - \vartheta_1)}{mL \cosh \vartheta_1 + \varphi (\vartheta_1 - \vartheta_2)} \right),
\]

which is included in $QF6$ in (C.2.9). This term was omitted by the calculation performed in [PT10]; its presence is critical for the cluster property.

### 6.2.2.3 Performing the $I_1, I_2$ sum and the large volume limit

We can write

\[
C_{22}^{\text{mndia}} = \sum_{I_1 > I_2} \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_2)}{\rho_2(\vartheta_1, \vartheta_2)} = \frac{1}{2} \left( \sum_{I_1, I_2} - \sum_{I_1 = I_2} \right) \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_2)}{\rho_2(\vartheta_1, \vartheta_2)} = \\
= \frac{1}{2} \sum_{I_1 > I_2} \int_{C_{I_1}} \int_{C_{I_2}} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_2)}{e^{iQ_1(\vartheta_1, \vartheta_2)} + 1} \left[ e^{iQ_2(\vartheta_1, \vartheta_2)} + 1 \right] \\
+ \frac{1}{2} \sum_{I_1 = I_2} \int_{C_{I_1}} \frac{d\vartheta_1}{2\pi} \left[ e^{iQ_1(\vartheta_1, \vartheta_1)} + 1 \right] \rho_1(\vartheta_1) \frac{\tilde{C}_{22}(\vartheta_1, \vartheta_1)}{\rho_2(\vartheta_1, \vartheta_1)}.
\]

Since $\tilde{C}_{22}$ doesn’t have any pole we open the contours in the usual way enclosing the real axis as illustrated in Figure 6.2.1. For the $L \to \infty$ it is necessary to examine the behaviour
of the \(Q\)-functions:

\[
iQ_1(\vartheta_1 + i\varepsilon_1, \vartheta_2 + i\varepsilon_2) = imL \sinh(\vartheta_1 + i\varepsilon_1) + i\delta(\vartheta_1 + i\varepsilon_1 - \vartheta_2 - i\varepsilon_2) = \]

\[
= imL \sinh \vartheta_1 - mL \cosh \vartheta_1 \sin \varepsilon_1 + i\delta(\vartheta_1 + i\varepsilon_1 - \vartheta_2 - i\varepsilon_2),
\]

and similarly for \(Q_2\) and \(Q'_1,2\). This results in the following limits:

\[
\lim_{L \to \infty} \frac{1}{e^{iQ_1(\vartheta_1 + i\varepsilon_1, \vartheta_2 + i\varepsilon_2)} + 1} = \begin{cases} 
1, & \varepsilon_i \in [0, \pi] + 2n\pi \\
0, & \varepsilon_i \in [\pi, 2\pi] + 2n\pi 
\end{cases}
\]

\[
\lim_{L \to \infty} \frac{1}{e^{iQ_1(\vartheta_1 + i\varepsilon_1, \vartheta_1 + i\varepsilon_1)} + 1} = \begin{cases} 
1, & \vartheta_1 \in [0, \pi] + 2n\pi \\
0, & \vartheta_1 \in [\pi, 2\pi] + 2n\pi 
\end{cases}
\]

\[
\lim_{L \to \infty} \frac{1}{e^{iQ'_1(\vartheta'_1 + i\varepsilon_1, \vartheta_1 + i\varepsilon_1)} + 1} = \begin{cases} 
1, & \varepsilon_i \in [0, \pi] + 2n\pi \\
0, & \varepsilon_i \in [\pi, 2\pi] + 2n\pi 
\end{cases}
\]

Therefore only the upper contours need to be kept, since all other terms vanish exponentially for large \(L\):

\[
\frac{1}{2} \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \tilde{C}_{22}(\vartheta_1 + i\varepsilon, \vartheta_2 + i\varepsilon) - \frac{1}{2} \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_1'}{2\pi} \tilde{C}_{22}(\vartheta_1 + i\varepsilon, \vartheta_1 + i\varepsilon) \frac{\rho_1(\vartheta_1 + i\varepsilon)}{\rho_2(\vartheta_1 + i\varepsilon, \vartheta_1 + i\varepsilon)}. 
\]

In addition, the integrals can be shifted to the real axis. However this leads to singularities in the contribution like \(QF5\) (C.2.6) due to the term containing

\[
\frac{K'_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1)}{[e^{iQ'_1(\vartheta'_1, \vartheta_1)} + 1] (\vartheta_1 - \vartheta_1')} 
\]

which can be treated using the identity

\[
\frac{1}{x \pm i\varepsilon} = \mathcal{P} \frac{1}{x} \mp i\pi \delta(x).
\]

### 6.2.3 End result for \(D_{22}\)

The terms divergent as \(L \to \infty\) drop out when including the contribution \(-Z_1 \tilde{D}_{11} - Z_2 C_{00}\).
We can also combine some terms by introducing the function

\[ F_{4ss}^O (\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2, \vartheta'_1) = \]
\[ = \frac{i}{\vartheta_1 - \vartheta'_2} (\vartheta_1 - \vartheta_2) - S(\vartheta'_2 - \vartheta'_1)) F_2^O (\vartheta_2 + i\pi, \vartheta'_1) \]
\[ + \frac{i (1 - S(\vartheta_1 - \vartheta_2) S(\vartheta'_2 - \vartheta'_1))}{\vartheta_2 - \vartheta'_1} F_2^O (\vartheta_1 + i\pi, \vartheta'_1) \]
\[ + F_{4tr}^O (\vartheta_1 + i\pi, \vartheta_2 + i\pi | \vartheta'_2, \vartheta'_1). \]

The end result is

\[
D_{22} = \frac{1}{4} \iiint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{d\vartheta'_1}{2\pi} F_4^O (\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1 + i\varepsilon, \vartheta'_2 + i\varepsilon) \]
\[ \times F_4^O (\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2 + i\varepsilon, \vartheta'_1 + i\varepsilon) K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta'_1 + i\varepsilon, \vartheta'_2 + i\varepsilon) \]
\[ + \iiint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \varphi \left( \frac{1}{2\pi} \int \frac{d\vartheta'_2}{2\pi} \right) F_4^O \{ F_{4ss}^O (\vartheta_1 + i\pi, \vartheta_2 + i\xi | \vartheta'_1, \vartheta'_1) F_2^O (\vartheta_2 + i\pi, \vartheta'_1) \}
\[ + F_{4ss}^O (\vartheta_1 + i\pi, \vartheta_2 + i\xi | \vartheta'_1, \vartheta'_1) \}
\[ \times [1 - S(\vartheta'_1 - \vartheta_1) S(\vartheta_1 - \vartheta_2)] (m x \cos \vartheta_1 - i m t \sinh \vartheta_1) \]
\[ - \varphi (\vartheta'_1 - \vartheta_1) S(\vartheta'_1 - \vartheta_1) S(\vartheta_1 - \vartheta_2) \]
\[ - \iiint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{d\vartheta'_1}{2\pi} F_2^O (\vartheta_2 + i\pi, \vartheta_1) K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_1) \]
\[ - \iiint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{d\vartheta'_1}{2\pi} F_2^O (\vartheta_2 + i\pi, \vartheta_1) K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_1) \]
\[ + \frac{1}{2} \iiint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} [ F_{4s}^O (\vartheta_1, \vartheta_2) \langle O_2 \rangle + F_{4s}^O (\vartheta_1, \vartheta_2) \langle O_1 \rangle ] K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2) \]
\[ + \iiint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{d\vartheta'_1}{2\pi} F_2^O (i\pi, 0) F_2^O (i\pi, 0) K_{t,x}^{(R)} (\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2) \]
\[ - \iiint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} [ F_2^O (i\pi, 0) \langle O_2 \rangle + F_2^O (i\pi, 0) \langle O_1 \rangle ] K_{t,x}^{(R)} (\vartheta_1, \vartheta_1 | \vartheta'_1, \vartheta'_1) \]

where \( \mathcal{P} \) denotes a principal value integral, \( K_{t,x}^{(R)} \) is defined in (6.2.1) and \( F_{4s} \) is symmetric evaluation of the form factor (2.3.8). Note that by introducing \( F_{4ss} \) we combined the terms (6.2.9) into the second integral, hence the need for the principal value.

In (6.2.11), the underlined pieces are the contributions that are different from the earlier calculation performed in [PT10]. The first underlined term only corrects a typo in the
previous work, where this piece was printed with the wrong sign. The second one comes from the subtraction of poles of the integrand in (6.2.7) and the careful evaluation of the principal value term (6.2.9), both of which occur in the manipulation of the QF5 contributions (C.2.6).

The third underlined term plays a crucial role in the cluster property. It is the leftover from the term

$$\frac{1}{2} \int\frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \left[ \cosh \vartheta_1 + \cosh \vartheta_2 \right]^2 \cosh \vartheta_1 \cosh \vartheta_2 \int_0^{\pi} \int_0^{\pi} F_2^{O_1} (i\pi, 0) F_2^{O_2} (i\pi, 0) e^{-mR(\cosh \vartheta_1 + \cosh \vartheta_2)},$$

present in the diagonal contribution (6.2.3), the dependence on

$$\frac{[\cosh \vartheta_1 + \cosh \vartheta_2]^2}{\cosh \vartheta_1 \cosh \vartheta_2} = \frac{\cosh \vartheta_1}{\cosh \vartheta_2} + \frac{\cosh \vartheta_2}{\cosh \vartheta_1} + 2,$$

is simplified by the inclusion of the contribution (6.2.8), coming from the second order pole terms collected in QF6 (C.2.9). In the large $L$ limit, the terms depending on the cosh ratios cancel, leaving us with the last underlined piece in (6.2.11). As mentioned before, one of the mistakes made in the evaluation of $D_{22}$ in [PT10] was the omission of this piece.

### 6.2.4 The full two-point function up to $D_{22}$

For completeness, we also give here the lower contributions to the two-point function. These are exactly the same as in [PT10], so we do not give the derivations here. The calculations are almost trivial with the exception of $D_{12}$, where one can use either the derivations presented in [PT10], or follow the steps outlined above, with slight modifications. The terms $D_{NM}$ with $N \leq M \leq 2$ are

\[
\begin{align*}
D_{00} &= \langle O_1 \rangle \langle O_2 \rangle, \\
D_{01} &= \int \frac{d\vartheta_1}{2\pi} F_1^{O_1} F_2^{O_2} e^{-imx \sinh \vartheta_1 - mt \cosh \vartheta_1}, \\
D_{02} &= \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F_2^{O_1} (\vartheta_1, \vartheta_2) F_2^{O_2} (\vartheta_2, \vartheta_1) e^{-imx(\sinh \vartheta_1 + \sinh \vartheta_2) - mt(\cosh \vartheta_1 + \cosh \vartheta_2)}, \\
D_{11} &= \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F_2^{O_1} (\vartheta_1 + i\pi, \vartheta_2) F_2^{O_2} (\vartheta_2 + i\pi, \vartheta_1) e^{imx(\sinh \vartheta_1 - \sinh \vartheta_2) - m(R-t) \cosh \vartheta_1 - mt \cosh \vartheta_2} \\
+ [\langle O_1 \rangle F_2^{O_2} + \langle O_2 \rangle F_2^{O_1}] \int \frac{d\vartheta_1}{2\pi} e^{-mR \cosh \vartheta_1},
\end{align*}
\]
\[ D_{12} = \frac{1}{2} \int \int d\vartheta'_1 d\vartheta'_2 \int d\vartheta''_1 d\vartheta''_2 F_3^{\mathcal{O}_1}(\vartheta''_1 + i(\pi + \varepsilon), \vartheta''_2) F_3^{\mathcal{O}_2}(\vartheta''_1 + i(\pi + \varepsilon), \vartheta''_2) \times K_{t,x}^{(R)}(\vartheta''_1 + i\varepsilon|\vartheta'_1, \vartheta'_2) \]

\[ + \int d\vartheta'_1 d\vartheta'_2 \left[ F_1^{\mathcal{O}_1} F_3^{\mathcal{O}_2}(\vartheta'_1 + i\pi|\vartheta'_2, \vartheta''_2) + F_1^{\mathcal{O}_2} F_3^{\mathcal{O}_1}(\vartheta'_1 + i\pi|\vartheta'_2, \vartheta''_1) \right] K_{t,x}^{(R)}(\vartheta'_1|\vartheta'_1, \vartheta'_2) \]

\[ + \int d\vartheta'_1 d\vartheta'_2 \left[ F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} K_{t,x}^{(R)}(\vartheta'_1|\vartheta'_1, \vartheta'_2) (S(\vartheta'_1 - \vartheta'_2) - 1) K_{t,x}^{(R)}(\vartheta'_1|\vartheta'_1, \vartheta'_2) \right] \]

\[ - \int d\vartheta'_1 F_1^{\mathcal{O}_1} F_1^{\mathcal{O}_2} K_{t,x}^{(R)}(\vartheta'_1|\vartheta'_1, \vartheta'_1) \]  

(6.2.12)

and \( D_{22} \) is given in (6.2.11). In \( D_{12} \) we defined the regular connected form factor function \( F_{3rc} \) via the following separation of the kinematical pole terms

\[ F_3^{\mathcal{O}_1}(\vartheta''_1 + i\pi, \vartheta'_1, \vartheta''_2) = \frac{i(1 - S(\vartheta'_1 - \vartheta'_2)) F_1^{\mathcal{O}_1}}{\vartheta''_1 - \vartheta'_1} + \frac{i(S(\vartheta'_2 - \vartheta''_2) - 1) F_1^{\mathcal{O}_2}}{\vartheta''_1 - \vartheta''_2} + F_{3rc}(\vartheta''_1 + i\pi|\vartheta'_1, \vartheta'_2), \]

and used the abbreviation

\[ K_{t,x}^{(R)}(\vartheta''_1|\vartheta'_1, \vartheta'_2) = e^{imx(\sinh \vartheta''_1 - \sinh \vartheta'_1 - \sinh \vartheta''_2) + m(R - t) \cosh \vartheta''_1 - mt(\cosh \vartheta'_1 + \cosh \vartheta'_2)}. \]

The other contributions \( D_{NM} \) for \( M > N \) can be obtained from \( D_{MN} \) by exchanging \( \mathcal{O}_1 \) with \( \mathcal{O}_2 \) and replacing \( t \rightarrow R - t \), \( x \rightarrow -x \).

### 6.2.5 The symmetry of the \( D_{22} \) term

The relation between the coefficients \( D_{NM} \) and \( D_{MN} \) stated above, when applied to \( D_{22} \) leads to the property that \( D_{22} \) must be symmetric under the following transformation:

\[ t \rightarrow R - t \]

\[ \mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \]

\[ x \rightarrow -x \]

This is the same as requiring that the result should be independent of which two-particle summation is performed first. However, when implementing such a transformation in (6.2.11), the signs of the \( \epsilon \) terms change, and therefore the contours must be pulled back to their original positions. The contour deformation encounters all the singularities on the real axis that were treated previously in this section, so the appropriate residue contributions must be
computed. This computation is relegated to Appendix Section C.3, where it is demonstrated that the required symmetry property indeed holds, providing the first nontrivial test of the result (6.2.11).

6.3 Numerical verification of the analytic results

The goal of this section is to validate the $D_{22}$ formula numerically. For this purpose we evaluated directly the sum for the two-particle states and compare it with the result of the contour integrals. For calculations we used the sinh-Gordon model described in Appendix A.

6.3.1 Evaluating the two-particle sum

Numerical evaluation of the sum is only possible at finite volume. The factors $K_{t,x}^{(R)}$ decrease exponentially at large rapidities, so it is possible to choose a rapidity cutoff and restrict the summation up to the corresponding Bethe-Yang quantum number. However for large volume this quantum number cutoff is still too big and it is practically impossible to evaluate the four particle sum. The compromise is to evaluate only the inner two particle sum with fixed outer rapidities at moderate volume. This is enough to check the validity of all the nontrivial contour deformations and residue manipulation in the $D_{22}$ calculation. We can write

$$C_{22}^{\text{mon diag}} = \sum_{I_1 > I_2} \tilde{C}_{22}(\vartheta_1, \vartheta_2),$$

with

$$\tilde{C}_{22}(\vartheta_1, \vartheta_2) = \sum_{J_1 > J_2} F_4^{O_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1, \vartheta'_2) F_4^{O_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2, \vartheta'_1) \frac{\rho_2(\vartheta_1, \vartheta_2)}{\rho_2(\vartheta'_1, \vartheta'_2)} \times K_{t,x}^{(R)}(\vartheta_1, \vartheta_2, \vartheta'_1, \vartheta'_2),$$

and evaluate $\tilde{C}_{22}$ for some given value of $\vartheta_{1,2}$, corresponding to a solution of the Bethe-Yang equations (6.2.2) with some quantum numbers $\{I_1, I_2\}$.

The parameters for the evaluation can be chosen to help with the convergence of the summation, while ensuring that the structure of the expression tested remains general. The exponential operators (A.1.5) in the sinh-Gordon model can be parametrised by the number $k$ that we chose for our evaluations as $k_1 = 2$ for $O_1$ and $k_2 = 4$ for $O_2$. The essential structure of the formula does not depend on this choice. The space-time parameters and the temperature were chosen as $m_x = 0.0$, $m_t = 0.4$, and $m_R = 0.8$. Setting $m_x$ to zero does not
hide any important structure of the equation, but makes the expression real and that helps in comparing the results with the contour integrals. The sum was evaluated with several values for the volume, sinh-Gordon coupling constant(A.1.3) and quantum numbers of the outer rapidities:

\[ mL = (10, 15, 20, 25, 30) \, , \]
\[ B = (0.1, 0.2, 0.3, 0.4, 0.55, 0.7, 0.9) \, , \]
\[ \{I_1, I_2\} \in \left\{ \left\{ \frac{5}{2}, \frac{1}{2} \right\}, \left\{ \frac{11}{2}, -\frac{5}{2} \right\}, \left\{ -\frac{5}{2}, -\frac{21}{2} \right\}, \left\{ \frac{7}{2}, -\frac{7}{2} \right\}, \left\{ \frac{1}{2}, -\frac{1}{2} \right\} \right\} \, . \]

The rapidity cutoff for the quantum numbers included in the sum was chosen as \( \vartheta = 3.0, 4.0, 5.0, 6.0 \), and the numerical results showed that for the value \( \vartheta = 6.0 \) the discrete sum was evaluated within a relative error of less than \( 10^{-14} \).

### 6.3.2 Evaluating the contour integrals

To compare the results of the contour integrals with the direct sum, the calculation must be performed at the same volumes. In this regime the exponential and power corrections in volume \( L \) are not negligible, so they must be taken into account. Exponential corrections come from the integration on the contours going under the real axis, while power corrections come from the total derivative contribution in the second order pole calculation and from the \( \{J_1, J_2\} = \{I_1, I_2\} \) point in the pole and the double integral contributions. The explicit formulas can be found in the Appendix Section C.2.

The integration contours run below and above the real axis, and it is important to find a choice that is optimal for numerical evaluation. The form factors and hence the integrands have poles on the real axis, so it would be better to integrate as far from the real axis as possible. However for rapidities with imaginary part larger than \( \frac{\pi}{2} \) the factor \( K_{t,x}^{(R)}(\vartheta_1, \vartheta_2, \vartheta'_1, \vartheta'_2) \) becomes oscillating and exponentially growing in the real part of the rapidities instead of decaying. Another issue is that the form factors and scattering matrices are also evaluated at rapidities that lie out of the physical strip. In the sinh-Gordon model the scattering matrix and hence the minimal form factor have poles out of the physical strip, with imaginary positions that are proportional to the coupling parameter \( B \) [FMS92]. Therefore the contour must be chosen to lie between these poles on the one hand and the poles on the real axis on the other hand. At the same time it must run as far away from all singularities as possible, and also to be closer to the real axis than \( \frac{\pi}{2} \). For small \( B \) this leaves little space for the contours so they run relatively close to the poles, resulting in a larger error in the
numerical integration. The integration itself was performed using Wolfram Mathematica and the Cuba library for multidimensional numerical integrations [Hah06].

6.3.3 Comparing the results

<table>
<thead>
<tr>
<th>$B\backslash mL$</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
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<td>$1.31 \times 10^{-8}$</td>
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<td>$1.4 \times 10^{-10}$</td>
<td>$1.4 \times 10^{-10}$</td>
</tr>
<tr>
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<td>$1.42 \times 10^{-10}$</td>
<td>$1.42 \times 10^{-10}$</td>
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<td>$1.3 \times 10^{-10}$</td>
<td>$1.3 \times 10^{-10}$</td>
</tr>
</tbody>
</table>

Table 6.3.1. Relative error of the difference between the direct sum and the contour integral evaluation of $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$ with $\{I_1, I_2\} = \left\{\frac{7}{2}, \frac{7}{2}\right\}$

Table 6.3.1 shows the relative deviation between the direct sum and the contour integral evaluation of $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$ with $\{I_1, I_2\} = \left\{\frac{7}{2}, \frac{7}{2}\right\}$. Note that the relative error decreases as $B$ grows which can be understood from the conditions for the choice of the integration contour mentioned above. Based on the above understanding of the deviations in the relative errors for different parameters, and the fact that this pattern of dependence was the same for every value of $I_1, I_2$ we checked, it can be inferred that the difference of the sum and the contour integration is only due to the numerical errors of integration.

To provide a further support for this conclusion, the above numerical test was repeated for $C_{12}$. The formula of $C_{12}$ is derived in two independent ways in [PT10] (depending on whether the one-particle or the two-particle summation is performed first), and therefore its validity is quite certain even without a numerical test. As in the case of $C_{22}$, let us denote by $\tilde{C}_{12}(\vartheta_1)$ the result of performing the two-particle summation first with fixed rapidity of the one-particle state. Table 6.3.2 shows the relative deviation between the direct sum and the contour integral evaluation of $\tilde{C}_{12}(\vartheta_1)$ with $I_1 = 17$ as the Bethe-Yang quantum number of the one-particle state. The relative deviation has the same pattern as for $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$, and is essentially of the same magnitude. Therefore this evaluation gives an independent support for the assertion that the deviations are caused by errors of numerical integration.

As the derivation of $D_{22}$ from $\tilde{C}_{22}(\vartheta_1, \vartheta_2)$ is almost trivial, the above numerical tests also confirm the details of our analytic result for $D_{22}$.

---

Table 6.3.2. Relative error of the difference between the direct sum and the contour integral evaluation of $\tilde{C}_{12}(\vartheta_1)$ with $I_1 = 17$

<table>
<thead>
<tr>
<th>$B\setminus mL$</th>
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<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
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<td>$2.03 \times 10^{-6}$</td>
<td>$3.56 \times 10^{-6}$</td>
<td>$3.41 \times 10^{-7}$</td>
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<td>$1.52 \times 10^{-9}$</td>
<td>$7.73 \times 10^{-10}$</td>
</tr>
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<td>$7.76 \times 10^{-11}$</td>
<td>$3.1 \times 10^{-11}$</td>
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<tr>
<td>0.4</td>
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<td>$5.81 \times 10^{-11}$</td>
<td>$6.77 \times 10^{-11}$</td>
<td>$6.59 \times 10^{-11}$</td>
</tr>
<tr>
<td>0.55</td>
<td>$5.87 \times 10^{-11}$</td>
<td>$6.61 \times 10^{-11}$</td>
<td>$6.87 \times 10^{-11}$</td>
<td>$6.9 \times 10^{-11}$</td>
<td>$6.96 \times 10^{-11}$</td>
</tr>
<tr>
<td>0.7</td>
<td>$2.96 \times 10^{-11}$</td>
<td>$5.32 \times 10^{-11}$</td>
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<td>$6.25 \times 10^{-11}$</td>
<td>$6.38 \times 10^{-11}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$6.01 \times 10^{-11}$</td>
<td>$6.48 \times 10^{-11}$</td>
<td>$6.52 \times 10^{-11}$</td>
<td>$6.51 \times 10^{-11}$</td>
<td>$6.52 \times 10^{-11}$</td>
</tr>
</tbody>
</table>

6.4 The cluster property of the two-point functions

Another important test of the results is provided by checking that the two-point function has the cluster property

$$\langle O_1(x,t)O_2(0,0)\rangle^R \sim \langle O_1(0,0)\rangle^R \langle O_2(0,0)\rangle^R$$

when the spatial separation $x$ grows large. Using the expansion up to $D_{22}$ one can write

$$\langle O_1(x,t)O_2(0,0)\rangle^R = \sum_{N,M} D_{NM}$$

$$= D_{00} + D_{01} + D_{10} + D_{11} + D_{02} + D_{20} + D_{12} + D_{21} + D_{22} + \ldots$$

For $mx \gg 1$ the terms containing

$$e^{imx(\sum_k \sinh \vartheta_k - \sum_i \sinh \vartheta_i')}$$

oscillate very fast, and therefore the support of the (multiple) rapidity integrals is restricted to the zero measure set

$$\sum_k \sinh \vartheta_k = \sum_i \sinh \vartheta_i'$$

and the integral vanishes. Although this argument looks simple, there is a possible problem. Namely, the argument only works if the integrands of the $x$-dependent terms are all regular.

A nontrivial example is the term

$$\frac{1}{2} \iint \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} \left\{ 2i \left( F_1^{(R)} F_2^{(R)} \right) (S(\vartheta_1 - \vartheta_2') - 1) K_{t,x}(\vartheta_1' | \vartheta_1', \vartheta_2') + (\vartheta_1' \leftrightarrow \vartheta_2') \right\}$$
in the contribution \( D_{12} \) (cf. equation (6.2.12)), which is in fact regular at \( \vartheta'_1 = \vartheta'_2 \) when the two terms inside the braces are added together. In the case of the principal value integral in \( D_{22} \) in equation (6.2.11), the regularity of the integrand is ensured by the principal value prescription itself.

As a result, one only needs to examine the terms that are \( x \)-independent. We denote these by putting a bar over the respective contribution \( D_{NM} \) and they read:

\[
\begin{align*}
\bar{D}_{00} &= \langle O_1 \rangle \langle O_2 \rangle, \\
\bar{D}_{01} &= \bar{D}_{10} = \bar{D}_{02} = \bar{D}_{20} = 0, \\
\bar{D}_{11} &= \left[ \langle O_1 \rangle F_2^{O_2} (i\pi, 0) + \langle O_2 \rangle F_2^{O_1} (i\pi, 0) \right] \int \frac{d\vartheta_1}{2\pi} e^{-mR \cosh \vartheta_1}, \\
\bar{D}_{12} &= \bar{D}_{21} = 0, \\
\bar{D}_{22} &= \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \int \frac{d\vartheta_2}{2\pi} \left[ F_4^{O_1} (\vartheta_1, \vartheta_2) \langle O_2 \rangle + F_4^{O_2} (\vartheta_1, \vartheta_2) \langle O_1 \rangle \right]
\times e^{-mR (\cosh \vartheta_1 + \cosh \vartheta_2)}
+ \int \frac{d\vartheta_1}{2\pi} F_2^{O_1} (i\pi, 0) F_2^{O_2} (i\pi, 0) e^{-mR (\cosh \vartheta_1 + \cosh \vartheta_2)}
- \int \frac{d\vartheta_1}{2\pi} \left[ F_2^{O_1} (i\pi, 0) \langle O_2 \rangle + F_2^{O_2} (i\pi, 0) \langle O_1 \rangle \right] e^{-2mR \cosh \vartheta_1}.
\end{align*}
\]

The one-point function up to two-particle order is \([PT08b]\):

\[
\langle O \rangle^R = \langle O \rangle + \int \frac{d\vartheta_1}{2\pi} F_2^{O} (i\pi, 0) e^{-mR \cosh \vartheta_1} - \int \frac{d\vartheta_1}{2\pi} F_2^{O} (i\pi, 0) e^{-2mR \cosh \vartheta_1}
+ \frac{1}{2} \int \frac{d\vartheta_1}{2\pi} \int \frac{d\vartheta_2}{2\pi} F_4^{O} (\vartheta_1, \vartheta_2) e^{-mR (\cosh \vartheta_1 + \cosh \vartheta_2)} + O \left( e^{-3mR} \right).
\]

As a result one obtains that

\[
\bar{D}_{00} + \bar{D}_{11} + \bar{D}_{22} = \langle O_1 \rangle^R \langle O_2 \rangle^R + O \left( e^{-3mR} \right)
\]

and therefore the cluster property is satisfied to the given order. Note that the same argument shows that the formula for \( D_{22} \) derived in \([PT10]\) violates the cluster property, providing another argument that it needs to be corrected.

### 6.5 Summary

In this chapter, we presented the form factor spectral expansion for the finite temperature
two-point function. We corrected the result for the $D_{22}$ term in the expansion appeared previously in [PT10], and performed numerical check for the contour manipulation of the finite volume formulas as well. Furthermore, we showed the symmetry property of the $D_{22}$ term, and proved the cluster property for the two-point function up to second order, that is a nontrivial check for the result of $D_{22}$. 
Chapter 7

Conclusion
This thesis covered the application of form factor spectral expansions to calculate thermal correlators and finite size matrix elements in integrable QFTs. After a short introduction, we reviewed the S-matrix and form factor bootstrap programs. We presented the effect of a finite volume geometry on the energy levels, and on the form factors up to exponential corrections in integrable QFTs with diagonal scattering and massive particles.

In Chapter 4, we presented the LeClair-Mussardo conjecture that expresses thermal one-point functions with form factors and the thermodynamic Bethe ansatz pseudoenergy. After reviewing the supporting argument of the conjecture, we turned to the numerical evaluation of the LM series and its verification against an independent numerical method, the renormalisation group improved Truncated Conformal Space Approach. To conduct the explicit numerical calculation, the so-called $T_2$ model was our choice, where both methods of calculation are available. The model contains operators for which the numerical evaluation is a nontrivial check. We found perfect agreement between the two numerical evaluations, and the LM conjecture has good convergence for large volume parameters, i.e. low temperature. For small volumes, the terms in the LM series have the same magnitude, hence the truncation of the series is not an efficient way of numerical evaluation anymore.

In Chapter 5, the focus was the finite size matrix element interpretation of the LM series, and Pozsgay’s generalisation of it to general diagonal finite volume matrix elements. We presented the generalised formula for arbitrary IQFTs with diagonal scattering and non-degenerate massive spectrum. The formula reproduces the previously known diagonal form factor formula for large volumes. We proved the generalised conjecture for diagonal finite volume matrix elements of the trace of the stress-energy tensor operator. After the analytic proof, the numerical verification of the formula for operators different from the trace of the energy-momentum tensor was shown. Similarly to the verification of the LM series verification, we checked the numerical result against RG improved TCSA in the $T_2$ model. In the domain where the series converges, there was perfect agreement between the form factor series and the TCSA results. Similarly to our investigation with the LM conjecture, the generalised series is not convergent for small volume parameters. Moreover, for states where the TBA equations have a nontrivial transition at some critical volume, the convergence of the generalised conjecture breaks down exactly at the critical volume. In general, this can happen for states with zero total momentum, and for volumes where the active singularities, specifying the state, coincide. Above the critical volume, the generalised conjecture is convergent, and it is an efficient way to evaluate the matrix element. Continuation of the result through these critical points is an unsolved issue: a reformulation of the series, similar in spirit to the so-called desingularisation of the excited TBA equations, seems to be needed. This is an
interesting an interesting direction to continue the investigation of the finite volume matrix elements.

Finally, in Chapter 6, we considered the evaluation of thermal two-point functions using spectral expansion and finite volume regularisation. We reproduced the previous calculation of Pozsgay and Takacs, and also corrected the $D_{22}$ term in the expansion. The analytic calculation was supported by numerical evaluation of the finite volume regularised expression in the sinh-Gordon model. After assuring ourselves in the validity of the formulas, we showed the cluster property of the two-point function up to second order in the spectral expansion, which is a nontrivial consistency check of the results. Further calculation of terms in the spectral expansion is a tedious work, and better understanding of the structure of the expansion is the subject of ongoing research.

An interesting extension of the results presented here, is the generalisation to theories with non-diagonal scattering, similarly to [BT14], or massless particles, however, that is not part of the thesis present.
Appendix A

Integrable models and their form factors
A.1 Sinh-Gordon model

The sinh-Gordon model is described by the Lagrangian density

\[ L = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \lambda \cosh g \Phi, \]  
(A.1.1)

where \( \Phi \) is a real scalar field. The quantum model is integrable [RT11], and the spectrum contains a single massive particle. Due to integrability, scattering in the sinh-Gordon theory factorises, and its two-particle scattering matrix is simple but nontrivial:

\[ S(\theta) = \frac{\sinh \theta - i \sin \frac{\pi B}{2}}{\sinh \theta + i \sin \frac{\pi B}{2}}, \]  
(A.1.2)

where

\[ B = \frac{2g^2}{8\pi + g^2}. \]  
(A.1.3)

An important class of local operators is the one of exponential fields

\[ e^{kg\Phi}, \]  
(A.1.4)

where \( k \) is a real number and whose form factors are explicitly known [FMS92, KM93]:

\[ F_n^{(k)} (\theta_1, \ldots, \theta_n) = G_k (B) H_n [k] Q_n (x_1, \ldots, x_n) \prod_{i < j} \frac{F_{\text{min}} (\theta_{ij}, B)}{x_i + x_j}, \]  
(A.1.5)

where

\[ [n] = \frac{\sin n \pi B}{\sin \pi \frac{B}{2}}, \quad x_i = e^{\theta_i}, \quad \theta_{ij} = \theta_i - \theta_j. \]  
(A.1.6)

The minimal two-particle form factor is

\[ F_{\text{min}} (\theta, B) = e^{I(\theta, B)}, \]  
(A.1.7)

with

\[ I(\theta, B) = 8 \int_0^\infty \frac{dx \sinh \frac{\pi B}{4} \sinh \frac{\theta}{2} (1 - \frac{B}{2}) \sinh \frac{\theta}{2} \sin^2 \left[ \frac{x(i\pi - \theta)}{2\pi} \right]}{x \sinh^2 x}, \]  
(A.1.8)
and normalisation

\[ N = \exp \left[ -4 \int \frac{dx \sinh \frac{x}{2} \sinh \frac{x}{2} \left( 1 - \frac{B}{2} \right) \sinh \frac{x}{2}}{\sinh^2 x} \right]. \]

For numerical purposes the integral has a more convergent representation

\[ I(\theta, B) = 8 \int_0^\infty \frac{dx \sinh \frac{x}{4} \sinh \frac{x}{2} \left( 1 - \frac{B}{2} \right) \sinh \frac{x}{2}}{\sinh^2 x} \left[ N + 1 - Ne^{-2x} \right] e^{-2Nx} \sin^2 \left[ \frac{x(i\pi - \theta)}{2\pi} \right] \]

\[ + \sum_{k=0}^{N-1} (k + 1) \left\{ \log \nu \left[ \theta, k + \frac{1}{2} \right] + \log \nu \left[ \theta, k + \frac{3}{2} - \frac{B}{4} \right] + \log \nu \left[ \theta, k + 1 + \frac{B}{4} \right] - \log \nu \left[ \theta, k + \frac{3}{2} \right] - \log \nu \left[ \theta, k + \frac{1}{2} + \frac{B}{4} \right] - \log \nu \left[ \theta, k + 1 - \frac{B}{4} \right] \right\}, \tag{A.1.9} \]

with

\[ \nu(\theta, a) = 1 + \frac{(i\pi - \theta)^2}{(2\pi)^2 a^2}, \tag{A.1.10} \]

and \( N \) being a positive integer. The \( Q_n \) polynomials are given in a determinant form

\[ Q_n(k) = \det M_{ij}(k), \]

\[ M_{ij}(k) = [i - j + k] \sigma_{2i-j}^{(n)} \]

\[ \sigma_{k}^{(n)} \]

\[ i, j = 1, \ldots, n - 1 \tag{A.1.11} \]

where the \( \sigma_{k}^{(n)} \) are the elementary symmetric polynomials of order \( k \) in the \( n \) variables \( x_1, \ldots, x_n \) defined by

\[ \prod_{i=1}^{n} (x + x_i) = \sum_k x^{n-k} \sigma_{k}^{(n)} (x_1, \ldots, x_n) \tag{A.1.12} \]

(this means in particular that \( \sigma_{n}^{(k)} = 0 \) for \( k > n \) or \( k < 0 \)). The normalisation factors read

\[ H_n = \left( \frac{4 \sin \frac{\pi B}{2}}{N} \right)^{n/2}, \tag{A.1.13} \]

and \( G_{k}(B) \) is the vacuum expectation value of the exponential operator. The expression for \( G_{k}(B) \) and the relation between the mass of the particles and the coupling constants \( \lambda \) and \( g \) are described in [FLZZ97].
A.2 Scattering theory and form factors of the $T_2$ model

The $T_2$ model is the perturbation of the $\mathcal{M}_{2,7}$ non-unitary minimal model by its $\Phi_{1,3}$ primary field with a formal action

$$\mathcal{A} = \mathcal{A}_{\mathcal{M}_{2,7}} + \lambda \int \! dx^2 \Phi_{1,3}. \tag{A.2.1}$$

The perturbation is integrable [Zam88] and it can also be described as a restriction of the sine-Gordon model at a particular value of the coupling [EY89, Smi90]. The central charge of the conformal theory is $c = -68/7$ and it contains two primary fields $\Phi_{1,2}$ and $\Phi_{1,3}$ with scaling dimensions $\Delta_{1,2} = \bar{\Delta}_{1,2} = -2/7$ and $\Delta_{1,3} = \bar{\Delta}_{1,3} = -3/7$. In the off-critical theory we keep the name primary for the operators, whose form factors has the mildest ultraviolet behaviour. These operators turn into the primary operators in the UV limit.

For $\lambda > 0$, the spectrum of the perturbed theory contains two massive particles. The coupling constant of the perturbation is proportional to the mass gap [Fat94]

$$\lambda = \kappa m_1^{2-2\Delta_{1,3}}, \tag{A.2.1}$$

with

$$\kappa = -0.04053795542..., \tag{A.2.2}$$

and the relation of the masses is [Zam95]

$$m_2 = 2m_1 \cos\left(\frac{\pi}{5}\right),$$

and the bulk energy density is given by

$$B = -\frac{m_1^2}{8 \sin \frac{\pi}{5}}.$$

From the reduced sine-Gordon point of view, the two massive particles of the theory are related to the first two breather states $B_1$ and $B_2$ of the sine-Gordon spectrum, and their
two-particle $S$-matrices take the form

\[
S_{11}(\theta) = \left\{ \begin{array}{c} \frac{2}{5} \end{array} \right\}_\theta, \quad S_{12}(\theta) = \left\{ \begin{array}{c} \frac{1}{5} \end{array} \right\}_\theta \left\{ \begin{array}{c} \frac{3}{5} \end{array} \right\}_\theta, \quad S_{22}(\theta) = \left\{ \begin{array}{c} \frac{2}{5} \end{array} \right\}_\theta \left\{ \begin{array}{c} \frac{4}{5} \end{array} \right\}_\theta,
\]

with

\[
\{x\}_\theta = \frac{\sinh \theta + i \sin \pi x}{\sinh \theta - i \sin \pi x}.
\]

In the $T_2$ model, every particle is its own antiparticle and the following fusion processes can happen

\[
B_1 \times B_1 \rightarrow B_2, \\
B_2 \times B_2 \rightarrow B_1,
\]

with their crossed versions. Two fusion angles are

\[
u_{11}^2 = \frac{2\pi}{5}, \\
u_{22}^1 = \frac{4\pi}{5},
\]

and the others can be calculated from the addition relation (2.2.4) and from the fact that the fusion angles are symmetric in their lower indices for this model. From (A.2.3) we see, that the $S$-matrices have poles at the appropriate fusion angles, and the double pole in $S_{22}$ can be understood as a multiscattering process.

The form factors of the $T_2$ model were constructed in [Kou94]. Form factors only containing type-1 particles are related to the sine-Gordon form factors only containing first breathers. The sine-Gordon form factors containing only first breathers are also related to the form factors of the sinh-Gordon theory by continuing the coupling constant of the sine-Gordon theory to pure imaginary values. As a consequence the form factors of the $T_2$ model with only type-1 particles are the same as the sinh-Gordon form factors with the coupling constant set to

\[
B = -\frac{4}{5}.
\]

In [Kou94], it was shown that the form factors of the two primary fields of the $T_2$ model $\Phi_{1,2}$ and $\Phi_{1,3}$ are the exponential form factors of the sinh-Gordon model with the above mentioned coupling constant and with $k = 2$ and $k = 1$ respectively. The vacuum expectation value of the form factors needs to be replaced by the exact vacuum expectation value of the
minimal model fields, derived in [FLZZ97]:

$$
\langle \Phi_{1,2} \rangle = -2.3251365527 \cdots \times i m_1^{-4/7},
$$

$$
\langle \Phi_{1,3} \rangle = 2.2695506880 \cdots \times m_1^{-6/7}.
$$

Form factors for type-2 particles can be efficiently calculated with the help of type-1 particle form factors and the bootstrap fusion written in the form

$$
F_{2\cdots}(\theta, \ldots) = \Gamma_{11}^2 F_{11\cdots}(\theta - i \bar{u}_{12}, \theta + i \bar{u}_{12}, \ldots),
$$

(A.2.5)

where $\bar{u}_{12} = \pi - u_{12} = \frac{\pi}{5}$, and $\Gamma_{11}^2 = \sqrt{2 \tan \left(\frac{2\pi}{5}\right)}$ is related to the pole of the S-matrix (2.2.3). This formula is the consequence of using the exchange axiom (2.3.4) in the dynamical pole axiom (2.3.7) for the fused particles of the form factor, and then evaluating the residue explicitly with the help of the pole structure of the S-matrix (2.2.3). According to this, the form factor containing $n$ type-1 particles and $m$ type-2 particles can be evaluated using a fundamental form factor (A.1.5) containing $n_1 + 2n_2$ type-1 particles; therefore the connected form factor in (4.3.1) can be evaluated from a $2n_1 + 4n_2$-particle fundamental form factor.

### A.3 Evaluation of connected diagonal form factors

Here we present a fast and efficient way to evaluate the connected diagonal form factors (2.3.9), which proceeds via the symmetric diagonal form factors (2.3.8).

#### A.3.1 Symmetric form factors

The symmetric form factor definition (2.3.8) for the $T_2$ model reads as

$$
F_{1,2\cdots}^Q_{\underbrace{1,\ldots,1,2,\ldots,2}_n}(\theta_1, \ldots, \theta_n, \theta_{n+1}, \ldots, \theta_{n+m})
$$

$$
= \lim_{\epsilon \to 0} F_{2\cdots}^Q_{\underbrace{2,1,\ldots,1,1,\ldots,1,2,2,\ldots,2}_m} \left(\theta_{n+m} + i\pi + \epsilon, \ldots, \theta_{n+1} + i\pi + \epsilon, \theta_n + i\pi + \epsilon, \ldots, \theta_1 + i\pi + \epsilon, \theta_{n+m}, \ldots, \theta_{n+1}, \ldots, \theta_{n+m}\right),
$$

(A.3.1)

where there are $n$ numbers of type-1 particles and $m$ numbers of type-2 particles. To compute the above expression, we use fusion (A.2.5) for type-2 particles, and calculate the limit in terms of a form factor with $2 (n + 2m)$ type-1 particles.
A.3.1.1 Denominator and minimal form factors

The denominator has the following form

\[ \prod_{i<j}^{2(n+2m)} (\tilde{x}_i - \tilde{x}_j), \tag{A.3.2} \]

where

\[ \tilde{x}_i = \begin{cases} -e^\epsilon x_{n+2m+1-i} & i \leq n+2m \\ x_{i-n-2m} & i > n+2m \end{cases}. \tag{A.3.3} \]

To leading order in \( \epsilon \), the denominator of the symmetric form factor takes the form

\[ (-\epsilon)^{n+2m} \left[ \prod_i^{n+2m} x_i \right] \left[ \prod_{i<j}^{n+2m} \left( x_i^2 - x_j^2 \right)^2 \right]. \tag{A.3.4} \]

From the minimal form factor part we get an \( F_{\text{min}}(i\pi) \) factor for every particle when the rapidities meet with their crossed version, i.e. a factor of \( [F_{\text{min}}(i\pi)]^{n+2m} \) altogether.

To simplify the other contribution we use the following relation for the sinh-Gordon minimal form factors [FMS92]

\[ F_{\text{min}}(i\pi + \vartheta) F_{\text{min}}(\vartheta) = \frac{\sinh(\vartheta)}{\sinh(\vartheta) + i \sin \left( \frac{\pi B}{2} \right)}. \tag{A.3.5} \]

There result for two type-1 particle including the denominator term is

\[ \frac{F_{\text{min}}(\vartheta_{ij}) F_{\text{min}}(\vartheta_{ji}) F_{\text{min}}(i\pi + \vartheta_{ij}) F_{\text{min}}(i\pi + \vartheta_{ji})}{(x_i^2 - x_j^2)^2} = \frac{1}{(x_i^2 - x_j^2)^2 + 4x_i^2x_j^2 \sin^2 \left( \frac{\pi B}{2} \right)}. \tag{A.3.6} \]

The result between one type-1 and a type-2 rapidity is

\[ \frac{1}{(x_i^2 - x_{j,+}^2)^2 + 4x_i^2x_{j,+}^2 \sin^2 \left( \frac{\pi B}{2} \right)} \times \frac{1}{(x_i^2 - x_{j,-}^2)^2 + 4x_i^2x_{j,-}^2 \sin^2 \left( \frac{\pi B}{2} \right)}, \tag{A.3.7} \]

where \( x_{j,\pm} = x_j e^{\pm i\delta_{12}} \). The result for rapidities from the same type-2 particle is \( (x_i = x e^{-i\delta_{12}}, x_j = x e^{i\delta_{12}}) \)

\[ \frac{\left[ F_{\text{min}}(\vartheta_{ij}) \right]^2 F_{\text{min}}(i\pi + \vartheta_{ij}) F_{\text{min}}(i\pi + \vartheta_{ji})}{(x_i^2 - x_j^2)^2} = \frac{-1}{16x^4 \sin^2 \left( \frac{\pi}{5} \right)}. \tag{A.3.8} \]
The result for rapidities from different type-2 particles is

\[
\frac{[F_{\min}(\theta_{ij}) F_{\min}(i\pi + \theta_{ij})]^2 [F_{\min}(\theta_{ij}) F_{\min}(i\pi + \theta_{ij})]^2}{(x_i^2 - x_j^2)^4 (x_i^2 e^{i2\tilde{u}} - x_j^2 e^{-i2\tilde{u}})^2 (x_j^2 e^{-i2\tilde{u}} - x_j^2 e^{i2\tilde{u}})^2} 
\times \left[ F_{\min}(\theta_{ij} - 2i\tilde{u}) F_{\min}(i\pi + \theta_{ij} - 2i\tilde{u}) \right] \left[ F_{\min}(\theta_{ij} + 2i\tilde{u}) F_{\min}(i\pi + \theta_{ij} + 2i\tilde{u}) \right] 
\times \left[ F_{\min}(\theta_{ij} - 2i\tilde{u}) F_{\min}(i\pi + \theta_{ij} - 2i\tilde{u}) \right] \left[ F_{\min}(\theta_{ij} + 2i\tilde{u}) F_{\min}(i\pi + \theta_{ij} + 2i\tilde{u}) \right] 
= \left[ (x_i^2 - x_j^2)^2 + 4x_i^2 x_j^2 \sin^2(2\tilde{u}) \right]^{-2} \left[ (x_i^2 e^{i2\tilde{u}} - x_j^2 e^{-i2\tilde{u}})^2 + 4x_i^2 x_j^2 \sin^2(2\tilde{u}) \right]^{-1} 
\times \left[ (x_i^2 e^{-i2\tilde{u}} - x_j^2 e^{i2\tilde{u}})^2 + 4x_i^2 x_j^2 \sin^2(2\tilde{u}) \right]^{-1}.
\]  

(A.3.9)

A.3.1.2 Symmetric polynomial part

There are \(2(n + 2m)\) type-1 rapidities due to the fusion, so the polynomial part of the form factor is a determinant of a \((2(n + 2m) - 1) \times (2(n + 2m) - 1)\) matrix

\[
M_{ij}(k) = [i - j + k] \sigma_{2n-j}^{(2(n+2m))}
\]  

(A.3.10)

In the \(\varepsilon \to 0\) limit the symmetric polynomials are

\[
\sigma_i^{(2p)}(x_1, x_2, \ldots, x_p, -\varepsilon x_1, -\varepsilon x, \ldots, -\varepsilon x_p) \rightarrow \begin{cases} 
\sigma_i^{(2p)}(x_1, x_2, \ldots, x_p, -x_1, -x_2, \ldots, -x_p) + O(\varepsilon) & l \text{ even} \\
\sum_{i=1}^p (-\varepsilon x_i) \sigma_{l-1}^{(2p-1)}(x_1, x_2, \ldots, x_i, \ldots, x_p) & l \text{ odd}
\end{cases}
\]  

(A.3.11)

Since every term in the determinant contains \((n + 2m)\) factors of odd symmetrical polynomials, the determinant is proportional to \(\varepsilon^{n+2m}\), which exactly cancels the \(\varepsilon\) powers in the denominator (A.3.4). From the definition (A.1.12) of the elementary symmetric polynomials it is easy to show that

\[
\sigma_i^{(2p)}(x_1, x_2, \ldots, x_p, -\varepsilon x_1, -\varepsilon x, \ldots, -\varepsilon x_p) \rightarrow \begin{cases} 
(-1)^{l/2} \sigma_{l/2}^{(p)}(x_1^2, x_2^2, \ldots, x_p^2) & l \text{ even} \\
\sum_{i=1}^p (-\varepsilon x_i) (-1)^{(l-1)/2} \sigma_{(l-1)/2}^{p-1}(x_1^2, x_2^2, \ldots, x_{i-1}^2, x_{i+1}^2, \ldots, x_p^2) & l \text{ odd}
\end{cases}
\]  

(A.3.12)
A.3.1.3 Result for symmetric form factors

Introducing the following definitions

\[ \hat{\sigma}_{p}^{l}(x_1, x_2, \ldots, x_p) = \begin{cases} \frac{(-1)^{l/2}}{\Gamma_{l/2}} \sigma_{l/2}(x_1^2, x_2^2, \ldots, x_p^2) & l \text{ even}, \\ \frac{(-1)^{(l-1)/2}}{\Gamma_{(l-1)/2}} \sigma_{(l-1)/2}^{p-1}(x_1^2, x_2^2, \ldots, x_{i-1}^2) & l \text{ odd}, \\ x_{i+1}^2, \ldots, x_p^2 & \end{cases} \]

\[ \hat{Q}_{n+2m}(k) = \det \hat{M}_{ij}(k) \quad \hat{M}_{ij}(k) = [i - j + k] \hat{\sigma}_{2i-j}^{(n+m)}, \]

\[ i, j = 1, \ldots, 2 (n + 2m) - 1 \] (A.3.13)

and

\[ F_{\min,\text{denom}}(x_1, x_2) = \begin{cases} F_{11,\text{denom}} \left( x_1, x_2 \right) & \text{type } 1 \leftrightarrow \text{type } 1 \\ F_{12,\text{denom}} \left( x_1, x_2 \right) & \text{type } 1 \leftrightarrow \text{type } 2 \\ F_{22,\text{dif}} \left( x_1, x_2 \right) & \text{type } 2 \leftrightarrow \text{type } 2 \end{cases} \]

\[ F_{11,\text{denom}}(x_1, x_2) = \frac{1}{(x_i^2 - x_j^2)^2 + 4x_i^2 x_j^2 \sin^2(2\bar{u})} \]

\[ F_{12,\text{denom}}(x_1, x_2) = F_{11,\text{denom}} \left( x_1, x_2 e^{i\bar{u}} \right) \times F_{11,\text{denom}} \left( x_1, x_2 e^{-i\bar{u}} \right) \]

\[ F_{22,\text{dif}}(x_1, x_2) = \left[ F_{11,\text{denom}} \left( x_1, x_2 \right) \right]^2 \times F_{11,\text{denom}} \left( x_1 e^{i\bar{u}}, x_2 e^{-i\bar{u}} \right) \times F_{11,\text{denom}} \left( x_1 e^{-i\bar{u}}, x_2 e^{i\bar{u}} \right) \]

\[ F_{22,\text{self}}(x) = \frac{-1}{16x^4 \sin^2(2\bar{u})} \] (A.3.14)

the symmetric form factor can be rewritten as:

\[ F_{\text{1,\ldots,1,\ldots,1}}^{O}(\theta_1, \ldots, \theta_n, \theta_{n+1}, \ldots, \theta_{n+m}) \]

\[ = [k] \left( -4 \sin \frac{2\pi}{5} \right)^{n+2m} \left( \Gamma_{11}^2 \right)^{2m} \hat{Q}_{n+2m}(x_1, \ldots, x_{n+2m}) \]

\[ \times \prod_{i<j}^{n+m} F_{\min,\text{denom}}(x_i, x_j) \prod_{i=1}^{n} x_i \prod_{j=n+1}^{m} \frac{F_{22,\text{self}}(\text{min,denom})(x_j)}{x_j^2} . \] (A.3.15)

For a large number of particles and/or large rapidities this formula is difficult to evaluate with the required numerical precision, because the determinant \( \hat{Q} \) is badly conditioned (the magnitude of its matrix elements differ by many orders). For a better precision it is necessary
balance the matrix the following way:

\[
\hat{Q}_{n+2m}(k) = \left[ \hat{\sigma}_1^{(n+2m)}/(n + 2m) \right]^{2(n+2m)^2-(n+2m)} \det \hat{M}_{ij}(k),
\]

\[
\hat{M}_{ij}(k) = [i - j + k] \hat{\sigma}_{2i-j}^{(n+m)}\left[ \hat{\sigma}_1^{(n+2m)}/(n + 2m) \right]^{2i-j},
\]

\[i, j = 1, \ldots, 2 (n + 2m) - 1. \tag{A.3.16}\]

A.3.2 Evaluation of the connected diagonal form factors

There are two ways to calculate the connected diagonal form factors using the symmetric form factors. One way is to use the symmetric-connected relations derived in [PT08b] in a recursive manner; this is a lengthy procedure for form factors with several variables and not very convenient for numerical calculations.

However, from the same relations it also follows that the connected diagonal form factor is the only part of the symmetric form factor that is fully periodic in all of its variables with period \(i\pi\). This is related to unitarity (2.2.1) and crossing invariance (2.2.2) of the S-matrix. As a result the kernel functions (the logarithmic derivative of the S-matrices (3.2.14)) have the anti-periodicity property

\[\varphi_{\alpha\beta}(\theta + i\pi) = -\varphi_{\alpha\beta}(\theta). \tag{A.3.17}\]

Applying this property to the connected-symmetric relations of [PT08b] leads to

\[
F_{2n,c}(\theta_1, \ldots, \theta_n) = \frac{1}{2^n} \sum_{\alpha_i=0,1} F_{2n,s}(\theta_1 + \alpha_1 i\pi, \theta_2 + \alpha_2 i\pi, \ldots, \theta_n + \alpha_n i\pi)
\]

\[
= \frac{1}{2^{n-1}} \sum_{\alpha_i=0,1} F_{2n,s}(\theta_1, \theta_2 + \alpha_2 i\pi, \ldots, \theta_n + \alpha_n i\pi), \tag{A.3.18}\]

which gives a faster and numerically much more stable evaluation\(^1\).

\(^1\)Extending the expression for several particle species is straightforward
Appendix B

Truncated Conformal Space Approach
In this appendix, we summarise the basic ingredients of the Truncated Conformal Space Approach (TCSA) for the $T_2$ model and its renormalisation group improvement for calculation of finite volume expectation values. Here, we are not delving into the details and notation of CFTs. For pedagogical introduction we refer the interested reader to the review [Gin88].

B.1 TCSA in the $T_2$ model

The TCSA method was introduced in [YZ90], and based on the idea to construct the Hamiltonian matrix of a model defined in finite volume, which is a perturbation of a given CFT, in the CFT basis. After truncating the CFT Hilbert space, the constructed Hamiltonian is a finite dimensional matrix, and hence numerical diagonalisation gives its eigenvalues and eigenvectors. By increasing the truncation level, the numerical spectrum and matrix elements, calculated numerically, are expected to converge to the exact result, at least for relevant perturbations.

As mentioned in Section A.2, the $T_2$ model is the perturbation of the $M_{2,7}$ conformal minimal model by its relevant operator $\Phi_{1,3}$ with a formal action

$$\mathcal{A} = \mathcal{A}_{M_{2,7}} + \lambda \int \text{d}x^2 \Phi_{1,3}.$$  

The $M_{2,7}$ minimal model has the central charge [BPZ84] $c = -\frac{68}{7}$, and has two nontrivial primary field $\Phi_{1,2}$ and $\Phi_{1,3}$ with conformal weights $\Delta_{1,2} = \bar{\Delta}_{1,2} = -2/7$ and $\Delta_{1,3} = \bar{\Delta}_{1,3} = -3/7$ and fusion rules

$$\Phi_{1,2} \times \Phi_{1,2} = 1 + \Phi_{1,3},$$
$$\Phi_{1,2} \times \Phi_{1,3} = \Phi_{1,2} + \Phi_{1,3},$$
$$\Phi_{1,3} \times \Phi_{1,3} = 1 + \Phi_{1,2} + \Phi_{1,3},$$

where it is helpful to note the identifications $\Phi_{1,2} \equiv \Phi_{1,5}$ and $\Phi_{1,3} \equiv \Phi_{1,4}$.

We consider the theory on a Euclidean space-time cylinder of circumference $L$ which can be mapped unto the punctured complex plane using

$$w = \exp \frac{2\pi}{L} (\tau - ix), \quad \bar{w} = \exp \frac{2\pi}{L} (\tau + ix),$$

(B.1.2)
under which primary fields transform as

\[ \Phi(\tau, x) = \left( \frac{2\pi w}{L} \right)^{\Delta} \left( \frac{2\pi \bar{w}}{L} \right)^{\bar{\Delta}} \Phi(w, \bar{w}). \] 

(B.1.3)

The fields \( \Phi_{1,k} \) are normalised as

\[ \langle 0 | \Phi_{1,k}(w, \bar{w}) \Phi_{1,k}(0, 0) | 0 \rangle = \frac{1}{w^{2h_{1,k}} \bar{w}^{2\bar{h}_{1,k}}}, \] 

(B.1.4)

and the Hilbert space is given by

\[ \mathcal{H}_{T_2} = \bigoplus_{\Delta = 0, -2/7, -3/7} \mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_\Delta, \] 

(B.1.5)

where \( \mathcal{V}_\Delta \) (\( \bar{\mathcal{V}}_\Delta \)) denotes the irreducible representation of the left (right) Virasoro algebra with highest weight \( \Delta \).

The Hamiltonian of the \( T_2 \) model takes the following form in the perturbed conformal field theory framework:

\[ H = H_0 + \lambda \int_0^L dx \Phi_{1,3}(0, x), \] 

(B.1.6)

where

\[ H_0 = \frac{2\pi}{L} \left( L \bar{L} - \frac{c}{12} \right), \] 

(B.1.7)

is the conformal Hamiltonian. For \( \lambda > 0 \), the spectrum of the perturbed theory contains two massive particles. The coupling constant of the perturbation is proportional to the mass gap [Fat94]

\[ \lambda = \kappa m_1^{2 - 2\Delta_{1,3}} = \kappa m_1^{20/7}, \] 

(B.1.8)

with

\[ \kappa = -0.04053795542..., \] 

(B.1.9)

and the relation of the masses is [Zam95]

\[ m_2 = 2m_1 \cos \frac{\pi}{5}, \] 

(B.1.10)
and the bulk energy density is given by

\[ B = -\frac{m_1^2}{8 \sin \frac{\pi}{5}}. \]  

Due to translational invariance of the Hamiltonian (B.1.6), the conformal Hilbert space \( \mathcal{H} \) can be split into sectors characterised by the eigenvalues of the total spatial momentum

\[ P = \frac{2\pi}{L} (L_0 - \bar{L}_0), \]  

where the operator \( L_0 - \bar{L}_0 \) generates spatial translations, and its eigenvalue is called the conformal spin. For a numerical evaluation of the spectrum, the Hilbert space is truncated by imposing a cut in the conformal energy. The truncated conformal space corresponding to a given truncation and fixed value \( s \) of the Lorentz spin reads

\[ \mathcal{H}_{TCS}(s, e_{\text{cut}}) = \left\{ |\psi\rangle \in \mathcal{H} | (L_0 - \bar{L}_0) |\psi\rangle = s |\psi\rangle, \left( L_0 + \bar{L}_0 - \frac{c}{12} \right) |\psi\rangle = e |\psi\rangle : e \leq e_{\text{cut}} \right\}. \]  

On this subspace, the dimensionless Hamiltonian matrix can be written as

\[ h_{ij} = \frac{2\pi}{l} \left( L_0 + \bar{L}_0 - \frac{c}{12} + \frac{k l^{2-2\ell_{1,3}}}{(2\pi)^{1-2\ell_{1,3}} G(s)^{-1} B_{\Phi_{1,3},(s)}} \right)_{ij}, \]  

where energy is measured in units of the particle mass \( m_1 \), \( l = m_1 L \) is the dimensionless volume parameter,

\[ G_{ij}^{(s)} = \langle i | j \rangle, \]  

is the conformal inner product matrix and

\[ B_{ij}^{\Phi_{1,3},(s)} = \langle i | \Phi_{1,3}(w, \bar{w}) | j \rangle \big|_{w=\bar{w}=1}. \]
twelve thousand vectors for even, and nine thousand vectors for odd values of the conformal spin. Once the eigenvectors are obtained, the matrix elements of local operators can be computed using the exponential mapping to evaluate matrix elements. Let us take two eigenvectors that are expressed on the conformal basis like

\[ |\{I_1, \ldots, I_m\}_L = \sum_i \Psi_i (I_1, \ldots, I_m; L) |i\rangle, \]

\[ |\{I_1', \ldots, I_k'\}_L = \sum_i \Psi_i (I_1', \ldots, I_k'; L) |i\rangle, \]

where \(I_i\) and \(I'_j\) are indexing the states according to their Bethe-Yang description for now. We denote their inner product with (B.1.15) like

\[ N = \sum_{i,j} \Psi_i (I_1, \ldots, I_m; L) G_{ij}^{(s)} \Psi_j (I_1, \ldots, I_m; L), \]

\[ N' = \sum_{i,j} \Psi_i (I_1', \ldots, I_k'; L) G_{ij}^{(s)} \Psi_j (I_1', \ldots, I_k'; L). \]

The finite volume matrix elements in these two state are calculated in TCSA as

\[ m_1^{-2\Delta_O} L \langle \{I_1', \ldots, I_k'\} | \mathcal{O} | \{I_1, \ldots, I_m\} \rangle_L = \left( \frac{2\pi}{l} \right)^{2\Delta_O} \frac{\sum_{i,j} \Psi_i (I_1, \ldots, I_k'; L) B_{ij}^{O(s)} \Psi_j (I_1, \ldots, I_m; L)}{\sqrt{NN'}}, \]

where \(B_{ij}^{O(s)} = \langle i|\mathcal{O}(w, \bar{w})|j\rangle_{w=\bar{w}=1}\) is the matrix element of \(\mathcal{O}\), and \(\Delta_O\) is its scaling dimension. For more details on the convention we refer the reader to [PT08a].

The spectrum and matrix elements calculated from TCSA depend on the cutoff \(e_{cut}\). With the RG improvement of the TCSA [GW11] and its application to expectation values [STW13], i.e. diagonal matrix elements, we can make the TCSA result more precise. Let us denote the expectation value of the operator \(\mathcal{O}_i\) in some given state as \(Q_i (n)\), where \(n\) is the cutoff Virasoro level\(^1\). Using conformal perturbation theory, we can express the difference of the expectation values for different cutoff levels\(^2\) in first order of the coupling constant \(\lambda\) as

\[ Q_i (n) - Q_i (n - 1) \sim \sum_A K_A n^{2\alpha_A - 3} (1 + \mathcal{O} (1/n)) + \mathcal{O} (\lambda^2), \]

\(^1\)There is no difference between the energy and cutoff level truncation in our case, since in the \(T_2\) model the differences of scaling dimensions of primary fields are smaller than 2.

\(^2\)The formula is valid for cutoff levels that satisfy the condition \(n \gg \frac{l}{4\pi}\).
where $A$ denotes the operators that are present in the operator product expansion of the perturbing operator $V$ and the considered operator $O_i$. The exponent depends on the combination of the scaling dimension of the operators, $\alpha_A = \Delta_{O_i} + \Delta_V - \Delta_A$. The solution of the difference equation is

$$Q_i(n) = Q_i(\infty) + \sum_A \tilde{K}_A n^{2\alpha_A - 2} (1 + \mathcal{O}(1/n)) + \mathcal{O}(\lambda^2), \quad (B.1.17)$$

where $Q_i(\infty)$ is the infinite cutoff value of the expectation value. So far we did not specify the state, in which we calculate the expectation value, because the difference between them in the extrapolation sense is only second order correction in $\lambda$. For our analysis first order extrapolation is sufficient, and hence we use the same exponents in (B.1.17) to extrapolate the expectation values in every state. For details of the derivation we refer the reader to [STW13]. With the fusion rules (B.1.1) we can easily calculate the exponents in the cutoff dependent expectation values (B.1.17) for any given operator $O_i$ and extrapolate the expectation values.
Appendix C

Supplementary calculations for $D_{22}$
C.1 Residue evaluations

In this section, we present useful formulas for the multidimensional residue calculations for Chapter 6.

C.1.1 First order poles

In this case, the two-dimensional residue formula

$$\int_\gamma \int_\delta \frac{dz_1 \, dz_2}{2\pi i} \frac{g(z_1, z_2)}{f_1(z_1, z_2) f_2(z_1, z_2)} = \frac{g(a, b)}{\det \left( \frac{\partial f_i}{\partial z_j} \right) \bigg|_{(z_1, z_2) = (a, b)},$$

can be applied directly as

$$\int_\gamma \int_\delta \frac{dz_1 \, dz_2}{2\pi i} \frac{g(z_1, z_2)}{[a - z_1] \left[ e^{i\mathbf{Q}_2(z_1, z_2)} + 1 \right]} = \frac{g(a, z_2^*)}{i \frac{\partial Q_2(a, z_2)}{\partial z_2} \bigg|_{z_2 = z_2^*}},$$

where the \((a - z_1)\) is the pole term coming from the appropriate form factor, and \(z_2^*\) is the root of

$$e^{i\mathbf{Q}_2(a, z_2^*)} + 1 = 0.$$

C.1.2 Second order poles

Residues with second order pole can be evaluated by successive integration, performing first the integral over the second order pole coming from the form factor. The fundamental formula to use is

$$\text{Res}_{z=a} \frac{h(z)}{g(z)} = \frac{2 h'(a)}{g''(a)} - \frac{2 g'''(a)}{3 [g''(a)]^2} h(a), \quad (C.1.1)$$

where for a second order pole \(g(a) = g'(a) = 0\) but \(g''(a) \neq 0\). The terms we need to evaluate have the form

$$\int_\gamma \int_\delta \frac{dz_1 \, dz_2}{\gamma \times \delta} \frac{g(z_1, z_2)}{[a - z_1]^2 \left[ e^{i\mathbf{Q}_2(z_1, z_2)} + 1 \right]}.$$

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Performing the $z_1$ integral leads to

$$\int \frac{dz_2}{2\pi i} \left\{ \frac{\partial g(z_1,z_2)}{\partial z_2} \bigg|_{z_2=a} - \frac{g(a,z_2)e^{iQ_2(a,z_2)}}{[e^{iQ_2(z_1,z_2)} + 1]^2} \right\} = \frac{i\partial g(z_1,z_2)}{\partial z_1} \bigg|_{z_1=a} - \int \frac{dz_2}{2\pi i} \frac{g(a,z_2)e^{iQ_2(a,z_2)}}{[e^{iQ_2(z_1,z_2)} + 1]^2} \bigg|_{z_1=a}.$$

For the second integral, introducing the notation $l(z_1,z_2) = g(z_1,z_2)i\frac{\partial Q_2(z_1,z_2)}{\partial z_2}$ and using (C.1.1) with $e^{iQ_2(a,z_2^*)} = -1$:

$$\int \frac{dz_2}{2\pi i} \frac{l(a,z_2)e^{iQ_2(a,z_2)}}{[e^{iQ_2(z_1,z_2)} + 1]^2} = \left. \frac{1}{\partial Q_2(a,z_2^*)} \frac{\partial}{\partial z_2} \left[ l(a,z_2) \frac{\partial Q_2(a,z_2)}{\partial z_2} \right] \right|_{z_2=z_2^*}.$$

Putting it together

$$\int \frac{dz_1}{2\pi i} \int \frac{dz_2}{2\pi i} \frac{g(z_1,z_2)}{a-z_1^2} \frac{e^{iQ_2(z_1,z_2)}}{[e^{iQ_2(z_1,z_2)} + 1]} = \left. \frac{1}{\partial Q_2(z_1,z_2^*)} \frac{\partial}{\partial z_1} \left[ \frac{g(a,z_2)e^{iQ_2(a,z_2)}}{[e^{iQ_2(z_1,z_2)} + 1]^2} \bigg|_{z_2=a} \right] \right|_{z_2=z_2^*},$$

where again $z_2^*$ is the root of

$$e^{iQ_2(a,z_2^*)} + 1 = 0.$$

### C.1.3 Useful formulas for practical evaluation

Using the form of the Bethe-Yang equations (3.2.1) for $\vartheta_1, \vartheta_2$

$$-e^{iQ_1(\vartheta_1,\vartheta_2)} = e^{imL \sinh \vartheta_1} S(\vartheta_1 - \vartheta_2) = 1,$$

$$-e^{iQ_2(\vartheta_1,\vartheta_2)} = e^{imL \sinh \vartheta_2} S(\vartheta_2 - \vartheta_1) = 1.$$
we can easily substitute their solutions into the Bethe-Yang equations for \( \vartheta_1', \vartheta_2' \)

\[
e^{iQ_1'(\vartheta_1', \vartheta_2')} = e^{imL \sinh \vartheta_1} \left( -S(\vartheta_1 - \vartheta_2') \right) = -S(\vartheta_2 - \vartheta_1)S(\vartheta_1 - \vartheta_2'),
\]

\[
e^{iQ_2'(\vartheta_2', \vartheta_1')} = e^{imL \sinh \vartheta_2} \left( -S(\vartheta_2 - \vartheta_1') \right) = -S(\vartheta_1 - \vartheta_2')S(\vartheta_2 - \vartheta_1'),
\]

\[
e^{iQ_1'(\vartheta_1', \vartheta_1')} = e^{imL \sinh \vartheta_1} \left( -S(\vartheta_1 - \vartheta_1') \right) = -S(\vartheta_1 - \vartheta_2)S(\vartheta_1 - \vartheta_1'),
\]

\[
e^{iQ_2'(\vartheta_1', \vartheta_2')} = e^{imL \sinh \vartheta_2} \left( -S(\vartheta_2 - \vartheta_2') \right) = -S(\vartheta_1 - \vartheta_2)S(\vartheta_2 - \vartheta_2').
\]

Denoting

\[
T(\vartheta_1, \vartheta_2, \vartheta_3) = 1 - S(\vartheta_1 - \vartheta_2)S(\vartheta_2 - \vartheta_3),
\]

we can also write

\[
e^{iQ_1'(\vartheta_1', \vartheta_2')} + 1 = T(\vartheta_2, \vartheta_1, \vartheta_2'),
\]

\[
e^{iQ_2'(\vartheta_2', \vartheta_1')} + 1 = T(\vartheta_1, \vartheta_2, \vartheta_1'),
\]

\[
e^{iQ_2'(\vartheta_1', \vartheta_1')} + 1 = T(\vartheta_2, \vartheta_1, \vartheta_1'),
\]

\[
e^{iQ_2'(\vartheta_1', \vartheta_2')} + 1 = T(\vartheta_1, \vartheta_2, \vartheta_1').
\]

### C.2 Pole terms for \( \tilde{C}_{22} \)

Here we list the complete result for the pole term subtractions that appear in the modification of (6.2.4)

\[
\tilde{C}_{22}(\vartheta_1, \vartheta_2) = \frac{1}{2} \oint \oint \frac{d\vartheta_1'}{2\pi} \frac{d\vartheta_2'}{2\pi} \left[ K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2') \times \right.
\]

\[
\left. \frac{F^Q_4(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta_1', \vartheta_2')}{e^{iQ_1'(\vartheta_1', \vartheta_2')} + 1} \right] \left[ e^{iQ_2'(\vartheta_1', \vartheta_2')} + 1 \right]
\]

\[
- QF1(\vartheta_1, \vartheta_2) - QF2(\vartheta_1, \vartheta_2) - QF3(\vartheta_1, \vartheta_2) - QF4(\vartheta_1, \vartheta_2)
\]

\[
- QF5(\vartheta_1, \vartheta_2) - QF6(\vartheta_1, \vartheta_2) - FF(\vartheta_1, \vartheta_2) - 2SQQ(\vartheta_1, \vartheta_2) \quad (C.2.1)
\]

where the \( QF \) are the contributions from the \( QF \) singularities, \( FF \) comes from the \( FF \) singularities and \( SQQ \) is the spurious \( QQ \) singularity term. As in the main text, the notation \( \Leftrightarrow \) corresponds to the straight line contours enclosing the real axis as illustrated in Figure
6.2.1. The explicit form of the individual contributions to (C.2.1) are as follows:

\[ QF1(\vartheta_1, \vartheta_2) = \int \frac{d\vartheta_1'}{2\pi} \left( \frac{F_{4\pi}^{O_1}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta_1', \vartheta_1) F_{2}^{O_2}(\vartheta_2 + i\pi, \vartheta_1') K^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{e^{iQ_1'(\vartheta_1', \vartheta_1)} + 1} \right. \]

\[ + \{ \vartheta_1 \leftrightarrow \vartheta_2, O^1 \leftrightarrow O^2 \} \]

\[ + \left( \frac{F_{4\pi}^{O_1}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta_2, \vartheta_1) F_{2}^{O_2}(i\pi, 0) K^{(R)}(\vartheta_1, \vartheta_2, \vartheta_1)}{mL \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} \right. \]

\[ + \{ \vartheta_1 \leftrightarrow \vartheta_2, O^1 \leftrightarrow O^2 \} \right) , \quad (C.2.2) \]

\[ QF2(\vartheta_1, \vartheta_2) = \int \frac{d\vartheta_1'}{2\pi} \left( \frac{F_{4\pi}^{O_1}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta_1', \vartheta_1) F_{2}^{O_2}(\vartheta_2 + i\pi, \vartheta_1') K^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)}{e^{iQ_1'(\vartheta_1', \vartheta_1)} + 1} \right. \]

\[ + \{ \vartheta_1 \leftrightarrow \vartheta_2, O^1 \leftrightarrow O^2 \} \]

\[ + \left( \frac{F_{4\pi}^{O_1}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta_2, \vartheta_1) F_{2}^{O_2}(i\pi, 0) K^{(R)}(\vartheta_1, \vartheta_2, \vartheta_1)}{mL \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} \right. \]

\[ + \{ \vartheta_1 \leftrightarrow \vartheta_2, O^1 \leftrightarrow O^2 \} \right) , \quad (C.2.3) \]

\[ QF3(\vartheta_1, \vartheta_2) = -i \int \frac{d\vartheta_1'}{2\pi} \left\{ \frac{K^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1) [S(\vartheta_1 - \vartheta_2) - S(\vartheta_1' - \vartheta_1)]}{[e^{iQ_1'(\vartheta_1', \vartheta_1)} + 1]} (\vartheta_2 - \vartheta_1) \right. \]

\[ \times [F_{2}^{O_1}(\vartheta_1 + i\pi, \vartheta_1') F_{2}^{O_2}(\vartheta_2 + i\pi, \vartheta_1') + \{ O_1 \leftrightarrow O_2 \}] + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \}

\[ - \left\{ \frac{K^{(R)}(\vartheta_1, \vartheta_2|\vartheta_2, \vartheta_1) [S(\vartheta_1 - \vartheta_2) - S(\vartheta_2 - \vartheta_1)]}{mL \cosh \vartheta_2 + \varphi (\vartheta_2 - \vartheta_1)} (\vartheta_2 - \vartheta_1) \right. \]

\[ \times [F_{2}^{O_1}(\vartheta_1 + i\pi, \vartheta_2) F_{2}^{O_2}(i\pi, 0) + \{ O_1 \leftrightarrow O_2 \}] + \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right\} , \quad (C.2.4) \]
\[ \begin{align*}
QF4(\vartheta_1, \vartheta_2) &= -i\int_\mathbb{S} \frac{d\vartheta_1'}{2\pi} \left\{ K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1) \left[ S(\vartheta_1' - \vartheta_1)S(\vartheta_1 - \vartheta_2) - 1 \right] \right. \\
&\left. \times \left[ F_2^{O_1}(i\pi, 0)F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1') + \{O_1 \leftrightarrow O_2\} \right] + \{\vartheta_1 \leftrightarrow \vartheta_2\} \right\} \\
&- \left\{ K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_2, \vartheta_1)\varphi(\vartheta_2 - \vartheta_1) \right. \\
&\left. \times \left[ F_2^{O_1}(i\pi, 0)F_2^{O_2}(i\pi, 0) + \{O_1 \leftrightarrow O_2\} \right] + \{\vartheta_1 \leftrightarrow \vartheta_2\} \right\}, \quad (C.2.5)
\end{align*} \]

\[ \begin{align*}
QF5(\vartheta_1, \vartheta_2) &= -i\int_\mathbb{S} \frac{d\vartheta_1'}{2\pi} \left\{ K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1) \left[ S(\vartheta_1' - \vartheta_1) - S(\vartheta_1 - \vartheta_2) \right] \right. \\
&\left. \times \left[ F_2^{O_1}(\vartheta_2 + i\pi, \vartheta_1')F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1) + \{O_1 \leftrightarrow O_2\} \right] + \{\vartheta_1 \leftrightarrow \vartheta_2\} \right\} \\
&- \left\{ K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_2, \vartheta_1)\varphi(\vartheta_2 - \vartheta_1) \right. \\
&\left. \times \left[ F_2^{O_1}(i\pi, 0)F_2^{O_2}(i\pi, 0) + \{O_1 \leftrightarrow O_2\} \right] + \{\vartheta_1 \leftrightarrow \vartheta_2\} \right\}, \quad (C.2.6)
\end{align*} \]

\[ \begin{align*}
QF6(\vartheta_1, \vartheta_2) &= -\int_\mathbb{S} \frac{d\vartheta_1'}{2\pi} \left( F_2^{O_1}(\vartheta_2 + i\pi, \vartheta_1')F_2^{O_2}(\vartheta_2 + i\pi, \vartheta_1')K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1) \right. \\
&\left. \times \left[ T(\vartheta_1', \vartheta_1, \vartheta_2) \left[ -m x \cosh \vartheta_1 + i m t \sinh \vartheta_1 \right] - m L \cosh(\vartheta_1) \right. \\
&\left. \times \varphi(\vartheta_1' - \vartheta_1)S(\vartheta_1' - \vartheta_1)S(\vartheta_1 - \vartheta_2) \right] + \{\vartheta_1 \leftrightarrow \vartheta_2\} \right) \\
&+ F_2^{O_1}(i\pi, 0)F_2^{O_2}(i\pi, 0)K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_2, \vartheta_1) \\
&\times \left( \left[ \frac{m L \cosh \vartheta_1 - \varphi(\vartheta_1 - \vartheta_2)}{m L \cosh \vartheta_2 + \varphi(\vartheta_2 - \vartheta_1)} \right] + \frac{m L \cosh \vartheta_2 - \varphi(\vartheta_2 - \vartheta_1)}{m L \cosh \vartheta_1 + \varphi(\vartheta_1 - \vartheta_2)} \right) \\
&\left. + \varphi(\vartheta_1' - \vartheta_1)S(\vartheta_1' - \vartheta_1)S(\vartheta_1 - \vartheta_2) \right],
\end{align*} \]

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\[ + \oint \frac{d\varphi'_1}{2\pi} \left( \frac{i}{e^{iQ'_1(\varphi'_1, \varphi_1)} + 1} \right) \]
\[ \times \frac{\partial}{\partial \varphi'_1} \left[ \frac{K^{(R)}_{t,x}(\varphi_1, \varphi_2|\varphi'_1, \varphi_1)T(\varphi'_1, \varphi_1, \varphi_2)\varphi(\varphi'_1 - \varphi_1)}{mL \cosh \varphi'_1 + \varphi(\varphi'_1 - \varphi_1)} \right] \]
\[ \times F_2^{O_1}(\varphi_2 + i\pi, \varphi'_1)F_2^{O_2}(\varphi_2 + i\pi, \varphi'_1) + \{ \varphi_1 \leftrightarrow \varphi_2 \} \]  
\[ + K^{(R)}_{t,x}(\varphi_1, \varphi_2|\varphi_1, \varphi_2)F_2^{O_1}(i\pi, 0)F_2^{O_2}(i\pi, 0)\varphi(\varphi_2 - \varphi_1)^2 \]
\[ \times \left( \frac{1}{[mL \cosh \varphi_2 + \varphi(\varphi_2 - \varphi_1)]^2} + \frac{1}{[mL \cosh \varphi_1 + \varphi(\varphi_1 - \varphi_2)]^2} \right), \]  
\[ (C.2.7) \]
\[ (C.2.8) \]
\[ (C.2.9) \]
\[ (C.2.10) \]
\[ (C.2.11) \]

We gave these contributions in their exact finite volume form (i.e. including the full volume dependence): albeit they simplify when taking the volume to infinity, and the $SQQ$ term does not even contribute in this limit, all terms must be kept in order for the numerical verification of Section 6.3 to work properly.

### C.3 Symmetry of $D_{22}$

We want to prove that $D_{22}$ is symmetric under
\[ t \rightarrow R - t, \]
\[ O^1 \leftrightarrow O^2, \]
\[ x \rightarrow -x. \]  
\[ (C.3.1) \]
First of all, notice that the diagonal terms contain no t and x factor, and are manifestly symmetric under exchanging $O_1$ and $O_2$. So it remains only to treat the non-diagonal part.

**C.3.1 The four-integral term**

First we treat the term in (6.2.11) that contains a fourfold integral. After the transformation we change the variables $\vartheta_{1,2} \leftrightarrow \vartheta'_{1,2}$, and shift every contour with $-2i\epsilon$. This results in the contour now running under the real axis:

\[
\frac{1}{4} \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta'_1 - i\epsilon, \vartheta'_2 - i\epsilon) \\
\times F_4^{C_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1 - i\epsilon, \vartheta'_2 - i\epsilon) F_4^{C_2}(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2 - i\epsilon, \vartheta'_1 - i\epsilon).
\]

By shifting the contour above the real axis we can transform this term back to its form in (6.2.11), but during this process we pick up some pole contributions from the poles in eqn. (6.2.6).

**C.3.1.1 First order pole terms containing $F_{4rc}$**

Using eqn. (6.2.6) we can identify a contribution containing $F_{4rc}$. In these contribution, all poles are of first order, so one can apply the Cauchy formula directly. One set of such terms is given by

\[
\frac{1}{4} \int_{C_-} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F_{4rc}^{C_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta'_1, \vartheta'_2) \left[ \frac{E}{\vartheta'_2 - \vartheta'_1} + \frac{F}{\vartheta'_1 - \vartheta'_1} + \frac{G}{\vartheta_2 - \vartheta'_2} + \frac{H}{\vartheta_1 - \vartheta'_1} \right] \\
\times K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2)
\]

\[
= -\frac{1}{4} \int_{C_+} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F_{4rc}^{C_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta'_1, \vartheta'_2) \left[ \frac{E}{\vartheta'_2 - \vartheta'_1} + \frac{F}{\vartheta'_1 - \vartheta'_1} + \frac{G}{\vartheta_2 - \vartheta'_2} + \frac{H}{\vartheta_1 - \vartheta'_1} \right] \\
\times K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2)
\]

\[
+ \frac{1}{4} \int \frac{d\vartheta'_2}{2\pi} F_{4rc}^{C_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta'_1, \vartheta_1) F_2^{C_2}^{(R)}(\vartheta_2 + i\pi, \vartheta'_1) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta_1) \\
\times [S(\vartheta_1 - \vartheta_2) - S(\vartheta_1 - \vartheta'_2)]
\]

\[
+ \frac{1}{4} \int \frac{d\vartheta'_2}{2\pi} F_{4rc}^{C_1}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta'_1, \vartheta'_2) F_2^{C_2}^{(R)}(\vartheta_2 + i\pi, \vartheta'_2) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2 | \vartheta_1, \vartheta'_2) \\
\times [S(\vartheta'_2 - \vartheta'_1) S(\vartheta_1 - \vartheta_2) - 1]
\]

\[\leftrightarrow\]
and a similar contribution from

\[
\begin{align*}
&+ \frac{1}{4} \int \frac{d\vartheta_1'}{2\pi} F_{4rc}^{\vartheta}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta_1', \vartheta_2') F_{2}^{\vartheta_2}(\vartheta_1 + i\pi, \vartheta_1') K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_2) \\
&\times [1 - S(\vartheta_1 - \vartheta_2)S(\vartheta_2 - \vartheta_1')]
\end{align*}
\]

\[
+ \frac{1}{4} \int \frac{d\vartheta_1'}{2\pi} F_{4rc}^{\vartheta}(\vartheta_2 + i\pi, \vartheta_1 + i\pi | \vartheta_1', \vartheta_2') F_{2}^{\vartheta_2}(\vartheta_1 + i\pi, \vartheta_1') K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_2', \vartheta_2') \\
\times [S(\vartheta_2' - \vartheta_2) - S(\vartheta_1 - \vartheta_2)]
\]

where \( C_{\pm} \) denote integration running above/below from the real axis, in the direction from left to right.

Since the exchange property (2.3.4) is valid for \( F_{4rc} \) in the first and last two variables, it can be used to simplify the total contribution to

\[
\frac{1}{2} \int \frac{d\vartheta_1'}{2\pi} \left\{ K_{t,x}^{(R)}(\vartheta_1, \vartheta_2 | \vartheta_1', \vartheta_1) \left\{ F_{4rc}^{\vartheta}(\vartheta_1 + i\pi, \vartheta_2 + i\pi | \vartheta_1', \vartheta_1) F_{2}^{\vartheta_2}(\vartheta_2 + i\pi, \vartheta_1') + \{ O^1 \leftrightarrow O^2 \} \right\} \\
+ \{ \vartheta_1 \leftrightarrow \vartheta_2 \} \right\}
\]

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Their evaluation is tedious, but straightforward. For example and one can evaluate the contributions resulting from the contour shift as

\[C.3.1.2 \text{ First order poles without } F_{4rc}\]

The form of these terms is

\[
\frac{1}{4} \int_{\mathcal{C}^{-}} \int_{\mathcal{C}^{+}} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2; \vartheta'_1, \vartheta'_2) \times
\]

\[
\left[ \frac{AE + DH}{(\vartheta_2 - \vartheta'_2)(\vartheta_1 - \vartheta'_1)} + \frac{AF + CH}{(\vartheta_2 - \vartheta'_1)(\vartheta_1 - \vartheta'_2)} + \frac{AG + BH}{(\vartheta_2 - \vartheta'_2)(\vartheta_1 - \vartheta'_1)} \right],
\]

Their evaluation is tedious, but straightforward. For example

\[AE + DH = -(S(\vartheta_2 - \vartheta_1) - S(\vartheta'_2 - \vartheta'_1))(S(\vartheta_1 - \vartheta_2) - S(\vartheta_2 - \vartheta'_1))\]

\[\times \left\{ F_{2}^{O_1}(\vartheta_1 + i\pi, \vartheta'_1) F_{2}^{O_2}(\vartheta_2 + i\pi, \vartheta'_2) + \{O_1 \leftrightarrow O_2\} \right\},\]

and one can evaluate the contributions resulting from the contour shift as

\[
\begin{align*}
&= +\frac{1}{4} \int_{\mathcal{C}^{-}} \int_{\mathcal{C}^{+}} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2; \vartheta'_1, \vartheta'_2) \left. \frac{AE + DH}{(\vartheta'_1 - \vartheta_1)(\vartheta'_2 - \vartheta_1)} \right|_{\vartheta'_2 = \vartheta_1} \\
&= +\frac{1}{4} \int_{\mathcal{C}^{-}} \int_{\mathcal{C}^{+}} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2; \vartheta'_1, \vartheta'_2) \left. \frac{AE + DH}{(\vartheta'_1 - \vartheta_2)(\vartheta'_2 - \vartheta_1)} \right|_{\vartheta'_1 = \vartheta_2} \\
&= -\frac{1}{4} \int_{\mathcal{C}^{-}} \int_{\mathcal{C}^{+}} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2; \vartheta'_1, \vartheta'_2) \left. \frac{AE + DH}{(\vartheta'_1 - \vartheta_1)(\vartheta'_2 - \vartheta_1)} \right|_{\vartheta'_2 = \vartheta_1} \\
&= -\frac{1}{2} \int_{\mathcal{C}^{-}} \int_{\mathcal{C}^{+}} \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \left. \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2; \vartheta'_1, \vartheta'_2) \left. \frac{AE + DH}{(\vartheta'_1 - \vartheta_2)(\vartheta'_2 - \vartheta_1)} \right|_{\vartheta'_1 = \vartheta_2} \right|_{\vartheta'_2 = \vartheta_1} \\
&= \left. \left\{ F_{2}^{O_1}(i\pi, 0) F_{2}^{O_2}(\vartheta_2 + i\pi, \vartheta'_2) + \{O_1 \leftrightarrow O_2\} \right\} \times [S(\vartheta_2 - \vartheta_1) - S(\vartheta'_2 - \vartheta'_1)] [S(\vartheta_1 - \vartheta_2) - S(\vartheta_1 - \vartheta'_1)] \right|_{\vartheta'_2 = \vartheta_1}.
\end{align*}
\]
and similarly for the remaining five cases.

C.3.1.3 Second order poles

We get four contributions which contain a second order pole. All four are the same after relabelling the rapidities:

\[
+ \frac{1}{4} \int \int \int \int_C C_+ \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \left\{ \frac{AH}{(\vartheta_2 - \vartheta'_1)^2} + \frac{BG}{(\vartheta_2 - \vartheta'_2)^2} + \frac{CF}{(\vartheta_1 - \vartheta'_1)^2} + \frac{DE}{(\vartheta_1 - \vartheta'_2)^2} \right\} \\
\times K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \\
= \int \int \int \int_C \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} D E K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \\
= \int \int \int \int_C \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{DE}{(\vartheta_1 - \vartheta'_2)^2} K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \\
+ i \int \int \int \int_C \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{DE}{(\vartheta_1 - \vartheta'_2)^2} K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \\
\times \left[ \partial \frac{DE K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2)}{\partial \vartheta'_2} \right]_{\vartheta'_2 = \vartheta_2}.
\]

After performing the differentiation, the final result is:

\[
\int \int \int \int_C \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \frac{DE}{(\vartheta_1 - \vartheta'_2)^2} K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta'_2) \\
+ \int \int \int \int_C \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F_2^{CO_1}(\vartheta_2 + i\pi, \vartheta'_1) F_2^{CO_2}(\vartheta_2 + i\pi, \vartheta'_1) K^{(R)}_{t,x}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) \\
\times \{ [mx \cosh \vartheta_1 - imt \sinh \vartheta_1] T(\vartheta_2, \vartheta_1, \vartheta'_1) T(\vartheta'_1, \vartheta_1, \vartheta_2) \\
+ [S(\vartheta_2 - \vartheta_1)S(\vartheta_1 - \vartheta'_1) - S(\vartheta'_1 - \vartheta_1)S(\vartheta_1 - \vartheta_2)] \varphi(\vartheta_1 - \vartheta'_1) \} ,
\]

where the notation

\[
T(\vartheta_1, \vartheta_2, \vartheta_3) = 1 - S(\vartheta_1 - \vartheta_2)S(\vartheta_2 - \vartheta_3),
\]

was used.

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C.3.1.4 Putting together the results

Putting together the result for the fourfold integral term one obtains

\[
\frac{1}{4} \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F^C_1 (\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1 - i\varepsilon, \vartheta'_2 - i\varepsilon) \\
\times F^C_2 (\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2 - i\varepsilon, \vartheta'_1 - i\varepsilon) K^{(R)}_{t,x} (\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2) \\
= \frac{1}{4} \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F^C_1 (\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1 + i\varepsilon, \vartheta'_2 + i\varepsilon) \\
\times F^C_2 (\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_2 + i\varepsilon, \vartheta'_1 + i\varepsilon) K^{(R)}_{t,x} (\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2) \\
+ \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} K^{(R)}_{t,x} (\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2) \left\{ F^C_{4 \vartheta} (\vartheta_1 + i\pi, \vartheta_2 + i\pi | \vartheta'_1, \vartheta'_2) F^C_2 (\vartheta_2 + i\pi, \vartheta'_1) \right\} \\
+ \left\{ \mathcal{O}^1 \leftrightarrow \mathcal{O}^2 \right\} \\
- \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \varphi \left\{ F^C_{4 \vartheta} (\vartheta_1 + i\pi, 0 | \vartheta'_1, \vartheta'_2) F^C_2 (\vartheta_2 + i\pi, \vartheta'_1) \right\} \\
+ \left\{ \mathcal{O}^1 \leftrightarrow \mathcal{O}^2 \right\} \left\{ S(\vartheta'_1 - \vartheta_1) - S(\vartheta_2 - \vartheta_1) \right\} \left\{ S(\vartheta_1 - \vartheta_2) - S(\vartheta_1 - \vartheta'_1) \right\} \\
- \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \varphi \left\{ F^C_{4 \vartheta} (\vartheta_1 + i\pi, \vartheta_1 | \vartheta'_1, \vartheta'_2) F^C_2 (\vartheta_2 + i\pi, \vartheta'_1) \right\} \\
+ \left\{ \mathcal{O}^1 \leftrightarrow \mathcal{O}^2 \right\} \left\{ 1 - S(\vartheta_2 - \vartheta_1) S(\vartheta'_1 - \vartheta_1) \right\} \left\{ S(\vartheta_1 - \vartheta'_1) - S(\vartheta_1 - \vartheta_2) \right\} \\
- \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} \varphi \left\{ F^C_{4 \vartheta} (\vartheta_1 + i\pi, \vartheta'_1 | \vartheta_1, \vartheta_2) F^C_2 (\vartheta_2 + i\pi, \vartheta'_1) \right\} \\
+ \left\{ \mathcal{O}^1 \leftrightarrow \mathcal{O}^2 \right\} \left\{ 1 - S(\vartheta_2 - \vartheta_1) S(\vartheta'_1 - \vartheta_1) \right\} \left\{ 1 - S(\vartheta'_1 - \vartheta_1) S(\vartheta_1 - \vartheta_2) \right\} \\
+ \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} F^C_1 (\vartheta_2 + i\pi, \vartheta'_1) F^C_2 (\vartheta_2 + i\pi, \vartheta'_1) K^{(R)}_{t,x} (\vartheta_1, \vartheta_2 | \vartheta'_1, \vartheta'_2) \\
\times \left\{ \left[ m x \cosh \vartheta_1 - i m t \sinh \vartheta_1 \right] T (\vartheta_2, \vartheta_1, \vartheta'_1) T (\vartheta'_1, \vartheta_1, \vartheta_2) \\
+ \left\{ S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta'_1) - S(\vartheta'_1 - \vartheta_1) S(\vartheta_1 - \vartheta_2) \right\} \varphi (\vartheta_1 - \vartheta'_1) \right\} .\]
C.3.2 Other terms

The following two terms in (6.2.11)

\[- \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F_2^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1) F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1, \vartheta_1)\]

\[- \int \frac{d\vartheta_1}{2\pi} \int \frac{d\vartheta_1'}{2\pi} F_2^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_1') F_2^{\mathcal{O}_2}(\vartheta_1 + i\pi, \vartheta_1') K_{t,x}^{(R)}(\vartheta_1, \vartheta_1'|\vartheta_1, \vartheta_1'),\]

transform to each other under the symmetry (C.3.1).

For the remaining terms in (6.2.11), after the transformation we redefine \( \vartheta_2 \leftrightarrow \vartheta_1' \) to have the same \( K \) factor as before and get:

\[- \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \int \frac{d\vartheta_1'}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)\]

\[\times \left( F_4^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_1' + i\pi|\vartheta_2, \vartheta_1) F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') + \{ \mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \} \right)\]

\[- i \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)\]

\[\times \left( \frac{1}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi} \right) \left( \vartheta_1' - \vartheta_1 \right)\]

\[\times \left( F_2^{\mathcal{O}_1}(\vartheta_1 + i\pi, \vartheta_2) F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') + \{ \mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \} \right)\]

\[- i \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)\]

\[\times \left( S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta_1') - 1 \right)\]

\[\times \left( F_2^{\mathcal{O}_1}(i\pi, 0) F_2^{\mathcal{O}_2}(\vartheta_2 + i\pi, \vartheta_1') + \{ \mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \} \right)\]

\[- i \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)\]

\[\times \left( S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta_1') \right)\]

\[\times \left( F_2^{\mathcal{O}_1}(\vartheta_2 + i\pi, \vartheta_1') F_2^{\mathcal{O}_2}(\vartheta_1' + i\pi, \vartheta_1) + \{ \mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \} \right)\]

\[- \int \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta_1'}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi} \frac{1}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta_1', \vartheta_1)\]

\[\times \left( T(\vartheta_2, \vartheta_1, \vartheta_1') [-m x \cosh \vartheta_1 + i m (R - t) \sinh \vartheta_1] + \varphi(\vartheta_2 - \vartheta_1) S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta_1') \right).\]
C.3.3 Collecting the terms

Using the results so far, the transformed $D_{22}$ is the following:

\[
D_{22}^{\text{trans}} = \frac{1}{4} \iint \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} \frac{d\vartheta'_2}{2\pi} K^{(R)}_{\ell,x}(\vartheta_1, \vartheta_2; \vartheta'_1 + i\varepsilon, \vartheta'_2 + i\varepsilon) \\
F^{O_1}_4(\vartheta_2 + i\pi, \vartheta_1 + i\pi, \vartheta'_1 + i\varepsilon, \vartheta'_2 + i\varepsilon) F^{O_2}_4(\vartheta_1 + i\pi, \vartheta_2 + i\pi, \vartheta'_1 + i\varepsilon, \vartheta'_2 + i\varepsilon) \\
+ (F_{4rc} \text{ terms}) + (\text{first order pole terms}) + (\text{second order pole terms}) \\
- \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} F^{O_1}_2(\vartheta_1 + i\pi, \vartheta_1) F^{O_2}_2(\vartheta_2 + i\pi, \vartheta_1) K^{(R)}_{\ell,x}(\vartheta_1, \vartheta_2; \vartheta_1, \vartheta_1) \\
- \int \frac{d\vartheta_1}{2\pi} \int \frac{d\vartheta'_1}{2\pi} F^{O_1}_2(\vartheta_1 + i\pi, \vartheta'_1) F^{O_2}_2(\vartheta_1 + i\pi, \vartheta'_1) K^{(R)}_{\ell,x}(\vartheta_1, \vartheta_1; \vartheta_1, \vartheta'_1) \\
+ D_{22}^{\text{diag}},
\]

where the $F_{4rc}$ terms are

\[
+ \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} K^{(R)}_{\ell,x}(\vartheta_1, \vartheta_2; \vartheta'_1, \vartheta_1) \left\{ F^{O_1}_{4rc}(\vartheta_1 + i\pi, \vartheta_2 + i\pi|\vartheta'_1, \vartheta_1) F^{O_2}_2(\vartheta_2 + i\pi, \vartheta'_1) \\
+ \{O^1 \leftrightarrow O^2\} \right\} \\
+ \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} K^{(R)}_{\ell,x}(\vartheta_1, \vartheta_2; \vartheta'_1, \vartheta_1) \left\{ F^{O_1}_{4rc}(\vartheta_1 + i\pi, \vartheta'_1 + i\pi|\vartheta_2, \vartheta_1) F^{O_2}_2(\vartheta_2 + i\pi, \vartheta'_1) \\
+ \{O^1 \leftrightarrow O^2\} \right\} \\
- \int \int \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} K^{(R)}_{\ell,x}(\vartheta_1, \vartheta_2; \vartheta'_1, \vartheta_1) \left\{ F^{O_1}_{4rc}(\vartheta_2 + i\pi, \vartheta_1 + i\pi|\vartheta_1, \vartheta'_1) F^{O_2}_2(\vartheta_2 + i\pi, \vartheta'_1) \\
+ \{O^1 \leftrightarrow O^2\} \right\},
\]

(C.3.2)
the first order pole terms are

\[- i \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \int \frac{d\theta'_1}{2\pi} \frac{K^{(R)}_{t,x}(\theta_1, \theta_2|\theta'_1, \theta_1)}{(\theta_2 - \theta'_1)} \left\{ F^{(C)}_2(i\pi, 0)F^{(C)}_2(\theta_2 + i\pi, \theta'_1) \right\} + \{ \mathcal{O}_1 \leftrightarrow \mathcal{O}_2 \} \right\} [S(\theta'_1 - \theta_1) - S(\theta_2 - \theta_1)] [S(\theta'_1 - \theta_2) - S(\theta_1 - \theta_2)]

while the second order pole terms are

\[+ \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{d\theta'_1}{2\pi} F^{(C)}_2(\theta_2 + i\pi, \theta'_1) F^{(C)}_2(\theta_2 + i\pi, \theta'_1) K^{(R)}_{t,x}(\theta_1, \theta_2|\theta'_1, \theta_1) \times \{ [mx \cosh \theta_1 - im \sinh \theta_1] T(\theta_2, \theta_1, \theta'_1) T(\theta'_1, \theta_1, \theta_2) \} + [S(\theta_2 - \theta_1)S(\theta'_1 - \theta_1) - S(\theta'_1 - \theta_1)S(\theta_1 - \theta_2)] \varphi(\theta_1 - \theta'_1) \right\} \]

\[+ \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{d\theta'_1}{2\pi} F^{(C)}_2(\theta_2 + i\pi, \theta'_1) F^{(C)}_2(\theta_2 + i\pi, \theta'_1) K^{(R)}_{t,x}(\theta_1, \theta_2|\theta'_1, \theta_1) \times \{ T(\theta_2, \theta_1, \theta'_1) [mx \cosh \varphi_1 + im (R - t) \sinh \theta_1] - \varphi(\theta_2 - \theta_1)S(\theta_2 - \theta_1)S(\theta_1 - \theta'_1) \} \right\} . \]
C.3.3.1 $F_{4rc}$ terms

The last two $F_{4rc}$ terms (C.3.2) cancel due to

$$F_{4rc}^{O_{1,2}}(\vartheta_1 + i\pi, \vartheta'_1 + i\pi|\vartheta_2, \vartheta_1) - F_{4rc}^{O_{1,2}}(\vartheta_2 + i\pi, \vartheta_1 + i\pi|\vartheta_1, \vartheta'_1) = 0,$$

which can be easily proven from the definition (6.2.5); the remaining one is combined with the first order pole terms below into the $F_{4ss}$ contributions in (6.2.11).

C.3.3.2 First order pole terms

The first order pole terms (C.3.3) can be rearranged into the form

$$-i\frac{\dd\vartheta_1}{2\pi} \frac{\dd\vartheta_2}{2\pi} \int \frac{\dd\vartheta'_1}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) \left\{ F_{2}^{O_{1}}(i\pi, 0) F_{2}^{O_{2}}(\vartheta_2 + i\pi, \vartheta'_1) + \{O^1 \leftrightarrow O^2\} \right\} [S(\vartheta'_1 - \vartheta_1) S(\vartheta_1 - \vartheta_2) - 1]$$

$$-i\frac{\dd\vartheta_1}{2\pi} \frac{\dd\vartheta_2}{2\pi} \int \frac{\dd\vartheta'_1}{2\pi} K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) \left\{ F_{2}^{O_{1}}(\vartheta_2 + i\pi, \vartheta_1) F_{2}^{O_{2}}(\vartheta_2 + i\pi, \vartheta'_1) + \{O^1 \leftrightarrow O^2\} \right\} [S(\vartheta'_1 - \vartheta_1) - S(\vartheta_1 - \vartheta_2)]$$

$$-i\frac{\dd\vartheta_1}{2\pi} \frac{\dd\vartheta_2}{2\pi} \frac{\dd\vartheta'_1}{2\pi} \left\{ F_{2}^{O_{1}}(\vartheta_1 + i\pi, \vartheta'_1) F_{2}^{O_{2}}(\vartheta_2 + i\pi, \vartheta'_1) + \{O^1 \leftrightarrow O^2\} \right\} [S(\vartheta_1 - \vartheta_2) - S(\vartheta'_1 - \vartheta_1)],$$

using the unitarity for the S-matrix (2.2.1). These terms are combined with the $F_{4rc}$ terms into the $F_{4ss}$ contributions in (6.2.11).

C.3.3.3 Second order pole terms

The second order pole terms (C.3.4) have a dependence on $R - t$ on the last line. One can make an integration by parts to transform it into a $t$-dependence as on the second line. Using

$$\frac{\partial K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1)}{\partial \vartheta_1} = -mR \sinh(\vartheta_1) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1),$$

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we can make the following transformation in the integrand

\[-iT(\vartheta_2, \vartheta_1, \vartheta'_1) mR \sinh \vartheta_1 K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) = iT(\vartheta_2, \vartheta_1, \vartheta'_1) \partial K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) \partial \vartheta_1 \]

(partial integration) \(\Rightarrow -iK_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) \frac{T(\vartheta_2, \vartheta_1, \vartheta'_1)}{\partial \vartheta_1} \]

\[= -K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) S(\vartheta_2 - \vartheta_1) S(\vartheta_1 - \vartheta'_1) [\varphi(\vartheta_1 - \vartheta'_1) - \varphi(\vartheta_2 - \vartheta_1)] , \]

and so the contribution is transformed into

\[+ \iint \frac{d\vartheta_1}{2\pi} \frac{d\vartheta_2}{2\pi} \frac{d\vartheta'_1}{2\pi} F_1^{(R)}(\vartheta_2 + i\pi, \vartheta'_1) F_2^{(R)}(\vartheta_2 + i\pi, \vartheta'_1) K_{t,x}^{(R)}(\vartheta_1, \vartheta_2|\vartheta'_1, \vartheta_1) \]
\[\times \{ [mx \cosh \vartheta_1 - imt \sinh \vartheta_1] T(\vartheta'_1, \vartheta_1, \vartheta_2) - S(\vartheta'_1 - \vartheta_1) S(\vartheta_1 - \vartheta_2) \varphi(\vartheta_1 - \vartheta'_1) \} . \]

With this the second order pole terms (C.3.4) reduce to the form of the second order pole term in (6.2.11).

C.3.4 End result

After putting together every term, using the definition of \(F^{ss}_{4}(6.2.10)\), and performing some simplifications one obtains

\[D_{22}^{trans} = D_{22} , \]

which is exactly what was to be proven.
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Finite size matrix elements and thermal correlators in integrable quantum field theories

István M. Szécsényi

Summary

In my PhD thesis, I study the evaluation of finite size matrix elements and thermal correlators in integrable quantum field theories (QFT). Correlation functions are fundamental quantities in QFTs, since the knowledge of all correlator of local operators contains all the information in the QFT. Thermal correlators play important role describing finite temperature quantum systems, e.g. understanding their phase transitions or relating theoretical predictions to experimental results. Finite size quantum systems appear many times as a way of regularisation, or numerical method in physics, e.g. in lattice QCD calculations. Understanding the volume dependence of matrix elements is of great importance to the infinite volume extrapolation process. In two dimensional relativistic theories, the finite size and finite temperature setup are related. For example, the thermal expectation value of a local operator is the same as the expectation value of the operator in the finite size vacuum state.

After a short introduction of the integrable S-matrix and form factor bootstrap program, I review the LeClair-Mussardo conjecture that is a form factor series description of thermal one-point functions. I present numerical verification of the conjecture and discussed its convergence property. Following the work of Pozsgay, who generalised the LeClair-Mussardo conjecture to finite size diagonal matrix elements, I state the general formula of his conjecture. I prove the general formula for the trace of the stress-energy tensor operator and performed nontrivial numerical verification of the formula as well. I observe breakdown in the convergence of the conjecture around volumes, where the thermodynamic Bethe ansatz equations of the state considered undergo a nontrivial transition.

In the last part of my thesis, I focus on the method, developed by Pozsgay and Takacs, to evaluate thermal two-point functions as a form factor spectral series using finite volume regularisation. I reproduce and correct previously known results and support the analytic calculations by numerical evaluation. I show the symmetry and cluster property of the expansion up to two-particle contributions, that serves as a nontrivial check of the results.
Véges méretű mátrixelemek és termális korrelátorok integrálható kvantumtérelméletekben

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Összefoglalás


I. A doktori értekezés adatai
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A doktori iskolán belüli doktori program neve: Részecskefizika és csillagászat
A témavezető neve és tudományos fokozata: Takács Gábor, MTA doktora
A témavezető munkahelye: BME Elméleti Fizikai Tanszék

II. Nyilatkozatok
A doktori értekezés szerzőjeként
a) hozzájárulok, hogy a doktori fokozat megszerzését követően a doktori értekezésem és a tézisek nyilvánosságágra kerüljenek az ELTE Digitális Intézményi Tudástárban. Felhatalmazom a Természettudományi Kar Dékáni Hivatalának Doktori, Habilitációs és Nemzetközi Ügyleg Csoportja ügyintézőjét, hogy az értekezést és a téziseket feltöltsön az ELTE Digitális Intézményi Tudástárban, és ennek során kitöltse a feltöltéshez szükséges nyilatkozatokat.
b) kérem, hogy a mellékelt kérelemben részletezett szabadalmi, illetőleg oltalmi bejelentés közzetételéig a doktori értekezést ne bocsássák nyilvánosságára az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban;
c) kérem, hogy a nemzetbiztonsági okból minősített adatot tartalmazó doktori értekezést a minősítés (dátum)-ig tartó időtartama alatt ne bocsássák nyilvánosságára az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban;
d) kérem, hogy a mű kiadására vonatkozó mellékelt kiadó szerződésre tekintettel a doktori értekezést a könyv megjelenéséig ne bocsássák nyilvánosságára az Egyetemi Könyvtárban és az ELTE Digitális Intézményi Tudástárban.

2. A doktori értekezés szerzőjeként kijelentem, hogy
a) az ELTE Digitális Intézményi Tudástárba feltöltendő doktori értekezés és a tézisek saját eredeti, önálló szellemi munkám és legjobb tudomásom szerint nem sértem vele senki szerzői jogait;
b) a doktori értekezés és a tézisek nyomtatott változatai és az elektronikus adathordozón benyújtott tartalmak (szöveg és ábrák) mindenben megegyeznek.

3. A doktori értekezés szerzőjeként hozzájárulok a doktori értekezés és a tézisek szövegének plágiumkereső adatbázisba helyezéséhez és plágiumellenőrző vizsgálatok lefuttatásához.
