Complex projective plane curves and low-dimensional topology

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## Contents

**Acknowledgement** i

**Introduction** v

- **Historical overview and motivation** v
- **Summary and organization of the thesis** vii

**Chapter 1. Plane curve singularities** 1

1.1. **Invariants of plane curve singularities** 1
1.2. **The semigroup counting function** 8
1.3. **The Upsilon function of algebraic knots** 11

**Chapter 2. Rational cuspidal plane curves** 15

2.1. **Introduction** 15
2.2. **Conjectures on cusp types** 17
2.3. **Superisolated singularities, Dehn surgeries** 19
2.4. **The semigroup distribution property** 20
2.5. **Unicuspidal curves** 22
2.6. **Curves with more cusps** 25
2.7. **Settling the original conjecture** 35
2.8. **Verifying the Weak Conjecture for known curves with $\nu \geq 3$** 48
2.9. **An invariance property** 50

**Chapter 3. Cuspidal curves of higher genus** 55

3.1. **Introduction: the generalized semigroup distribution property** 55
3.2. **An equation for unicuspidal curves with torus knot link** 58
3.3. **Generalized Pell equation** 75
3.4. **Curve construction** 81

**Chapter 4. Topological invariants of Dehn surgeries** 95

4.1. **Introduction** 95
4.2. **Motivation: normal surface singularities** 99
4.3. **Dehn surgeries and their universal abelian covers** 101
4.4. The Covering Additivity Property 106
4.5. Examples and applications 111
List of computer programs 119
Bibliography 121
Summary 127
Összefoglalás (Summary in Hungarian) 129


Introduction

Historical overview and motivation

In this thesis, we will mostly deal with complex projective plane curves and some related questions in low-dimensional topology.

Complex plane curves are one of the oldest, yet still fascinating subjects of algebraic geometry. They have a long history, with some very interesting recent results and open problems.

A complex projective plane curve is a zero set in the complex projective plane of some irreducible homogeneous complex polynomial in three variables. It is called cuspidal if all the local singularities (i.e. points where it is not smoothly embedded) are locally irreducible plane curve singularities. A complex projective cuspidal plane curve is homeomorphic to a two-dimensional real oriented surface of some genus. A very old question is to enumerate all the possible embedded local topological types of singularities which can occur on a cuspidal plane curve of some fixed genus or degree.

An early result goes back to Namba [45], where he classified all projective plane curves (not just cuspidal ones) up to degree 5.

The case of rational cuspidal curves, i.e. curves of genus 0 (homeomorphic to the two-sphere $S^2$) was studied extensively. In the spirit of Namba’s work, Fenske in [17] classified the degree 6 rational cuspidal curves up to projective equivalence.

There is a huge machinery behind the results on rational cuspidal curves. It turned out that the logarithmic Kodaira dimension (see [28, 44]) $\pi \in \{-\infty, 0, 1, 2\}$ of the complement of the curve plays a crucial role in the classification of curves ([78], see also [19] for an overview). Yoshihara in [33] proved that for unicuspidal curves (curves with one single cusp only) the property $\overline{\pi} = -\infty$ is equivalent to the inequality $\overline{C^2} \geq -1$, where $\overline{C^2}$ is the self-intersection of the strict transform of the curve in the resulting ambient surface after a minimal good resolution of its singularity. The advantage of this characterization is that $\overline{C^2}$ can be computed easily from the degree of the curve and the local topological type of the singularity. Moreover, Wakabayashi in [78] classified completely the possible local cusp types of all unicuspidal curves with $\overline{\pi} = -\infty$.

Tsunoda in [77] proved that the case $\overline{\pi} = 0$ can not happen.
Tono in [73] proved that \( \pi = 1 \) implies that the number of cusps is at most 2 and gave a complete classification of possible local types in this case. More recently, he also classified rational curves with \( \pi = 2 \) having one, two and three cusps and maximal possible \( C^2 \) in [76, 75].

The difference between the degree \( d \) of the curve and the maximal multiplicity \( m \) among the multiplicities of the cusps turned out to be an other important guiding principle in the classification of rational plane curves. Flenner and Zaidenberg listed the possible local cusp types of tricuspidal curves with \( d - m = 2 \) in [22] and with \( d - m = 3 \) in [23]. The work in this direction was continued by Fenske, who classified unicuspidal and bicuspidal curves with \( d - m = 2 \) and 3 in [17]. In [18], he discussed the case of \( d - m = 4 \) under some additional constraints.

In [21], J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández and A. Némethi gave a full list of possible singularity types on rational unicuspidal curves (that is, cuspidal curves of genus \( g = 0 \) with one singular point) whose singularity has one Puiseux pair only (that is, its link is a torus knot). This will be one of our model results, in fact, in Chapter 3 we will prove some kind of generalization of this classification for certain positive genera.

Recently, using birational geometry and the minimal model program, Koras and Palka obtained results addressing old conjectures about rational cuspidal curves. In [31], as a continuation of [68], they proved the famous Coolidge-Nagata conjecture stating that every rational cuspidal curve is birationally equivalent to a line. Palka has important results regarding the rigidity conjecture of Flenner and Zaidenberg and the number of nodes in the minimal good resolution of singularities of a cuspidal curve as well (see [69]).

There are very few results on cuspidal curves of higher genus. As pointed out in [10], one of the main difficulties in this case is that the complement of the curve is not a rational homology ball (not a \( \mathbb{Q} \)-acyclic surface). In [74], using the Hodge index theorem, Tono proved an upper bound on the number of cusps depending on the genus only.

The concept of superisolated singularities introduced in [37] by Luengo connects the theory of cuspidal curves with the theory of complex normal surface singularities. In fact, every complex projective curve has a corresponding normal surface singularity. Its link is a rational homology sphere if and only if the projective curve is rational and cuspidal.

Motivated by different variants of the Seiberg–Witten invariant conjecture of A. Némethi and L. Nicolaescu (see [56, 57, 58]) valid for some normal surface singularities, in [20] J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández and A. Némethi stated a conjecture about the product of Alexander polynomials of links of local plane curve singularities on rational cuspidal curves. It was checked for many series of rational cuspidal curves in [20].
In the special case of unicuspidal rational curves, the property can be interpreted in terms of the \textit{semigroup} of the singularity (see e.g. [19, §6]). The property from this point of view can be called the \textit{semigroup distribution property}.

A generalization of the semigroup distribution property for rational cuspidal curves with arbitrary number of cusps was proved by M. Borodzik and C. Livingston in [11]. The proof uses tools coming from low-dimensional topology: Heegaard Floer theory and the correction terms of P. Ozsváth and Z. Szabó [65].

The semigroup distribution property provides a strong necessary condition on the possible cusp types of a rational cuspidal curve. T. Liu in his Ph.D. thesis [35] deeply exploited this result, proving strong constraints on the possible cusp types of rational cuspidal curves. Among other nice achievements, he reproved the result of [21] on unicuspidal rational curves with torus knot link (one Newton pair singularity) and he almost classified the possible cusp types with two Newton pairs on unicuspidal rational curves (see [35, Theorem 1.1]) using the semigroup distribution property only.

In a joint paper [6] with Daniele Celoria and Marco Golla, we proved a generalization of the semigroup distribution property for cuspidal curves of higher genus. This result was obtained independently by M. Borodzik, M. Hedden and C. Livingston as well [10]. The tools for the proof are similar to the tools used in the original proof for rational curves in [11].

Although the proofs of the above generalizations of the semigroup distribution property use the embedded topological type of the cuspidal curve in the complex projective plane, in the case of rational cuspidal curves the result has a nice interpretation in the language of links of the corresponding superisolated singularities as well (see [19, §7], cf. also [50]). We strongly believe that the interplay between the theory of cuspidal curves and superisolated singularities is still not fully exploited.

Superisolated singularities were further studied by Luengo, Melle-Hernández and Némethi in [38] and by Stevens in [71]. It turned out that they provide counterexamples to many famous conjectures in the theory of normal surface singularities (see [38]). In particular, they provide counterexamples to the Seiberg–Witten invariant conjecture of A. Némethi and L. Nicolaescu mentioned earlier, see [38, §3.1].

\textbf{Summary and organization of the thesis}

In Chapter 1, mostly based on classical books [14, 15, 79] we start with a short introduction of the main invariants of local plane curve singularities. We will be particularly interested in the question how the invariants change under the basic algebraic operation called \textit{blow-up}. From this point of view, the vast majority of invariants is presented in the
classical literature and can be considered well-known. The behaviour of the semigroup is discussed in [2], see also [4].

On the other hand, the change of the semigroup counting function (see (1.1.7) in Section 1.1 for the definition) under blow-up was not discussed in the literature. In Proposition 1.2.6 we describe the behaviour of the semigroup counting function and relate it to the infimal convolution, an operation which appeared in the main result of [11] and which plays a crucial role in the generalizations of the semigroup distribution property in [6, 10] as well. Implicitly, it is also present in the lattice cohomological computations by Némethi and Román in [59].

The new results describing the semigroup counting function using the infimal convolution (in Section 1.2) were published as a part of a joint work [7] with András Némethi.

As an application of Theorem 1.2.2 on the semigroup counting function involving the infimal convolution, using the recent result in [9] of Borodzik and Hedden, we give an alternative formula for the Upsilon function (introduced by Ozsváth, Stipsicz and Szabó in [63]) of algebraic knots.

Chapter 2 is devoted to the study of rational cuspidal curves. The majority of the results in this chapter was obtained in a joint work with András Némethi. We recall the relation of cuspidal curves to superisolated singularities and the motivation of the Original Conjecture (see Conjecture 2.2.1) of J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández and A. Némethi from [20]. We recall the generalized semigroup distribution property (Theorem 2.4.3) for rational curves from [11].

We show that T. Liu’s result in [35, Theorem 1.1] based on the generalized semigroup distribution property is almost a complete classification of rational unicuspidal curves with two Newton pairs (see Theorem 2.5.3). This observation is published as a short article [5].

In Section 2.6 we give some examples on how to construct rational cuspidal curves by series of birational transformations (Proposition 2.6.1) and recall all the currently known rational cuspidal curves with at least three cusps (Proposition 2.6.3).

In the next section we discuss the relation between the generalized semigroup distribution property proved in [11] by Borodzik and Livingston and the Original Conjecture raised in [20] by J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández and A. Némethi. We show that the Original Conjecture fails in general, but it is true for curves with at most two cusps (Theorem 2.7.1). For the proof of this latter fact, we use the lattice cohomological computations of Némethi and Román from [59]. We also formulate a weaker conjecture than the original one (see Conjecture 2.3.1 and Conjecture 2.7.5) and we check it for all currently known rational cuspidal curves (Theorem 2.8.1). All the above considerations fit well into the language of lattice cohomologies of links of the corresponding
superisolated singularities, or more generally, negative definite plumbed 3-manifolds obtained by integral Dehn surgeries along a connected sum of algebraic knots. Theorem 1.2.2 from Chapter 1 will have an other interesting corollary on the zeroth lattice cohomology of these manifolds (Corollary 2.9.2). The above results are published in a joint work [7] with András Némethi.

In Chapter 3, we turn our attention to cuspidal curves of higher genus. The results of this chapter are from a joint work with Daniele Celoria and Marco Golla. Our main tool will be the generalized semigroup distribution property for such curves (Theorem 3.1.1), published in a joint work [6] with Daniele Celoria and Marco Golla and obtained independently also by M. Borodzik, M. Hedden and C. Livingston [10].

As a nice application of this result, we prove a numerical constraint on the possible cusp types with torus knot link on unicuspidal curves of higher genus (Theorem 3.1.2). As a corollary, we obtain an almost complete classification of such cusps for certain genera (see Theorem 3.1.7).

All the results in Chapter 3 are published in a joint article [6] with Daniele Celoria and Marco Golla.

Chapter 4 is from a joint work with András Némethi. In this chapter, we examine certain topological invariants of negative definite plumbed 3-manifolds obtained by Dehn surgery along connected sum of algebraic knots. The main motivation is the fact that all links of superisolated singularities corresponding to rational cuspidal curves are of this form. As we mentioned earlier, superisolated singularities provide counterexamples to many famous conjectures about normal surface singularities. In particular, the generalized Seiberg–Witten invariant conjecture of Némethi (see Formula (4.2.2) in Subsection 4.2.1 from [49, 55], asserting that the so called equivariant geometric genera should be equal to certain modifications of the Seiberg–Witten invariants, is not true in general for superisolated singularities. Still, surprisingly, the ‘covering additivity property’ (see Definition 4.1.1 in Subsection 4.1.2) of the normalized Seiberg–Witten invariants turns out to be true for links of superisolated singularities with rational homology sphere links provided its universal abelian cover is also a rational homology sphere. In fact, Theorem 4.1.2 is even more general, stating the covering additivity property (under the same assumption about the universal abelian cover) for any plumbed 3-manifold obtained by negative rational Dehn surgery along a connected sum of algebraic knots.

This covering additivity property is suggested by an identity valid for the equivariant geometric genera and it relates the normalized Seiberg–Witten invariants of the manifold to the canonical Seiberg–Witten invariant of its universal abelian covering manifold.
The material of this last chapter is available in a joint manuscript [8] with András Némethi.
CHAPTER 1

Plane curve singularities

1.1. Invariants of plane curve singularities

In this section, we present some definitions and invariants characterizing local types of complex plane curve singularities. We refer to [14, 15, 24, 79] for further details.

By a local plane curve singularity we mean a holomorphic function germ $f \in \mathbb{C}\{x,y\}$ such that $f(0) = 0$, where $0 = (0,0) \in \mathbb{C}^2$ is the origin. Strictly speaking, we say it is singular only if $\frac{\partial f}{\partial x}(0) = \frac{\partial f}{\partial y}(0) = 0$. We say it is locally irreducible if $f$ is irreducible in $\mathbb{C}\{x,y\}$. We will deal only with locally irreducible singularities in this work, and will often use the word cusp for them. For some sufficiently small neighborhood $B$ of the origin in $\mathbb{C}^2$ (such that the derivative of $f$ does not vanish in $B \setminus 0$) we will call the zero set defined by $\{ (x,y) \in B : f(x,y) = 0 \}$ a singular set, or singularity, with a slight abuse of terminology.

Two singularities defined by functions $f_1$ and $f_2$ are said to be locally topologically equivalent if there exist neighborhoods $B_1, B_2 \ni 0$ of the origin of $\mathbb{C}^2$ and a homeomorphism $\varphi : B_1 \to B_2$ between them such that its restriction induces a homeomorphism between the zero sets $\{f_1 = 0\} \cap B_1$ and $\{f_2 = 0\} \cap B_2$ inside neighborhoods as well. We will use only this concept of equivalence, although one can study contact equivalence as well, where $\varphi$ is required to be a local biholomorphism.

Although we allow $f$ to be a power series, in fact, one can assume $f$ is a polynomial: every local plane curve singularity is equivalent topologically (and even analytically) to a singularity given by a polynomial $f \in \mathbb{C}[x,y]$ (see e.g. [14, §8.3, Theorem 15], [24, §2.2]).

Every local singularity defined by some function $f \in \mathbb{C}\{x,y\}$ can be parametrized by some power series $x(t), y(t) \in \mathbb{C}\{t\}$. The local parametrization means that $f(x(t), y(t)) \equiv 0$ and $t \mapsto (x(t), y(t))$ is a bijection between some neighborhood of the origin in $\mathbb{C}$ and a neighborhood of the origin in the zero set of $f$. Up to local topological equivalence, we can assume that the first power series is of form $x(t) = t^a$ for some integer $a > 1$ and the other power series is in fact a polynomial:

**Proposition 1.1.1** (see e.g. [24, Chapter I, Section 3, Theorem 3.3], [14, §8.3]).

*Every local plane curve singularity is topologically equivalent to a singularity having local*
parametrization given by
\begin{equation}
(1.1.1) \quad x(t) = t^a, \quad y(t) = t^{b_1} + t^{b_2} + \cdots + t^{b_r},
\end{equation}
where \(a < b_1 < \cdots < b_r, \ a > \gcd(a, b_1) > \gcd(a, b_1, b_2) > \cdots > \gcd(a, b_1, b_2, \ldots, b_r) = 1\).

In the above case we say that the multiplicity of the singularity is \(a\) and the line \(\{y = 0\}\) is the tangent line of the singularity (the local intersection multiplicity of the singular germ with this line is strictly larger than the local intersection multiplicity with any other line going through the singular point).

The multiplicity is an invariant of the local topological type. However, we should note here that although a tangent line always exists, the local intersection multiplicity with it is not an invariant of the local topological type (as we allow non-linear local homeomorphisms in the concept of topological equivalence, local line segments are not necessarily preserved).

The numbers \(\{a; b_1, \ldots, b_r\}\) are called the characteristic exponents of the singularity.

We can write formally \(y\) as a polynomial of \(t = x^{1/a}\), leading to fractional powers (not necessarily in reduced form):
\[
y = x^{\frac{b_1}{a}} + x^{\frac{b_2}{a}} + \cdots + x^{\frac{b_r}{a}}.
\]

One can rewrite the above fractional power sum as
\begin{equation}
(1.1.2) \quad y = x^{\frac{u_1}{v_1}} + x^{\frac{u_2}{v_2}} + \cdots + x^{\frac{u_r}{v_r}},
\end{equation}
where \(u_i, v_i\) are coprime for all \(i = 1, \ldots, r\) and \(1 < \frac{v_1}{u_1} < \frac{v_2}{u_1u_2} < \cdots < \frac{v_r}{u_1u_2\cdots u_r}\). The pairs of numbers
\[
(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)
\]
are called the Puiseux pairs of the singularity.

Another possibility is to write
\begin{equation}
(1.1.3) \quad y = x^{\frac{q_1}{p_1}} \left(1 + x^{\frac{q_2}{p_1p_2}} \left(1 + x^{\frac{q_3}{p_1p_2p_3}} \left(1 + \cdots \right)\right)\right),
\end{equation}
where \(p_1 < q_1\) and \(p_i, q_i\) are coprime for all \(i = 1, \ldots, r\). The pairs of numbers
\[
(p_1, q_1), (p_2, q_2), \ldots, (p_r, q_r)
\]
are called the Newton pairs of the singularity.

Notice that having one Puiseux pair is equivalent to having one Newton pair, and in this case, the two pairs are the same.

From knot theoretical point of view, the basic invariant of the local topological type is the link of the singularity. It is defined as the isotopy type of the knot (embedded circle \(S^1\)) in the three-sphere \(S^3\) given by the intersection of the singular set with a sphere of
small enough radius, i.e. $K := \{f(x, y) = 0\} \cap \{|x|^2 + |y|^2 = \varepsilon^2\}$ for the singularity $f$ and for a sufficiently small $\varepsilon > 0$. By ‘sufficiently small’ we mean that we get the same knot isotopy type for every smaller $\varepsilon$. The isotopy types of knots arising this way are called algebraic knots (see [15] or [79, §5]).

It is not hard to see that the link of a singularity with one Puiseux pair $(a, b)$ is a torus knot of type $(a, b)$. More generally, given the Newton pairs of a singularity as above, set $a_1 = q_1$ and $a_{i+1} = q_{i+1} + a_ip_ip_{i+1}$ for $1 \leq i < r$. Then the link of the singularity is the $(p_r, a_r)$-cable of the $(p_{r-1}, a_{r-1})$-cable ... of the $(p_1, a_1)$-cable of the unknot (see [14, Appendix to Chapter 1]). In particular, every algebraic knot is an iterated torus knot, obtained by iterating the cabling operation with some parameters.

All algebraic knots are completely characterized by their Alexander polynomial ([33]). The Alexander polynomial is closely related to an other complete invariant of the topological type, namely the semigroup of the singularity (see [79, §4.3]). It is given by the set of possible local intersection multiplicities of the defining equation of the singularity with other function germs:

$$\Gamma = \{\text{ord}_t g(x(t), y(t)) : g \in \mathbb{C}\{x, y\}, f \nmid g\}.$$  

This is a cofinite subset of non-negative integers: $\#(\mathbb{Z}_{\geq 0} \setminus \Gamma) = \delta$ and $\max(\mathbb{Z}_{\geq 0} \setminus \Gamma) = 2\delta - 1$, where $\delta$ is the so called delta invariant of the singularity. The semigroup has the following antisymmetry property:

$$n \not\in \Gamma \iff 2\delta - 1 - n \in \Gamma, \ \forall n \in \mathbb{Z}.$$  

For further details we refer to [79, §4.3].

It is related to the algebraic knot as follows (see [25]). Set

$$\Delta(t) = (1 - t) \sum_{k \in \Gamma} t^k.$$  

We will call this polynomial $\Delta(t) \in \mathbb{Z}[t]$ the Alexander polynomial of the singularity. It is the ‘normalized’ version of the usual Alexander polynomial of the link, obtained by multiplying it with $t^\delta$ to eliminate negative powers of the variable $t$. Additionally, $\delta$ is also the Seifert genus of the algebraic knot $K$.

The semigroup counting function $H$ is given by

$$H(n) = \# \Gamma \cap [0, n)$$  

for integers $n > 0$. We define $H(n) = 0$ for non-positive integers $n$.

In the literature, the gap counting function $I$ is often used. The set of gaps is the complement of the semigroup, and $I$ is defined by
The semigroup and the gap counting function are related as follows (cf. (1.1.5)):

\[(1.1.9)\]
\[H(m) = m - \delta + I(m).\]

We will also need the multiplicity sequence of the singularity (see [18, §1], [79, §3.5]), which can be defined inductively as follows. Assume that \(f \in \mathbb{C}[x,y]\) is a representative of the given topological type which admits a parametrization given by (1.1.1). Then we can write

\[f(x_1, x_1 y_1) = x_1^a f_1(x_1, y_1)\]

for some \(f_1 \in \mathbb{C}[x,y]\) such that \(x_1\) does not divide \(f_1\). Then the multiplicity sequence of \(f\) is \([m_1, m_2, \ldots, m_s]\), where \(m_1 = a\) and \([m_2, \ldots, m_s]\) is the multiplicity sequence of the topological type of the singularity given by equation \(f_1(x, y) - f_1(0,0) = 0\). The multiplicity sequence of a smooth local branch is by definition the empty sequence \([\ ]\). It can be proved that the multiplicity sequence is well-defined and \(m_1 \geq m_2 \geq \cdots \geq m_s > 1\). The first member \(m_1\) is also called the multiplicity of the singularity. For brevity, we will often write \([n_k]\) for \(k\) consecutive copies of the number \(n\) in a multiplicity sequence. For example, we can write \([4,2,3]\) instead of \([4,4,2,2,2]\).

The delta invariant of the singularity with multiplicity sequence \([m_1, m_2, \ldots, m_s]\) equals (see [18, §3], [42, §2.2])

\[(1.1.10)\]
\[\delta = \sum_{j=1}^s \frac{m_j(m_j - 1)}{2}.\]

Every singularity can be resolved by a series of blow-ups (see e.g. [79, §3]). By a blow-up we mean a local modification of the ambient space as follows (cf. [24, Chapter I, Definition 3.16]). First we define the blow-up of \(\mathbb{C}^2\) at the origin \(0 = (0,0) \in \mathbb{C}^2\). Consider the complex subspace

\[B_0\mathbb{C}^2 = \{(z, L) \in \mathbb{C}^2 \times \mathbb{CP}^1 : z \in L\} = \{(x, y, [s : t]) \in \mathbb{C}^2 \times \mathbb{CP}^1 : tx - sy = 0\} \subset \mathbb{C}^2 \times \mathbb{CP}^1,\]

where we identified the projective line \(\mathbb{CP}^1\) with complex lines passing through the origin and \([s : t]\) are homogeneous coordinates in \(\mathbb{CP}^1\).

Let \(\pi : B_0\mathbb{C}^2 \to \mathbb{C}^2\) be the restriction of the projection \(\mathbb{C}^2 \times \mathbb{CP}^1 \to \mathbb{C}^2\). The map \(\pi\) is an isomorphism over \(\mathbb{C}^2 \setminus 0\). Observe that \(\pi^{-1}(0) \cong \mathbb{CP}^1\). This preimage is usually denoted by \(E\) and is called the exceptional divisor of the blow-up. Consider any neighborhood \(U \ni 0\) and the restriction of \(\pi\) onto its preimage \(V := \pi^{-1}(U)\). Usually the map \(\pi|_V : V \to U\) is called the blow-up of the point \(0 \in U\). Notice that it can be performed on any complex
two dimensional manifold at any point \( p \): a small enough neighborhood of the point \( p \) we want to blow up plays the role of \( U \) and as \( \pi \) is isomorphism over \( U \setminus p \), it can be extended as an isomorphism over the rest of the manifold.

Given a local plane curve singularity at \( 0 \in \mathbb{C}^2 \), after the blow-up at \( 0 \) we can consider the strict transform of the zero set \( \{ f = 0 \} \), that is, the algebraic closure of \( \pi^{-1}(\{ f = 0 \} \setminus 0) \) in the preimage of the neighborhood. The strict transform may be still singular, but we can perform another blow-up (of a neighborhood) at the singular point on the new ambient complex surface, and repeat this process while the strict transform is still singular. By a finite number of consecutive blow-ups, we can achieve that the strict transform is smooth and all the exceptional divisors produced by the consecutive blow-ups are in normal crossing position: any two intersect each other transversely and no more than two divisors share a common point, additionally, the final strict transform of the originally singular zero set intersects only the last exceptional divisor and this intersection is transverse as well.

The intersection pattern of the exceptional divisors and the strict transform can be encoded by a tree: the vertices represent the strict transforms of exceptional divisors produced by the consecutive blow-ups, an arrowhead vertex represents the strict transform of the original singularity, and there is an edge between two vertices if and only if the corresponding divisors intersect each other. Additionally, each vertex possesses a negative integer (decoration) giving the Euler number of the normal bundle of the corresponding divisor in the ambient surface obtained by the blow-ups. This number is also called the self-intersection of the divisor.

The decorated graph described above is called the embedded resolution graph of the plane curve singularity. For more details on the resolution process, we refer to \([14, \S 8.4]\).

The list of characteristic exponents, Puiseux pairs and Newton pairs, the multiplicity sequence, the Alexander polynomial, the semigroup, the semigroup counting function and the embedded resolution graph are all complete invariants of the embedded local topological type of a singularity and consequently any of them determines all the others (see e.g. \([79, \text{Proposition 4.3.8}]\)).

The blow-up and blow-down process will be used in Section \([2.6]\) of Chapter \([2]\) and Section \([3.4]\) of Chapter \([3]\) to construct cuspidal curves by Cremona transformations. For further details on the blow-up process and Cremona transformations we refer to \([79, \S 4, \S 5]\) and \([42, \S 5]\). We briefly recall here the most important facts. The topological type of the singularity of the strict transform after a blow-up depends only on the topological type of the original singularity. More concretely, blowing up a singular point with multiplicity
sequence \([m_1, m_2, \ldots, m_s]\) we obtain a singularity with multiplicity sequence \([m_2, \ldots, m_s]\) (in particular, we get a smooth curve if the latter sequence is empty).

If a smooth curve had a local intersection multiplicity \(m\) with this singular curve, after the blow-up, the strict transforms of the two curves will have local intersection multiplicity \(m - m_1\) (in particular, they will be disjoint if \(m = m_1\)). The new exceptional divisor will have intersection multiplicity \(m_1\) with the strict transform of the curve, and self-intersection \(-1\). Blowing up a smooth point of any divisor decreases its self-intersection by \(1\). More generally, if a curve with self-intersection \(e\) on a complex surface has a singular point with multiplicity \(m\), then blowing up that point leads to a strict transform with self-intersection \(e - m^2\).

**Example 1.1.2.** We present the computation of the multiplicity sequence and the resolution graph on a simple example. Consider the local plane curve singularity at the origin \((0, 0)\) given by equation \(\{f(x, y) = 0\}\), where

\[
f(x, y) = y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7.
\]

One can check that \(x = t^4, y = t^6 + t^7\) is a parametrization of the singularity (see [79, Example 2.1.1, Lemma 2.3.1]).

Therefore, we can write

\[
y = x^\frac{3}{2} + x^\frac{7}{2} = x^\frac{3}{2}(1 + x^\frac{1}{2}),
\]

so the Puiseux pairs are \((2, 3), (2, 7)\), the Newton pairs are \((2, 3)(2, 1)\). The link of the singularity is the \((2, 2 \cdot 2 \cdot 3 + 1) = (2, 13)\) cable of the \((2, 3)\) torus knot.

To obtain the multiplicity sequence from the equation, observe the following. The single lowest degree summand in \(f\) is \(y^4\), meaning that the line \(\{y = 0\}\) is tangent to the singularity (has local intersection multiplicity \(6\) with the singular locus, which is higher than \(4\), the intersection multiplicity with any other line through the origin). So we can perform a blow-up, represented by introducing new variables such that \(x_1 = x, x_1y_1 = y\).

\[
f(x_1, x_1y_1) = x_1^4f_1(x_1, y_1),
\]

therefore \(m_1 = 4\) and

\[
f_1(x_1, y_1) = y_1^4 - 2x_1y_1^2 - 4x_1^2y_1 + x_1^2 - x_1^3.
\]

The lowest degree summand here is \(x_1^2\), meaning that the tangent line is \(\{x_1 = 0\}\). We can perform a linear coordinate change \(\overline{x}_1 = y_1, \overline{y}_1 = x_1\), obtaining \(f_1(x_1, y_1) = f_1(\overline{x}_1, \overline{y}_1)\), then introduce the new variables such that \(x_2 = \overline{x}_1, x_2y_2 = \overline{y}_1\). Now

\[
\overline{f}_1(x_2, x_2y_2) = x_2^2f_2(x_2, y_2),
\]
so \( m_2 = 2 \) and

\[
f_2(x_2, y_2) = x_2^2 - 2x_2y_2 - 4x_2y_2^2 + y_2^2 - x_2y_2^3 = (x_2 - y_2)^2 - 4x_2y_2^2 - x_2y_2^3.
\]

The lowest degree terms are collected into \((x_2 - y_2)^2\), therefore, the tangent line is \(\{x_2 - y_2 = 0\}\). We can perform the linear coordinate change \(\overline{x_2} = y_2, \overline{y_2} = x_2 - y_2\) to get \(f_2(x_2, y_2) = f_2(\overline{x_2}, \overline{y_2})\). Then the usual variable change \(x_3 = \overline{x_2}, x_3y_3 = \overline{y_2}\) leads to

\[
\overline{f}_2(x_3, x_3y_3) = x_3^2f_3(x_3, y_3),
\]

so \(m_3 = 2\) and

\[
f_3(x_3, y_3) = y_3^2 - 4x_3 - 4x_3y_3 - x_3^2 - x_3^2y_3,
\]

which is already a smooth branch (the partial derivative with respect to \(x_3\) is non-zero at the origin).
So the multiplicity sequence is \([4, 2, 2]\). From this, the pattern of the resolution process can be easily recovered. Figure 1.1 shows the configuration at each step of the resolution and how the final good resolution is encoded by the resolution graph.

1.2. The semigroup counting function

In this section (which appeared in a joint work with András Némethi [7, §5]), we want to study how the semigroup counting function changes during the operation of a blow-up. The main goal is to prove Theorem 1.2.2 below, but first we need to define the infimal convolution of two functions.

**Definition 1.2.1.** The infimal convolution \(C = A \diamond B\) of two functions \(A\) and \(B\) is given by

\[
C(m) = \inf_{i+j=m} \{A(i) + B(j)\}
\]

for every \(m \in \mathbb{Z}\), resp. \(m \in \mathbb{R}\) as \(i, j\) run through the domain of \(A\) and \(B\).

In all the cases we are going to apply this construction, the infimum in the above definition could be replaced by minimum. It is easy to see that the infimal convolution is an associative operation on functions bounded from below.

**Theorem 1.2.2.** Let \(H\) be the semigroup counting function of a singularity with multiplicity sequence \([n_1, n_2, \ldots, n_r]\). For any \(n \geq 2\), denote by \(H[n]\) the semigroup counting function of the singularity with multiplicity sequence \([n]\). Then

\[
H = H[n_1] \diamond H[n_2] \diamond \cdots \diamond H[n_r].
\]

**Proof.** For any multiplicity sequence \([n', n'_2, \ldots, n'_{r'}]\), denote by \(H[n'_1, n'_2, \ldots, n'_{r'}]\) the semigroup counting function of the corresponding singularity. Due to the obvious associativity of the infimal convolution, for the statement of the theorem it is enough to show that

\[
H[n_1, n_2, \ldots, n_r] = H[n_1] \diamond \cdots \diamond H[n_r].
\]

This will be proved in Proposition 1.2.6 below. \(\square\)

Before giving the precise statement of Proposition 1.2.6, we set up some notation.

Let \(\Gamma_1\) and \(\Gamma_2\) be semigroups of plane curve singularities. We will assume that \(\Gamma_1\) is a single blow-up of \(\Gamma_2\). Let \(H_1(i)\) and \(H_2(i)\) be the corresponding counting functions, i.e. \(H_\ell(i) = \#\{s \in \Gamma_\ell : s < i\}\). Our goal is to compare \(H_1\) and \(H_2\).
Denote by \( m \) the *multiplicity* of the second singularity, i.e. \( m = \min \{ s \in \Gamma_2 : 0 < s \} \).

The *Apéry set* of a numerical semigroup with respect to one of its elements is a standard invariant commonly used in semigroup theory. It consists of the smallest elements of the semigroup from each (nonempty) residue class modulo the given element. We consider the Apéry set of \( \Gamma_2 \) with respect to \( m \), that is, \( \text{Ap}(m, \Gamma_2) = \{ b_0, b_1, \ldots, b_{m-1} \} \), where \( 0 = b_0 < b_1 < \cdots < b_{m-1} \). It is a complete residual system modulo \( m \), and by the definition, for each \( i \) (\( 0 \leq i \leq m-1 \)) we have \( b_i \in \Gamma_2 \) but \( b_i - m \notin \Gamma_2 \).

The definition guarantees that \( \Gamma_2 = \text{Ap}(m, \Gamma_2) + m \cdot \mathbb{Z}_{\geq 0} \). In fact, for every element \( s \in \Gamma_2 \) there exist uniquely \( j \in \mathbb{Z} \) and \( u \in \mathbb{Z} \) such that \( s = b_j + mu \), \( 0 \leq j \leq m-1 \), \( 0 \leq u \).

**Lemma 1.2.3.** If \( \Gamma_1 \) is the blow-up of \( \Gamma_2 \), then \( m \in \Gamma_1 \) as well.

**Proof.** The strict transform of the singular curve after the blow-up and the reduced exceptional divisor of the blow-up have intersection multiplicity \( m \). \( \square \)

Therefore, we can consider the Apéry set with respect to \( m \) of \( \Gamma_1 \) as well: \( \text{Ap}(m, \Gamma_1) = \{ a_0, a_1, \ldots, a_{m-1} \} \) is a complete residual system modulo \( m \) such that \( a_i \in \Gamma_1 \) but \( a_i - m \notin \Gamma_1 \) for all \( 0 \leq i \leq m-1 \), and \( 0 = a_0 < a_1 < \cdots < a_{m-1} \). Again, \( \Gamma_1 = \text{Ap}(m, \Gamma_1) + m \cdot \mathbb{Z}_{\geq 0} \), i.e. for any \( s \in \Gamma_1 \) there exist unique \( j, u \in \mathbb{Z} \) such that \( s = a_j + mu \), \( 0 \leq j \leq m-1 \), \( 0 \leq u \).

**Proposition 1.2.4 (Apéry, [2] Lemme 2, see also [4] Proposition 2.3).** The *blow-up* of a *semigroup* is described by the two *Apéry sets* with respect to the original multiplicity \( m \) in the following way:

\[
a_j + jm = b_j \quad (j = 0, \ldots, m-1).
\]

**Remark 1.2.5.** The previous proposition implies that ‘the order is preserved’, i.e. if \( \Gamma_2 \) is a semigroup of a plane curve singularity which has multiplicity \( m \) and the *ordered* Apéry set with respect to this multiplicity is \( \text{Ap}(m, \Gamma_2) = \{ b_0, b_1, \ldots, b_{m-1} \} \) with \( 0 = b_0 < b_1 < \cdots < b_{m-1} \), then the series of inequalities \( 0 = b_0 - 0 \cdot m < b_1 - m < b_2 - 2m < \cdots < b_{m-1} - (m-1)m \) must be satisfied. This is a nontrivial necessary condition for an algebraic numerical semigroup to be a semigroup of a plane curve singularity.

Let \( \Gamma_{[m]} \) be the semigroup of the plane curve singularity with multiplicity sequence \([m]\), it is generated as a semigroup by \( m \) and \( m + 1 \). Denote its counting function by \( H_{[m]} \).

The counting functions \( H_1 \), \( H_2 \) and \( H_{[m]} \) are related as follows.

**Proposition 1.2.6.** For all \( l \geq 0 \) one has

\[
H_2(l) = \min_{0 \leq j \leq l} \{ H_1(l - j) + H_{[m]}(j) \}.
\]
Proof. We need to prove that for all \( l \geq 0 \), and for all \( j \) with \( 0 \leq j \leq l \) one has 
\[
H_2(l) - H_1(l - j) \leq H_{[m]}(j);
\]
furthermore, that for all \( l \geq 0 \) equality holds for some \( j \).

It will be useful to view the semigroups as unions of ‘layers’ according to the Apéry
sets. Namely, for \( i = 0, 1, \ldots, m - 1 \) set \( \Gamma_{i}^{(j)} := a_i + m\mathbb{Z}_{\geq 0} \). Then \( \Gamma_1 = \sqcup_{i=0}^{m-1} \Gamma_{i}^{(j)} \) as a disjoint
union. Similarly, set \( \Gamma_{2}^{(i)} := b_i + m\mathbb{Z}_{\geq 0} \), hence \( \Gamma_2 = \sqcup_{i=0}^{m-1} \Gamma_{2}^{(i)} \) as a disjoint
union. Then
\[
H_1(l - j) = \sum_i \# \{ s \in \Gamma_{1}^{(i)} : s < l - j \}, \quad H_2(l) = \sum_i \# \{ s \in \Gamma_{2}^{(i)} : s < l \}.
\]
By Proposition 1.2.4 we get that the \( i \)-th layer \( \Gamma_{2}^{(i)} \) of the semigroup \( \Gamma_2 \) just has to be
shifted to the left by \( im \) to get the \( i \)-th layer \( \Gamma_{1}^{(i)} \) of the semigroup \( \Gamma_1 \). Hence,
\[
H_2(l) = \sum_i \# \{ s \in \Gamma_{1}^{(i)} : s < l - im \}.
\]
Now for a fixed \( l \) the difference which has to be (sharply) bounded from above can be
written as a difference of set–cardinalities, the sets being differences of subsets of the
semigroup (layers of) \( \Gamma_1 \):
\[
H_2(l) - H_1(l - j) = \# \{ A_{j,l} \} - \# \{ B_{j,l} \},
\]
where \( A_{j,l} = \sqcup_{i=0}^{m-1} A_{j,l}^{(i)} \) and \( B_{j,l} = \sqcup_{i=0}^{m-1} B_{j,l}^{(i)} \) as disjoint unions, with
\[
A_{j,l}^{(i)} = \{ s \in \Gamma_{1}^{(i)} : l - j \leq s < l - im \}, \quad B_{j,l}^{(i)} = \{ s \in \Gamma_{1}^{(i)} : l - im \leq s < l - j \}.
\]
(Note that for all \( i \), at least one of \( A_{j,l}^{(i)} \) and \( B_{j,l}^{(i)} \) is empty.)

Hence, we need to prove that \( \# \{ A_{j,l} \} - \# \{ B_{j,l} \} \leq H_{[m]}(j) \), and for each \( l \) equality
holds for some \( j = 0, 1, \ldots, l \).

The inequality follows from \( \# \{ B_{j,l} \} \geq 0 \) and \( \# \{ A_{j,l} \} \leq H_{[m]}(j) \), where the second
inequality is not straightforward.

First we check it for the multiples of \( m \), i.e. for \( j \)’s of form \( j = \omega m \):
\[
\# \{ A_{\omega m,l} \} = \sum_{i=0}^{m-1} \# \{ A_{\omega m,l}^{(i)} \} \leq \sum_{i=0}^{m-1} \max \{ \omega - i, 0 \} = \sum_{i=0}^{\omega} \min \{ i, m \} = H_{[m]}(\omega m).
\]
This is true because \( \# \{ A_{\omega m,l}^{(i)} \} = \# \{ r \in \mathbb{Z}_{\geq 0} : l - \omega m \leq a_i + rm < l - im \} \leq \max \{ \omega - i, 0 \} \).
(This upper bound is valid even if \( \omega m > l \).)

For \( j \)’s not of the form \( \omega m \), write \( j = \omega m + \gamma \) with \( 0 < \gamma < m \).

First assume that \( \gamma \leq \omega + 1 \). In this case, \( H_{[m]}((\omega + 1)m) = H_{[m]}(\omega m) + \omega + 1 \) and
\( H_{[m]}(\omega m + \gamma) = \min \{ \gamma, \omega + 1 \} + H_{[m]}(\omega m) \). Observe also that \( 0 \leq \# \{ A_{j+1,l} \} - \# \{ A_{j,l} \} \leq 1 \)
(for \( \# \{ A_{j+1,l}^{(i)} \} - \# \{ A_{j,l}^{(i)} \} \in \{ 0, 1 \} \), and except for at most one \( i \), the difference is 0, as
elements of \( A_{j,l}^{(i)} \) for different \( i \)’s have different residues modulo \( m \) ). Therefore, on one
hand,
\[ \#\{A_{\omega m+\gamma}\} \leq \#\{A_{\omega m}\} + \gamma \leq H_{[m]}(\omega m) + \gamma, \]
on the other hand,
\[ \#\{A_{\omega m+\gamma}\} \leq \#\{A_{(\omega+1)m}\} \leq H_{[m]}((\omega + 1)m) = H_{[m]}(\omega m) + \omega + 1. \]
In this way,
\[ \#\{A_{\omega m+\gamma}\} \leq H_{[m]}(\omega m) + \min\{\gamma, \omega + 1\} = H_{[m]}(\omega m + \gamma) \]
as desired.

Now assume that \( \omega + 1 < \gamma < m \), then we have \( H_{[m]}(\omega m + \gamma) = H_{[m]}((\omega + 1)m) \geq \#\{A_{(\omega+1)m}\} \geq \#\{A_{\omega m+\gamma}\} \) (here we have used (1.2.1) for \( (\omega + 1)m \) instead of \( \omega m \)).

Next we show that for any \( l \) there exists a \( j \) for which equality holds. From the above, it is clear which conditions do we want to be satisfied. We will choose a \( j \) such that \( 0 \leq j \leq l, j \leq (m-1)m, B_{j,l} = 0 \) and \#\{\( r \in \mathbb{Z}_{\geq 0} : l - j \leq a_i + rm < l - im \)\} = \#\{A_{j,l}^{(i)}\} = \max\{\lfloor \frac{j}{m} \rfloor - i, 0\} \) for all \( i = 0, 1, \ldots, m - 1 \).

For any \( l \), let \( i_0 \) be the smallest index \( i \) among \( 0, 1, \ldots, m - 1 \) for which \( l - im \leq a_i \) is already valid, if such index exists. If not, take \( i_0 = m - 1 \).

It is not hard to see that \( j = \min\{i_0m, l\} \) is a good choice, i.e. for \( j = \min\{i_0m, l\} \) we will have equality in the upper bound. For if \( j = i_0m \), then by the choice of \( i_0 \) we have \( B_{i_0m,l} = \emptyset \) and \( A_{i_0m,l}^{(i)} = \max\{i_0 - i, 0\} \) for all possible \( i \), and these two conditions (via (1.2.1)) are enough to guarantee the equality \#\{\( A_{i_0m,l} \)\} = \#\{\( B_{i_0m,l} \)\} = \( H_{[m]}(i_0m) \). If \( j = l \) happens to be the case (i.e. if \( l < i_0m \)), then by a similar argument as above, we have \( H_{[m]}(i_0m) = A_{i_0m,l} \), hence \( A_{l,l} \leq H_{[m]}(l) \leq H_{[m]}(i_0m) = A_{i_0m,l} = A_{l,l} \), which implies \#\{\( A_{l,l} \)\} = \#\{\( B_{l,l} \)\} = \( H_{[m]}(l) \) again, as \( B_{l,l} = \emptyset \).

\[ \Box \]

1.3. The Upsilon function of algebraic knots

As an application of Theorem 1.2.2 obtained above, we give an alternative way to compute the Upsilon function \( \Upsilon_K(t) \) for algebraic knots. The \( \Upsilon \) function is a knot invariant defined in [63] by Ozsváth, Stipsicz and Szabó. With any knot \( K \) one can associate a function \( \Upsilon_K : [0, 2] \to \mathbb{R} \). It can be used to give bounds on the four-genus of knots ([63, Theorem 1.11]), for one example of its usage, see e.g. [36].

There is an algorithm for the computation of the \( \Upsilon \) function of algebraic knots, or more generally, for \( L \)-space knots in terms of their Alexander polynomial ([63, Theorem 6.2]). For torus knots \( K_n \) of type \( (n, n+1) \), the formula for the \( \Upsilon \) function is particularly simple.
Proposition 1.3.1 (Ozsváth, Stipsicz, Szabó, [63, Proposition 6.3]). The values of the \( \Upsilon \) function of the \((n, n+1)\) torus knot are given by the formula

\[
\Upsilon_{[n]}(t) = -i(i+1) - \frac{1}{2}n(n-2i-1)t
\]

for \( t \in \left[\frac{2i}{n}, \frac{2i+2}{n}\right] \), \( i = 0, 1, \ldots, n-1 \).

Our goal is to express the function \( \Upsilon \) of any algebraic knot as a sum of functions of the form above. Denote the Upsilon function of an algebraic knot corresponding to a plane curve singularity with multiplicity sequence \([m_1, \ldots, m_s]\) by \( \Upsilon_{[m_1,\ldots,m_s]}(t) \). In particular, the Upsilon function of an \((n, n+1)\) torus knot will be denoted by \( \Upsilon_{[n]} \), consistently with the notations in the above proposition. We will prove the following formula:

Proposition 1.3.2.

(1.3.1) \( \Upsilon_{[m_1,\ldots,m_s]}(t) = \Upsilon_{[m_1]}(t) + \cdots + \Upsilon_{[m_s]}(t) \).

The proof has two main ingredients. The first is Theorem 1.2.2, which allows us to write the semigroup counting function of any singularity as an infimal convolution of semigroup counting functions of singularities with one single multiplicity in their multiplicity sequence. The second is [9], where Borodzik and Hedden proved that the Upsilon function of an algebraic knot is the Legendre transform of some simple variant of the semigroup counting function and the Legendre transform takes infimal convolution of functions into the sum of the Legendre transforms of the functions.

As a preparation, we recall some results and set up the notation based on [9].

Definition 1.3.3 ([9, Definition 2.1]). For a continuous function \( f : \mathbb{R} \to \mathbb{R} \), its Legendre transform is the function \( f^* : D \to \mathbb{R} \) defined by

\[
f^*(t) = \sup_{x \in \mathbb{R}} \{tx - f(x)\}
\]

and the domain \( D \) of \( f^* \) is the set of those values of \( t \) for which the above supremum is finite.

As in Definition 1.2.1 for any two real-valued functions \( f, g \) defined on \( \mathbb{R} \), set

\[
(f \diamond g)(t) = \inf_{u+v=t} \{f(u) + g(v)\}
\]

for their infimal convolution.

Lemma 1.3.4 ([9, Lemma 2.6]). For any two continuous functions \( f, g : \mathbb{R} \to \mathbb{R} \) we have

\[
(f \diamond g)^*(t) = f^*(t) + g^*(t)
\]
for each \( t \) where both sides are defined.

**Proof.**

\[
(f \circ g)^*(t) = \sup_x \{ tx - (f \circ g)(x) \} = \sup_x \{ tx - \inf_{u+v=x} \{ f(u) + g(v) \} \} = \\
= \sup_x \sup_{u+v=x} \{ tx - f(u) - g(v) \} = \sup_x \sup_u \sup_v \{ tu + tv - f(u) - g(v) \} = f^*(t) + g^*(t).
\]

□

Let \( K \) be an algebraic knot and \( H_K \) the semigroup counting function of the corresponding singularity. Denote by \( \delta_K \) its delta invariant, and recall that this is the same as the Seifert genus of the knot \( K \). Define a function \( h_K \) on \( \mathbb{R} \) as follows. For any integer \( n \in \mathbb{Z} \) set

\[
h_K(n) = 2H_K(n + \delta_K)
\]

and extend it to real numbers linearly between two integers, for \( t \in [n, n+1] \), \( n \in \mathbb{Z} \) being

\[
h_K(t) = (h_K(n+1) - h_K(n))(t - n) + h_K(n).
\]

Then for any real value \( t \), the Upsilon function of the knot \( K \) can be expressed as follows:

**Proposition 1.3.5** (Borodzik, Hedden, [9, Theorem 4.7]).

\[
\Upsilon_K(t) = h^*_K(t).
\]

**Proof.** We need to prove that \( \Upsilon_K \) is the Legendre transform of \( h_K \). In [9, Theorem 4.7], it is proved that it is the Legendre transform of the function \( x \mapsto 2J(-x) \), where \( J(-x) = I(-x + \delta_K) \) is the shifted version of the gap counting function. But this is exactly the function \( h_K \) defined above, see (1.1.5).

\[
\text{□}
\]

Now we are ready to prove Proposition 1.3.2.

**Proof of Proposition 1.3.2.** For any singularity with multiplicity sequence \([m_1, \ldots, m_s]\), denote by \( H_{[m_1, \ldots, m_s]} \) its semigroup counting function, by \( \delta_{[m_1, \ldots, m_s]} \) its delta invariant and set \( h_{[m_1, \ldots, m_s]} \) for the function defined above as the extension over the real numbers of the function \( n \mapsto 2H_{[m_1, \ldots, m_s]}(n + \delta_{[m_1, \ldots, m_s]}) \).

By Theorem 1.2.2 we have

\[
H_{[m_1, \ldots, m_s]} = H_{[m_1]} \circ \cdots \circ H_{[m_s]},
\]

and, since \( \delta_{[m_1, \ldots, m_s]} = \delta_{[m_1]} + \cdots + \delta_{[m_s]} \) (see Equation (1.1.10)), it follows that

\[
h_{[m_1, \ldots, m_s]} = h_{[m_1]} \circ \cdots \circ h_{[m_s]}.
\]

By Lemma 1.3.4 we have

\[
h^*_{[m_1, \ldots, m_s]} = h^*_{[m_1]} + \cdots + h^*_{[m_s]},
\]

which by Proposition 1.3.5 is exactly the desired identity for the Upsilon functions. \( \square \)
Remark 1.3.6. Recently, in [16], Feller and Krcaovich, also using the work [9] of Borodzik and Livingston, proved an inductive formula for the computation of the Upsilon function of torus knots (see [16] Proposition 6). As an \((a,b)\)-torus knot with \(1 < a < b\) is always an algebraic knot such that the corresponding multiplicity sequence starts with \(a\), the above mentioned inductive formula of [16] can be considered as a special case of Proposition 1.3.2.

Example 1.3.7. Let us consider the example presented in Example 1.1.2. Take a singularity with multiplicity sequence \([4, 2, 2]\), which corresponds to Newton pairs \((2, 3)(2, 1)\). Its link is the \((2, 13)\)-cable on the \((2, 3)\) torus knot. From the Newton pairs, one can compute the Alexander polynomial as in [20], §2.1 and get

\[
\Delta(t) = \frac{(t - 1)(t^{12} - 1)(t^{26} - 1)}{(t^4 - 1)(t^6 - 1)(t^{13} - 1)} = t^{16} - t^{15} + t^{12} - t^{11} + t^{10} - t^9 + t^8 - t^7 + t^6 - t^5 + t^4 - t + 1,
\]

therefore, the symmetrized Alexander polynomial is

\[
t^8 - t^7 + t^4 - t^3 + t^2 - t + 1 - t^{-1} + t^{-2} - t^{-3} + t^{-4} - t^{-7} + t^{-8}.
\]

Using [63] Theorem 6.2, one obtains that for any \(t \in [0, 2]\)

\[
\Upsilon_{[4, 2, 2]}(t) = \max \{-8t, -2 - 4t, -4 - 2t, -6, -8 + 2t, -10 + 4t, -16 + 8t\}.
\]

On the other hand, by Proposition 1.3.2, it is enough to compute \(\Upsilon_{[4]}\) and \(\Upsilon_{[2]}\) based on Proposition 1.3.1 then take the sum over the multiplicities:

\[
\Upsilon_{[4, 2, 2]}(t) = \Upsilon_{[4]}(t) + 2\Upsilon_{[2]}(t).
\]

Figure 1.2. The graph of functions \(\Upsilon_{[4, 2, 2]}(t)\) (black), \(\Upsilon_{[4]}(t)\) (thin grey) and \(\Upsilon_{[2]}(t)\) (thick grey).
CHAPTER 2

Rational cuspidal plane curves

2.1. Introduction

A complex projective plane curve is given as a zero set in the complex projective plane of an irreducible homogeneous polynomial in three variables with complex coefficients:

\[ C = \{[x : y : z] : h(x, y, z) = 0\} \subset \mathbb{CP}^2 \]

for some irreducible homogeneous polynomial \( h \in \mathbb{C}[x, y, z] \). The degree \( d \) of the polynomial is called the degree of the curve.

A point \( P \in C \) is called singular if the gradient vector of the defining equation vanishes at that point, i.e.

\[ \frac{\partial h}{\partial x}(P) = \frac{\partial h}{\partial y}(P) = \frac{\partial h}{\partial z}(P) = 0. \]

Denote by \( \text{Sing } C = \{P_1, P_2, \ldots, P_\nu\} \) the (finite) set of singular points. At each singular point \( P \in \text{Sing } C \) the defining equation \( h \) is a representative of a singular function germ. Therefore, it determines a local plane curve singularity.

The curve \( C \) is called cuspidal if all of its singularities are locally irreducible plane curve singularities. We will often refer to the number of cusps by prefixes, that is, curves will be called unicuspidal, bicuspidal or tricuspidal if \( \nu = 1, 2 \) or 3, respectively. The cuspidal curve \( C \) is called rational if it is homeomorphic to the two-dimensional real sphere \( S^2 \) (equivalently, if it can be parametrized by the complex projective line \( \mathbb{CP}^1 \)). By the degree-genus formula (see e.g. [3], Section II.11) one obtains that \( C \) is rational if and only if

\[ \frac{(d - 1)(d - 2)}{2} = \sum_{j=1}^{\nu} \delta_j, \]

where \( \delta_j \) is the delta invariant of the singularity at \( P_j \).

We want to study which topological types of local plane curve singularities can occur on rational cuspidal curves. The main task will be to prove some necessary conditions on the invariants of the local singularities.

As we mentioned in the introduction, the study of cuspidal curves, especially the rational ones has a long history and several methods for their study come from different
areas of mathematics. Nevertheless, we will restrict ourselves here to certain topological
methods, and we will not discuss some very strong and widely used tools, such as the
spectrum semicontinuity, the Hodge index theorem, the minimal model program and the
theory of linear systems.

In [20] Fernández de Bobadilla, Luengo, Melle-Hernández and Némethi formulated
a conjecture (which will be called the **Original Conjecture**) on the topological types of
irreducible singularities of a rational cuspidal projective plane curve \( C \subset \mathbb{CP}^2 \). Recently
in [11] Borodzik and Livingston, mostly motivated by [20], proved a necessary condition
which we will call here the **semigroup distribution property (SDP)** satisfied by the topologi-
cal types of cusps of rational cuspidal plane curves. The Original Conjecture and the SDP
will be reviewed in Subsection 2.2.1 and Section 2.4, respectively. Both of them cover
some deep connection with low–dimensional topology. Indeed, the Original Conjecture
in [20] was motivated by the Seiberg–Witten Invariant Conjecture of Némethi and Nico-
laescu [56], which for a normal surface singularity connects the Seiberg–Witten invariant
of the link with the geometric genus; while the proof of the main result (SDP) of Borodzik
and Livingston from [11] is based on the properties of the \( d \)–invariants of Heegaard Floer
theory.

The semigroup distribution property imposes strong restrictions on the possible local
topological types of singularities occurring on rational cuspidal plane curves. It can be
used very effectively for unicuspidal curves, but it becomes less restrictive for curves with
more cusps (cf. Corollary 2.9.5). As a short digression, we briefly present some results on
rational curves with one, two and more cusps. In Section 2.5 we collect some classification
results on unicuspidal rational curves illustrating the power of the semigroup distribution
property. In Section 2.6 we give some constructions of bicuspidal curves in the proof of
Proposition 2.6.1 and we recall all the currently known rational curves with at least three
cusps in Proposition 2.6.3.

Our next goal is to clarify the possible interactions between the Original Conjecture and
the semigroup distribution property by examples and conceptual theoretical explanations.
It turns out that this can be done using the theory of lattice cohomology.

In the comparison of the Original Conjecture from [20] and the SDP from [11], the
number of cusps plays a crucial role. When there is only one cusp, then the two state-
ments are equivalent; in particular, in the unicuspidal case the theorem of [11] proves the
conjecture of [20]. However, in the case of at least two cusps, the connection between the
two conditions is less transparent. Although the SDP proved in [11] contains equalities,
while the Original Conjecture in [20] contains inequalities and thus it is seemingly ‘less
2.2. Conjectures on Cusp Types

2.2.1. Notations and the Original Conjecture from [20]. Let \( C \subset \mathbb{CP}^2 \) be a rational cuspidal curve of degree \( d \) with \( \nu \) cusps (that is, with locally irreducible singularities) at points \( P_1, P_2, \ldots, P_\nu \). Recall that by the local embedded topological type of the singularity at a point \( P_i \) we mean the isotopy type of the algebraic knot \( K_i = C \cap S_i \subset S_i \),

precise’, we will see that it is not a combinatorial corollary of the former one if the number of cusps is at least three.

Nevertheless, after we reformulate all the statements in the language of lattice cohomology (Section 2.7), we show that for bicuspidal curves the conjecture is implied by the results of Borodzik and Livingston [11] and by the lattice cohomology formulae of [59] (see Theorem 2.7.1 (1)).

Furthermore, we show that for curves with at least three cusps, the ‘original conjecture’ from [20] is not true, in general (Theorem 2.7.1 (2), cf. Example 2.7.9).

However, we formulate a weakened version of the conjecture (Conjecture 2.3.1), more in the spirit of the motivation of the Seiberg–Witten Invariant Conjecture, which intended to connect Euler characteristic type invariants instead of cohomology groups. This weaker version, quite surprisingly, turns out to be true for all known rational cuspidal curves, even for those, which have at least three cusps (Theorem 2.8.1). This is proved in Section 2.8.

In Section 2.9 we present a procedure which makes certain 0th lattice cohomological computations a lot easier: it proves a stability property of the lattice cohomology with respect to some kind of ‘surgery manipulations’ with the multiplicity sequences of the local singularities. These computations are closely related to the infimal convolution formula of Theorem 1.2.2 and the lattice cohomological reformulation of the SDP from [11], and enlarge drastically and conceptually those geometric situations where the SDP is valid (showing e.g. that the global analytic realization of the local cusp types is ‘less’ important among the conditions of the main theorem of [11]). Accordingly, this also shows that the SDP as a test for the analytic realizability of the rational cuspidal curves, is less restrictive.

More precisely, the SDP is a combinatorial condition on the collection of topological types of cusps of existing rational cuspidal projective plane curves. This necessary condition can be applied as a criterion when one wants to list the possible cusp types of rational cuspidal curves. It turns out that this criterion is less restrictive when the number of local topological cusp types is larger (see Corollary 2.9.5 and Remark 2.9.6).

The new results of this chapter are mostly from a joint work with András Némethi [7]. I am grateful for his work and revision on Subsection 2.2.1, Section 2.3 and Section 2.7, as well as for his help with some curve constructions in the proof of Proposition 2.6.1.
where $S_i$ is a 3-sphere centered at $P_i$ with sufficiently small radius. It is completely determined by the semigroup $\Gamma_i \subset \mathbb{Z}_{\geq 0}$ of the plane curve singularity $(C, P_i)$, or, equivalently (see (2.2.1)), by the Alexander polynomial $\Delta_i(t)$ of $K_i \subset S_i = S^3$. In our convention $\Delta_i$ is indeed a polynomial, and it is normalized by $\Delta_i(1) = 1$.

Recall from Formula [1.1.6] that by [25], the semigroup $\Gamma_i$ and the Alexander polynomial $\Delta_i$ of the singularity at $P_i$ are related as follows ($i = 1, \ldots, \nu$):

\begin{equation}
\Delta_i(t) = (1 - t) \cdot \sum_{k \in \Gamma_i} t^k.
\end{equation}

(2.2.1)

Recall also that by (1.1.7), the semigroup counting functions are defined as

\begin{equation}
H_i(k) := \# \{ s \in \Gamma_i : s < k \}.
\end{equation}

(2.2.2)

Set $\delta := \delta_1 + \cdots + \delta_\nu$ for the sum of delta invariants. The degree-genus formula 2.1.1 for singular curves provides the following necessary condition for the existence of a degree $d$ rational cuspidal curve with cusps of given topological type:

\begin{equation}
2\delta = (d - 1)(d - 2).
\end{equation}

(2.2.3)

Consider the product of Alexander polynomials: $\Delta(t) := \Delta_1(t)\Delta_2(t)\cdots\Delta_{\nu}(t)$. There is a unique polynomial $Q$ for which $\Delta(t) = 1 + \delta(t - 1) + (t - 1)^2Q(t)$. Write

\begin{equation}
Q(t) = \sum_{j=0}^{2\delta-2} q_j t^j.
\end{equation}

(2.2.4)

The definition of $Q$ was motivated by the expression (2.2.7) below.

For $\nu = 1$, using (2.2.1) and properties of $\Delta_1$, one shows that (cf. [59] §2])

\begin{equation}
Q(t) = \sum_{s \notin \Gamma_1} (1 + t + \cdots + t^{s-1}), \text{ hence } q_j = \# \{ s \notin \Gamma_1 : s > j \} \quad \text{(if } \nu = 1\text{)}.
\end{equation}

(2.2.5)

For arbitrary $\nu$, the dependence of the coefficients of $Q$ in terms of $\Gamma_i$ will be given in (2.7.2). Notice that $q_0 = \delta$ and $q_{2\delta-2} = 1$ [59 (2.4.4)]. From the symmetry of $\Delta$ one also gets

\begin{equation}
q_{2\delta-2-j} = q_j + j + 1 - \delta \quad \text{for } 0 \leq j \leq 2\delta - 2.
\end{equation}

(2.2.6)

Next, set the rational function

\begin{equation}
R(t) := \frac{1}{d} \sum_{\xi \in \Gamma_1} \frac{\Delta(\xi t)}{(1 - \xi t)^2} - \frac{1 - t^d}{(1 - t^d)^3}.
\end{equation}

(2.2.7)
In [20 (2.4)] is proved that \( R(t) \) is a symmetric polynomial \((R(t) = t^{d(d-3)} R(1/t))\), and

\[
(2.2.8) \quad R(t) = \sum_{j=0}^{d-3} \left( q(d-3-j) d - \frac{(j + 1)(j + 2)}{2} \right) t^{(d-3-j)d}.
\]

Since several conjectures will appear later it is convenient to give names to them. The conjecture from [20] we wish to discuss will be called the ‘Original Conjecture’.

**Conjecture 2.2.1 (Original Conjecture, Fernández de Bobadilla, Luengo, Melle-Hernández, Némethi, [20]).** For any rational cuspidal plane curve \( C \subset \mathbb{CP}^2 \) of degree \( d \) the coefficients of \( R(t) \) are non-positive.

Later, (see Example 2.4.1, Subsections 2.7.5 and 2.7.6) we will show that for \( \nu \leq 2 \) the computations of [59] reduce the conjecture to the statement of [11], and for \( \nu \geq 3 \), in general, it is false, see Example 2.7.9.

### 2.3. Superisolated singularities, Dehn surgeries

The main motivation for the expression \( R(t) \), and for the formulation of the conjecture was a weaker version of the statement, a comparison of an analytic invariant (the geometric genus \( p_g \)) and a topological invariant (the Seiberg–Witten invariant of the link) of the superisolated hypersurface singularity associated with \( C \). The authors of [20] were led to it via the Seiberg–Witten Invariant Conjecture (SWIC) of Némethi and Nicolaescu [56].

More precisely, let \( f_d \) be the homogeneous equation of degree \( d \) of \( C \), and set a generic homogeneous function \( f_{d+1} \) of degree \( (d+1) \). Then \( f = f_d + f_{d+1} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) defines an isolated hypersurface singularity. It is called ‘superisolated’ since a single blow-up at the origin resolves \( \{ f = 0 \} \), see [37] or [38] §2.2. The singularity \( \{ f = 0 \} \) has geometric genus \( p_g = d(d-1)(d-2)/6 \). Let \( M \) denote its oriented link. One shows that it is the surgery manifold \( S^3_{\text{can}}(K) \), where \( K \) is the connected sum \( \#_i K_i \). If \( \widetilde{X} \to \{ f = 0 \} \) is a resolution, we denote the canonical class of \( \widetilde{X} \) by \( K_{\text{can}} \) and rank \( H_2(\widetilde{X}) \) by \( s_{\widetilde{X}} \). Then \( K_{\text{can}}^2 + s_{\widetilde{X}} \) is an invariant of the link (in this case it equals \(-(d-1)(d^2-3d+1)\)) and one also has (20)

\[
(2.3.1) \quad R(1) = -\text{sw}_{\text{can}}(M) - (K_{\text{can}}^2 + s_{\widetilde{X}})/8 - p_g.
\]

Here \( \text{sw}_{\text{can}}(M) \) is the Seiberg–Witten invariant of \( M \) associated with the canonical spin\(^c\) structure. Here we adopt the sign convention of later articles, e.g. of [13], which is the opposite of [20]. The integer \( \text{sw}_{\text{can}}(M) + (K_{\text{can}}^2 + s_{\widetilde{X}})/8 \) is usually called the ‘normalized Seiberg–Witten’ invariant. \( \text{sw}_{\text{can}}(M) \) can be determined (at least) by two ways, the first goes via Turaev torsion (as in [20]), or one can rely on the surgery formula of [13]. In
both cases the key term is the sum from (2.2.7). For more details see also Subsection 2.7.3 and Remark 2.7.17 below.

The SWIC [56] predicts that for certain singularities \( R(1) = 0 \). Later it turned out that this is not true e.g. for all superisolated singularities [38, 20]. Nevertheless, in all the counterexamples of [38, 20] the inequality \( R(1) \leq 0 \) holds. In fact, the computations and examples of [20] suggested that one might expect even the stronger set of inequalities, namely that all the coefficients of \( R(t) \) are non–positive: this fact was formulated in the above (Original) Conjecture 2.2.1. As we will show, this stronger expectation was too optimistic: it fails for certain curves with \( \nu > 2 \). Therefore, it is natural to return to the weaker version motivated by the SWIC, namely to the form \( R(1) \leq 0 \). This is the present reformulated version, what we will call ‘Weak Conjecture’.

**Conjecture 2.3.1. (Weak Conjecture)**

\[
R(1) \leq 0, \quad \text{that is,} \quad p_g \geq -\text{sw}_{\text{can}}(M) - (K^2_{\text{can}} + s_{\tilde{X}})/8.
\]

Its analytic interpretation is the following. For several analytic structures of normal surface singularities (e.g. for rational, minimally elliptic, weighted homogeneous, splice quotient) \( R(1) = 0 \), that is, the geometric genus equals the topological invariant \(-\text{sw}_{\text{can}}(M) - (K^2_{\text{can}} + s_{\tilde{X}})/8\) of the link. The above conjecture predicts that for superisolated singularities, though \( p_g \) might be different than this topological invariant, it cannot be smaller. This, reinterpreted in terms of the projective curve \( C \) produces serious restrictions on the topology of local singularities and the degree \( d \).

### 2.4. The semigroup distribution property

As the origin of the semigroup distribution property goes back to the study of unicuspidal curves ([20, §3]), we start with this case as an example.

**Example 2.4.1. (The case \( \nu = 1 \))** In this case \( q_j = \#\{s \not\in \Gamma_1 : s > j\} \), cf. (2.2.5) or [59] §2. By the symmetry of \( \Gamma_1 \) (that is, \( s \in \Gamma_1 \) if and only if \( 2\delta - 1 - s \not\in \Gamma_1 \)) one also has

\[
q_{2\delta-2-k} = H_1(k + 1) \quad \text{for} \quad k = 0, \ldots, 2\delta - 2.
\]

Hence, the \( q \)-coefficient needed in (2.2.8) is \( q_{d-3-j} = \#\{s \in \Gamma_1 : s \leq jd\} = H_1(jd + 1) \).

Furthermore, the coefficients of \( R(t) \) from equation (2.2.8) can be reinterpreted geometrically by a very nice idea of Fernández de Bobadilla, Luengo, Melle-Hernández and Némethi using Bézout’s theorem as follows (for details see [20, Proposition 3.2.1]). The dimension of the vector space \( V \) of homogeneous polynomials \( h \) of degree \( j \) in three variables
is \((j + 1)(j + 2)/2\). Fix \(j < d\). The number of conditions for \(h \in V\) to have with \(C\) at \(P_1\) intersection multiplicity > \(jd\) is \(\#\{s \in \Gamma_1 : s \leq jd\}\). Hence, \(H_1(jd + 1) < (j + 1)(j + 2)/2\) would imply the existence of a curve with equation \(\{h = 0\}\), which would contradict Bézout’s theorem. Therefore, if \(C \subset \mathbb{CP}^2\) is a rational unicuspidal curve of degree \(d\), then the counting function \(H_1\) of the local topological type of its singularity for each \(j = 0, 1, \ldots, d-3\) satisfies

\[
(2.4.2) \quad q_{(d-3-j)d} = H_1(jd + 1) \geq \frac{(j + 1)(j + 2)}{2}.
\]

In particular, this inequality and (2.2.8) show that for \(\nu = 1\) the Original Conjecture 2.2.1 is equivalent to the vanishing of \(R(t)\). Furthermore, they are also equivalent with the Weak Conjecture 2.3.1 since if \(R(1) \leq 0\) then necessarily \(R(t) = 0\).

For arbitrary \(\nu\), in terms of our present notation, the above inequality (2.4.2) provided by Bézout theorem transforms into the following general form.

**Lemma 2.4.2** (Fernández de Bobadilla, Luengo, Melle-Hernández, Némethi, [20], Proposition 3.2.1). Let \(C \subset \mathbb{CP}^2\) be a rational cuspidal curve of degree \(d\) with \(\nu\) cusps. Then the counting functions \(H_i\) \((i = 1, \ldots, \nu)\) of the local singularities satisfy

\[
(2.4.3) \quad \min_{j_1+j_2+\cdots+j_\nu=jd+1} \{H_1(j_1) + H_2(j_2) + \cdots + H_\nu(j_\nu)\} \geq \frac{(j + 1)(j + 2)}{2}
\]

for each \(j = 0, 1, \ldots, d - 3\).

This inequality was improved by Borodzik and Livingston, proving in [11, Theorem 5.4] that in fact

\[
(2.4.4) \quad \min_{j_1+j_2+\cdots+j_\nu=jd+1} \{H_1(j_1) + H_2(j_2) + \cdots + H_\nu(j_\nu)\} = \frac{(j + 1)(j + 2)}{2}.
\]

To state it in a more compact form in the language of the infimal convolution, define function \(H\) by

\[
(2.4.5) \quad H = H_1 \circ \cdots \circ H_\nu,
\]

where \(H_1, \ldots, H_\nu\) are the semigroup counting functions of the local topological types of the singularities of a rational cuspidal curve.

**Theorem 2.4.3** (Borodzik, Livingston, [11, Theorem 6.5]). Let \(H\) be the infimal convolution of the semigroup counting functions of local cusp types on a rational cuspidal curve of degree \(d\). Then for every \(j = 0, \ldots, d - 3\) we have

\[
(2.4.6) \quad H(jd + 1) = \frac{(j + 1)(j + 2)}{2}.
\]
We will call this property the \textit{semigroup distribution property} (SDP), although it is more general than the corresponding statement for unicuspidal curves, where the terminology originates from. The term \textit{generalized SDP} will be preserved for the statement valid for curves of arbitrary genus, see Theorem 3.1.1.

The proof is based on some topological invariants, namely the so called \textit{correction terms} or \textit{d}-invariants (see \cite{65}) of the 3-manifold $Y$ which is the boundary of the tubular neighborhood of the curve $C \subset \mathbb{C}P^2$. The main idea is that the correction terms of $Y$ can be computed from the semigroups of the singularities when looking at $Y$ as the boundary of the tubular neighborhood of the curve $C$. On the other hand, $Y$ is also the boundary of the complement in $\mathbb{C}P^2$ of the tubular neighborhood, and this complement is a rational homology ball, because $C$ is rational. And since $Y$ then bounds a rational homology ball, certain correction terms must vanish. The details can be found in \cite{11}.

2.5. Unicuspidal curves

In \cite{21}, Fernández de Bobadilla, Luengo, Melle-Hernández and Némethi proved a nice classification result on possible local singularity types with torus knot link occurring on rational unicuspidal curves. Let $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}, n \geq 1$ be the Fibonacci numbers.

\textbf{Theorem 2.5.1} (Fernández de Bobadilla, Luengo, Melle-Hernández, Némethi, \cite{21}, Theorem 1.1]). \textit{A local plane curve singularity with one Newton pair $(a, b)$ can occur on a unicuspidal rational plane curve of degree $d$ if and only if the triple $(a, b, d)$ of integers is present on the following list:}

\begin{enumerate}
  \item (a) $(a, b) = (d - 1, d), d \geq 3$,
  \item (b) $(a, b) = (d/2, 2d - 1), d \geq 4$ even,
  \item (c) $(a, b) = (F_{j-2}^2, F_j^2), d = F_{j-1}^2 + 1, j \geq 5$ odd,
  \item (d) $(a, b) = (F_{j-2}, F_{j+2}), d = F_j, j \geq 5$ odd,
  \item (e) $(a, b) = (3, 22), d = 8$,
  \item (f) $(a, b) = (6, 43), d = 16$.
\end{enumerate}

We mention here that this will be our model theorem in Chapter 3 when studying higher genus curves under similar restrictions (cf. Theorem 3.1.7).

By \cite{2.4.6} if $H$ is the semigroup counting function of the only singularity on a unicuspidal curve, then

\begin{enumerate}
  \item (2.5.1) $H(jd + 1) = (j + 1)(j + 2)/2$
\end{enumerate}

for every $j = 0, 1, \ldots, d - 3$. 
Although the proof of Theorem 2.5.1 above in [21] does not use this semigroup distribution property in its full strength (it was not proved in full generality that time), T. Liu in [35] showed that one can obtain Theorem 2.5.1 relying on (2.5.1) only. We will also see this fact, cf. Remark 3.2.11 and Remark 3.2.18 in Chapter 3.

Now we want to discuss the case of unicuspidal curves whose singularity has two Newton pairs. Liu in [35] used (2.5.1) to give a list of possible Newton pairs, but he did not really discuss the realizability of Newton pairs on his list. We will show that he almost classified all possible types with two Newton pairs: all but two sporadic pairs on his list are realizable. This observation (the rest of this section) is published in a short article [5].

Recall that if a local plane curve singularity \( \{ f(x, y) = 0 \} \) has two Newton pairs \((p_1, q_1)(p_2, q_2)\), then by definition, after a local homeomorphism the singularity can be parametrized by

\[
    x(t) = t^{p_1}p_2, \quad y(t) = t^{q_1}p_2 + t^{q_1}p_2 + q_2.
\]

In this case we simply say that the singularity is of type \((p_1, q_1)(p_2, q_2)\), where \(\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1, p_1, p_2 \geq 2\) and, by convention, \(p_1 < q_1\). Sometimes we will allow \(p_1 = 1\), in such case we say that the Newton pair ‘degenerates’ to one Newton pair \((p_2, p_2q_1 + q_2)\).

We recall some results regarding the numerical invariants of plane curves in this special case, relying mostly on [19, 20]. In the case of a local singularity of type \((p_1, q_1)(p_2, q_2)\), the delta invariant can be expressed (see [15]) as

\[
    \delta = (p_1q_1p_2^2 + p_2q_2 - p_1p_2 - q_1p_2 - q_2 + 1)/2.
\]

Also, the additive semigroup \(\Gamma\) is generated (over \(\mathbb{Z}_{\geq 0}\)) by the following three elements \(g_0, g_1, g_2\) (see e.g. [20, §2.1]):

\[
    g_0 = p_1p_2, \quad g_1 = q_1p_2, \quad g_2 = p_1p_2q_1 + q_2.
\]

There are several invariants guiding the classification of projective plane curves. One can take the strict transform \(\overline{C}\) under the local embedded minimal good resolution \(X \to \mathbb{CP}^2\) of the singularities (only one singularity in our case) of \(C\). (That is, we blow up \(\mathbb{CP}^2\) several times until we resolve the singularities and obtain a normal crossing configuration: the exceptional divisors and the strict transform of the curve, which is smooth, intersect each other transversely and no three of them goes through the same point.) We denote by \(\overline{C}^2\) the self-intersection of this strict transform \(\overline{C}\) in \(X\).

For a unicuspidal rational curve with two Newton pairs (see [19, §4.3], [15, §17]):

\[
    \overline{C}^2 = d^2 - p_2q_2 - p_1q_1p_2^2.
\]
This, using the degree genus formula for rational unicuspidal curves (that is, \((d-1)(d-2) = 2\delta\)), turns into the following simpler expression:

\[
\overline{C^2} = 3d - 1 - p_1p_2 - q_1p_2 - q_2.
\]

Denote by \(\pi\) the logarithmic Kodaira dimension of \(\mathbb{C}P^2 \setminus C\) (see [28, 44]). This turns out to be an extremely important invariant of a plane curve. Based on [20, §1, (a), (b), (c)], recall the following. By the work [77] of Tsunoda, \(\pi \neq 0\), therefore, the possibilities are \(\pi \in \{\infty, 1, 2\}\). Notice that in the case of unicuspidal curves, by the results of Yoshihara [83], \(\pi = \infty\) is equivalent with \(\overline{C^2} \geq -1\). Recall that all rational curves with \(\pi = \infty\) are classified by Kashiwara in [29]; see also [41]. Also, rational unicuspidal curves with \(\pi = 1\) are classified by Tono in [72, 73]. This gives us a guiding principle where to look for the description of a unicuspidal curve with given numerical invariants.

Further facts and useful observations can be found in [20, 19].

**Remark 2.5.2.** We wish to emphasize that since in [20] the property (2.5.1) was checked for all rational cuspidal curves with \(\pi \leq 1\), all the possible local types of cusps on such curves are explicitly listed in [20]. Moreover, the only known possible local types with two Newton pairs on a rational unicuspidal curve with \(\pi = 2\) are those occurring on Orevkov’s curves from [62] ([vii] and [viii] in Theorem 2.5.3 below). For those, (2.5.1) was also checked in [20] Theorem 2 (c)]. In particular, all the local types from Theorem 2.5.3 are already explicitly listed by Fernández de Bobadilla, Luengo, Melle-Hernández and Némethi in [20] with the appropriate references to their original constructions.

Based on Liu’s list in [35], we give the following complete list of singularity types. Set \(F_0 = 0, F_1 = 1, F_{k+1} = F_k + F_{k-1}, k \geq 1\) for the Fibonacci numbers.

**Theorem 2.5.3 (Based on T. Liu, [35, Theorem 1.1]).** Let \(C\) be a rational unicuspidal curve of degree \(d\) whose singularity can be characterized by two Newton pairs. Then its type \((p_1, q_1)(p_2, q_2)\) is present in the list below. Conversely, for any member \((p_1, q_1)(p_2, q_2)\) of this list, there exists a rational unicuspidal curve \(C\) with this type of singularity.

(i) \((1F_{2k-1}^2 + F_{2k-3}^2, 1F_{2k+1}^2 + F_{2k-1}^2 + 2)(F_{2k-1}^2 + 1F_{2k-1}^2 + F_{2k-3}^2),\)

\[d = F_{2k-1}F_{2k+1}(1F_{2k-1}^2 + F_{2k-3}^2), k \geq 2, l \geq 0.\]

\((k = 2, l = 0\) degenerates to one Newton pair type.)

(ii) \((1F_{2k-1}^2 + F_{2k-3}^2, 1F_{2k+1}^2 + F_{2k-1}^2 + 2)(F_{2k-1}^2 + 1F_{2k-1}^2 + F_{2k-5}^2),\)

\[d = F_{2k+1}(1F_{2k-1}^2 + F_{2k-3}^2), k \geq 3, l \geq 0.\]

(iii) \((n - 1, n)(m, nm - 1), d = nm, n \geq 3, m \geq 2.\)

(iv) \((n, 4n - 1)(m, nm - 1), d = 2nm, n, m \geq 2.\)

(v) \((n - 1, n)(n, (n + 1)^2), d = n^2 + 1, n \geq 3.\)
2.6. CURVES WITH MORECUSPS

(vi) \((n, 4n + 1)\)\((4n + 1, (2n + 1)^2)\), \(d = 8n^2 + 4n + 1, n \geq 2\).

(vii) \((F_{4k}/3, F_{4k+4}/3)(3, 1)\), \(d = F_{4k+2}, k \geq 2\).

(viii) \((F_{4k}/3, F_{4k+4}/3)(6, 1)\), \(d = 2F_{4k+2}, k \geq 2\).

Proof. First we show that being on the above list is necessary for the realizability.

Assume that \((p_1, q_1)(p_2, q_2)\) is the local cusp type of a unicuspidal rational curve of degree \(d\). Then by [11, Theorem 1.1] \((2.5.1)\) must hold. In this case, by [35, Theorem 1.1], the Newton pair either equals to \((2, 7)(4, 17)\) with \(d = 17\) or to \((2, 3)(6, 31)\) with \(d = 20\), or it is present in the above list. \((2, 7)(4, 17)\) is excluded in [19, §6.10] by the spectrum semicontinuity (SS) criterion (SS fails at \(l = 12\) with the notations therein). The other pair \((2, 3)(6, 31)\) can be excluded similarly, now SS fails at \(l = 13\).

Now we show that any element of the above list is realizable.

In case (i) and (ii) one computes that \(C^2 = 0\) and \(-1\), respectively. Therefore, in both cases, \(\kappa = -\infty\). In [20], based on [29] (cf. also [41]), a list of numerical invariants was presented for these curves. One finds that curves from (i) and (ii) are exactly those listed in [20, §6.2.1, §6.2.2 with \(N = 1\)], respectively.

The existence of the case (iii), as also Liu notices, follows from [18, Theorem 1.1, 1a] \((C^2 = m, \kappa = -\infty)\), cf. also [20, §4, (9)].

The existence of (iv) is proved in [72, Theorem 1.1, (iv)] (set \(k_1 = 1, d_1 = nm, k_2 = n - 1, d_2 = m, k_3 = m - 1\) to match the two notations). These are of Abhyankar–Moh type (cf. again [20, §4]), that is, the tangent line to its singular point has no other intersection with the curve; as it is easily seen by comparing the semigroup generators \((2.5.2)\) with the degree of the curves and then using Bézout’s theorem. We have \(C^2 = m\) and \(\kappa = -\infty\).

For the existence of (v) (with \(C^2 = -n + 1\)) and (vi) (with \(C^2 = -n\)), see [73, Theorem 2, (i), Type I, \(s = 2\) and (ii), Type II] (cf. also [20, §7.1]). In these cases, \(\kappa = 1\).

The existence of (vii) and (viii) follows from [62, Theorem C, b), c)], respectively (cf. [20, §9.1, §9.2]). In these cases, one has \(C^2 = -2\) and \(\kappa = 2\). (Notice that Orevkov uses characteristic sequences rather than the Newton pairs; for the definition and comparison, see [62, §3]).

\(\Box\)

2.6. Curves with more cusps

As we will discuss later in Section 2.9, the semigroup distribution property (Theorem 2.4.3) is less restrictive when the number of cusps is higher than one. In this section, we do not want to discuss any restrictions on numerical invariants in the case of curves with
more than one cusp. Nevertheless, we wish to present some results on what kinds of bicuspidal curves exist and describe the construction by birational (Cremona) transformations for some of them.

**Proposition 2.6.1** (Partially based on Borodzik–Zenądek, [12]). The following rational bicuspidal curves exist (the singularity types are given by multiplicity sequences):

(a1) A curve of degree \(d = u(l - 1)m + m - u + 1\) (\(u \geq 2, l \geq 2, m \geq 2\)) with the following cusp types:
\[
((u - 1)(l - 1)m + m - u + 1, (m(l - 1) - 1)_{u-1}, m_{l-2}, m - 1],
\]
\[
([(l - 1)m]_{u}, m_{l-1}].
\]

(a2) A curve of degree \(d = ulm + m + 1\) (\(u \geq 2, l \geq 1, m \geq 2\)) with the following cusp types:
\[
[(lm + 1)_{u}, m_{l}],
\]
\[
[(u - 1)ml + m, (lm)_{u-1}, m_{l}].
\]

(a3) A curve of degree \(d = ulm + um + 1\) (\(u \geq 2, l \geq 1, m \geq 2\)) with the following cusp types:
\[
[((l + 1)m + 1)_{u-1}, lm + 1, m_{l}],
\]
\[
[(u - 1)(l + 1)m, ((l + 1)m)_{u-1}, m_{l+1}].
\]

(a4) A curve of degree \(d = ulm - u + 1\) (\(u \geq 2, l \geq 2, m \geq 2\)) with the following cusp types:
\[
[(u - 1)(lm - 1), (lm - 1)_{u-1}, m_{l-1}, m - 1],
\]
\[
[(lm)_{u-1}, (l - 1)m, m_{l-1}].
\]

(b) A curve of degree \(d = 2k + 1, k \geq 2\) with the following cusp types:
\[
[k_{1}]\text{ and } [2k].
\]

(c) A curve of degree \(d = 4k + 1, k \geq 1\) with the following cusp types:
\[
[(2k)_{3}, 2k]\text{ and } [(2k), 2k].
\]

(d1) A curve \(C_{k}\) of degree \(d = 8k + 2, k \geq 1\) with the following cusp types:
\[
[4k + 2, 4k - 2, 4k_{-1}, 2_{2}]\text{ and } [(4k)_{2}, 4k].
\]

(d2) A curve \(D_{k}\) of degree \(d = 8k + 6, k \geq 1\) with the following cusp types:
\[
[4k + 4, 4k, 4k]\text{ and } [(4k + 2)_{2}, 4k, 2_{2}].
\]

(e) A curve of degree \(d = 3k + 4, k \geq 1\) with the following cusp types:
\[
[3k, 3k]\text{ and } [4k, 2_{3}].
\]

(f) A curve of degree \(d = 14\) with cusp types [8, 4, 4, 2, 2] and [6, 6, 3, 3].

**Proof.** For curves under (a1), consider the following construction. Let \(C_{l,m} := \{(y^{l-1}z - x^{l})^{m} - x^{lm-1}y = 0\} \subset \mathbb{CP}^{2}\) for any two integers \(l \geq 2, m \geq 2\). This is a bicuspidal rational projective plane curve. Its degree is \(d = lm\) and the self-intersection of
the minimal good resolution is $C^2 = 0$. The two singularities are at points $p = [0:1:0]$ and $q = [0:0:1]$.

At $p$ we have a singularity with one Newton (or Puiseux) pair $(m, lm - 1)$, or, with multiplicity sequence $[m, lm - 1, m - 1]$. We will need the local intersection multiplicity with its tangent as well, it is $(T_pC_{l,m} \cdot C_{l,m})_p = lm - 1$.

At $q$ we have a singularity with two Newton pairs $(l - 1, l)(m, 1)$, or, with multiplicity sequence $[(l - 1)m, ml - 1]$.

In what follows, for the sake of simplicity, when considering blow-ups and blow-downs, we will use the same notation for curves and divisors and their strict transforms. We perform a quadratic Cremona transformation with two proper basepoints (see [42 §5.3.4]): one is the transverse intersection point of $C_{l,m}$ and $T_pC_{l,m}$, the other point is $p$. The third (non-proper) basepoint is the one infinitely near to $p$, and lying at the intersection of the exceptional divisor of the blow-up at $p$ and the strict transform under this blow-up of the line $pq$.

More concretely, the following is happening. Denote by $L_1$ the tangent $T_pC_{l,m}$ and by $L_2$ the line $pq$. Blow up at $p$ obtaining an exceptional divisor $E$. Then blow up at the intersection point of (the strict transforms of) the line $L_1$ and the curve $C_{l,m}$ different from the singular point of the curve, obtaining exceptional divisor $L'_1$ and blow up at $E \cap L_2$, obtaining exceptional divisor $L'_2$ (see Figure 2.1). Notice that this configuration can be
blown down in a different way: first blow down $L_1$ and $L_2$, then $E$. The result is an other bicuspidal curve with longer multiplicity sequences at the cusps.

We repeat similar quadratic Cremona transformation with two proper basepoints. We get a three-parameter series ($u \geq 2, l \geq 1, m \geq 2$) of bicuspidal rational curves, with two cusps of topological type as described in the proposition (one quadratic Cremona transformation described above increases $u$ by 1).

For curves under (a2), consider the following construction. Let $C_{l,m} := \{x(yx^l + z^{l+1})^m - z^{(l+1)m+1} = 0\} \subset \mathbb{CP}^2$ for any two integers $l \geq 1, m \geq 2$. This is a bicuspidal rational projective plane curve. Its degree is $d = lm + m + 1$ and the self-intersection of the minimal good resolution is $C^2 = 0$. The two singularities are at points $p = [0 : 1 : 0]$ and $q = [1 : 0 : 0]$.

At $p$ we have a singularity with one Newton (or Puiseux) pair $(lm+1, (l+1)m+1)$, or, with multiplicity sequence $[lm+1, ml]$. We will need the intersection multiplicity with its tangent as well, it is $(T_p C_{l,m} \cdot C_{l,m})_p = (l+1)m$.

At $q$ we have a singularity with one Newton (or Puiseux) pair $(m, (l+1)m+1)$, or, with multiplicity sequence $[ml+1]$. We will need the intersection multiplicity with its tangent as well, it is $(T_q C_{l,m} \cdot C_{l,m})_q = (l+1)m$.

For curves under (a3), consider the following construction. Let $C_{l,m} := \{x(yx^l + z^{l+1})^m - z^{(l+1)m+1} = 0\} \subset \mathbb{CP}^2$ for any two integers $l \geq 1, m \geq 2$. This is a bicuspidal rational projective plane curve. Its degree is $d = lm + m + 1$ and the self-intersection of the minimal good resolution is $C^2 = 0$. The two singularities are at points $p = [0 : 1 : 0]$ and $q = [1 : 0 : 0]$.

At $p$ we have a singularity with one Newton (or Puiseux) pair $(lm+1, (l+1)m+1)$, or, with multiplicity sequence $[lm+1, ml]$. We will need the intersection multiplicity with its tangent as well, it is $(T_p C_{l,m} \cdot C_{l,m})_p = (l+1)m$.

At $q$ we have a singularity with one Newton (or Puiseux) pair $(m, (l+1)m+1)$, or, with multiplicity sequence $[ml+1]$. We will need the intersection multiplicity with its tangent as well, it is $(T_q C_{l,m} \cdot C_{l,m})_q = (l+1)m$. 
2.6. CURVES WITH MORE CUSPS

We perform a quadratic Cremona transformation with two proper basepoints: one is the smooth transverse intersection point of $C_{l,m}$ and the tangent line $L_q = T_q C_{l,m}$, the other point is the intersection point $L_q \cap L_p$ of the tangents at singularities ($L_p = T_p C_{l,m}$). The third (non-proper) basepoint is the one infinitely near to $L_p \cap L_q$, and lying at the intersection of the exceptional divisor of the blow-up at $L_p \cap L_q$ and the strict transform under this blow-up of the line $L_p$.

We repeat similar quadratic Cremona transformation with two proper basepoints. We get a three-parameter series ($u \geq 2, l \geq 1, m \geq 2$) of bicuspidal rational curves, with two cusps of topological type as described in the proposition (one quadratic Cremona transformation described above increases $u$ by 1).

For curves under (a4), consider the following construction. Let $C_{l,m} := \{(y^{l-1}z - x^m - x^{m-1}y = 0) \subset \mathbb{CP}^2$ for any two integers $l \geq 2, m \geq 2$. This is a bicuspidal rational projective plane curve. Its degree is $d = lm$ and the self-intersection of the minimal good resolution is $C^2 = 0$. The two singularities are at points $p = [0 : 1 : 0]$ and $q = [0 : 0 : 1]$.

At $p$ we have a singularity with one Newton (or Puiseux) pair $(m, lm - 1)$, or, with multiplicity sequence $[m_{l-1}, m - 1]$. We will need the intersection multiplicity with its tangent as well, it is $(T_p C_{l,m} \cdot C_{l,m})_p = lm - 1$.

At $q$ we have a singularity with two Newton pairs $(l - 1, l)(m, 1)$, or, with multiplicity sequence $[(l - 1)m, m_{l-1}]$. We will need the intersection multiplicity with its tangent as well, it is $(T_q C_{l,m} \cdot C_{l,m})_q = lm$.

We perform a quadratic Cremona transformation with two proper basepoints: one is the transverse intersection point of $C_{l,m}$ and the tangent line $L_p = T_p C_{l,m}$; the other point is the intersection point $L_p \cap L_q$ of the tangents at singularities ($L_q = T_q C_{l,m}$). The third (non-proper) basepoint is the one infinitely near to $L_p \cap L_q$, and lying at the intersection of the exceptional divisor of the blow-up at $L_p \cap L_q$ and the strict transform under this blow-up of the line $L_q$.

We repeat similar quadratic Cremona transformation with two proper basepoints. We get a three-parameter series ($u \geq 2, l \geq 2, m \geq 2$) of bicuspidal rational curves, with two cusps of topological type as described in the proposition (one quadratic Cremona transformation described above increases $u$ by 1).

For curves under (b), take a configuration of two conics $C_1$ and $C_2$ and a line $L$ as follows: $C_1$ and $C_2$ has only one intersection point $p = C_1 \cap C_2$ of local intersection multiplicity 4; $C_1$ and $L$ has also one intersection point only, namely $q = C_1 \cap L$ which is a touching point with local intersection multiplicity 2; further, $\{r, s\} = C_2 \cap L$ are two distinct transverse intersection points.
Such a configuration exists, consider for example the set of curves given by equations \(\{x^2 + y^2 - yz = 0\}, \{x^2 + y^2 - yz = 0\}, \{y - z = 0\}\) in \(\mathbb{CP}^2\) equipped with homogeneous coordinates \([x : y : z]\).

Again, for simplicity, we will use the same notation for curves, respectively divisors and their strict transforms. Blow up \(p\) obtaining an exceptional divisor \(E_1\), then blow up the intersection point of (the strict transforms of) \(C_1\) and \(C_2\) three more times, obtaining exceptional divisors \(E_2, E_3, E_4\) in this order. Then blow up the point \(C_1 \cap E_4\) resulting in exceptional divisor \(C_1'\) and blow up \(s\), one of the intersection points of \(L\) and \(C_2\), resulting in exceptional divisor \(C_2'\).

Notice that one can now blow down \(C_1, C_2, E_4, E_3, E_2\) and \(E_1\) in this order, then the strict transform of \(L\) will be a unicuspidal rational curve with cusp type \([2]\) at a point to be called \(q\) from now on, \(C_1'\) a conic going through the cusp at \(q\) and touching \(L\) at a point to be called \(r\) from now on with local intersection multiplicity 4, \(C_2'\) a conic going through \(r\) having local intersection multiplicity 5 with \(L\) at \(r\) and a transverse intersection point with \(L\) at some other, smooth point which we will call \(s\).

Now one can repeat a similar process: rename \(C_1'\) to \(C_1\), \(C_2'\) to \(C_2\), blow up four times the intersection point of (the strict transforms of) \(C_1\) and \(C_2\) (keeping calling it \(r\) at each step) resulting in exceptional divisors \(E_1, E_2, E_3, E_4\) in this order, then blow up at \(C_1 \cap E_4\) obtaining \(C_1''\). Now further blow up at the intersection point of \(L\) and \(C_2\) not lying on \(E_4\).

\[\text{Figure 2.2. The configuration of divisors after blowing up and before blowing down at the intermediate step of constructing curves of type as listed in (b).}\]
After this one has the following configuration (see Figure 2.2) for $k = 1$ (in a complex surface obtained from $\mathbb{CP}^2$ by the blow-ups): A curve $L$ with one cusp of type $[2k]$ at a point to be called $q$, a divisor $C_1$ going through $q$ (having intersection multiplicity 2 with $L$) and intersecting $C_1'$, $C_1''$ intersecting $E_4$, $E_4$ touching $L$ at a point to be called $r$ with local intersection multiplicity $k$, $C_2$ going through $r$ and intersecting $C_2'$, $C_2''$ intersecting $L$ in a point to be called $s$, $E_i$ intersecting $E_{i+1}$ for $i = 1, 2, 3$. Divisors $C_1, C_1', C_2, C_2'$ have self-intersection $-1$, $E_i$, $i = 1, 2, 3, 4$ have self-intersection $-2$.

Notice that this configuration can be blown down to get $\mathbb{CP}^2$ in two different ways: blowing down $C_1', C_2', E_4, E_3, E_2, E_1$ in this order or blowing down $C_1, C_2, E_4, E_3, E_2, E_1$ in this order. In the first case, the strict transform of $L$ is a bicuspidal curve with cusp types $[k_4]$ (smooth for the degenerating starting case $k = 1$) and $[2k]$, in the second case it is a bicuspidal curve with cusp types $[(k + 1)_4]$, $[2k+1]$.

This, together with the construction of the starting case $k = 1$ above, gives an inductive construction of curves claimed in (b).

For curves under (c), consider the following configuration of two conics $C_1$ and $C_2$: 

\[ \{p, q\} = C_1 \cap C_2 \] such that $p$ is a point with local intersection multiplicity 3 and $q$ is a transverse intersection point, $L$ is a line tangent to $C_1$ at point $r$ and tangent to $C_2$ at point $s$.

Such a configuration exists, consider for example the set of curves given by equations

\[ \{x^2 + y^2 + xy - yz = 0\}, \{x^2 + y^2 - xy - yz = 0\}, \{3y - 4z = 0\}. \]

Now blow up three times the intersection point of (the strict transforms of) $C_1$ and $C_2$ distinct from $q$, resulting in exceptional divisors $E_1, E_2, E_3$, in this order. Blow up at point $q$ as well, obtaining the exceptional divisor $E_0$.

Blow up $E_3 \cap C_1$, obtaining $C_1'$ and $E_0 \cap C_2$, obtaining $C_2'$. Notice that one can blow down $C_1, C_2, E_0, E_3, E_2, E_1$ in this order to obtain a curve with 2 cusps of type $[2_4]$ at point called $p$ and $[2_2]$ at point called $q$, respectively, such that $C_1'$ and $C_2'$ are both going through the cusps at $p$ and $q$ and are having no further intersection points with the bicuspidal curve (the strict transform of $L$). $C_1'$ with $L$ has local intersection multiplicity 6 at $p$ and 4 at $q$; $C_2'$ with $L$ has local intersection multiplicity 8 at $p$ and 2 at $q$.

Now rename $C_1'$ to $C_1$ and $C_2'$ to $C_2$ and repeat a similar process: blow up at the intersection point of (strict transforms of) $C_1$ and $C_2$ distinct from $q$ three times, obtaining $E_1, E_2, E_3$, respectively, then blow up at $q$ obtaining $E_0$. Blow up $C_1 \cap E_3$ obtaining $C_1''$ and $C_2 \cap E_0$ obtaining $C_2''$. Notice that this leads to a configuration (see Figure 2.3) with $k = 1$ as follows: $E_i$ is intersecting $E_{i+1}$ for $i = 1, 2, 3$ is intersecting $E_2, C_1'$ and $C_2, E_0$ is intersecting $C_1$ and $C_2'$, $C_i$ is intersecting $C_i'$ for $i = 1, 2$. $L$ has a cusp of type $[2_k]$ at $C_2 \cap E_3$, $L$ having local intersection multiplicity $2k$ with $E_3$ and 2 with $C_2$ at that point;
has a cusp type $[2_k]$ at $C_1 \cap E_0$ as well, $L$ having local intersection multiplicity $2k$ with $E_0$ and $2$ with $C_1$ at that point.

Notice that one can blow down this configuration in two different ways to obtain $\mathbb{CP}^2$ as ambient space: $C_1, C_2, E_0, E_3, E_2, E_1$ in this order, or $C_1, C_2, E_0, E_3, E_2, E_1$ in this order. The first option takes $L$ to a bicuspidal curve with cusp types $[(2k)_3, 2k]$ and $[(2k), 2k]$, the second takes $L$ to a bicuspidal curve with cusp types $[(2k + 2)_3, 2k+1]$ and $[(2k + 2), 2k+1]$. In this way, this construction together with obtaining the starting case $k = 1$ as above, leads to an inductive construction of curves as claimed in (c).

For curves under (d1) and (d2), one can check that these are described in [12, Main Theorem, (s)], where a parametrization is given for them.

As mentioned in [12], according to M. Koras, this series was first constructed by Pierrette Cassou-Noguès. One can obtain these curves recursively by birational transformation as follows. We just sketch the construction. Start with a conic $C$ and a line $L$ in $\mathbb{CP}^2$ intersecting in two distinct points $p$ and $q$. Blow up at point $p$ to obtain $E_1$, then blow up at point $E_i \cap C$ to obtain $E_{i+1}$ for $i = 1, 2, 3$, then blow up at $L \cap C$ to obtain $F$. This configuration can be blown down in another way to get $\mathbb{CP}^2$ as ambient space again: first
blow down $C$, then $L$, then $E_1, E_2, E_3$ in this order. The strict transform of $E_4$ will be a line to be renamed to $L$ and the strict transform of $F$ will be a conic to be renamed to $C$. This is the birational transformation to obtain $D_k$ from $C_k$ and $C_{k+1}$ from $D_k$. The starting configuration is a conic $C'$ touching $C$ at a point $r \neq p, q$ with local intersection multiplicity 4 and $L$ at a point $s \neq p, q$ with local intersection multiplicity 2.

Curves under (e) correspond to those given in [12, Main Theorem, (i)]. To give a series of birational transformations producing them, consider the following. Equation $y^4 - 2xy^2z + x^2z^2 - yz^3 = 0$ determines a rational unicuspidal curve of degree 4 and cusp type $[2_3]$ (cf. [43, §3.2, curve $C_3$]). Let $p$ be a smooth inflection point on the curve and $L_1$ be the tangent to the curve at $p$, with local intersection multiplicity 3. Let $r$ be the other intersection point of the curve and $L_1$. Let $L_2$ be the line having local intersection multiplicity 4 with the curve at its cusp at point called $q$. $L_2$ has no other intersection point with the curve. Let $s$ be the intersection point of $L_1$ and $L_2$. Blow up $s$ to produce exceptional divisor $E$, then blow up $r$ to produce $L'_1$ and $E \cap L_2$ to produce $L'_2$. Now blow down $L_1$ and $L_2$, then $E$. The strict transform of the curve will be a bicuspidal curve as described in (e) for $k = 1$. Rename $L'_1$ to $L_1$ and $L'_2$ to $L_2$ and repeat a similar process, that is, perform a quadratic Cremona transformation with two proper basepoints: one proper basepoint being the point at the cusp of type $[3k, 3k]$, the other proper basepoint being the transverse intersection point of $L_1$ and the curve and the third, non-proper basepoint being the one infinitely near to the first basepoint and lying at the intersection of (the strict transform of) $L_2$ and the exceptional divisor produced by the blow-up of the first cusp (the one of type $[3k, 3k]$). After this process, we get a curve with cusp types $[3(k+1), 3k+1]$ and $[4k+1, 2_3]$ and with local intersection multiplicities with two lines as needed to continue the induction.

For the curve in (f), a parametrization (by $[t : s] \in \mathbb{CP}^1$) is given in [12, Main Theorem, (t)]:

$$x = 3t^{14} + 3st^{13} + 2s^2t^{12}, \quad y = 3s^{12}t^2 - 3s^{13}t + s^{14}, \quad z = 3s^6t^8.$$ 

□

Remark 2.6.2. From curves under (a1), taking $u = 2$, we get Tono’s first bicuspidal 2-parameter series with $C^2 = -1$ ($a_{\text{Tono}} = l - 1$, [75, Theorem 2, No. 1]). Taking $l = 2$, we get Fenske’s 6th series ($a_{\text{Fenske}} = u, d_{\text{Fenske}} = m - 1$, [17, Theorem 1.1, 6]).

From curves under (a2), taking $u = 2$ we get Tono’s second bicuspidal 2-parameter series with $C^2 = -1$ ($a_{\text{Tono}} = l$, [75, Theorem 2, No. 2]). For $l = 1$, we get Fenske’s 5th series ($d_{\text{Fenske}} = m, a_{\text{Fenske}} = u$, [17, Theorem 1.1, 5]).
From curves under (a3), taking \( u = 2 \) we get Tono's third bicuspidal 2-parameter series with \( C^2 = -1 \) (\( a_{\text{Tono}} = l + 1, [75] \) Theorem 2, No. 3). For \( l = 0 \), we get Fenske’s 4th series (\( d_{\text{Fenske}} = m, a_{\text{Fenske}} = u - 1, [17] \) Theorem 1.1, 4). From curves under (a4), taking \( u = 2 \) we get Tono’s 4th bicuspidal 2-parameter series with \( C^2 = -1 \) (\( a_{\text{Tono}} = l - 1, [75] \) Theorem 2, No. 4). In Fenske’s terminology [18], a cuspidal curve is of type \((d,d-k)\) if \( \deg(C) = d \) and the maximal multiplicity of the cusps is \( d-k \). Flenner, Zaidenberg and Fenske classified all rational cuspidal curves with \( k = 2,3 \), and assuming the rigidity conjecture, also for \( k = 4 \), see [17], [18], [22], [23]. For this series (a4), \( k = \min\{lm, (u-1)(lm-1)\} \).

Curves under (a1)-(a4) should also be compared with [12] Main Theorem, (c)-(f).

The list of bicuspidal curves in Proposition 2.6.1 above is not intended to be complete by any means. On the other hand, it is conjectured (see e.g. [70]) that we already know all the cuspidal curves with more than two cusps.

The currently known rational cuspidal curves with more than two cusps are the following. We present the cusp types by multiplicity sequences and Newton pairs.

**Proposition 2.6.3 (Flenner, Zaidenberg and Fenske; see [22] §3.5, [23] §1.1, [18], cf. also [43] 2.4.5, [70].) The following rational cuspidal curves exist:**

(1) A curve \( C_{d,u} \) (with \( d \geq 4 \) and \( 1 \leq u \leq d-3 \)) is of degree \( d \) and has the following cusp types:

- \([d-2] \), alternatively \((d-2,d-1)\)
- \([2_u] \), alternatively \((2,2u+1)\)
- \([2_{d-2-u}] \), alternatively \((2,2d-2u-3)\).

(Here it is enough to take \( u \leq \left\lfloor \frac{d-2}{2} \right\rfloor \) or \( \left\lceil \frac{d-2}{2} \right\rceil \leq u \), as \( C_{d,u} = C_{d,d-2-u} \).)

(2) A curve \( D_l \) (with \( l \geq 1 \)) is of degree \( d = 2l + 3 \) and has the following cusp types:

- \([2l,2l] \), alternatively \((l,l+1)(2,1)\) (case \( l = 1 \) degenerates to one Newton pair)
- \([3l] \), alternatively \((3,3l+1)\)
- \([2] \), alternatively \((2,3)\).

(3) A curve \( E_l \) (with \( l \geq 1 \)) is of degree \( d = 3l + 4 \) and has the following cusp types:

- \([3l,3l] \), alternatively \((l,l+1)(3,1)\) (case \( l = 1 \) degenerates to one Newton pair)
- \([4l,2_2] \), alternatively \((2,2l+1)(2,1)\)
- \([2] \), alternatively \((2,3)\).

(4) A tricuspidal curve of degree \( d = 5 \) with cusp types \([2_2], [2_2], [2_2] \).

(5) A curve of degree \( d = 5 \) with four cusps of type \([2_3], [2], [2], [2] \).
2.7. Settling the original conjecture

The material of this section is mostly taken from a joint work with András Némethi [7, §2, §3]. I am grateful to him for his work, as well as to the referees of [7] for their comments and suggestions which were very helpful to improve the overall exposition.

Now we return to the comparison of the Original Conjecture and the semigroup distribution property.

It is clear that for $\nu = 1$ (when $H = H_1$) Theorem 2.4.3 implies the Original Conjecture 2.2.1 (see Example 2.4.1), hence the Weak Conjecture 2.3.1 as well. By Lemma 2.4.2, in fact, the statements of the Original Conjecture 2.2.1 and Theorem 2.4.3 are equivalent, provided that $\nu = 1$.

In the last point of [11, Remark 5.5] the authors ask about the relation of the two statements for $\nu \geq 2$. In this section, we will completely clarify this relation. Namely, we will prove the following:

**Theorem 2.7.1.** Consider $\nu$ topological types of plane curve singularities and set $\delta$ for the sum of their delta invariants. Let $H$ be the infimal convolution of the semigroup counting functions of the given singularities defined as in (2.4.5) and $q_k$ be the coefficients of the polynomial obtained from the Alexander polynomials of the given singularities defined as in Formula (2.2.4). Then the following assertions hold:

1. If $\nu = 2$, then $q_{2\delta-2-k} \leq H(k+1), k = 0, 1, \ldots, 2\delta-2$. In particular, for bicuspidal curves Theorem 2.4.3 implies the Original Conjecture 2.2.1.
2. If $\nu \geq 3$, then the above inequality does not hold in general, not even for $k = jd$ ($j = 0, 1, \ldots, d-3$), where $d$ is the degree of a cuspidal curve with $\nu$ cusps of the given local singularity types.

Notice that (1) in the above theorem is true regardless of the realizability of the given topological singularity types on a rational cuspidal curve, cf. Subsection 2.7.6.

First of all, we state the following symmetry property of $H$, an analogue of (2.2.6).

**Lemma 2.7.2.** $H(2\delta - 2 - j + 1) = H(j + 1) - j - 1 + \delta$ for every $j \in \mathbb{Z}$.

**Proof.** By the symmetry (1.1.5) of each semigroup one gets for each counting function $H_i(j_i) = H_i(2\delta_i - j_i) + j_i - \delta_i$ for any $j_i \in \mathbb{Z}$. Then use the definition of $H$. \qed

**2.7.1. Reformulation of the Original Conjecture.** Conjecture 2.2.1 and the coefficients in equation (2.2.8) resemble the identity (2.4.6). Let us emphasize the difference.

We can start in both cases with the counting functions $H_i$. In the Borodzik–Livingston theorem one has to take the infimal convolution $H = H_1 \diamond H_2 \diamond \cdots \diamond H_\nu$ and (2.4.6) says that $H(jd+1)$ equals $(j+1)(j+2)/2$ under the assumption of realizability.
On the other hand, in the Original Conjecture \[2.2.1\] first one determines \( \Delta_i \) from \( H_i \) by \[2.2.1\] and \[2.2.2\]. Then one takes the product of all \( \Delta_i \), and finally one takes the coefficients of \( Q \) as in Formula \[2.2.4\], which are compared with number of form 

\[(j + 1)(j + 2)/2.\]

Next, we make explicit these last steps and provide the combinatorial formula for \( q_j \).

Define sequences \( \{h_j^{(i)}\}_{j=0}^{\infty} \) by \( h_j^{(i)} := \delta H_i(j + 1) \) (notice the shift by one). For any sequence \( a = \{a_j\}_{j=0}^{\infty} \) denote by \( \partial a \) its difference sequence, i.e. \( (\partial a)_j = a_j - a_{j-1} \) \( j \geq 0 \) with the convention that the ‘\((-1)\)st element’ of a sequence is always zero, i.e. \( a_{-1} = 0 \). Similarly, we will denote by \( \Sigma a \) the sequence of partial sums, i.e. \( (\Sigma a)_j = a_0 + \cdots + a_j \).

Of course, \( \Sigma \partial a = a \) and \( \partial \Sigma a = a \) for any sequence \( a \).

By \[2.2.1\] and \[2.2.2\], the coefficient \( c_j^{(i)} \) of \( t^j \) in \( \Delta_i(t) \) can be written as \( c_j^{(i)} = (\partial \partial h^{(i)})_j \).

The coefficient sequence of a polynomial product is the usual convolution of coefficient sequences of the factors. Hence, the coefficient \( c_j \) of \( t^j \) in \( \Delta(t) \) is

\[ c_j = \sum_{j_1 + \cdots + j_\nu = j} c_{j_1}^{(1)} \cdots c_{j_\nu}^{(\nu)} . \]

Denoting the convolution of two sequences \( a = \{a_j\}_{j=0}^{\infty} \) and \( b = \{b_j\}_{j=0}^{\infty} \) by \( a \ast b \), i.e. \( (a \ast b)_j = \sum_{k=0}^{j} a_k b_{j-k} \), we get \( c_j = (\partial \partial h^{(1)} \ast \cdots \ast \partial \partial h^{(\nu)})_j \). Let us define:

\[ F(j) := (\Sigma \Sigma (\partial \partial h^{(1)} \ast \cdots \ast \partial \partial h^{(\nu)}))_j . \]

If \( A(t) = \sum_j a_j t^j \) and \( B(t) = \sum_j b_j t^j \) satisfy \( A(t) = A(1) + (t - 1)B(t) \), then \( (\Sigma a)_j = A(1) - b_j \). This applied twice for \( \Delta \) gives \( (\Sigma \Sigma c)_j = j + 1 - \delta + q_j \). Hence, the definition of \( Q \) and \[2.2.6\] provides

\[ q_{2\delta - 2 - j} = (\Sigma \Sigma (\partial \partial h^{(1)} \ast \cdots \ast \partial \partial h^{(\nu)}))_j = F(j) \quad \text{for} \quad 0 \leq j \leq 2\delta - 2 . \]

**Example 2.7.3.** Let us exemplify these steps. Consider three singularities given by one Newton pair each: \((3, 4)\), \((2, 5)\) and \((2, 3)\), respectively. The sum of delta invariants is \( \delta = \delta_1 + \delta_2 + \delta_3 = 3 + 2 + 1 = 6 \). The Alexander polynomials are

\[ \Delta_1(t) = \frac{(t - 1)(t^{12} - 1)}{(t^3 - 1)(t^4 - 1)} = 1 - t + t^3 - t^5 + t^6 , \]

\[ \Delta_2(t) = \frac{(t - 1)(t^{10} - 1)}{(t^2 - 1)(t^5 - 1)} = 1 - t + t^2 - t^3 + t^4 , \quad \Delta_3(t) = \frac{(t - 1)(t^6 - 1)}{(t^2 - 1)(t^3 - 1)} = 1 - t + t^2 . \]

Consequently,

\[ \Delta(t) = \Delta_1(t) \Delta_2(t) \Delta_3(t) = 1 + 6(t - 1) + (t - 1)^2(6 + 3t + 5t^2 + 2t^3 + 3t^4 + t^5 + 2t^6 + 2t^8 - t^9 + t^{10}) , \]

that is,

\[ Q(t) = 6 + 3t + 5t^2 + 2t^3 + 3t^4 + t^5 + 2t^6 + 2t^8 - t^9 + t^{10} . \]
Recall that the semigroups are generated over $\mathbb{Z}_{\geq 0}$ by the two elements of the corresponding Newton pairs. That is, the values $H_i(k)$ of the semigroup counting functions for $k = 0, 1, 2, \ldots$ are $0, 1, 1, 2, 3, 3, 4, 5, \ldots$; $0, 1, 1, 2, 2, 3, 4, \ldots$ and $0, 1, 1, 2, 3, \ldots$, respectively ($i = 1, 2, 3$).

\[
\begin{array}{c|cccc}
 i & 1 & 2 & 3 & 4 \\
 \hline
 k & 0 & 1 & 2 & 3 \\
 H_i(k) & 0 & 1 & 1 & 2 \\
\end{array}
\]

The sequences $h^{(i)}$ and their difference sequences are as follows.

\[
\begin{array}{c|cccc}
 i & 1 & 2 & 3 & 4 \\
 \hline
 h^{(i)} & 1 & 0 & 1 & 1 \\
 \partial h^{(i)} & 1 & 0 & 1 & 1 \\
 \partial \partial h^{(i)} & 1 & -1 & 0 & 1 \\
\end{array}
\]

Notice that one can read off the coefficients of the corresponding Alexander polynomials from the last row. Then one can compute the convolution $p = \partial \partial h^{(1)} \ast \partial \partial h^{(2)} \ast \partial \partial h^{(3)}$, then twice the sequence of partial sums to obtain the values of $F$ (compare the last row containing the values of $Q$ above):

\[
\begin{array}{c|cccc}
 p & 1 & 1 & 1 & 2 \\
 \Sigma p & 1 & 1 & 1 & 2 \\
 \Sigma \Sigma p & 1 & -1 & 2 & 0 \\
\end{array}
\]

Now we can reformulate the inequalities of the Original Conjecture 2.2.1 as follows.

**Conjecture 2.7.4. (Original Conjecture, alternative form)**

Let $C \subset \mathbb{CP}^2$ be a rational cuspidal curve of degree $d$ with $\nu$ cusps of given topological types (in particular, $d(d - 3) = 2\delta - 2$). Set $F(j) := (\Sigma \Sigma (\partial \partial h^{(1)} \ast \cdots \ast \partial \partial h^{(\nu)}))$, where $h^{(i)}_j = H_i(j + 1)$, and $H_i$ is the semigroup counting function of the $i$-th singularity. Then

\[
(2.7.3) \quad F(jd) \leq \frac{(j + 1)(j + 2)}{2} \quad \text{for all } j = 0, 1, \ldots, d - 3.
\]

The Weak Conjecture is obtained by taking sum.

**Conjecture 2.7.5. (Weak Conjecture, first alternative form)**
Under the conditions and with the notation of Conjecture 2.7.4,
\[\sum_{j=0}^{d-3} F(jd) \leq \frac{d(d-1)(d-2)}{6}.\]

2.7.2. Examples and counterexamples. Let us summarize the situation. Starting from the semigroups of \(\nu\) local singularities we can define functions \(H\) and \(F\) with integer values depending only on the local topological types of the singularities.

**Definition 2.7.6.** If the sum of delta invariants of the local singularity types, \(\delta\), is of form \(2\delta = (d-1)(d-2)\) for some integer \(d\), we say that these \(\nu\) local topological types are candidates to be the \(\nu\) singularities of a rational cuspidal plane curve of degree \(d\).

If such a curve exists, then Theorem 2.4.3 from [11] prescribes ‘each \(d\)–th value’ of \(H\) by \(H(jd+1) = (j+1)(j+2)/2\). Furthermore, the Original Conjecture 2.7.4 would give an upper bound on ‘each \(d\)–th value’ of \(F\), namely \(F(jd) \leq (j+1)(j+2)/2\). As we already mentioned, if \(\nu = 1\) then \(F(k) = H(k+1)\) for each \(k \in \mathbb{Z}_{\geq 0}\) (not just for \(k \in d \cdot \mathbb{Z}_{\geq 0}\)), and the theorem implies the conjecture (and the inequalities are equalities).

However, for \(\nu > 1\) the values \(F(k)\) and \(H(k+1)\) become different. If one starts to play with two singularities, one can notice that \(F(k) \leq H(k+1)\) seems to be true for every integer \(k \geq 0\), not just for the multiples of \(d\). Later, using lattice cohomology interpretations, we will prove that this is indeed true for \(\nu = 2\), cf. § 2.7.6. See also [46] for an elementary proof.

With these facts in mind, it is tempting to conjecture that maybe the inequality \(F(k) \leq H(k+1)\) is always true – even independently of \(d\) as a property of local singularity types – which would be an interesting, completely combinatorial statement making the Original Conjecture 2.2.1 a corollary of Theorem 2.4.3. But, for \(\nu \geq 3\) there is no such relation between functions \(F\) and \(H\), as we will demonstrate next.

**Example 2.7.7.** Take \(\nu = 3\), and assume that all local singularities are ‘simple’ cusps, that is cusps with multiplicity sequence \([2]\) (or, equivalently, with one Newton pair \((2,3)\), or with semigroup \(\langle 2,3 \rangle = \{0,2,3,4,\ldots\}\)). Then the functions \(F\) and \(H\) are as follows:

<table>
<thead>
<tr>
<th>(k)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H(k+1))</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(F(k))</td>
<td>1</td>
<td>-1</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>(H(k+1) - F(k))</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Notice that for \(k = 2\) the desired inequality \(F(k) \leq H(k+1)\) fails. Hence the inequality \(F(k) \leq H(k+1)\) cannot be true for every integer \(k\). (By the way, this collection of
local cusp types can be realized on a rational tricuspidal curve of degree four, cf. Proposition 2.6.3.)

Example 2.7.8. (Example 2.7.3 revisited.) Consider now a tricuspidal rational projective plane curve of degree $d = 5$ such that each of its singularities has one Newton pair, namely $(3, 4)$, $(2, 5)$ and $(2, 3)$, already considered in Example 2.7.3. For its realizability see Proposition 2.6.3. As we already have shown the function values are as follows:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(k+1)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$F(k)$</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>$H(k+1) - F(k)$</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Since the data are provided by an existing curve of degree 5, the $H(jd+1)$-values at $jd = k = 0, 5, 10$ are the corresponding triangular numbers (1, 3, 6, respectively), as predicted by Theorem 2.4.3. Notice also that again, the desired inequality $F(k) \leq H(k+1)$ fails at $k = 2, 8$. However, the inequalities needed for the Original Conjecture 2.2.1 corresponding to $k = 0, 5, 10$ (multiples of $d$) are true. (This example appears in [20], supporting Conjecture 2.2.1.) So one could still hope in the inequality $F(k) \leq H(k+1)$ in the case of existing curves and for $k \in d \cdot \mathbb{Z}$, where $d$ is the degree of the curve.

Example 2.7.9. (Counterexample to the Original Conjecture 2.2.1)

Consider three semigroups given by two generators each as follows: $\Gamma_1 = \langle 6, 7 \rangle$, $\Gamma_2 = \langle 2, 9 \rangle$ and $\Gamma_3 = \langle 2, 5 \rangle$. These are semigroups of plane curve singularities characterized by multiplicity sequences $[6]$, $[2, 4]$ and $[2, 2]$, respectively. There exists a rational tricuspidal curve of degree $d = 8$ with three singularities exactly of this topological type (cf. Proposition 2.6.3). The values of functions $H$ and $F$ are as follows – we are interested here only in the values at the multiples of $d$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>…</th>
<th>8</th>
<th>…</th>
<th>16</th>
<th>…</th>
<th>24</th>
<th>…</th>
<th>32</th>
<th>…</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(k+1)$</td>
<td>1</td>
<td>…</td>
<td>3</td>
<td>…</td>
<td>6</td>
<td>…</td>
<td>10</td>
<td>…</td>
<td>15</td>
<td>…</td>
<td>21</td>
</tr>
<tr>
<td>$F(k)$</td>
<td>1</td>
<td>…</td>
<td>4</td>
<td>…</td>
<td>5</td>
<td>…</td>
<td>9</td>
<td>…</td>
<td>16</td>
<td>…</td>
<td>21</td>
</tr>
<tr>
<td>$H(k+1) - F(k)$</td>
<td>0</td>
<td>…</td>
<td>-1</td>
<td>…</td>
<td>1</td>
<td>…</td>
<td>1</td>
<td>…</td>
<td>-1</td>
<td>…</td>
<td>0</td>
</tr>
</tbody>
</table>

Of course, as the data are realized by a curve, Theorem 2.4.3 is satisfied: note the triangular numbers in the second row. The condition $\sum_{j=0}^{d-3} F(jd) \leq \sum_{j=0}^{d-3} (j+1)(j+2)/2$ asked by the first alternative version of the Weak Conjecture 2.7.5 is also satisfied, in fact, by equality: this can be seen immediately by summation of the last row of the above table.

However, for $j = 1$ and $j = 4$ the inequality $F(jd) \leq (j+1)(j+2)/2$ fails, hence this is a counterexample to the Original Conjecture 2.2.1.
This example was not checked in [20], as it was not clear at that time that the number
of cusps was crucial. Some series with \( \nu = 1 \) were checked and other examples as well
(also with \( \nu \geq 3 \)), but only up to degree 7 (note that a complete classification of cuspidal
curves exists only up to degree 6). As we will see later in Remark 2.8.2 the smallest degree
where the Original Conjecture 2.2.1 fails among currently known rational cuspidal curves
is exactly 8.

This example also shows that the inequalities of the Original Conjecture 2.2.1 are not
combinatorial consequences of the equalities of Theorem 2.4.3. Moreover, the inequalities
\( F(k) \leq H(k + 1) \) are not true in general, not even for existing curves of degree \( d \) and
setting \( k \in d \cdot \mathbb{Z} \).

Example 2.7.10. The following example will show that the Weak Conjecture (version 2.7.5 of it) is not a combinatorial consequence of the equalities of Theorem 2.4.3 either.

Consider three semigroups given by their generators: \( \Gamma_1 = \langle 3, 5 \rangle \), \( \Gamma_2 = \langle 2, 3 \rangle \), \( \Gamma_3 = \langle 2, 3 \rangle \). The corresponding multiplicity sequences are \([3, 2], [2], [2] \), respectively. The sum of delta invariants is \( 4 + 1 + 1 = (5 − 1)(5 − 2)/2 \), so these three topological types of local singularities are possible candidates for three cusps of a tricuspidal rational projective plane curve of degree \( d = 5 \). Since the complex projective quintics are completely classified (see e.g. [45] or [42] Chapter 6), it is known that such a curve does not exist. However, Theorem 2.4.3 does not exclude the existence of this curve, as the values \( H(k + 1) \) are again 1, 3, 6 at \( k = 0, 5, 10 \), respectively.

\[
\begin{array}{cccccccccc}
  k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  H(k + 1) & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 6 \\
  F(k) & 1 & -1 & 2 & 1 & 0 & 4 & 1 & 3 & 5 & 3 & 6 \\
  H(k + 1) - F(k) & 0 & 2 & -1 & 1 & 2 & -1 & 2 & 1 & -1 & 2 & 0 \\
\end{array}
\]

On the other hand, the inequality of the Original Conjecture 2.2.1 fails at \( k = 5 \).
(Recall however that there does not exist a quintic with the above singularities and that
Conjecture 2.2.1 can not be used to show this fact, as it already failed in Example 2.7.9.)
But, additionally, for this candidate the Weak Conjecture 2.7.5 also fails, as \( 1 + 4 + 6 > 5 \cdot 4 \cdot 3/6 \).

Therefore, if the Weak Conjecture 2.7.5 would be proved – independently of the classification of projective curves –, it would provide an independent tool for checking whether a given collection of local topological singularity types can be realized as the collection of cusp types of a rational cuspidal projective plane curve.
2.7.3. Lattice cohomological interpretation. Now we show the lattice-cohomological meaning of the values of functions $H$ and $F$. From this point of view, it will be obvious that, on one hand, for $\nu = 2$ inequalities $F(k) \leq H(k+1)$ hold, but on the other hand, it is odd to expect such a relation for $\nu \geq 3$. The necessary computations are done in [59] by Némethi and Román, but in that note they were not analyzed from the present point of view.

The lattice cohomology of a weight function.

For the definition of lattice cohomology, see [52]. There is a detailed description in [59] §3 as well. In short, the construction is the following.

 Usually one starts with a lattice $\mathbb{Z}^s$ with fixed base elements \{E_i\}. This automatically provides a cubical decomposition of $\mathbb{R}^s = \mathbb{Z}^s \otimes \mathbb{R}$: the 0–cubes are the lattice points $l \in \mathbb{Z}^s$, the 1–cubes are the ‘segments’ with endpoints $l$ and $l + E_i$, and more generally, a q–cube $\square = (l, I)$ is determined by a lattice point $l \in \mathbb{Z}^s$ and a subset $I \subset \{1, \ldots, s\}$ with $\# I = q$, and it has vertices at the lattice points $l + \sum_{j \in J} E_j$ for different $J \subset I$.

 One also takes a weight function $w : \mathbb{Z}^s \to \mathbb{Z}$ bounded below, and for each cube $\square = (l, I)$ one defines $w(\square) := \max\{w(v), \nu \text{ vertex of } \square\}$. Then, for each integer $n \geq \min(w)$ one considers the simplicial complex $S_n$, the union of all the cubes $\square$ of any dimension with $w(\square) \leq n$. Then the lattice cohomology associated with $w$ is $\{H^q(\mathbb{Z}^s, w)\}_{q \geq 0}$, defined by $H^q(\mathbb{Z}^s, w) := \oplus_{n \geq \min(w)} H^q(S_n, \mathbb{Z})$. Each $H^q$ is graded (by $n$) and it is a $\mathbb{Z}[U]$–module, where the $U$–action consists of the restriction maps induced by the inclusions $S_n \hookrightarrow S_{n+1}$. Similarly, one defines the reduced cohomology associated with $w$ by $H^q_{\text{red}}(\mathbb{Z}^s, w) := \oplus_{n \geq \min(w)} H^q(S_n, \mathbb{Z})$. In all our cases $H^q_{\text{red}}(\mathbb{Z}^s, w)$ has finite $\mathbb{Z}$–rank. The normalized Euler characteristic of $H^*(\mathbb{Z}^s, w)$ is $\text{eu } H^* := -\min(w) + \sum_{q \geq 0} (-1)^q \text{rank}_\mathbb{Z} H^q_{\text{red}}$. Formally, we also set $\text{eu } H^0 := -\min(w) + \text{rank}_\mathbb{Z} H^0_{\text{red}}$.

The lattice cohomology of a negative definite plumbing graph.

Let $G$ be a connected negative definite plumbing graph, and let $(\cdot, \cdot)$ be the negative definite intersection form associated with it. In this chapter we will always assume that in all cases $G$ is a tree (unless otherwise stated), and all the genus decorations of the vertices are zero, that is, the corresponding plumbed 3–manifold $M = M(G)$ is a rational homology sphere. We set $L$ for the lattice generated by elements $E_v$ indexed by the vertices of the graph, which is naturally endowed with the form $(\cdot, \cdot)$. Set $L' := \text{Hom}(L, \mathbb{Z})$ regarded as a subgroup of $L \otimes \mathbb{Q}$. Then $L'/L$ is identified with $H_1(M(G), \mathbb{Z})$, and $\text{Spin}^c(M)$ is an $L'/L$–torsor (and it will be identified with $L'/L$, too). The canonical spin$^c$ structure correspond to the class of zero. Let $S'$ be the anti–nef cone $\{l' \in L' : (L', E_v) \leq 0 \text{ for all } v\}$. 


Let $K_{\text{can}} \in L'$ be the canonical characteristic element defined by the adjunction formulæ $(K_{\text{can}}, E_v) = -(E_v, E_v) - 2$ for any vertex $v$. Moreover, for any class $\sigma \in L'/L$ let $l'_\sigma$ be the unique minimal element of $S'$ whose class is $\sigma$. Then one defines the weight function

$$\chi_\sigma : L \rightarrow \mathbb{Z}, \quad \chi_\sigma(l) := -(l, l + K_{\text{can}} + 2l'_\sigma)/2.$$ 

For each spin$^c$ structure $\sigma \in L'/L$ of $M$ one considers the lattice $L$ with the weight function $\chi_\sigma$. This pair determines the lattice cohomology $H^*(M, \sigma)$ and the whole lattice cohomology package as presented above. It depends only on the 3–manifold $M$ and $\sigma \in \text{Spin}^c(M)$. For more details see e.g. [52, 59].

For the canonical spin$^c$ structure $\sigma = [0]$, $l'_\sigma = 0$, and the corresponding lattice cohomology is denoted by $H^*_{\text{can}}$.

In this language the invariant $K_{\text{can}}^2 + s_\chi$ considered in (2.3.1) is the sum of $(K_{\text{can}}, K_{\text{can}})$ and the number of vertices $s$. For a different interpretation of eu $H^*$ see Remark 2.7.17.

The lattice cohomology of the surgery 3–manifold $S^3_{-d}(K)$.

In [59] Némethi and Román computed lattice cohomologies of the surgery 3–manifold obtained as follows. The input consists of $\nu$ local topological plane curve singularity types, as in our case above, and a positive integer $d$. The surgery 3–manifold is $S^3_{-d}(K)$, the manifold obtained by a $(-d)$–surgery along the connected sum $K$ of knots of the given plane curve singularities ($K = K_1 \# \cdots \# K_\nu \subset S^3$). For motivation see Subsection 2.2.1. However, we do not assume here that $(d - 1)(d - 2) = 2\delta$.

$S^3_{-d}(K)$ is a plumbed 3–manifold represented by a negative definite plumbing graph whenever $d > 0$, hence the above constructions of lattice cohomology run. Moreover, $L'/L = \mathbb{Z}_d$, accordingly we parametrize the spin$^c$ structures by $\sigma = [a] \in \mathbb{Z}_d$, and for each of them one considers the weight function $\chi_\sigma$. For details see [59].

However, in [59], the lattice cohomologies $H^*(S^3_{-d}(K), a)$ are not computed by the definition presented above, but by a powerful reduction machinery, called ‘lattice reduction’, cf. [32, 59]. This allows to express the cohomology modules in a lattice (in fact, in a ‘rectangle’) of rank $\nu$, and its newly defined weight function is determined directly from the semigroups (or counting functions) of the given local topological singularity types. The needed facts are summarized in the proof of the next theorem and in Theorem 2.7.12.

For more details see again [59].

The main result of this section is the following theorem.

**Theorem 2.7.11.** Assume that $K = \#_i K_i$, where $\{K_i\}_i$ are algebraic knots. Define functions $H$ and $F$ from the corresponding semigroups as in previous sections (see (2.4.5),
2.7. Settling the original conjecture

(2.7.1) Assume that $d$ is any positive integer. Then

$$\text{eu} \mathbb{H}^0 \left( S^3_d(K), a \right) = \sum_{j \equiv a \pmod{d}} \left( H(j + 1) + \delta - 1 - j \right),$$

(2.7.4)

$$\text{eu} \mathbb{H}^* \left( S^3_d(K), a \right) = \sum_{j \equiv a \pmod{d}} \left( F(j) + \delta - 1 - j \right),$$

(2.7.5)

PROOF. We will recall several needed statements from [59].

Let $f_i$ be a local equation of the local plane curve singularity corresponding to the algebraic knot $K_i$, and let $m_i$ be the multiplicity along the unique irreducible exceptional divisor with self-intersection $(-1)$ of the pull back of $f_i$ in the minimal good embedded resolution of $(C, P_i)$. It is a topological invariant, and $m_i > 2\delta_i$. In fact, if the topological type of the singularity is given by the multiplicity sequence $\{n_1^{(i)}, n_2^{(i)}, \ldots, n_{s_i}^{(i)}\}$ ($n_{s_i}^{(i)} > 1$), then the above invariant equals $m_i = \sum_{j=1}^{s_i} (n_{s_i}^{(i)})^2 + n_{s_i}^{(i)}$ (cf. [15, \S 4.3]).

We consider the lattice points in the rank-$\nu$ multirectangle $R := [0, m_1] \times \cdots \times [0, m_\nu]$. We denote them by $x = (x_1, \ldots, x_\nu)$, and we also write $|x| := \sum_{i=1}^\nu x_i$.

For any $a$ with $0 \leq a \leq d - 1$ we set the weight function on $R$ by

$$w_a(x) = \sum_{i=1}^\nu H_i(x_i) + \min\{0, 1 + a - |x|\}.$$

It is convenient to define another weight function too, which is independent of $d$ and $a$:

$$W(x) = \sum_{i=1}^\nu \# \{s \in \Gamma_i : s \geq x_i\} = \sum_{i=1}^\nu \left( \delta_i - x_i + H_i(x_i) \right) = \delta - |x| + \sum_{i=1}^\nu H_i(x_i).$$

For any $j \geq 0$ denote the ‘diagonal hyperplane intersections’ of the multirectangle by

$$T_j := \{ x \in R : |x| = j + 1 \}.$$

Note that $T_j = \emptyset$ whenever $j > M := m_1 + \cdots + m_\nu$.

Next, as in [59], we define lattice cohomologies on the ‘diagonal’ sets $T_j$ as well, considering the cohomologies of the intersection of simplicial level sets of the lattice rectangle and the $(\nu - 1)$-dimensional hyperplane of $T_j$, i.e. $\mathbb{H}^q_{\text{red}}(T_j, W) := \oplus_{n \geq \min(W)} \tilde{H}^q(S_n \cap T_j, \mathbb{Z})$ (and similarly for the non-reduced version; cf. [59] (6.1.10)), where the simplicial complex (level set) $S_n$ is the union of all cubes $\Box$ with $W(\Box) \leq n$.

THEOREM 2.7.12 (Némethi, Román, [59] formulae (6.1.15) and (6.1.16)). For any $d > 0$ one has:

$$\text{eu} \mathbb{H}^0 \left( S^3_d(K), a \right) = \sum_{j \equiv a \pmod{d}} \min W|_{T_j},$$

(2.7.6)
Clearly \( \min W|_{T_j} = \delta - j - 1 + H(j + 1) \), which equals \( H(2\delta - 1 - j) \) by Lemma 2.7.2, thus it is zero for \( j \not\in 2\delta - 2 \). Hence the identity (2.7.4) follows.

Next, fix some \( j \geq 0 \), and apply Theorem 2.7.12 for an auxiliary large \( D > M \) (substituted for \( d \)), and for \( a = j \). By [59, Proposition 5.3.4, Corollary 5.3.7, Theorem 6.1.6 e]), for such \( D > M \), one has

\[
\mathbb{H}^*(S^3_{-D}(K), j) \cong \mathbb{H}^*[0, m_1] \times \cdots \times [0, m_\nu], w_j).
\]

Moreover, by [59, Proposition 7.1.3], for \( D > M \) the normalized Euler characteristic of this cohomology can be compared with the coefficients of the polynomial \( Q \). Namely,

\[
q_j = \mathbb{H}^*([0, m_1] \times \cdots \times [0, m_\nu], w_j).
\]

Then (2.7.7), (2.7.8) and (2.7.9) combined give \( q_j = -\mathbb{H}^*(T_j, W) \) for any \( j \). Notice that \( q_j = 0 \) if \( j \) is not in the interval \([0, 2\delta - 2]\), and for these values by (2.2.6) one also has \( q_j = q_{2\delta - 2 - j} + \delta - j - 1 \), which equals \( F(j) + \delta - j - 1 \) by (2.7.2). Hence (2.7.7) (now applied with the original \( d \)) implies (2.7.5).

Remark 2.7.13. In fact, the integer \( d \), the sum of delta invariants \( \delta \) and the function \( H \) completely determine the whole \( \mathbb{H}^0 \) as a graded \( \mathbb{Z}[U] \)-module (and not just its Euler characteristic). For this fact, we refer to [59, Lemma 6.1.1, Theroem 6.1.6].

Corollary 2.7.14. Assume that \( d(d - 3) = 2\delta - 2 \), cf. (2.2.3). Then

\[
\begin{align*}
\mathbb{H}^0(S^3_{-d}(K), a) &= \sum_{j \equiv -a \pmod{d}} H(j + 1), \\
\mathbb{H}^*(S^3_{-d}(K), a) &= \sum_{j \equiv -a \pmod{d}} F(j).
\end{align*}
\]

The value \( a = 0 \) corresponds to the canonical spin\(^c\) structure. Denote the corresponding lattice cohomology of \( S^3_{-d}(K) \) by \( \mathbb{H}^*_{\text{can}}(S^3_{-d}(K)) \). Then the above identities read as:

\[
\begin{align*}
\mathbb{H}^0_{\text{can}}(S^3_{-d}(K)) &= \sum_{0 \leq j \leq d - 3} H(jd + 1), \\
\mathbb{H}^*_{\text{can}}(S^3_{-d}(K)) &= \sum_{0 \leq j \leq d - 3} F(jd).
\end{align*}
\]

Proof. Use the symmetry property (2.2.6) and Lemma 2.7.2 \( \square \)
2.7.4. Reformulations of Theorem 2.4.3 and the Weak Conjecture 2.3.1.

Because of the inequalities (implied by Bézout’s theorem) of Lemma 2.4.2 from \[20\], Proposition 3.2.1, Theorem 2.4.3 of Borodzik and Livingston is true if and only if the corresponding sums over \(j\) are equal. This combined with (2.7.10) provides the following equivalent form of the semigroup distribution property.

**Theorem 2.7.15.** (Equivalent form of Theorem 2.4.3 of Borodzik and Livingston) For a link \(M = S^3_d(K)\) of a superisolated surface singularity corresponding to a rational cuspidal projective plane curve of degree \(d\) we have:

\[
eu \mathbb{H}^0_{\text{can}} \left( S^3_d(K) \right) = d(d-1)(d-2)/6.
\]

This form is also present in the recent article \[60\] (in Example 2.4.3 (a) and Section 3); cf. also \[19\], Theorem 8.9 for the unicuspidal case.

Next, using (2.7.11) we give an equivalent formulation of the Weak Conjecture 2.3.1 in terms of lattice cohomology, see also its alternative version, Conjecture 2.7.5.

**Conjecture 2.7.16.** (Weak Conjecture, second alternative form)

For a link \(M = S^3_d(K)\) of a superisolated surface singularity corresponding to a rational cuspidal projective plane curve of degree \(d\) we have:

\[
eu \mathbb{H}^*_{\text{can}} \left( S^3_d(K) \right) \leq d(d-1)(d-2)/6.
\]

Alternatively, in the light of the previous theorem:

\[
eu \mathbb{H}^*_{\text{can}} \left( S^3_d(K) \right) \leq \eu \mathbb{H}^0_{\text{can}} \left( S^3_d(K) \right).
\]

In this context, the Weak Conjecture 2.7.5 is much more natural than the Original Conjecture 2.2.1 which would require the validity of \(F(jd) \leq (j+1)(j+2)/2\) for every single \(j = 0, 1, \ldots, d-3\), i.e. an inequality for the lattice cohomological Euler characteristic of each diagonal set \(T_{jd}\) (see the proof of Theorem 2.7.11).

**Remark 2.7.17.** (a) (Connection with the Seiberg–Witten invariant)

Let \(G\) be a connected negative definite plumbing graph, \(M = M(G)\) the corresponding plumbed 3–manifold, and \(\sigma \in \text{Spin}^c(M)\), cf. the beginning of Subsection 2.7.3. In \[53\] it is proved that \(\eu \mathbb{H}^* (M, \sigma)\) equals the normalized Seiberg–Witten invariant of \(M\) associated with the spin\(^c\) structure \(\sigma\). In particular, for the canonical spin\(^c\) structure, one has \(\eu \mathbb{H}^*_{\text{can}} (M) = -\text{sw}_{\text{can}}(M) - (K^2_{\text{can}} + s)/8\), compatibly with the Weak Conjecture 2.3.1.

(b) (Connection with the Heegaard Floer homology)

Let \(HF^+(M, \sigma)\) denote the Heegaard Floer homology of a plumbed 3–manifold \(M\) associated with a connected negative definite graph \(G\) and \(\sigma \in \text{Spin}^c(M)\). Then one has
a graded \( \mathbb{Z}[U] \)-module isomorphism \( HF^+(M, \sigma) = T_{d(M, \sigma)}^+ \oplus HF^+(M, \sigma)_{\text{red}} \), where the reduced Heegaard Floer homology \( HF^+(M, \sigma)_{\text{red}} \) has a finite \( \mathbb{Z} \)-rank and an absolute \( \mathbb{Z}_2 \)-grading (even, odd), \( T_{d(M, \sigma)}^+ \) is isomorphic to \( \mathbb{Z}[U, U^{-1}] / U \cdot \mathbb{Z}[U] \) with shifted grading such that \( d(M, \sigma) \) is the minimal value among \( \mathbb{Z} \)-gradings of its elements and \( d(M, \sigma) \) denotes the \( d \)-invariant, see [67]. Then

\[
\text{sw}(M, \sigma) = \text{rank}_\mathbb{Z} HF^+_{\text{red,even}}(M, \sigma) - \text{rank}_\mathbb{Z} HF^+_{\text{red,odd}}(M, \sigma) - d(M, \sigma)/2.
\]

In particular, via part (a), \( \text{eu} \mathbb{H}^*(M, \sigma) \) can also be interpreted as the normalized Euler characteristic of the Heegaard Floer homology.

In fact, in [52] Némethi conjectured that up to a degree shift by \( d(M, \sigma) \)

\[
\begin{align*}
HF^+_{\text{red,even}}(-M, \sigma) &= \bigoplus_{q \text{ even}} \mathbb{H}_q^\text{red}(M, \sigma) \\
HF^+_{\text{red,odd}}(-M, \sigma) &= \bigoplus_{q \text{ odd}} \mathbb{H}_q^\text{red}(M, \sigma).
\end{align*}
\]

If \( M = S^3_{-d}(K) \) as above, and \( \nu \leq 2 \), then the conjecture is true: if \( \nu = 1 \) then the graph is ‘almost rational’, hence the statement follows from [52], if \( \nu = 2 \) then one can use [64]. In these cases, in fact, \( HF^+_{\text{red,even}}(-M) = \mathbb{H}_0^\text{red}(M), HF^+_{\text{red,odd}}(-M) = \mathbb{H}_1^\text{red}(M) \). Furthermore, \( \mathbb{H}_q^\text{red}(M) = 0 \) for \( q \geq 2 \) since the graph has only \( \nu \leq 2 \) ‘bad vertices’, cf. [59, 54]. Nevertheless, if \( \nu \geq 3 \) then \( \mathbb{H}^3 \) in principle can be nontrivial, hence \( \mathbb{H}^0 \) cannot be determined from the Heegaard Floer homology: the additional \( \mathbb{Z} \)-grading of the lattice cohomology is a finer invariant.

Remark 2.7.18. We wish to emphasize that it is essential that in the above Weak Conjecture, alternative form 2.7.16, we talk about the lattice cohomologies corresponding to the canonical spin\(^c\) structure only. Using formulae of Corollary 2.7.14 one can check easily that for superisolated singularity link \( M \) coming from the existing curve of Example 2.7.9 choosing spin\(^c\) structure corresponding to \( a = 4 \) we have \( \text{eu} \mathbb{H}^*(M, a = 4) = 45 > \text{eu} \mathbb{H}^0(M, a = 4) = 42 \). Also, for many curves from series (1) in Proposition 2.6.3 one can find spin\(^c\) structures for which the inequality fails, e.g. \( \text{eu} \mathbb{H}^*(M(C_{4,1}), a = 2) = 3 > \text{eu} \mathbb{H}^0(M(C_{4,1}), a = 2) = 2 \), cf. Example 2.7.7.

Remark 2.7.19. It is well-known that the link \( M \) of a superisolated singularity corresponding to a projective plane curve is a rational homology sphere (\( \mathbb{Q}HS^3 \)) if and only if the curve is rational and cuspidal, see [19, §7.1] and the references therein.

The fact that \( M \) is \( \mathbb{Q}HS^3 \) is probably also essential in the above conjecture. To see this, consider the following example. It is not hard to see by a construction using Cremona transformations that there exists a rational projective curve \( C \) of degree \( d = 5 \) with three singular points which are of the following type. One singularity is a simple transverse self-intersection: a reducible \( A_1 \)-singularity. The other two are locally irreducible singularities,
with multiplicity sequences \([3, 2]\), resp. \([2]\), alternatively, with Newton pairs \((3, 5)\), resp. \((2, 3)\). From the embedded resolution graphs of plane curve singularities, it is easy to construct the plumbing graph of the link \(M\) of the corresponding superisolated singularity. Due to the locally reducible singularity, it has one cycle, so it is not a tree. One checks that for this link \(\text{eu } H^*\left(\frac{S^3 - D}{K}, a\right) = 11 > \text{eu } H^0\left(\frac{S^3 - D}{K}, a\right) = 10\). Note that although \(M\) is not a rational homology sphere, we can still speak about the corresponding lattice cohomologies with the same definition as in the \(\mathbb{Q}HS^3\) case.

2.7.5. Proof of the second alternative form Conjecture 2.7.16 of the Weak Conjecture for \(\nu = 2\).

First we recall that \(H^q(S^3_d(K), a) = 0\) for any \(q \geq \nu\). This follows from the fact that the non-compact simplicial subcomplexes \(S_n\) of \(\mathbb{R}^\nu\) (in the reduced lattices) have no nonzero cohomologies \(H^q(S_n, \mathbb{Z})\) for \(q \geq \nu\); or just apply [32] or [54, Theorem 6.2.1]. Then, for \(\nu = 2\), we have \(\text{eu } H^*(S^3_d(K), a) = \text{eu } H^0(S^3_d(K), a) - \text{rank}_\mathbb{Z} H^1(S^3_d(K), a)\), hence the second alternative form transforms into \(\text{rank}_\mathbb{Z} H^1_{\text{can}}(S^3_d(K)) \geq 0\), which is certainly true.

Notice also that for \(\nu \geq 3\) similar argument does not work. From this point of view, it is even more surprising that in all the known cases, the Weak Conjecture holds, cf. Section 2.8.

2.7.6. Proof of the alternative form Conjecture 2.7.4 of the Original Conjecture for \(\nu = 2\).

In fact, essentially by the same argument, in case of \(\nu = 2\) one can prove the Original Conjecture 2.7.4 as well. Using Formula (2.4.6) of Theorem 2.4.3 and comparing it with (2.7.3), it is enough to prove that for \(\nu = 2\) the inequality \(F(k) \leq H(k + 1)\) holds for any \(0 \leq k \leq 2\delta - 2\). This inequality is purely combinatorial, completely independent of the parameter \(d\), and has nothing to do with the realizability of cusp types on an existing rational projective curve (neither with the validity or failure of equalities (2.4.6)).

Indeed, similarly as in the proof of Theorem 2.7.11, set \(D > 2\delta - 2\). Then, by (2.7.4) and (2.7.5), the inequality \(F(k) \leq H(k + 1)\) turns into \(\text{eu } H^*(S^3_d(K), k) \leq \text{eu } H^0(S^3_d(K), k)\) which is again true, since due to the vanishing following from the reduction principle, the difference is the only summand \(\text{rank}_\mathbb{Z} H^1_{\text{red}}(S^3_d(K), k) \geq 0\).

For a different, elementary proof, see [46].
2.8. Verifying the Weak Conjecture for known curves with $\nu \geq 3$

The results of this section can be found in the joint work with Andrś Némethi [7 §4]. In this section we show that the Weak Conjecture 2.7.16 is true for all the rational cuspidal curves with at least three cusps currently known (by the author). A list of these curves was given in Proposition 2.6.3. There are three infinite series of tricuspidal curves; one is a two-parameter family, the other two series have one parameter each (the curve degree). There are two ‘sporadic’ curves not contained in any of the three series. Both of them is of degree 5; one is tricuspidal, the other has four cusps (this curve is conjectured to be the only rational cuspidal curve with more than three cusps).

**Theorem 2.8.1.** Let $H$ and $F$ be the functions as defined in Formulae (2.4.5) and (2.7.1) corresponding to the singularities of the curves given in Proposition 2.6.3 in the corresponding item. Then we have:

1. \[\sum_{j=0}^{d-3} H(jd + 1) - F(jd) = \begin{cases} l(l-1) & \text{if } d = 2l + 1, \\ (u-l)(u-l+1) & \text{if } d = 2l, u \geq l-1. \end{cases}\]

In particular, $\sum_{j=0}^{d-3} (H(jd + 1) - F(jd)) \geq 0$ in all cases.

2. \[\sum_{j=0}^{d-3} H(jd + 1) - F(jd) = \begin{cases} 4p(3p-1) + 2 & \text{if } l = 3p-1, \\ 4p(3p-1) & \text{if } l = 3p, \\ 12p(p+1) + 2 & \text{if } l = 3p+1. \end{cases}\]

3. \[\sum_{j=0}^{d-3} H(jd + 1) - F(jd) = \begin{cases} 60p^2 - 2p & \text{if } l = 4p, \\ 60p^2 + 46p + 10 & \text{if } l = 4p+1, \\ 60p^2 + 62p + 16 & \text{if } l = 4p+2, \\ 60p^2 + 100p + 42 & \text{if } l = 4p+3. \end{cases}\]

4. \[\sum_{j=0}^{d-3} H(jd + 1) - F(jd) = 6.\]

5. \[\sum_{j=0}^{d-3} H(jd + 1) - F(jd) = 8.\]
In particular, the Weak Conjecture 2.7.5 is satisfied in each case, as we always get a non-negative number.

Proof. Since in each case we know explicitly the singularities, we also know explicitly the product \( \Delta(t) \) of their Alexander polynomials. Therefore, it is convenient to work with Formula (2.2.7), as (2.2.8) reads as:

\[
R(1) = \text{eu } \mathbb{H}_{\text{can}}^* - \text{eu } \mathbb{H}_{\text{can}}^0 = \sum_{j=0}^{d-3} F(jd) - H(jd + 1).
\]

Recall also Theorem 2.7.15 and Formulae (2.7.10), (2.7.11).

We do not give the computations here, just present the form of the polynomial \( \Delta(t) = \Delta_1(t)\Delta_2(t)\Delta_3(t) \) in terms of the parameters in cases (1), (2) and (3) (see the Newton pairs given in Proposition 2.6.3 and \[20 \S 2.1\] for the Alexander polynomial in terms of the Newton pairs).

(1)

\[
\Delta^{(1)}(t) = \frac{(t - 1)(t'(d-2)(d-1) - 1) (t - 1)(t^2(2u+1) - 1) (t - 1)(t^{2d-2u-3} - 1)}{(t^d - 1)(t^{d-1} - 1) (t^2 - 1)(t^{2u+1} - 1) (t^2 - 1)(t^{2d-2u-3} - 1)}
\]

(2)

\[
\Delta^{(2)}(t) = \frac{(t - 1)(t^2(l+1) - 1)(t^2+4(l+1) - 1) (t - 1)(t^{2(3l+1)} - 1) (t - 1)(t^6 - 1)}{(t^2 - 1)(t^{2l+1} - 1) (t^3 - 1)(t^{3l+1} - 1) (t^3 - 1)(t^3 - 1)}
\]

(3)

\[
\Delta^{(3)}(t) = \frac{(t - 1)(t^{3l(l+1)} - 1)(t^{3+9l(l+1)} - 1) (t - 1)(t^{4l(l+1)} - 1)(t^{2+8(2l+1)} - 1) (t - 1)(t^6 - 1)}{(t^{3l - 1})(t^{3(l+1)} - 1)(t^{1+3l(l+1) - 1}) (t^4 - 1)(t^{2(2l+1)} - 1)(t^{1+4(2l+1)} - 1) (t^2 - 1)(t^3 - 1)}
\]

Then, in each case, use Formula (2.2.7) with the corresponding \( d \) to obtain the result.

Finally, check the conjecture for the two exceptional curves. The tricuspidal one in (4) of degree \( d = 5 \) has cusp types \([2_2],[2_2],[2_2]\). The numerical values of functions \( F \) and \( H \) are as follows:

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H(k+1) )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>( F(k) )</td>
<td>1</td>
<td>-1</td>
<td>3</td>
<td>-3</td>
<td>6</td>
<td>-3</td>
<td>7</td>
<td>-1</td>
<td>6</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>( H(k+1) - F(k) )</td>
<td>0</td>
<td>2</td>
<td>-1</td>
<td>5</td>
<td>-3</td>
<td>6</td>
<td>-3</td>
<td>5</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Hence \( \text{eu } \mathbb{H}_{\text{can}}^0 - \text{eu } \mathbb{H}_{\text{can}}^* = 0 + 6 + 0 > 0 \).

The single known rational cuspidal curve with four cusps in (5) has degree \( d = 5 \). The cusp types are \([2_3],[2],[2],[2]\). The detailed data:
<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(k+1)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>$F(k)$</td>
<td>1</td>
<td>-2</td>
<td>5</td>
<td>-5</td>
<td>8</td>
<td>-5</td>
<td>9</td>
<td>-3</td>
<td>8</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>$H(k+1) - F(k)$</td>
<td>0</td>
<td>3</td>
<td>-3</td>
<td>7</td>
<td>-5</td>
<td>8</td>
<td>-5</td>
<td>7</td>
<td>-3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Hence $eu \mathbb{H}^0_{can} - eu \mathbb{H}^*_{can} = 0 + 8 + 0 > 0$. □

**Remark 2.8.2.** We present a table containing the detailed data $H jd + 1 - F jd$, $j = 0, \ldots, d-3$ for the first few members of the first series (curves $C_{d,u}$ from Proposition 2.6.3):

<table>
<thead>
<tr>
<th>Curve</th>
<th>Degree</th>
<th>Cusp types</th>
<th>$H jd + 1 - F jd$</th>
<th>$eu \mathbb{H}^0 - eu \mathbb{H}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{4,1}$</td>
<td>$d = 4$</td>
<td>[2], [2], [2]</td>
<td>0 0</td>
<td>0</td>
</tr>
<tr>
<td>$C_{5,1}$</td>
<td>$d = 5$</td>
<td>[3], [2], [2]</td>
<td>0 2 0</td>
<td>2</td>
</tr>
<tr>
<td>$C_{6,1}$</td>
<td>$d = 6$</td>
<td>[4], [2], [2]</td>
<td>0 0 0 0</td>
<td>0</td>
</tr>
<tr>
<td>$C_{6,2}$</td>
<td>$d = 6$</td>
<td>[4], [2], [2]</td>
<td>0 0 0 0</td>
<td>0</td>
</tr>
<tr>
<td>$C_{7,1}$</td>
<td>$d = 7$</td>
<td>[5], [2], [2]</td>
<td>0 3 0 3 0</td>
<td>6</td>
</tr>
<tr>
<td>$C_{7,2}$</td>
<td>$d = 7$</td>
<td>[5], [2], [2]</td>
<td>0 3 0 3 0</td>
<td>6</td>
</tr>
<tr>
<td>$C_{8,1}$</td>
<td>$d = 8$</td>
<td>[6], [2], [2]</td>
<td>0 0 1 1 0 0</td>
<td>2</td>
</tr>
<tr>
<td>$C_{8,2}$</td>
<td>$d = 8$</td>
<td>[6], [2], [2]</td>
<td>0 -1 1 1 -1 0</td>
<td>0</td>
</tr>
<tr>
<td>$C_{8,3}$</td>
<td>$d = 8$</td>
<td>[6], [2], [2]</td>
<td>0 -1 1 1 -1 0</td>
<td>0</td>
</tr>
<tr>
<td>$C_{9,1}$</td>
<td>$d = 9$</td>
<td>[7], [2], [2]</td>
<td>0 3 1 4 1 3 0</td>
<td>12</td>
</tr>
<tr>
<td>$C_{9,2}$</td>
<td>$d = 9$</td>
<td>[7], [2], [2]</td>
<td>0 4 0 4 0 4 0</td>
<td>12</td>
</tr>
<tr>
<td>$C_{9,3}$</td>
<td>$d = 9$</td>
<td>[7], [2], [2]</td>
<td>0 4 0 4 0 4 0</td>
<td>12</td>
</tr>
</tbody>
</table>

We see that the smallest degree where the Original Conjecture 2.2.1 fails is degree 8. The general pattern for larger $d$’s seems to be that the conjecture fails only at even degrees and when $d - 3 > u > 1$ (so it still seems to be true when the degree is odd or the degree is even and $u = 1$ or $u = d - 3$).

Computations suggest that the other two series satisfy even the Original Conjecture 2.2.1.

### 2.9. An invariance property

In [11, Example 6.16], the difficulty of obtaining bounds on the number of cusps of a cuspidal curve is briefly discussed, by presenting a list of 5 local topological types such that, among other criteria, Theorem 2.4.3 cannot obstruct the existence of a corresponding hypothetical cuspidal curve. In fact, we will show that Theorem 2.4.3 alone cannot provide any restrictions on the number of cusps of projective curves (see Remark 2.9.6 (b)).
The key point is that the equality (2.4.6) depends only on the infimal convolution $H$ and many different combinations of semigroup counting functions can have the same infimal convolution, as Theorem 1.2.2 suggests.

In this section, we present some examples and discuss further corollaries of Theorem 1.2.2 in terms of lattice cohomologies of certain Dehn surgery manifolds, including links of superisolated singularities.

2.9.1. Dependence of $H^0(S^3_d(K))$ and $eu H^0(S^3_d(K))$ on the multiplicity sequences. In this subsection we present an effective way to compute $eu H^0(S^3_d(K), a)$, where the setting is as in Subsection 2.7.3 and in [59], i.e. $K = \#_{i=1}^\nu K_i$, $d$ is an arbitrary positive integer, and $a$ stands for a spin$^c$ structure of $S^3_d(K)$, hence $a \in \{0, \ldots, d-1\}$. In this discussion we prefer to fix the integers $d$, $a$ and $\delta$. Hence, by (2.7.4), $eu H^0(S^3_d(K))$ (and, by Remark 2.7.13 the $\mathbb{Z}[U]$-module $H^0(S^3_d(K))$ as well) is completely determined by the infimal convolution $H = H_1 \circ \cdots \circ H_\nu$. We focus on the dependence of $H$ on the multiplicity sequences of plane curve singularities corresponding to knots $K_i$.

Let $[n_1^{(1)}, \ldots, n_{r_1}^{(1)}], \ldots, [n_1^{(\nu)}, \ldots, n_{r_\nu}^{(\nu)}]$ be the multiplicity sequences of the local singularities.

Then the sum of delta invariants of the singularities (see Formula (1.1.10)) is

\begin{equation}
\delta = \sum_{i=1}^\nu \sum_{j=1}^{r_i} n_j^{(i)} (n_j^{(i)} - 1) / 2.
\end{equation}

We will see that the infimal convolution $H = H_1 \circ \cdots \circ H_\nu$ depends only on the multiset of multiplicities $\{n_1^{(1)}, \ldots, n_{r_1}^{(1)}, \ldots, n_1^{(\nu)}, \ldots, n_{r_\nu}^{(\nu)}\}$. By a multiset we mean a set, where the same element might be repeated and we keep track the number of appearances; hence a multiset with integer entries basically is an element of the group ring $\mathbb{Z}[\mathbb{Z}]$.

**Theorem 2.9.1.** Assume we are given two collections of plane curve singularity types with their multiplicity sequences:

$[n_1^{(1)}, \ldots, n_{r_1}^{(1)}], \ldots, [n_1^{(\nu)}, \ldots, n_{r_\nu}^{(\nu)}]$

and

$[\overline{n}_1^{(1)}, \ldots, \overline{n}_{r_1}^{(1)}], \ldots, [\overline{n}_1^{(\nu)}, \ldots, \overline{n}_{r_\nu}^{(\nu)}]$.

Denote the counting functions of their semigroups by $H_1, \ldots, H_\nu$ and $\overline{H}_1, \ldots, \overline{H}_\nu$, respectively.

If

$\{\{n_1^{(1)}, \ldots, n_{r_1}^{(1)}, \ldots, n_1^{(\nu)}, \ldots, n_{r_\nu}^{(\nu)}\}\} = \{\{\overline{n}_1^{(1)}, \ldots, \overline{n}_{r_1}^{(1)}, \ldots, \overline{n}_1^{(\nu)}, \ldots, \overline{n}_{r_\nu}^{(\nu)}\}\}$
as multisets, then
\[ H_1 \diamond \cdots \diamond H_* = H_1 \diamond \cdots \diamond H_. \]

**Proof.** This is an immediate corollary of Theorem 1.2.2 and the associativity and commutativity of the infimal convolution. \( \square \)

**Corollary 2.9.2.** Assume that \( K = K_1 \# \ldots \# K_* \) and \( \overline{K} = K_1 \# \ldots \# K_. \) are connected sums of algebraic knots with summands as above. If the collections of numbers coming from multiplicity sequences corresponding to the algebraic knots are equal as multisets, then
\[ \mathbb{H}^0(S^3_{-d}(K), a) \cong \mathbb{H}^0(S^3_{-d}(\overline{K}), a) \]
for any integer \( d > 0 \) and any spin\(^c\) structure \( a \in \{0, \ldots, d-1\} \).

The same is true for \( \text{eu} \mathbb{H}^0 \) as well.

**Proof.** Use (2.9.1) and Remark 2.7.13 (resp. Formula (2.7.4)). \( \square \)

**Remark 2.9.3.** Note that a similar statement is not true for \( \mathbb{H}^q \) \( (q \geq 1) \), not even for the numerical value \( \text{eu} \mathbb{H}^* \); see e.g. the links of superisolated singularities corresponding to the curves (4), (5) of Proposition 2.6.3. The two curves have the same degree, the same multiset of multiplicities, but different \( F\)-functions and different lattice cohomologies with respect to the canonical spin\(^c\) structure (see the end of proof of Theorem 2.8.1).

**Remark 2.9.4.** It is quite surprising that from the point of view of the zeroth lattice cohomology only the collection of multiplicities ‘put together’ is important. This fact makes a lot easier to compute \( \mathbb{H}^0(S^3_{-d}(K), a) \) in many cases. We can view this result in the following way as well: the zeroth lattice cohomology shows a stability with respect to the ‘combinatorial surgery’ of moving multiplicity numbers from one multiplicity sequence to another (if allowed, i.e. if we get valid multiplicity sequences). We illustrate this by a simple example from [20, table in Section 2.3].

There exist cuspidal curves of degree 5 with the following cusp data (we present the multiplicity sequences):
- \([3, 2] \) \( (\nu = 2) \)
- \([3, 2], [2] \) \( (\nu = 2) \)
- \([3], [2], [2] \) \( (\nu = 3) \).

Of course the corresponding surgery manifolds have \( \mathbb{H}^0(S^3_{-d}(K), 0) \) as prescribed by Theorem 2.7.15. From Corollary 2.9.2 we see immediately without any computation that these manifolds must have identical zeroth lattice cohomology and not only for the canonical spin\(^c\) structure \( a = 0 \), but for the other values \( a = 1, \ldots, 4 \) as well.

Notice that curves of degree 5 with the following cusp data do not exist:
• \([3, 2], [2], [2] \ (\nu = 3)\)
• \([3], [2], [2], [2] \ (\nu = 4)\)

However, the corresponding surgery manifolds also have \(H^0(S^3_{d}(K), a)\) as above, and to see this we do not need any further computations, since it is obvious from the multiset of multiplicities.

Notice that these manifolds are all different: one can construct their plumbing graphs from the embedded resolution graphs of the plane curve singularities and the Dehn surgery coefficient \(d\) (see \([59] \, \text{§2.3}\)), then observe that the plumbing graphs are all different and they are in reduced form (see \([61] \, \text{§4}\)). Alternatively, one can also compute \(e_U H^*\) by \((2.7.5)\) from the Alexander polynomials of the given singularities to distinguish some of these manifolds.

In general, we have the following statement.

**Corollary 2.9.5.** Assume that we have local singularity types which are candidates to be the singularities of a rational cuspidal plane curve of degree \(d\) in the sense of Definition 2.7.6, and they satisfy the necessary condition given by Theorem 2.4.3. Then any other collection of local singularity types, such that the multiset of the occurring multiplicities is the same as in the case of the original collection of singularities, as a new candidate satisfies the necessary condition given by Theorem 2.4.3 as well.

**Remark 2.9.6.** (a) This result enlarge the applicability of the criterion provided by Borodzik–Livingston Theorem 2.4.3 drastically: if we reorganize the multiplicity numbers of a rational cuspidal curve candidate (by keeping the multiset), then the new combinatorial candidate satisfies the output of the Borodzik–Livingston Theorem 2.4.3 if and only if the original candidate satisfied it (regardless of the algebraic realizability on a rational cuspidal curve).

This shows that although in the Borodzik–Livingston Theorem 2.4.3 the algebraic realizability is important, in reality it matters ‘less’, and presumably it can be replaced by a much weaker assumption. E.g., one could possibly require only the smooth realizability of the curve (near the singular points a smooth model of the singular local embeddings, otherwise a smooth embedding). In fact, analyzing the proof of \([11]\), only this data is used: the fact that the smooth boundary \(\partial T\) of a tubular neighborhood \(T\) of the curve \(\{f_d = 0\}\) of degree \(d\) in the projective plane \(\mathbb{CP}^2\) bounds a rational homology ball, namely \(\mathbb{CP}^2 \setminus T\), the determination of the spin\(^c\) structures of these manifolds, and properties of the \(d\)–invariants of \(\partial T\).
It would be interesting to prove that two candidates with equivalent data in the sense of Theorem 2.9.1 can/cannot be simultaneously smoothly embedded in \( \mathbb{CP}^2 \). See also [10, Question 4].

(b) As there exist rational cuspidal curves with arbitrarily long multiplicity sequences (even in the unicursipdal case, see e.g. Orevkov’s curves in [62]), the above corollary also shows that Theorem 2.4.3 cannot provide any restriction on the number of cusps of rational cuspidal curves. Recall that it is conjectured that the number of cusps is always less than five, i.e. \( \nu \leq 4 \), see e.g. [70]. A result of Tono shows that \( \nu \leq 8 \), see [74]; cf. also [11, Example 6.16].
CHAPTER 3

Cuspidal curves of higher genus

3.1. Introduction: the generalized semigroup distribution property

In this chapter we deal with cuspidal complex projective plane curves. Such a curve is defined as the zero set in the complex projective plane of an irreducible homogeneous polynomial with complex coefficients in three variables. Recall that by cuspidal we mean that all the singularities on the curve are locally irreducible plane curve singularities.

Every cuspidal complex projective plane curve is homeomorphic to a real oriented surface of some genus \( g \). If the curve is smooth, this genus can be computed from the degree \( d \) of the curve: \( g = \frac{(d - 1)(d - 2)}{2} \). If there are \( \nu \) locally irreducible singularities on the curve with delta invariants \( \delta_1, \ldots, \delta_\nu \), then the relation between the degree and the genus is (see e.g. [3, Section II.11], cf. (2.1.1) in the rational case)

\[
(g + 1) = \frac{(d - 1)(d - 2)}{2} + \sum_{j=1}^{\nu} \delta_j.
\]

Cuspidal curves of higher genus in the complex projective plane were not studied as much as the rational ones. As we already mentioned, the main difficulty in this case is that the complement of the curve is not a rational homology ball, alternatively, not a \( \mathbb{Q} \)-acyclic surface (see [10, §1]).

Nevertheless, the semigroup distribution property (Theorem [2.4.3]) can be generalized to this case. This was done in a joint work [6] with Daniele Celoria and Marco Golla, and independently by Maciej Borodzik, Matthew Hedden and Charles Livingston in [10].

**Theorem 3.1.1** (see [6, Theorem 1.1 and Remark 5.4] with Celoria and Golla and [10, Theorem 1] by Borodzik, Hedden and Livingston). Let \( H \) be the infimal convolution of the semigroup counting functions of local cusp types on a cuspidal curve of degree \( d \) and genus \( g \). Then the inequalities

\[
0 \leq H(jd + 1 - 2k) + k - \frac{(j + 1)(j + 2)}{2} \leq g
\]

hold for every \( j = -1, 0, \ldots, d - 2 \) and \( k = 0, \ldots, g \).

We will call this statement the generalized semigroup distribution property.
The main idea of the proof is similar to the one of the proof of Theorem 2.4.3 in [11]. One considers the boundary \( Y \) of the tubular neighborhood of the cuspidal curve and computes its correction terms (\( d \)-invariants) in terms of the invariants of the cusp types. However, the complement of the tubular neighborhood in \( \mathbb{CP}^2 \) is now not a rational homology ball. Nevertheless, its homologies can be described and inequalities for the computed correction terms (from [65]) can be applied. For the details, see [6] §3, §4, §5 and [10].

Our goal is to apply this generalized semigroup distribution property to curves with one cusp only whose link is a torus knot. We would like to obtain some variant of Theorem 2.5.1 taken from [21].

In this chapter, we will call a cuspidal curve 1-unicuspidal if it has only one singularity and that singularity is locally irreducible having one Puiseux pair only (equivalently, its link is a torus knot rather than an iterated torus knot). If this Puiseux pair is \((a, b)\), we say that the curve is \((a, b)\)-unicuspidal. As the delta invariant of a cusp with one Puiseux pair \((a, b)\) equals \((a - 1)(b - 1)/2\) (see e.g. [79] §4.3), the degree-genus formula (3.1.1) in this case specializes to the following identity:

\[
(d - 1)(d - 2) = (a - 1)(b - 1) + 2g.
\]

In Section 3.2 we will prove the following theorem.

**Theorem 3.1.2.** Fix a positive integer \( g \geq 1 \). Let \( C \) be a 1-unicuspidal curve of genus \( g \) and degree \( d \) whose singularity is of type \((a, b)\). Then, if \( d \) is sufficiently large,

\[
a + b = 3d
\]

or, equivalently (see the variant (3.1.3) of the degree-genus formula),

\[
\left( \frac{7b - 2a}{3} \right)^2 - 5b^2 = 4(2g - 1).
\]

Theorem 3.1.2 imposes strong restrictions on pairs \((a, b)\) that can be realized as Puiseux pairs of the singularity of a plane unicuspidal curve of genus \( g \), in the spirit of [21]. In particular, we obtain the following two corollaries of Theorem 3.1.2.

**Corollary 3.1.3.** For all genera \( g \) with \( g \equiv 2 \) (mod 5) or \( g \equiv 4 \) (mod 5) there are only finitely many 1-unicuspidal, genus-\( g \) curves up to equisingularity.

**Proof.** If \( g \equiv 2 \) or \( g \equiv 4 \) modulo 5, the congruence \( x^2 \equiv 4(2g - 1) \) (mod 5) has no solution, since 2 and 3 are not quadratic residues modulo 5. Hence Equation (3.1.5) has no solution. Therefore, there are only finitely many possible degrees for a 1-unicuspidal genus-\( g \) curve, and for any fixed degree and genus, by (3.1.3), the number of possible pairs \((a, b)\) is finite. \( \square \)
The second corollary is a degree-multiplicity inequality in the spirit of Matsuoka–Sakai \cite{40} and Orevkov \cite{62}. For convenience let \( \phi \) denote the golden ratio \( \phi = \frac{1+\sqrt{5}}{2} \).

**Corollary 3.1.4.** Let \( g \geq 1 \). Then there exists a constant \( c \) such that

\[
\phi^2a - c < d < \phi^2a + c
\]

for every 1-unicuspidal genus-\( g \) curve of degree \( d \) with Puiseux pair \((a,b)\).

**Proof of Corollary 3.1.4.** Plugging the relation \( b = 3d - a \) obtained from (3.1.4) into (3.1.3) we get that for almost all pairs

\[
(d-1)(d-2) = (a-1)(3d-a-1) + 2g.
\]

Recall that \( a \) is the multiplicity of the singularity, i.e. the local intersection multiplicity of the singular branch with a generic line through the singular point. Therefore, due to Bézout’s theorem, it can not be larger than \( d \). So from the above equation one can compute

\[
a = \frac{3d - \sqrt{5d^2 + 4(2g-1)}}{2}
\]

and notice that \( \phi^2a - d \) is bounded for \( a \geq 0, d \geq 0 \). \( \square \)

**Remark 3.1.5.** The case \( g = 0 \) is excluded in Theorem 3.1.2: singularities of 1-unicuspidal rational curves have been classified in [21] (see Theorem 2.5.1), and the result does not hold in this case. However, applying Theorem 3.1.1 (which in the case of \( g = 0 \) is the main theorem of [11], see Theorem 2.4.3) we can recover the four infinite families of singularities (Theorem 2.5.1 (a)–(d)) obtained in [21] (see Remarks 3.2.11 and 3.2.18).

The proof of Theorem 3.1.2 relies almost exclusively on Theorem 3.1.1 except when \( g = 1 \). In that case, Theorem 3.1.1 alone cannot exclude the family \((a,b) = (l,9l+1)\) with \( d = 3l \) (Case VII in the proof of Proposition 3.2.2 in Subsection 3.2.1). This family can be excluded using an inequality due to Orevkov \cite{62}, as pointed out by Borodzik, Hedden and Livingston in [10].

**Remark 3.1.6.** Corollary 3.1.4 in particular shows that an analogue of Orevkov’s asymptotic inequality between the multiplicity and the degree holds for any fixed genus \( g \) in the special case of 1-unicuspidal curves. Even more surprisingly, in the 1-unicuspidal case, for any fixed genus \( g \geq 1 \) an asymptotic inequality *in the opposite direction* holds as well.

In Section 3.3 we discuss the solutions of (3.1.5) and describe them completely for certain values of \( g \) (see Remark 3.3.13). This is a preparation for the proof of Theorem 3.1.7 stated below.
It turns out that for some of such genera (see the last paragraph of Theorem 3.1.7 below), almost all solutions of \((3.1.5)\) can be realized as Puiseux pairs of a 1-unicuspidal curve. This construction is given in Section 3.4, where we construct an infinite family of 1-unicuspidal curves for each triangular genus.

Before stating the main result of the last section, we set up some notation first. Given an integer \(k\), denote by \((L^k_n)_{n \in \mathbb{Z}}\) the generalized Fibonacci or Lucas sequence defined by setting \(L^k_0 = k - 1, L^k_1 = 1\) and the recurrence \(L^k_{n+1} = L^k_n + L^k_{n-1}\). Notice that \(n\) varies among integers rather than positive integers.

It is easy to check that for every \(i \geq 2\) the pair \((a, b) = (L^k_{4i-3}, L^k_{4i+1})\) is a solution of \((3.1.5)\) if \(g = k(k - 1)/2\). In this case, the degree is \(d = L^k_{4i-1}\). Also, for \(j \geq 1\) the pair \((a, b) = (-L^k_{-4j+1}, -L^k_{-4j-3})\) is a solution of \((3.1.5)\) if \(g = k(k - 1)/2\). In this case, the degree is \(d = -L^k_{-4j-1}\).

**Theorem 3.1.7.** Let \(k \geq 2\) be an integer, and define \(g = k(k - 1)/2\). For each \(i \geq 2\) there exists a unicuspidal projective plane curve of genus \(g\) and degree \(d = L^k_{4i-1}\) such that the singularity has one Puiseux pair \((a, b) = (L^k_{4i-3}, L^k_{4i+1})\). Similarly, for each \(j \geq 1\) there exists a unicuspidal projective plane curve of genus \(g\) and degree \(d = -L^k_{-4j-1}\) such that the singularity has one Puiseux pair \((a, b) = (-L^k_{-4j+1}, -L^k_{-4j-3})\).

Moreover, if \(k \equiv 2 \pmod{3}\) and \(2g - 1\) is a power of a prime, then any 1-unicuspidal curve of genus \(g\) and sufficiently large degree has one of the singularities listed above.

Thus, for certain genera, namely for genera of form \(g = k(k - 1)/2\) for some \(k \equiv 2 \pmod{3}\) such that \(2g - 1\) is a prime power, we obtain an almost complete classification (i.e., a classification up to finitely many exceptions) of possible torus knot types of 1-unicuspidal genus-\(g\) curves.

The results of this chapter were achieved in a joint work with Daniele Celoria and Marco Golla and are published in the joint article [6]. We should note here that in [10], Borodzik, Hedden and Livingston used Theorem 3.1.1 to classify cusp types with one Puiseux pair of genus-1 curves, up to finitely many possible exceptions.

### 3.2. An equation for unicuspidal curves with torus knot link

This section was published in a joint article with D. Celoria and M. Golla [6, §6], thus it is a product of several revisions. I am particularly grateful to Marco Golla and the referee of [6] for many useful suggestions and improvements on the content. I would like to thank Marco Golla for the two figures in Subsection 3.2.2 as well.
In this section our goal is to prove Theorem 3.1.2. We fix a positive integer \( g \) and we restrict ourselves to unicuspidal curves of genus \( g \) whose singularity has only one Puiseux pair \( (a, b) \), \( 1 < a < b \). Recall that in this case, we say that the curve is \((a, b)\)-unicuspidal.

Notice that by the degree-genus formula (3.1.3), once we fix \( g \), the pair \( (a, b) \) determines uniquely the degree \( d \) (unless \( g = 0 \) and \( a = 1 \), in which case we have no singularity).

**Definition 3.2.1.** We say that a pair \((a, b)\) with \( a < b \) is a candidate (to be the Puiseux pair of a unicuspidal genus \( g \) curve) if \( a \) and \( b \) are coprime and there is a positive integer \( d \) such that the degree-genus formula (3.1.3) holds.

If the corresponding semigroup counting function \( H \) of the singularity with one Puiseux pair \((a, b)\) satisfies (3.1.2) with the given genus \( g \) for all possible values of \( j \) and \( k \) as specified in Theorem 3.1.1, we say that the pair \((a, b)\) is an admissible candidate.

We are going to say that a certain property holds for almost all elements in a set if there are finitely many elements for which it does not hold.

Theorem 3.1.2 is a consequence of the following two propositions. Recall that we fixed the genus \( g \geq 1 \) of the curves we consider.

**Proposition 3.2.2.** If \( g \geq 1 \), then for almost all admissible candidates \((a, b)\) the ratio \( b/a \) lies in the interval \((6, 7)\).

**Proposition 3.2.3.** If \( g \geq 1 \), then for almost all admissible candidates \((a, b)\) such that \( b/a \in (6, 7) \) we have \( a + b = 3d \).

**Remark 3.2.4.** We note here that in the proof of Proposition 3.2.2 we use the recent work of Borodzik, Hedden and Livingston [10] to exclude the family \((a, b) = (l, 9l + 1)\) in the case \( g = 1 \).

**Proof of Theorem 3.1.2.** Combining the two propositions above, we get that almost all admissible pairs \((a, b)\) satisfy \( a + b = 3d \). If we substitute \( 3d = a + b \) in the degree-genus formula (3.1.3) we readily obtain Equation (3.1.5). \( \square \)

We prove Proposition 3.2.2 in Subsection 3.2.1 and Proposition 3.2.3 in Subsection 3.2.2.

**3.2.1. The proof of Proposition 3.2.2.** Before diving into the actual proof, we set up some notation and some preliminaries.

We are going to denote by \( \mathbb{N} \) the set of non-negative integers, \( \mathbb{N} = \{0, 1, \ldots\} \). Recall that the semigroup \( \Gamma \subset \mathbb{N} \) associated with the singularity with Puiseux pair \((a, b)\) is generated by \( a \) and \( b \): \( \Gamma = \langle a, b \rangle \). For any positive integer \( n \) let us denote by \( \Gamma(n) \) the \( n \)-th smallest element (with respect to the natural ordering of integers) of the semigroup.
CUSPIDAL CURVES OF HIGHER GENUS

Γ; for example, Γ(1) is always 0 and Γ(2) is always a. For the sake of brevity, for the rest of this chapter set \( R_n = H(n) \) for the values of the semigroup counting function of Γ; that is, \( R_n \) is the number of non-negative integers \( 0 \leq m < n \) such that \( m = va + ub \) for some non-negative integers \( u, v \).

We introduce the notation \( \Delta_j \) for the triangular number \( \frac{(j+1)(j+2)}{2} \). Setting \( k = 0 \) and using the lower bound in (3.1.2), for every \( j = 0, 1, \ldots, d-2 \) we get the inequalities:

\[
\Delta_j \leq R_{jd+1}, \quad \text{or, equivalently,} \quad \Gamma(\Delta_j) \leq jd;
\]

while setting \( k = g \) and using the upper bound, for every \( j = 0, 1, \ldots, d-2 \) we get:

\[
R_{jd+1-2g} \leq \Delta_j, \quad \text{or, equivalently,} \quad \Gamma(\Delta_j + 1) > jd - 2g.
\]

Every semigroup element can be expressed as \( ub + va \) for some non-negative integers \( u \) and \( v \). Writing \( \Gamma(\Delta_j) = ub + va \), \((\ast_j)\) reads \( ub/j + va/j \leq d \), and substituting this into the degree-genus formula (3.1.3) we get:

\[
2g + (a-1)(b-1) \geq \left( \frac{u}{j} b + \frac{v}{j} a - 1 \right) \left( \frac{u}{j} b + \frac{v}{j} a - 2 \right).
\]

Analogously, if we write \( \Gamma(\Delta_j + 1) = ub + va \), \( (\ast\ast_j) \) reads \( ub/j + va/j > d - 2g/j \), and substituting this into (3.1.3) we get:

\[
2g + ab - a - b + 1 < \left( \frac{u}{j} b + \frac{v}{j} a + 2g/j \right) \left( \frac{u}{j} b + \frac{v}{j} a + 2g - 2j \right).
\]

Equations (3.2.1) and (3.2.2) give a prescribed region for admissible candidate pairs \((a,b)\); since the two inequalities are quadratic in \( a \) and \( b \), the boundary of such region is a conic, typically a hyperbola. We are interested in the equation of the asymptotes of these hyperbolae, especially their slope. This motivates the following definition.

**Definition 3.2.5.** We say that for a set \( \mathcal{P} \subset \mathbb{N}^2 \) of pairs \((a,b)\) the asymptotic inequality \( \frac{b}{a} \preceq \alpha \) (respectively \( \frac{b}{a} \succeq \alpha \)) holds, if there is a constant \( C \) such that \( b \leq aa + C \) (resp. \( b \geq aa + C \)) for almost all pairs in \( \mathcal{P} \). We say that \( \frac{b}{a} \approx \alpha \) if both \( \frac{b}{a} \preceq \alpha \) and \( \frac{b}{a} \succeq \alpha \).

**Remark 3.2.6.** In this language, the Matsuoka–Sakai inequality means that \( d/a \preceq 3 \) and Orevkov’s sharper result [62] means that \( d/a \preceq \phi^2 \). Notice that these results hold without any restrictions on the number of cusps and the Puiseux pairs: if we let \( m_p \) be the minimal positive element in the semigroup of the singularity at \( p \), then \( a \) is replaced by \( \max_p m_p \).

We now compute the slopes and equations of asymptotes of region boundaries arising from (3.2.1) and (3.2.2).
**Lemma 3.2.7.** Fix real constants $p, q, c_1, c_2, c_3$ such that $1 - 4pq > 0$. If $p \neq 0$, let 
\[ \lambda_{\pm} = \frac{(1 - 2pq \pm \sqrt{1 - 4pq})}{2p^2} \]
and let $((a_n, b_n))_{n \geq 1}$ be a sequence of pairs of non-negative integers $a_n \leq b_n$ with $a_n \to \infty$. If almost all pairs $(a_n, b_n)$ belong to $D$ then:

- if $p = 0$, then $b_n/a_n \geq q^2$;
- if $p \neq 0$, then 
  \[ \lambda_- \leq \frac{b_n}{a_n} \leq \lambda_+ . \]

On the other hand, if almost all pairs $(a_n, b_n)$ do not belong to $D$, then:

- if $p = 0$, then $b_n/a_n \leq q^2$;
- if $p \neq 0$, then 
  either $\frac{b_n}{a_n} \leq \lambda_-$ or $\lambda_+ \leq \frac{b_n}{a_n}$, 
  in the sense that the pairs can be divided into two subsets such that for the pairs in the first, resp. in the second subset the first, resp. the second asymptotic inequality holds.

**Proof.** If $p = 0$, the conic has a vertical asymptote; the other asymptote is defined by the equation $(qx + c_1)(qx + c_2) = (y - 1)(x - 1) + c_3$ and has slope $q^2$, from which we immediately obtain that $b_n/a_n \geq q^2$ if almost all pairs $(a_n, b_n)$ are in $D$, and $b_n/a_n \leq q^2$ if almost all pairs are outside $D$.

A similar argument applies when $p \neq 0$. In this case, both asymptotes are non-vertical and their slopes are the solutions of the equation $p^2\lambda^2 + (2pq - 1)\lambda + q^2 = 0$, which are precisely $\lambda_{\pm}$. The analysis of the two cases is straightforward. \[\square\]

**Remark 3.2.8.** In some cases we will also need to compute (in terms of $p, q, c_1, c_2, c_3$) the largest constant $C_l$, respectively the smallest constant $C_s$, for which the following property holds: for every $\varepsilon > 0$, for almost all pairs $(a, b)$ satisfying the assumptions of Lemma 3.2.7, the suitable combination of inequalities (depending on the actual applicable statement of the lemma) of type

\[ \lambda_{\pm} a + C_l - \varepsilon < b, \quad \text{resp.} \quad b < \lambda_{\pm} a + C_s + \varepsilon \]

holds. We will call such constants optimal. Rather than a priori computing the explicit constant, we will do it only when needed. Observe that the optimal constant is in fact the constant term in the normalized equation $y = Ax + C$ of the line which is the asymptote
of the (relevant branch of the) hyperbola described by

\[(py + qx + c_1)(py + qx + c_2) = (y - 1)(x - 1) + c_3.\]

We set out to prove that \(b/a\) lies in the interval \((6, 7)\) for almost all admissible candidates \((a, b)\).

**Lemma 3.2.9.** For every \(M > 0\) there are finitely many admissible candidates with \(a < M\).

**Proof.** Suppose that there are infinitely many admissible candidates with \(a < M\). Then for infinitely many candidates \(b > 3M > 3a\) holds. From Equation (3.1.3) it follows that \(d > 3M + 2g\) for infinitely many candidates; if \(b > 3a\), however, the fourth semigroup element is \(3a\). By \([**_1]\) we get \(3a > d - 2g > 3M\), contradicting the assumption \(a < M\). \(\Box\)

We handle the problem in seven cases, depending on the integer part of \(b/a\). The general pattern of the proof in each case is the following. First, we choose an appropriate \(j\) and we determine \(u\) and \(v\) such that \(\Gamma(\Delta_j) = ub + va\) (respectively \(\Gamma(\Delta_j + 1) = ub + va\)). We then apply \([*_j]\) or \([**_j]\) to get a quadratic inequality of type (3.2.1) or (3.2.2).

Of course, we can apply \([*_j]\) and \([**_j]\) only when \(j \leq d - 2\), but since for any bounded \(d\) there are only finitely many admissible candidates by the degree-genus formula (3.1.3), any result obtained by applying \([*_j]\) or \([**_j]\) with \(j\) bounded will be valid for almost all admissible candidates. We wish to emphasize here that the actual upper bound for \(j\) may (and in many cases indeed will) depend on the fixed genus \(g\).

Finally, we apply Lemma 3.2.7 and Remark 3.2.8 to obtain from (3.2.1) and (3.2.2) inequalities of the type

\[\alpha a + C_1 - \varepsilon < b < \beta a + C_2 + \varepsilon,\]

valid for all but finitely many relevant admissible candidates for any choice of \(\varepsilon > 0\).

In most of the cases, \(\alpha\) and \(\beta\) will be rational. If needed, we repeat the above process and choose new values of \(j\) to get better estimates, until we get estimates with \(\alpha = \beta\), i.e. we arrive at a bound of type

\[ra + C_1 \leq sb \leq ra + C_2\]

with some constants \(r, s, C_1, C_2\), where \(r\) and \(s\) are integers and \(C_1\) and \(C_2\) might depend on \(g\). That is, we have an asymptotic equality \(b/a \approx r/s\) rather than two asymptotic inequalities.

In this way, we reduce each case to a finite number of possible linear relations between \(a\) and \(b\), that is, relations of form \(ra + C = sb\) with integral coefficients.
As soon as we have such a relation, we can ask the following question: is it possible for a pair \((a,b)\) satisfying this relation to be a candidate in the sense of Definition 3.2.1? Solving the degree-genus formula (3.1.3) as a quadratic equation in \(d\), we see that there is an integral solution for \(d\) if and only if 
\[4(a-1)(b-1) + 8g + 1 = K^2,\]
where we write \(K\) instead of \((2d - 3)\). Plugging in the linear relation between \(a\) and \(b\), we can show that the equation has very few solutions.

We are now ready to prove Proposition 3.2.2. Recall that \(g \geq 1\) is an arbitrary genus but it is fixed during the proof. Also, by Lemma 3.2.9, for any fixed bound on \(a\), there are at most finitely many admissible candidates \((a,b)\) (with degree \(d\)) for a genus \(g_1\)-unicuspidal curve singularity; likewise, by the degree-genus formula (3.1.3), for any fixed bound on the degree \(d\), there are at most finitely many admissible candidates \((a,b)\).

**Proof of Proposition 3.2.2**

**Case I:** \(1 < b/a \leq 2\). Choose \(j = 1\). Now \(b = \Gamma(\Delta) = \Gamma(3)\) and by \((\star_1)\) we have \(b \leq d\), and Lemma 3.2.7 with Remark 3.2.8 implies \(b < a + 1 + \varepsilon\) for any \(\varepsilon > 0\) for almost all admissible candidates. So eventually \(b = a + 1\) for almost all admissible pairs (that is, \(b/a \approx 1^2\)). Plugging this relation into the degree-genus formula (3.1.3) we obtain

\[4(a-1)a + 8g + 1 = K^2 \iff (2a - 1)^2 + 8g = K^2\]

and this is possible for infinitely many \(a\) only if \(g = 0\).

**Case II:** \(2 < b/a \leq 4\). Choose \(j = 1\), and observe that \(2a = \Gamma(\Delta)\). By \((\star_1)\) \(2a \leq d\), so by Lemma 3.2.7 and Remark 3.2.8 we have \(4a - 1 - \varepsilon < b\), so \(b = 4a - 1\) for almost all admissible pairs \((b/a \approx 2^2)\).

\[4(a-1)(4a - 2) + 8g + 1 = K^2 \iff (4a - 3)^2 + 8g = K^2,\]

and this equation has infinitely many solutions only if \(g = 0\).

**Case III:** \(4 < b/a \leq 5\). Choose \(j = 2\) and apply \((\star_2)\) we obtain \(b = \Gamma(\Delta) \leq 2d\), that is \(b < 4a + 1 + \varepsilon\), so \(b = 4a + 1\) for almost all admissible pairs \((b/a \approx 2^2)\).

\[4(a-1)4a + 8g + 1 = K^2 \iff (4a - 2)^2 + 8g - 3 = K^2,\]

and this is not possible for infinitely many \(a\) for any non-negative \(g\), as \(8g - 3 \neq 0\).

**Case IV:** \(5 < b/a \leq 6\). We have \(5a = \Gamma(\Delta)\). Choose \(j = 2\) and apply \((\star_2)\), so that \(5a \leq 2d\), hence \(b/a \geq 25/4 > 6\), so there are at most finitely many admissible pairs \((a,b)\) in this case.

**Case V:** \(7 < b/a \leq 8\). This needs a longer examination.
First choose $j = 3$, and apply \((\star_3)\) this yields $8a = \Gamma(\Delta_3) \leq 3d$, from which we obtain that for any $\varepsilon > 0$, for almost all admissible candidates belonging to this case,

\[(3.2.3) \quad 64a/9 + 1/9 - \varepsilon < b.\]

Set $j = 4$ and notice that $11a = \Gamma(\Delta_4 + 1)$; using \((\star_4)\) we obtain that $b/a \leq (11/4)^2$.

We will now compare $2b$ with $15a$.

Assume first $2b > 15a$. Then from \((\star_7)\) we get $2b + 4a = \Gamma(\Delta_7 + 1) > 7d - 2g$, hence (by Lemma 3.2.7) $\frac{33}{8} + \frac{7}{8}\sqrt{17} \leq \frac{b}{a}$ (notice that the other asymptotic inequality is irrelevant in this case). Since $\frac{112}{37} < \frac{33}{8} + \frac{7}{8}\sqrt{17}$, combining the two asymptotic inequalities we just obtained, we proved that there are only finitely many admissible pairs $(a, b)$ in this subcase.

So we can assume $2b < 15a$. Again, from \((\star_7)\) we get $19a = \Gamma(\Delta_7 + 1) > 7d - 2g$, hence $b/a \leq (19/7)^2$.

**Claim 3.2.10.** The 45th element of the semigroup is $2b + 7a$, that is, $2b + 7a = \Gamma(45)$.

**Proof.** Notice that the elements preceding $2b + 7a$ are exactly the following:

$$0, a, \ldots, 21a; b, b + a, \ldots, b + 14a; 2b, 2b + a, \ldots, 2b + 6a,$$

as $2b + 7a < 3b$ and $2b + 7a < 22a$. \(\square\)

From the claim above and \((\star_8)\) we get $2b + 7a = \Gamma(\Delta_8) \leq 8d$, which implies $b/a \leq \frac{9}{2} + \frac{4}{\sqrt{2}}$.

In particular, we can assume $b/a < 22/3$, and hence $6b + 2a < 3b + 24a$. This means that the elements $44a, b + 37a, 2b + 30a, 3b + 23a, 4b + 16a, 5b + 9a, 6b + 2a$ all precede $3b + 24a$. So $ub + va < 3b + 24a$ if $7u + v \leq 44$. There are 168 semigroup elements of this form. In addition, the elements $45a < b + 38a < 2b + 31a$ also precede $3b + 24a$. So $3b + 24a$ is at least the 172nd semigroup element. Using \((\star_17)\) we get

$$3b + 24a \geq \Gamma(\Delta_{17} + 1) > 17d - 2g \implies b/a \leq 64/9 = (8/3)^2.$$ 

Notice again that the other asymptotic inequality obtained by Lemma 3.2.7 is irrelevant since we are in the case $7 < b/a \leq 8$.

Coupled with \((3.2.3)\), this means that $64a + 1 - \varepsilon \leq 9b \leq 64a + C$ for some constant $C$. Thus, for any given positive integer $k$ and for any sufficiently large $a$, \((\star_{6k+17})\) reads

$$3b + (16k + 24)a = \Gamma(\Delta_{6k+17} + 1) > (6k + 17)d - 2g,$$

which shows that

$$9b \leq 64a + \frac{96g + 6k + 17}{6k + 1} + \varepsilon$$

holds for any $\varepsilon > 0$ for almost all admissible candidates belonging to this case.
The fraction on the right-hand side tends to 1 as \( k \to \infty \), so we can chose a large enough \( k \) (depending on the given fixed genus \( g \)) and a small enough \( \varepsilon > 0 \) such that 
\[
\frac{96g + 6k + 17}{6k + 1} + \varepsilon < 2.
\]
Combining this inequality with (3.2.3) applied with an \( \varepsilon < 1/9 \), we see that for all but finitely many admissible pairs \( (a, b) \) belonging to this case the inequalities 
\[ 64a < 9b < 64a + 2 \]
hold, that is, \( 64a + 1 = 9b \) for almost all admissible pairs in this case. Plugging this into the degree-genus formula (3.1.3) we get:
\[
4(a - 1) \left( \frac{64a}{9} + \frac{1}{9} - 1 \right) + 8g + 1 = K^2 \iff (16a - 9)^2 + 72g - 40 = 9K^2
\]
and this has finitely many solutions \( a \), as \( 72g - 40 \neq 0 \).

**Case VI:** \( 8 < b/a \leq 9 \). Since \( b + 3a = \Gamma(\Delta_4 + 1) \), from (3.2.3) we get \( b + 3a > 4d - 2g \), and by Lemma 3.2.7 we have \( 9a - 6g - 2 - \varepsilon < b \Rightarrow b/a \approx 9 = 3^2 \).

So \( 9a - C < b < 9a \), and from this it is not hard to see that for any positive integer \( k \) for every sufficiently large \( a \) (and \( b \)) we have \( \Gamma(\Delta_{6k+4} + 1) = (2k + 1)b + 3a \). This means that using (3.2.3) we get \( (2k + 1)b + 3a > (6k + 4)d - 2g \Rightarrow 9a - \frac{6a + 3k + 2}{3k + 1} - \varepsilon \leq b \).

The lower bound tends to \( 9a - 1 - \varepsilon \) as \( k \to \infty \), so fixing a large enough \( k \) depending on \( g \) only and a small enough \( \varepsilon > 0 \), we obtain that \( 9a - 1 = b \) holds for almost all admissible pairs in this case.

\[
4(a - 1)(9a - 2) + 8g + 1 = K^2 \iff (18a - 11)^2 + 72g - 40 = (3K)^2.
\]
This is not possible for infinitely many \( a \) for any non-negative \( g \) as \( 72g - 40 \neq 0 \).

**Case VII:** \( 9 < b/a \). Choose \( j = 1 \) and notice that \( 3a = \Gamma(\Delta_1 + 1) \); by (3.2.1) we get \( 3a > d - 2g \), hence, by Lemma 3.2.7 \( b/a \leq 9 \Rightarrow b/a \approx 9 = 3^2 \).

So we obtained \( 9a < b < 9a + C \), and from this it is not hard to see that for any positive integer \( k \) for every sufficiently large \( a \) (and \( b \)) we have \( \Gamma(\Delta_{6k+4} + 1) = (18k + 12)a \), leading via (3.2.3) to \( (18k + 12)a > (6k + 4)d - 2g \Rightarrow b \leq 9a + \frac{6a + 3k + 2}{3k + 1} + \varepsilon \).

The upper bound tends to \( 9a + 1 + \varepsilon \) as \( k \to \infty \), so (fixing again a large enough \( k \) and a small enough \( \varepsilon > 0 \)) we have \( 9a + 1 = b \) for almost all admissible pairs in this case.

\[
4(a - 1)9a + 8g + 1 = K^2 \iff (6a - 3)^2 + 8g - 8 = K^2,
\]
which is possible for infinitely many \( a \) only if \( g = 1 \). This family \( (a, b) = (a, 9a + 1) \) with \( d = 3a \) is however excluded in Example 9.3.

This concludes the proof.

**Remark** 3.2.11. Notice that we used the assumption \( g \geq 1 \) only in Cases I and II. If \( g = 0 \), from the proof above we get that almost all admissible candidates \((a, b)\)
satisfying \( b/a \notin (6,7) \) are either of the form \((a,b) = (l, l + 1)\) for some \( l \geq 2 \) or of the form \((a,b) = (l, 4l - 1)\) for some \( l \geq 2 \). These are the infinite families (a) and (b) of Theorem 2.5.1 ([21, Theorem 1.1]).

### 3.2.2. The proof of Proposition 3.2.3

Before turning to the proof of Proposition 3.2.3, we recall some basic facts about the Fibonacci numbers. The interested reader is referred to [62, Section 6] for further details.

Recall that we denote with \( \phi \) the golden ratio, \( \phi = \frac{1+\sqrt{5}}{2} \). The Fibonacci numbers are defined by recurrence as \( F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1} \); more explicitly, one can write \( F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \).

We collect in the following proposition some useful identities about the Fibonacci sequence, easily proved either by induction or by substituting the explicit formula above.

**Proposition 3.2.12.** The following identities hold for any integers \( k \geq 2 \) and \( l \geq 1 \):

\[
\begin{align*}
\text{(3.2.4)} & \quad \gcd(F_{2l-1}, F_{2l+1}) = \gcd(F_{2l-1}, F_{2l+3}) = 1 \\
\text{(3.2.5)} & \quad F_k^2 - F_{k-2}F_{k+2} = (-1)^k \\
\text{(3.2.6)} & \quad F_{k-2} + F_{k+2} = 3F_k \\
\text{(3.2.7)} & \quad F_{2l+3}^2 F_{2l-1}^2 - F_{2l+1}^2 (F_{2l+1}^2 + 2) = 1 \\
\text{(3.2.8)} & \quad F_{2l+3}^2 + F_{2l+1}^2 - 3F_{2l+1}F_{2l+3} = -1 \\
\text{(3.2.9)} & \quad \lim_{l \to \infty} \frac{F_{2l-1}^2}{F_{2l+1}^2} \left( \phi^4 - \frac{F_{2l+1}^2}{F_{2l-1}^2} \right) = \frac{2}{5} \left( \phi^4 - 1 \right) \\
\text{(3.2.10)} & \quad \lim_{l \to \infty} \frac{F_{2l+1}^2}{F_{2l-1}^2} \left( \phi^{-4} - \frac{F_{2l+1}^2}{F_{2l-1}^2} \right) = \frac{2}{5} \left( 1 - \phi^{-4} \right).
\end{align*}
\]

In this subsection, \((a,b)\) will always denote a pair such that \( 6 < b/a < 7 \). We want to prove that an admissible pair \((a,b)\) and the corresponding degree \( d \) are tied by the relation \( a + b = 3d \), with at most finitely many exceptions.

In the course of the proof, we will state several lemmas, systematically postponing their proof to the next Subsection 3.2.3.

**Proof of Proposition 3.2.3.** Using \((\ast_3)\) and the fact that \( a + b = \Gamma(\Delta_3) \leq 3d \), which in turn by Lemma 3.2.7 implies \( b/a \leq \phi^4 \).

More precisely (see Remark 3.2.8), the relevant asymptote of the hyperbola \( \gamma_0 \) determined by \( a + b = 3d \) has equation \( b = \phi^4 a \).

In particular, for any \( \varepsilon > 0 \), for almost all admissible pairs we have

\[
\text{(3.2.11)} \quad b \leq \phi^4 a + \varepsilon.
\]
One easily verifies that for every \( D \geq 1 \) the hyperbola \( \gamma_D \) determined (via the degree-genus formula (3.1.3)) by \( a + b = 3d - D \) lies below \( \gamma_0 \) in the relevant region \( \{2 \leq a < b\} \).

As \( a + b \leq 3d \), we only need show that only finitely many admissible pairs satisfy \( a + b < 3d \). It will turn out that the line \( b = \phi^4a \) plays a crucial role in the proof. We divide the region below it into infinitely many sectors cut out by lines of slope \( F_{2l+1}/F_{2l-1} \) \((l \geq 2)\). Notice that the sequence \( (F_{2l+1}/F_{2l-1})_l \) is increasing in \( l \) and tends to \( \phi^4 \) as \( l \to \infty \). See Figure 3.1.

**Definition 3.2.13.** The \( l \)-th sector \( S_l \) \((for \ l \geq 2)\) is the open region in the positive quadrant bounded by lines \( F_{2l+1}a = F_{2l-1}b \) and \( F_{2l+3}a = F_{2l+1}b \). That is,

\[
S_l = \{(a,b) : F_{2l+1}a/F_{2l-1} < b < F_{2l+3}a/F_{2l+1}\}.
\]

The \( l \)-th punctured sector \( S_l^* \) is defined as \( S_l^* = S_l \setminus \{(F_{2l-1}, F_{2l+3})\} \).

The following lemma takes care of the region below all sectors.

**Lemma 3.2.14.** There are at most finitely many admissible candidates such that \( b < 25a/4 = F_{5}^2a/F_{2}^2 \).

The next lemma ensures that almost all admissible candidates below the line \( b = \phi^4a \) live in the sectors \( S_l, \ l \geq 2 \) if \( g \geq 1 \). In fact, we prove more, namely that almost all admissible candidates below that line are in the punctured sectors.

**Lemma 3.2.15.** For \( g \geq 1 \), there are only finitely many candidates of the form \( (a,b) = (F_{2l-1}, F_{2l+3}) \) and \( (a,b) = (F_{2l-1}, F_{2l+1}) \), \((l \geq 2)\).

For admissible candidates inside the sectors, we prove the following.
Lemma 3.2.16. For any admissible candidate \((a, b) \in S_l\) at least one of the following upper bounds hold:

\[(3.2.12)\]
\[a \leq 2(2g - 1)F_{2l+1} + 2,\]

\[(3.2.13)\]
\[b \leq 2(2g - 1)\frac{F_{2l+1}^2}{F_{2l-1}} + 2.\]

In particular, there are finitely many admissible candidates in each sector.

Recall that at the beginning of the proof we already obtained \(a + b \leq 3d\) for almost all admissible candidates, so we want to prove that there are in fact only finitely many admissible candidates such that \(a + b \leq 3d - 1\). Notice that all pairs \((a, b)\) satisfying \(a + b \leq 3d - 1\) and the degree-genus formula \((3.1.3)\) lie on or below \(\gamma_1\), which has an asymptote (in the relevant region \(\{0 < a < b\}\)) with equation \(b = \phi^4 a - 2\phi^2 / \sqrt{5}\).

To finish the proof, we need one final lemma.

Lemma 3.2.17. There is a decreasing, infinitesimal sequence \((C_l)_{l \geq 2}, C_l \to 0\) of real numbers (which further depends on \(g\)) such that for every \(l \geq 2\) and for every admissible candidate \((a, b) \in S_l:\)

\[0 \leq \phi^4 a - b \leq C_l.\]

Now we can show that almost all admissible candidates lie above the line \(b = \phi^4 a - 1\). This is obviously true for pairs such that \(b \geq \phi^4 a\). To handle pairs below the line \(b = \phi^4 a\), first apply Lemmas 3.2.14 and 3.2.15 and conclude that almost all admissible candidates in this case lie in the union of punctured sectors \(S_l^*\) for \(l \geq 2\). Now choose \(l_0\) such that \(C_{l_0} < 1 < (2\phi^2) / \sqrt{5}\). From Lemma 3.2.16 above, we know that there are only finitely many admissible candidates in sectors \(S_l\) with \(l \leq l_0\).

So almost all admissible candidates are in sectors \(S_l\) with \(l > l_0\). For these, by Lemma 3.2.17, the inequality \(\phi^4 a - b < C_{l_0} < 1\) holds. Notice that for \(g \geq 1\) the relevant branch (i.e. the branch falling into the sector \(\{0 < a < b\}\)) of the hyperbola \(\gamma_1\) lies above its asymptote, and recall that the latter has equation \(b = \phi^4 a - 2\phi^2 / \sqrt{5}\).

Denote by \((a_1, b_1)\) the intersection point of \(\gamma_1\) and the line \(b = \phi^4 a - 1\) in the positive quadrant, i.e. \(0 < a_1 < b_1\). One can easily compute that \(a_1 = 2g\).

We proved that almost all admissible candidates with \(a + b < 3d\) lie on or below \(\gamma_1\) and above the line \(b = \phi^4 a - 1\) (by the argument involving Lemma 3.2.17, Lemma 3.2.16 and a suitable choice of \(l_0\)), and as these two intersect (see Figure 3.2), almost all admissible candidates not satisfying \((3.1.5)\) lie in a bounded region \(a \leq a_1 = 2g\) and \(b \leq b_1\).

Therefore, almost all admissible candidates satisfy \(a + b = 3d\). \(\square\)
Remark 3.2.18. By some small modifications of the argument, we are able to recover (up to finitely many candidates in a bounded region, which after working out a concrete bound, can be checked one by one by computer) the classification result of [21, Theorem 1.1] for \( g = 0 \) as well. This is particularly interesting because our method uses the semigroup distribution property of Theorem 3.1.1 only (which, for \( g = 0 \) is a result of [11]). As we mentioned earlier, Tiankai Liu in his PhD thesis among other results also reproved this classification based on the semigroup distribution property only, see [35, Theorem 2.3].

So assume for the moment \( g = 0 \). Recall that according to Remark 3.2.11 we can deal with candidates \((a, b)\) such that \(b/a \notin (6, 7)\). For the case \(b/a \in (6, 7)\), one can make the following changes to the proof above:

- The pairs listed in Lemma 3.2.15 are admissible candidates; in fact, 1-unicuspidal rational curves with those singularities exist: these are families (c) and (d) of Theorem 2.5.1 ([21, Theorem 1.1]).
- The asymptote of \(\gamma_0\) still has equation \(b = \phi^4 a\), but \(\gamma_0\) now lies below it. So it is enough to deal with pairs below this line. This region is divided into sectors \(S_l\).
- From the proof of Lemma 3.2.16 we obtain that there are no admissible candidates in the punctured sectors \(S_l^*\). So, in fact, almost all admissible candidates are those already enumerated in Lemma 3.2.15.

Figure 3.2. The hyperbolae \(\gamma_0\) and \(\gamma_1\) in the \((a, b)\)-plane, together with the lines \(b = \phi^4 a\), \(b = \phi^4 a - 1\) and \(b = a\)
3. CUSPIDAL CURVES OF HIGHER GENUS

In this way, we obtain an almost complete classification in the rational case: families (c) and (d) of Theorem 2.5.1 ([21, Theorem 1.1]) are obtained above, and families (a) and (b) were obtained in Remark 3.2.11. We also see that almost all admissible candidates for \( g = 0 \) belong to one of these families.

### 3.2.3. Technical proofs.

In this subsection we deal with all the lemmas stated above. The following claim will be useful in the proof of several lemmas.

**Claim 3.2.19.** Assume that we have three reduced fractions

\[
0 < \frac{m_1}{n_1} < \frac{b}{a} < \frac{m_2}{n_2}
\]

and set \( P = m_2n_1 - m_1n_2 \). Then

\[
b \geq \frac{m_1 + m_2}{P} \quad \text{and} \quad a \geq \frac{n_1 + n_2}{P}.
\]

**Proof.** Write \( b = \lambda_1m_1 + \lambda_2m_2 \) and \( a = \lambda_1n_1 + \lambda_2n_2 \). Since \( b/a \) falls between the two endpoints, \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \). As \( m_2n_1 - m_1n_2 = P \), due to Cramer's rule, the coefficients can be written in a form \( \lambda_1 = \ell_1/P \) and \( \lambda_2 = \ell_2/P \), for some integers \( \ell_1 \) and \( \ell_2 \), not necessarily coprime with \( P \). Since they are also positive, \( \ell_1 \geq 1 \) and \( \ell_2 \geq 1 \), so \( b \geq (m_1 + m_2)/P \) and \( a \geq (n_1 + n_2)/P \).

**Proof of Lemma 3.2.14.** Apply \([*2]\): \( 5a = \Gamma(\Delta_2) \leq 2d \) from which (via Lemma 3.2.7 and Remark 3.2.8) \( 25a/4 - 1/4 - \varepsilon < b < 25a/4 \). Therefore, for almost all admissible candidates \( 4b = 25a - 1 \) holds. For the candidates lying on this line, however, \( 5a = 2d \) cannot hold, as then \( a \) would be even; a contradiction. So actually \( 5a \leq 2d - 1 \), but the hyperbola determined by \( 5a \leq 2d - 1 \) already intersects the line \( 4b = 25a - 1 \), providing for \( a \) the following upper bound: \( a \leq 4g/5 + 1/5 \).

**Proof of Lemma 3.2.15.** For every \( l \geq 2 \) the triples \((a, b, d_0) = (F_{2l-1}, F_{2l+3}, F_{2l+1})\) and \((a, b, d_0) = (F^2_{2l-1}, F^2_{2l+1}, F_{2l+1}F_{2l-1})\) satisfy the degree-genus formula \((3.1.3)\) with \( g = 0 \): this is a consequence of Proposition 3.2.12 above. In particular, \( 4(a-1)(b-1) = (2d_0 - 3)^2 - 1 \). (Notice that these triples are realized by families (c) and (d) of Theorem 2.5.1.)

In general, solving \((3.1.3)\) as a quadratic equation for \( d \), one sees that for a given \( g \), \((a, b)\) is a candidate if and only if \( 4(a-1)(b-1) + 8g + 1 = (2d-3)^2 \). Comparing this with the above relation \( 4(a-1)(b-1) = (2d_0 - 3)^2 - 1 \) we get that

\[
(2d_0 - 3)^2 + 8g = (2d - 3)^2
\]

which has only finitely many solutions for \( d \neq d_0 \) if \( g \neq 0 \).

Before the proof of Lemma 3.2.16 we need some preparation.
CLAIM 3.2.20. Let \( \frac{b}{a} \neq \frac{F_{2l+3}}{F_{2l-1}} \) be a reduced fraction in the open interval \( \left( \frac{F_{2l+1}}{F_{2l-1}}, \frac{F_{2l+3}}{F_{2l+1}} \right) \) (where \( l \geq 2 \)). Then \( a > F_{2l-1} \) and \( (a - 1)(b - 1) \geq F_{2l+1}(F_{2l+1} + 1) \).

PROOF. Together with

(3.2.14) \[ a \geq F_{2l-1} + 1, \]

we are going to prove that

(3.2.15) \[ b \geq \frac{F_{2l+1}^2 + F_{2l+3}}{F_{2l-1}}. \]

In fact, the above two inequalities imply that

\[(b - 1)(a - 1) \geq F_{2l+1}^2 + F_{2l+3} - F_{2l-1} \geq F_{2l+1}^2 + F_{2l+1},\]

where the last inequality follows from \( F_{2l+3} = F_{2l+2} + F_{2l+1} > 2F_{2l+1} \).

We split the proof into four cases:

(i) \( \frac{b}{a} \in \left( \frac{F_{2l+1}^2 + F_{2l+3}}{F_{2l-1}}, \frac{F_{2l+3}}{F_{2l-1}} \right) \)

(ii) \( \frac{b}{a} \in \left( \frac{F_{2l+1}^2 + F_{2l+3}}{F_{2l-1}}, \frac{F_{2l+3}}{F_{2l-1}} + 2 \right) \)

(iii) \( \frac{b}{a} = \frac{F_{2l+1}^2 + 2}{F_{2l-1}} \)

(iv) \( \frac{b}{a} \in \left( \frac{F_{2l+1}^2 + 2}{F_{2l-1}}, \frac{F_{2l+3}}{F_{2l+1}} \right) \)

Notice that each fraction above is in reduced form (see (3.2.4), (3.2.7)).

(i) Using Equation (3.2.5), we get that, in the notation of Claim 3.2.19, \( P = F_{2l-1}F_{2l+3} - F_{2l+1}F_{2l-1} = F_{2l-1} \), so via Claim 3.2.19 we immediately obtain (3.2.15) and (3.2.14).

(ii) If \( b/a < (F_{2l+1}^2 + 2)/F_{2l-1} \), using Claim 3.2.19 we compute \( P = F_{2l-1} \), hence (3.2.15) and (3.2.14) both hold (the estimate for \( b \) is much larger than needed).

(iii) When \( b/a = (F_{2l+1}^2 + 2)/F_{2l-1} \), (3.2.15) reads:

\[ b = F_{2l+1}^2 + 2 \geq \frac{F_{2l+1}^2 + F_{2l+3}}{F_{2l-1}}, \]

which follows from rearranging

\[ F_{2l+1}^2 (F_{2l-1} - 1) \geq F_{2l}^2 \geq 3F_{2l} = F_{2l+3} - 2F_{2l-1}. \]

On the other hand, the inequality (3.2.14) is obvious.
(iv) If \((F_{2l+1}^2 + 2)/F_{2l-1}^2 < b/a\), then using Claim 3.2.19, we get \(P = 1\). This leads to
\[ b \geq F_{2l+3}^2 + F_{2l+1}^2 + 2 > \frac{F_{2l+1}^2 + F_{2l+3}^2}{F_{2l-1}^2} \]
and
\[ a \geq F_{2l+1}^2 + F_{2l-1}^2 > F_{2l-1} + 1, \]
which show both (3.2.15) and (3.2.14).

In particular, the above claim says that for a pair \((a, b) \in S_l^*\) the assumptions of the next Lemma 3.2.21 hold automatically.

**Lemma 3.2.21.** If for an admissible candidate \((a, b)\) we have \(F_{2l+1} \leq d - 2\), \(F_{2l-1} < a\) and \((F_{2l+1}/F_{2l-1})^2 < b/a\), then one of the following two inequalities hold:

\[ F_{2l+3}a \leq F_{2l+1}d \quad \text{or} \quad F_{2l-1}b \leq F_{2l+1}d. \]

**Proof.** The key point is that due to the assumption \(F_{2l+1} \leq d - 2\) we can apply \((\ast)_{F_{2l+1}}\)

We count how many semigroup elements \(ub + va\) can precede \(F_{2l+3}a\). Since we assumed \((F_{2l+1}/F_{2l-1})^2 < b/a\), we can prove that \(ub + va > F_{2l+3}a\) as soon as \(uF_{2l+1}^2 > F_{2l+1}^2(F_{2l+3} - v)\). So there is a chance to have \(ub + va < F_{2l+3}a\) only if

\[ uF_{2l+1}^2 \leq F_{2l-1}^2(F_{2l+3} - v). \]

So bounding the number of semigroup elements that precede \(F_{2l+3}a\) turns into the question of how many integer pairs \((u, v)\) satisfy \(0 \leq u, 0 \leq v\), and \(uF_{2l+1}^2 + vF_{2l-1}^2 \leq F_{2l-1}^2F_{2l+3}\). Denote the set of these pairs by \(H_l\) and its cardinality by \(N_l\). Notice that the pair \((u, v) = (0, F_{2l+3})\) in \(H_l\) corresponds to \(F_{2l+3}a\). Later, it will be important that \((u, v) = (F_{2l-1}, 0) \in H_l\) as well (see (3.2.5)), i.e. the corresponding element, \(F_{2l-1}b\) can precede \(F_{2l+3}a\).

**Claim 3.2.22.** The cardinality of \(H_l\) is \(N_l = \Delta_{F_{2l+1}} + 1\).

**Proof.** Notice that \(N_l\) is the number of integral lattice points on the boundary or in the interior of the triangle \(T\) with vertices given by coordinates \(O = (0, 0)\), \(A = (0, F_{2l+3})\) and \(C = (F_{2l+3}F_{2l-1}^{2l+1}, 0) = (F_{2l-1} + F_{2l-1}^{2l+1}, 0)\) (use (3.2.5)).

We will count the integral lattice points in the interior or on the boundary of a smaller triangle \(T'\) with integral lattice point vertices given by coordinates \(O = (0, 0)\), \(A = (0, F_{2l+3})\) and \(B = (F_{2l-1}, 0)\) instead. This number will be \(N_l\) as well, as there is no lattice point in the closure of the difference \(T \setminus T'\) (triangle \(ABC\)) except points \(A\) and \(B\). To see this, assume that there is such a lattice point \(P\) with coordinates \((u, v)\) in the triangle
ABC. Set \(0 \leq s := F_{2l+3} - v \leq F_{2l+3}\) and \(r := u\), and compare the slopes of lines \(AC\), \(AB\) and \(AP\): the existence of the point \(P\) would mean that the slope of \(AP\) (which is \(-s/r\)) is either strictly between the slopes of \(AB\) and \(AC\) (being \(-F_{2l+3}/F_{2l-1}\) and \(-F_{2l+1}^2/F_{2l-1}^2\), respectively), or coincides with one of them. But this is a contradiction, since the fractions \(P\) that is the first inequality we were looking for.

(0

there is no rational number \(r/s\) such that \(s \leq F_{2l+3}\) and

\[
\frac{F_{2l-1}}{F_{2l+3}} < \frac{r}{s} < \frac{F_{2l-1}^2}{F_{2l+1}^2}
\]

To prove the above fact, use Claim 3.2.19 and notice that, with the notation therein,

\[P = F_{2l+3}F_{2l-1}^2 - F_{2l-1}F_{2l+1}^2 = F_{2l-1}\] (use (3.2.5)), and get \(s \geq (F_{2l+3}F_{2l+1}^2)/F_{2l-1} > F_{2l+3}\).

(For this last inequality use again (3.2.5) and the trivial fact that \(F_{2l+3} > 1\).

Since \(F_{2l+1}\) and \(F_{2l+3}\) are coprime, there are no lattice points on the hypotenuse of the triangle \(T'\) other than the endpoints. In this way, \(N_l = 1 + \frac{1}{2}(F_{2l-1} + 1)(F_{2l+3} + 1)\) (half of the number of the lattice points in the appropriate closed rectangle, plus one endpoint of the hypotenuse), which, using (3.2.6) further equals \(1 + \frac{1}{2}(F_{2l+1} + 1)(F_{2l+1} + 2) = 1 + \Delta F_{2l+1}\). □

This means that at most \(\Delta F_{2l+1}\) semigroup elements can precede \(F_{2l+3}a\) (remember that \((0, F_{2l+3}) \in H_l\). So \(F_{2l+3}a\) is at most the \((\Delta F_{2l+1} + 1)\)-th element: \(F_{2l+3}a \leq \Gamma(\Delta F_{2l+1} + 1)\).

If \(F_{2l+3}a\) was not the \((\Delta F_{2l+1} + 1)\)-th, then by \([\star F_{2l+1}]\) we would have

\[F_{2l+3}a \leq \Gamma(\Delta F_{2l+1}) \leq F_{2l+1}d\]

that is the first inequality we were looking for.

On the other hand, if \(F_{2l+3}a\) is the \((\Delta F_{2l+1} + 1)\)th element, i.e. \(F_{2l+3}a = \Gamma(\Delta F_{2l+1} + 1)\), then all the semigroup elements corresponding to integer pairs in \(H_l\) have to be smaller than \(F_{2l+3}a\) (except of course \(F_{2l+3}a\) itself). In particular, \(F_{2l-1}b < F_{2l+3}a\) (equality here can not hold for \(a = F_{2l-1}\) due to coprimality), so \(F_{2l-1}b\) is at most the \(\Delta F_{2l+1}\)-th element. In this case, applying \([\star F_{2l+1}]\) we have

\[F_{2l-1}b \leq \Gamma(\Delta F_{2l+1}) \leq F_{2l+1}d\]

Thus the proof of Lemma 3.2.21 is completed. □

**Proof of Lemma 3.2.16** First notice that both estimates are true for \((a, b) = (F_{2l-1}, F_{2l+3})\), therefore we can assume that \((a, b) \in S_1^*\), thus we can apply Claim 3.2.20 and Lemma 3.2.21.

Assume that for an admissible candidate \((a, b)\) in the \(l\)-th sector the first inequality of Lemma 3.2.21 holds: that is, the quantity \(r = F_{2l+1}d - F_{2l+3}a\) is non-negative. Let

\[s = F_{2l+3}^2 - F_{2l+1}^2 b \geq 1\]. A direct computation of the intersection of the line \(F_{2l+3}^2 - F_{2l+1}^2 b\) –
\[ F_{2l+1}^2 b = s = \text{constant} \] and the hyperbola \( F_{2l+1}^2 d - F_{2l+3}^2 a = r = \text{constant} \) yields

\[(3.2.16) \quad a = (2g - 1) \frac{F_{2l+1}^2}{s + 2F_{2l+3}^2 - 1} + \frac{s + 3F_{2l+1}^2 r - r^2}{s + 2F_{2l+3}^2 - 1}.\]

(To obtain this, from the two equations defining \( s \) and \( r \) express \( b \) and \( d \) in terms of \( a, r, s \), then substitute into the degree-genus formula \((3.1.3)\); express \( a \) in terms of \( r, s \) and finally use the identity \((3.2.8)\).)

If \( r = 0 \), then, as \( F_{2l+1} \) and \( F_{2l+3} \) are coprime, \( F_{2l+1} \) divides \( a \), so \( s \) is divisible by \( F_{2l+1} \) as well. In particular, \( s \geq F_{2l+1} \). So we can estimate the right hand side of the above expression as follows:

\[ a \leq (2g - 1) \frac{F_{2l+1}^2}{F_{2l+1}^2 - 1} + 2. \]

If \( r \geq 1 \), then (using \( s \geq 1 \) as well) we get the following upper bound:

\[ a \leq (2g - 1) \frac{F_{2l+1}^2}{2F_{2l+3}^2} + 1. \]

The upper bound given in \((3.2.12)\) is a generous upper estimate for both of the above bounds.

If the second case of Lemma \(3.2.21\) holds, introduce notations \( s = F_{2l+1}^2 b - F_{2l+1}^2 a \) and \( r = F_{2l+1}^2 d - F_{2l+1}^2 b \). In a similar way as above, we see that \((3.2.13)\) is a (rather generous) upper bound for \( b \). □

Observe that in the case \( g = 0 \) the statement of Lemma \(3.2.16\) does not hold; however, a similar computation shows that for any admissible candidate in the punctured sector \((a, b) \in S_l^*\) either \( a < 2 \) or \( b < 2 \) holds (or both): in fact, the first summand of the expression \((3.2.16)\) is negative in this case and the second is at most 2. So the above proof also shows that there are no admissible candidates in \( S_l^* \) for \( g = 0 \). See also Remark \(3.2.18\).

**Proof of Lemma \(3.2.17\)**. It is obvious that \( 0 \leq \phi^4 a - b \), as the pair \((a, b)\) is assumed to be in the sector \( S_l \) which is below the line \( b = \phi^4 a \). To obtain the upper bound, we use Lemma \(3.2.16\). When \((3.2.12)\) holds, using \((3.2.9)\) we can write:

\[ 0 \leq \phi^4 a - b \leq \phi^4 a - \frac{F_{2l+1}^2}{F_{2l-1}^2} a = a \left( \phi^4 - \frac{F_{2l+1}^2}{F_{2l-1}^2} \right) \]

\[ \leq \left( 2(2g - 1) \frac{1}{F_{2l+1}^2} F_{2l+1}^2 + 2 \right) \left( \phi^4 - \frac{F_{2l+1}^2}{F_{2l-1}^2} \right) \to 0. \]
3.3. GENERALIZED PELL EQUATION

On the other hand, when \((3.2.13)\) holds, using \((3.2.10)\) we can write:

\[
0 \leq \phi^4a - b \leq \phi^4 \frac{F_{2l-1}}{F_{2l+1}}b - b = \phi^4b \left( \frac{F_{2l-1}}{F_{2l+1}} - \phi^{-4} \right) \\
\leq \phi^4 \left( 2(2g - 1) \frac{1}{F_{2l-1}} F_{2l+1}^2 + 2 \right) \left( \frac{F_{2l-1}}{F_{2l+1}} - \phi^{-4} \right) \to 0.
\]

□

3.3. Generalized Pell equation

This section is taken from the joint work [6, §7]. However, some parts of the argument (especially around Proposition 3.3.10) are given in a slightly different presentation.

This section is a short trip in number theory, in which we study the solutions of the generalized Pell equation

\[(\spadesuit_n)\]

\[x^2 - 5y^2 = n\]

for certain values of \(n\). We will be interested mainly in the case \(n = 4(2g - 1)\). This is closely related to Equation \((3.1.5)\). In particular, we will be determining the values of \(n = 4(2g - 1)\) for which there exists a solution \((x, y)\) to \((\spadesuit_n)\) where \(x\) and \(y\) are coprime; for these \(g\) there are infinitely many such pairs, and this allows us to generalize Corollary 3.1.3 and prove Theorem 3.1.7. Along the way, we will also introduce some notation that we will use in the next section.

We will work in the ring \(O = \mathcal{O}_K\) of integers of the real quadratic field \(K = \mathbb{Q}(\sqrt{5})\); there is an automorphism on \(O\), that we call conjugation and denote with \(\alpha \mapsto \overline{\alpha}\), that is obtained by restricting the automorphism of \(K\) (as a \(\mathbb{Q}\)-algebra) that maps \(\sqrt{5}\) to \(-\sqrt{5}\).

Given \(\alpha = x + y\sqrt{5} \in O\) we will call \(N(\alpha) = \alpha \cdot \overline{\alpha} = x^2 - 5y^2\) the norm of \(\alpha\). The element \(\phi = \frac{1 + \sqrt{5}}{2} \in O\) has norm \(-1\), hence it is a unit.

We begin by collecting some well-known facts about \(O\). (See, for example, [39, Chapters 2 and 3].)

**Theorem 3.3.1 ([39, Chapter 2]).** The ring \(O\) has the following properties:

- \(O\) is generated (as a ring) by \(\phi\), i.e. \(O = \mathbb{Z}[\phi]\);
- \(O\) is a Euclidean ring, hence it is a principal ideal domain (PID);
- \(\phi\) is a unit in \(O\). The group of units \(O^*\) of \(O\) is isomorphic to \(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\), and the isomorphism maps \(\phi\) to \((1, 0)\) and \(-1\) to \((0, 1)\). In particular, elements of norm 1 are exactly those of the form \(\pm \phi^{2h}\) for some integer \(h\).

Notice that if \(N(\alpha)\) is prime in \(\mathbb{Z}\), then \(\alpha\) itself is prime in \(O\). Since \(O\) contains \(\mathbb{Z}[\sqrt{5}]\), whose additive group is isomorphic to \(\mathbb{Z}^2\), in what follows we will frequently identify a
pair of integers \((x, y)\) with the algebraic integer \(x + y\sqrt{5}\). In particular, we will identify a solution \((x, y)\) of Equation \((\spadesuit_n)\) with the associated element \(x + y\sqrt{5}\) of \(\mathcal{O}\) of norm \(n\).

We start by looking at equation \((\spadesuit_p)\) when \(p\) is a prime (see Corollary 3.3.4). We recall a standard result about factorization of primes in \(\mathcal{O}\).

**Theorem 3.3.2** ([39, Theorem 25]). Given a prime \(p \in \mathbb{Z}\), consider the ideal \(P = p\mathcal{O}\):

(i) \(P\) is prime if and only if \(p \equiv \pm 2 \pmod{5}\);
(ii) \(P = Q^2\) for some prime ideal \(Q \subset \mathcal{O}\) if and only if \(p = 5\);
(iii) \(P = QQ'\) for two distinct prime ideals \(Q, Q' \subset \mathcal{O}\) if and only if \(p \equiv \pm 1 \pmod{5}\); moreover, in this case \(Q' = \overline{Q}\). Additionally: if \((5) = Q_1Q_2\) for some prime ideals \(Q_1, Q_2\), then \(Q_1 = Q_2 = (\sqrt{5})\); if \(p \equiv \pm 1 \pmod{5}\) and \(Q_1Q_1 = Q_2Q_2 = p\mathcal{O}\) are two prime factorisations of \(p\mathcal{O}\), then either \(Q_1 = Q_2\) or \(Q_1 = \overline{Q_2}\).

In particular, since \(\mathcal{O}\) is a PID, the ideal \(Q\) of point (iii) above is generated by an element \(\alpha \in \mathcal{O}\), whose norm \(N(\alpha)\) is \(\pm p\). Since \(\phi\) is a unit of norm \(-1\), up to multiplication by \(\phi\), we can assume that \(N(\alpha) = p\).

**Claim 3.3.3.** The ideal \(Q\) in the above Theorem 3.3.2 is generated by an element of norm \(p\) in \(\mathbb{Z}[\sqrt{5}] \subset \mathcal{O}\).

**Proof.** Let \(\alpha\) be a generator of \(Q\) with norm \(p\). Write \(\alpha = u + v\phi\). If \(v\) is even, there is nothing to prove; therefore, let us suppose that \(v\) is odd.

If \(u\) is even, consider \(\alpha' = \phi^2\alpha\); \(N(\alpha') = N(\alpha)N(\phi)^2 = p\), and, since \(\phi^2 = \phi + 1\), we have
\[
\alpha' = \phi^2 \cdot (u + v\phi) = u\phi + u + 2\phi v + v = u + v + (u + 2v)\phi,
\]
and \(u + 2v\) is even.

Analogously, if \(u\) is odd, consider \(\alpha' = \overline{\phi^2}\alpha\); as above, \(N(\alpha') = p\), and, since \(\overline{\phi} = 1 - \phi\) and \(\overline{\phi}^2 = 2 - \phi\), we have
\[
\alpha' = \overline{\phi^2}(u + v\phi) = 2u - u\phi - v(1 - \phi) = 2u - v + (v - u)\phi,
\]
and \(v - u\) is even.

In either case, \(\alpha' \in \mathbb{Z}[\sqrt{5}]\) and has norm \(p\). \(\square\)

**Corollary 3.3.4.** For every prime number \(p \equiv 0, \pm 1 \pmod{5}\) there exist two integers \(x, y \in \mathbb{Z}\) such that \(x^2 - 5y^2 = p\).

We set \(\alpha_p = x + y\sqrt{5}\) for an arbitrary fixed choice of such element \(\alpha_p \in \mathcal{O}\) and we call it a fundamental solution of the generalized Pell equation \((\spadesuit_p)\). If \(p \neq 5\), we also have that \(\alpha_p\) and \(\overline{\alpha_p}\) are coprime. Set also \(\alpha_5 = 5 + 2\sqrt{5}\) and notice that \(\alpha_5\) and \(\overline{\alpha_5}\) are associates.
The above considerations allow us to characterize the primes of $O$ in the following way. For any prime $p \equiv 0, \pm 1 \pmod{5}$ in $\mathbb{Z}$, fix some choice of a fundamental solution $\alpha_p$ as above.

**Proposition 3.3.5.** Every prime in $O$ is an associate of exactly one of the following:  
- $q \in \mathbb{Z} \subset O$, where $q \equiv \pm 2 \pmod{5}$ is a prime among integers;  
- $\alpha_5 \in \mathbb{Z}[\sqrt{5}] \subset O$;  
- $\alpha_p \in \mathbb{Z}[\sqrt{5}] \subset O$ or $\alpha_p \in \mathbb{Z}[\sqrt{5}] \subset O$ (defined as in the proof of Claim 3.3.3 and Corollary 3.3.4, but with some fixed choice) for every prime $p \equiv \pm 1 \pmod{5}$ among integers.

**Corollary 3.3.6.** Let $\alpha$ be an element of $O$, $p, q$ two primes in $\mathbb{Z}$ such that $p \equiv \pm 1 \pmod{5}$ and $q \equiv \pm 2 \pmod{5}$.  
- $N(\alpha) = 1 \iff \alpha = \pm \phi^{2h}, h \in \mathbb{Z}$.  
- $N(\alpha) = 5 \iff \alpha = \pm 5\phi^{2h}, h \in \mathbb{Z}$.  
- $N(\alpha) = p \iff \alpha = \pm \alpha_p \phi^{2h}$ or $\alpha = \pm \bar{\alpha}_p \phi^{2h}, h \in \mathbb{Z}$.  
- $N(\alpha) = q^2 \iff \alpha = \pm q\phi^{2h}$.  
- $N(\alpha) = q$ has no solution for $\alpha \in O$.  

The values of $\alpha$ enumerated above are pairwise different.

Recall that we are looking for coprime solutions $(a, b)$, $0 < a < b$ of the equation (3.1.5). Notice that if $(a, b)$ is an integral solution, then $\left(\frac{7b-2a}{3}\right)^2 = 5b^2 + 4(2g - 1)$ is automatically an integer, therefore 3 divides $7b-2a$, i.e. we have integral solution of the generalized Pell equation

\[(3.3.1) \quad x^2 - 5y^2 = 4(2g - 1).\]

This is equivalent to the following:

\[(3.3.2) \quad N\left(\frac{x}{2} + \frac{y}{2}\sqrt{5}\right) = 2g - 1.\]

**Definition 3.3.7.** We say that a pair $(a, b)$ of integers corresponds to a pair of integers $(x, y)$ if $x = (7b - 2a)/3$ and $y = b$. In this case we also say that $(a, b)$ corresponds to the element $\zeta = x + y\sqrt{5} \in O$.

Notice that if $(a, b)$ is a solution of (3.1.5), then it corresponds in the above sense to some element $2\alpha$, where $\alpha \in O$ is a solution of $N(\alpha) = 2g - 1$, cf. Equation (3.3.2).

**Definition 3.3.8.** We call an integer pair $(x, y)$ almost coprime if $\gcd(x, y) \mid 2$. 
Proposition 3.3.9. Let \((a, b)\) be a coprime solution of \((3.1.5)\). Then the pair \((x, y)\) to which it corresponds is almost coprime, \(3 \nmid y\) and satisfies \((3.3.1)\). Conversely, let \((x, y)\) be a pair of almost coprime positive integers such that \(3 \nmid y\) and \((3.3.1)\) holds. Then the pair \((a, b)\) which corresponds to it is a coprime solution of \((3.1.5)\).

Proof. Assume we have a coprime solution \((a, b)\) of \((3.1.5)\). Observe that
\[
\gcd(x, y) | \gcd(3x, y) = \gcd(7b - 2a, b) = \gcd(2a, b) | 2
\]
therefore \(3 \nmid y\) and \(\gcd(x, y) = \gcd(3x, y) = \gcd(2a, b)\). Moreover:

(i) either \(\gcd(x, y) = 1\), and this holds if and only if \(b\) is odd; in this case \((x, y)\) is a coprime solutions of the equation \(x^2 - 5y^2 = 4(2g - 1)\);

(ii) or \(\gcd(x, y) = 2\), and this holds if and only if \(b\) is even and \(a\) is odd; in this case \((x', y')\) is a coprime solution of \(x'^2 - 5y'^2 = 2g - 1\), where \(x = 2x', y = 2y'\).

Conversely, assume we have an almost coprime pair \((x, y)\) satisfying \((3.3.1)\) such that \(3 \nmid y\).

Assume first \(y\) is odd, hence \(x, y\) are in fact coprime. Then we get a pair \((a, b)\) which is a solution of \((3.1.5)\), with \(a = (7y - 3x)/2\) and \(y = b\). Since \(y\) is odd, parity considerations (via \(x^2 - 5y^2 = 4(2g - 1)\)) imply that so is \(x\). Hence \(a\) is automatically an integer. Notice also that since \(y = b\) is odd, \(\gcd(a, b) = \gcd(2a, b) = \gcd(3x, y) = \gcd(x, y) = 1\).

Assume now \(y\) is even and set \(y' = y/2\). Then \((x/2)^2 - 5y'^2 = 2g - 1\), hence \(x\) must be even as well, write \(x = 2x'\). Then we get a pair \((a, b)\) which is a solution of \((3.1.5)\), with \(a = 7y' - 3x'\) and \(b = 2y'\). As \(x' + y' \equiv x'^2 - 5y'^2 = 2g - 1 \equiv 1 \pmod{2}\), \(x'\) and \(y'\) have different parity. Thus, \(\gcd(x', y') = 1\), which implies that \(a\) and \(b\) are also coprime.

\[\square\]

Thus when looking for coprime solutions of \((3.1.5)\), we are looking for some almost coprime solutions of \((3.3.1)\), equivalently, Equation \((3.3.2)\).

Proposition 3.3.10. Equation \((3.3.2)\) has a solution \(\alpha = \frac{x}{2} + \frac{y}{2}\sqrt{5} \in \mathcal{O}\) with almost coprime \((x, y)\) if and only if \(2g - 1\) has no prime factors congruent to \(\pm 2\) modulo 5 and \(25 \nmid 2g - 1\).

In the case it has a solution \(\alpha = \frac{x}{2} + \frac{y}{2}\sqrt{5}\) with almost coprime \((x, y)\), then writing
\[
2g - 1 = 5^e p_1^{\gamma_1} \cdots p_\omega^{\gamma_\omega}
\]
for the prime decomposition (in \( \mathbb{Z} \)) of \( 2g - 1 \) (with \( \varepsilon \in \{0, 1\} \), \( \gamma_i \) positive integer), we have that all solutions \( \alpha = \frac{x}{2} + \frac{y}{2} \sqrt{5} \) of (3.3.2) are of form

\[
\alpha = \pm \phi^{2h} \alpha_5^{\delta_5} \alpha_{p_1}^{\delta_1} \ldots \alpha_{p_\omega}^{\delta_\omega} \alpha_\gamma^{-\gamma_1} \ldots \alpha_\delta^{-\delta_\omega}
\]

with \( h \in \mathbb{Z} \), where \( \alpha_{p_i} \) can be either \( \alpha_{p_i} \) or \( \bar{\alpha}_{p_i} \), and these choices can be made independently for all \( i = 1, \ldots, \omega \). The above almost coprime solutions \((x, y)\) are all different.

In other words, all almost coprime solutions \((x, y)\) of (3.3.1) are of form

\[
x + y \sqrt{5} = 2\alpha = \pm 2\phi^{2h} \alpha_5^{\delta_5} \bar{\alpha}_{p_1}^{\gamma_1} \ldots \alpha_{p_\omega}^{\gamma_\omega} \bar{\alpha}_\gamma^{-\gamma_1} \ldots \bar{\alpha}_{\delta}^{-\delta_\omega}, \quad h \in \mathbb{Z}.
\]

**Proof.** Write the prime decomposition of \( 2g - 1 \) in \( \mathbb{Z} \) and the prime decomposition of \( \alpha \) in \( \mathcal{O} \). Recall Proposition 3.3.5 and Corollary 3.3.6 and use the multiplicativity of the norm.

If \( 2g - 1 \) had an odd prime factor \( q \equiv \pm 2 \pmod{5} \), then the prime \( q \in \mathcal{O} \) would divide \( \alpha \in \mathcal{O} \), therefore \( x \) and \( y \) would not be almost coprime.

If \( 25 \mid 2g - 1 \), then \( \alpha_5^2 \) would divide \( \alpha \) in \( \mathcal{O} \), but then \( 5 \) would divide \( \alpha \in \mathcal{O} \) (as it divides \( \alpha_5^2 \) in \( \mathcal{O} \)), so again, \( x \) and \( y \) would not be almost coprime.

Now writing \( 2g - 1 = 5^\varepsilon p_1^{\gamma_1} \ldots p_\omega^{\gamma_\omega} \) for the prime decomposition (in \( \mathbb{Z} \)) of \( 2g - 1 \) with \( p_i \) being odd primes \( \equiv \pm 1 \pmod{5} \), we get that \( \alpha \) must have the following prime decomposition in \( \mathcal{O} \):

\[
\alpha = \pm \phi^{2h} \alpha_5^{\delta_5} \alpha_{p_1}^{\delta_1} \ldots \alpha_{p_\omega}^{\delta_\omega} \alpha_\gamma^{-\gamma_1} \ldots \alpha_{\delta}^{-\delta_\omega}.
\]

If \( 0 < \delta_i < \gamma_i \) for some \( i \), then \( p_i = \alpha_{p_i} \bar{\alpha}_{p_i} \mid \alpha \in \mathcal{O} \), therefore again, \( p \) would divide \( \alpha \) in \( \mathcal{O} \), so \( x \) and \( y \) would not be almost coprime.

Conversely, if \( \alpha \) is of the above form, no odd prime \( p \equiv \pm 1 \pmod{5} \) can divide \( \alpha \in \mathcal{O} \), because \( p = \alpha_{p_i} \bar{\alpha}_{p_i} \) is not a divisor of \( \alpha \) (which is obvious from its prime decomposition in \( \mathcal{O} \)). Also, \( 5 \) does not divide \( \alpha \), as \( 5 = \alpha_5^2 \phi^{-6} \) does not divide \( \alpha \). There are no odd primes \( q \equiv \pm 2 \pmod{5} \) in the prime decomposition of \( \alpha \) over \( \mathcal{O} \), so these can not divide \( \alpha \) either. Finally, \( 4 \nmid \gcd(x, y) \), otherwise \( 2 \mid \alpha \) in \( \mathcal{O} \), therefore \( N(2) = 4 \) would divide \( N(\alpha) = 2g - 1 \), which is impossible.

The fact that all the solutions listed above are different, follows from the fact that \( \alpha_p \) and \( \bar{\alpha}_p \) are not associates for \( p \equiv \pm 1 \pmod{5} \).

The above proposition says that if \( 2g - 1 \) is a number such that (3.3.1) has an almost coprime solution \((x, y)\), and \( 2g - 1 \) has \( \omega \) distinct prime factors different from \( 5 \), then we have \( 2^\omega \) distinct choices for \( x + y \sqrt{5} = 2\alpha \in \mathbb{Z} [\sqrt{5}] \) (up to associates). If \( \omega > 0 \), up to conjugation, these reduce to \( \Omega := 2^{\omega - 1} \) different choices. That is to say, we can select \( \Omega \) different elements \( \beta_1, \ldots, \beta_\Omega \in \mathcal{O} \) of norm \( 2g - 1 \) in a way that for every solution \( \beta \) of \( N(\beta) = 4(2g - 1) \) there exist uniquely an index \( j \) and an integer \( h \) such that \( \beta = \pm 2\phi^{2h} \beta_j \).
or \( \beta = \pm 2\phi^{2h} \beta_j \). If \( 2g-1 = 5 \), then every solution of (3.3.1) is of form \( x + y\sqrt{5} = \pm 2\phi^{2h} \alpha_5 \), that is, we do not need conjugation (cf. Example 3.4.12).

**Definition 3.3.11.** We call a set \( \mathcal{F}_{2g-1} = \{ \beta_1, \ldots, \beta_\Omega \} \) described above a generating set of solutions of Equation (3.3.1).

That is, by Proposition 3.3.10 for every almost coprime solution \((x, y)\) of Equation (3.3.1), there exists a unique generating solution \( \beta_j \in \mathcal{F}_{2g-1} \) and a unique integer \( h \) such that \( x + y\sqrt{5} = \pm 2\phi^{2h} \beta_j \) or \( \pm 2\phi^{2h} \beta_j \).

Recall that by Proposition 3.3.9 all coprime solutions \((a, b)\) of (3.1.5) correspond to an almost coprime solution \((x, y)\) of (3.3.1) such that \( 3 \nmid y \).

Now we examine when does \( 3 \mid y \) happen. Suppose that \( x + y\sqrt{5} \) has norm \( N(x + y\sqrt{5}) \equiv 2 \) (mod 3): since 2 is not a quadratic residue modulo 3, then 3 does not divide \( y \). On the other hand, suppose that \( N(x + y\sqrt{5}) \equiv 1 \) (mod 3): in this case there are solutions with \( y \equiv 0 \) (mod 3), and this subset is acted upon by \( \phi^4 = \frac{7+3\sqrt{5}}{2} \). Likewise, the subset of solutions with \( y \not\equiv 0 \) (mod 3) is acted upon by \( \phi^4 \), and multiplication by \( \phi^2 \) takes one family to the other.

We collect all results about coprime solutions of Equation (3.1.5) in the following proposition.

**Proposition 3.3.12.** Let \( \mathcal{F}_{2g-1} \) be a generating set of solutions of Equation (3.1.5), in the sense of Definition 3.3.11. If \((a, b)\) is a coprime solution of Equation (3.1.5), then \((a, b)\) corresponds to either \( \pm 2\phi^{2h} \beta \) or \( \pm 2\phi^{2h} \beta_j \) for some \( \beta \in \mathcal{F}_{2g-1} \) and \( h \in \mathbb{Z} \).

Conversely, given a solution \( \beta \in \mathcal{F}_{2g-1} \):

- if \( g \equiv 0 \) (mod 3), then for every integer \( h \in \mathbb{Z} \) both \( \pm 2\phi^{2h} \beta \) and \( \pm 2\phi^{2h} \beta_j \) are elements such that there are coprime solutions of (3.1.5) corresponding to them;
- if \( g \equiv 1 \) (mod 3), then either for all even integers \( h \), or for all odd integers \( h \), both \( \pm 2\phi^{2h} \beta \) and \( \pm 2\phi^{2h} \beta_j \) are elements such that there exist coprime solutions of (3.1.5) corresponding to them;
- if \( g \equiv 2 \) (mod 3), 3 divides \( 2g-1 \), and there are no coprime solutions of (3.1.5).

We can think of the statement above in the following way. Let \( g \) be a positive integer such that \( 2g-1 \) is either \( n' \) or \( 5n' \), where \( n' \) is a product of primes congruent to \( \pm 1 \) modulo 5. There are \( \Omega = 2^{\omega(n')} - 1 \) families of solutions of Equation (3.1.5), and in each family any two members differ, up to sign and conjugation, by a power of \( \phi^2 \); if \( g \equiv 1 \) (mod 3), this power is always an even power, i.e. a power of \( \phi^4 \).

**Remark 3.3.13.** If we also assume that \( 2g-1 \) is 5 or a power of a prime different from 5, then \( |\mathcal{F}_{2g-1}| = 1 \), hence for any two coprime solutions \((a, b), (a', b')\) of Equation (3.1.5)
there is an integer \( h \) such that the corresponding elements \( \zeta, \zeta' \in \mathcal{O} \) satisfy \( \zeta = \pm \phi^{2h} \zeta' \) or \( \bar{\zeta} = \pm \phi^{2h} \zeta' \) for some integer \( h \). Moreover, if \( g \equiv 1 \mod 3 \), then \( h \) is even.

Recall that we are interested in triangular numbers, which correspond to genera of smooth plane curves, and that we are going to construct curves of triangular genus (Theorem 3.1.7).

**Example 3.3.14.** Let \( k \) be an integer, and define \( g \) as \( g := \frac{k(k-1)}{2} \) and \( n := 2g-1 = k(k-1)-1 \). Notice that, by Proposition 3.3.10, \( n \) has no prime factors congruent to \( \pm 2 \) modulo 5, since \((2k-1,1)\) is a coprime solution of Equation (3.3.1). Also, \( 25 \nmid n = 2g-1 \).

For small triangular numbers, the size \( \Omega \) of the generating set of solutions is often 1. In fact, the smallest case in which \( \Omega > 1 \) is \( g = 105 \), when \( 2g-1 = 209 = 11 \cdot 19 \); we then have two integers \( \alpha, \beta \in \mathcal{O} \) with \( N(\alpha) = 11 \), \( N(\beta) = 19 \), and these give rise to the two solutions \( \alpha \cdot \beta \) and \( \alpha \cdot \bar{\beta} \), which can be chosen to be the two elements of the generating set of solutions \( \mathcal{F}_{2g-1} \).

### 3.4. Curve construction

This section is from a joint work [6, §8] and contains several joint ideas with Marco Golla. I am particularly grateful for his suggestions on how to improve the exposition of the proof of Lemma 3.4.4 and for the idea of using Bertini’s Theorem in the proof of Lemma 3.4.5 and Lemma 3.4.19.

In this section we will construct curves and examples and prove Theorem 3.1.7. Unless otherwise stated, curves will be of genus \( g \geq 1 \).

**3.4.1. Cremona transformations and the proof of Theorem 3.1.7.** In this subsection we examine which solutions of the generalized Pell equation (3.1.5) are realizable by 1-unicuspidal curves.

We use a construction due to Orevkov [62] (see also [10, Proposition 9.12]). Let \( N \) be a nodal cubic; denote by \( B_1 \) and \( B_2 \) the two smooth local branches at the node. Define a birational transformation \( f_1^N : \mathbb{CP}^2 \to \mathbb{CP}^2 \) as follows. Blow up seven points infinitely close to the node of \( N \) at branch \( B_1 \) (resulting in a chain of seven exceptional divisors \( E_1, \ldots, E_7 \), where the divisors are indexed by the order of appearance); then, in the resulting configuration of divisors, blow down the proper transform of \( N \) and six more exceptional divisors \( (E_1, \ldots, E_6 \text{ in this order}) \). The birational map \( f_2^N \) is defined analogously, only we blow up at points on the branch \( B_2 \) instead of \( B_1 \). In both cases, after the last blow-down, the image of the exceptional divisor appearing at the last blow-up (that is, \( E_7 \)) is a nodal cubic [62, Section 6]: we denote them by \( N^1 \) and \( N^2 \) respectively,
and we say that they are associated to \( f_1^N \) and \( f_2^N \), respectively, and each of them is associated to \( N \).

To state the main technical result of this section, we introduce some terminology. In this section, we want to allow the concept of a “smooth Puiseux pair”, by formally allowing pairs \((1, b)\) for \( b > 1 \); notice that this is not a Puiseux pair in the usual sense, since it corresponds to a smooth point on the curve. By extension, a smooth curve will be a \((1, b)\)-unicuspidal curve for every \( b > 1 \).

**Definition 3.4.1.** Let \( N \) be a nodal cubic with node \( p \), and denote by \( B_1, B_2 \) the two branches of \( N \) at the point \( p \). We say that an \((a, b)\)-unicuspidal curve \( C \) of degree \( d \) sweeps \( N \) if:

1. the cusp of \( C \) is at \( p \);
2. the branch of \( C \) at \( p \) has intersection multiplicity \( a \) with \( B_1 \) and \( b \) with \( B_2 \);
3. the only intersection point of \( C \) and \( N \) is \( p \).

Notice that Bézout’s theorem implies that, assuming (1) and (2) in the definition above, (3) is equivalent to the condition \( a + b = 3d \); when \( C \) is smooth, i.e. when \( a = 1 \), this determines \( b = 3d - 1 \).

Finally, observe that \( C \) defines an ordering of the two branches of \( N \) at the node, where the first branch \( B_1 \) is the one with the lower multiplicity of intersection with \( C \).

Now we set up some notation and recall some classical facts about how Puiseux pairs behave under blowing up and blowing down.

Given two curves \( C_1 \) and \( C_2 \) in a surface \( X \) and a point \( r \in X \), denote by \((C_1 \cdot C_2)_r\) the local intersection multiplicity of \( C_1 \) and \( C_2 \) at \( r \). We use the same notation when \( C_1 \) and \( C_2 \) are just local curve branches rather than curves.

**Lemma 3.4.2** (see [79, Theorem 3.5.5], [62, Proposition 3.1]). Let \( \sigma : X \to \mathbb{C}^2 \) be the blow-up of \( \mathbb{C}^2 \) at a point \( q \) with exceptional divisor \( E \); let \( C \) and \( B \) be a local irreducible singular branch and a smooth local branch at \( q \) respectively, with strict transforms \( \tilde{C} \) and \( \tilde{B} \); let \( T \) be a smooth curve branch in \( X \) intersecting \( E \) transversely. Denote by \( p \) the intersection point of \( \tilde{C} \) and \( E \).

If \( C \) has one Puiseux pair \((a, b)\) at \( q \), then \( \tilde{C} \) has a singularity at \( p \) of type:

- \((a, b - a)\), if \( b \geq 2a \);
- \((b - a, a)\), if \( b < 2a \).

In both cases, \((E \cdot \tilde{C})_p = a\) and \((\tilde{B} \cdot \tilde{C})_p = (B \cdot C)_q - a\).

If \( \tilde{C} \) has one Puiseux pair \((a, b)\) at \( p \), then \( C \) has a singularity at \( q \) of the following type:
• one Puiseux pair \((a, a+b)\), if \((E \cdot \bar{C})_p = a\);
• one Puiseux pair \((b, a+b)\), if \((E \cdot \bar{C})_p = b\);
• two Puiseux pairs, if \(a < (E \cdot \bar{C})_p < b\).

In any case, \((T \cdot \bar{C})_p + (E \cdot \bar{C})_p = (\sigma(T) \cdot C)_q\). When \(\bar{C}\) is smooth (that is, when \(a = 1\)), then in case of \((E \cdot \bar{C})_p = a = 1\), \(C\) is also smooth (consistently with the formal fact that its “Puiseux pair” begins with \(a = 1\)); and in case of \((E \cdot \bar{C})_p = b > 1\), \(C\) is singular and has singularity type \((b, a+b) = (b, b+1)\).

The main technical result of this section is the following:

**Proposition 3.4.3.** Let \(C\) be an \((a, b)\)-unicuspidal curve of genus \(g\) and degree \(d = (a+b)/3\) that sweeps the nodal cubic \(N\); suppose that \((a, b)\) corresponds to \(\zeta = x + y\sqrt{5} \in \mathcal{O}\) (in the sense of Definition 3.3.7). Denote by \(f_1^N\), \(f_2^N\) the rational transformations associated to \(N\), and let \(N^1\) and \(N^2\) be the associated nodal cubics, as described above; here we require that the labelling of the branches is the one induced by \(C\). Then:

1. \(f_1^N(C)\) is a \((b, 7b-a)\)-unicuspidal curve of genus \(g\) sweeping \(N^1\); moreover, \((b, 7b-a)\) corresponds to \(\zeta \phi^4\);
2. if \(b < 7a\), \(f_2^N(C)\) is a \((7a-b, a)\)-unicuspidal curve of genus \(g\) sweeping \(N^2\); moreover, \((7a-b, a)\) corresponds to \(\zeta \phi^{-4}\);
3. if \(b > 7a\), \(f_2^N(C)\) is a \((b-7a, 7b-48a)\)-unicuspidal curve of genus \(g\) sweeping \(N^2\); moreover, \((b-7a, 7b-48a)\) corresponds to \(\zeta \phi^{-12}\).

**Proof.** We first observe that the genus is invariant under birational transformations, hence \(f_i^N(C)\) has genus \(g\) for \(i = 1, 2\).

Using the previous lemma, it is not hard to follow what happens with a Puiseux pair \((a, b)\) under the blow-ups and blow-downs giving the birational transformations \(f_1^N\), \(f_2^N\).

Notice that we can assume the inequality \(2a < b\) when we consider \(f_1^N\) and inequality \(13a < 2b\) when we consider \(f_2^N\) (these are needed to conveniently analyse the occurring blow-ups and blow-downs). We will argue that \(b/a > \phi^4\); this is enough, since \(\phi^4\) is larger than both \(13/2\) and \(2\). In fact, \(a + b = 3d\) because \(C\) sweeps \(N\); moreover, \((a, b)\) is a solution of the degree-genus formula (3.1.3), hence it also satisfies (3.1.5). In particular, the ratio \(b/a\) is larger than \(\phi^4\), as desired; this is represented in Figure 3.2. \((a, b)\) belongs to the hyperbola \(\gamma_0\), which lies above its asymptote \(b/a = \phi^4\). Notice that here we are using the assumption \(g \geq 1\), since in the rational case the hyperbola \(\gamma_0\) lies below its asymptote. (See Subsection 3.2.2 and Remark 3.2.18)

Notice also that in the third case, i.e. when applying \(f_2^N\) to an \((a, b)\)-unicuspidal curve with \(b > 7a\), the proof differs slightly according to whether \(b < 8a\) or \(b \geq 8a\); in both cases, however, the final outcome is \((b-7a, 7b-48a)\).
For example, the proof of case (1) goes as follows. (For simplicity, through the whole series of blow-ups and blow-downs, we will use the same notation for a curve/branch and for its strict transform in the other surface.) As \( C \) with a cusp type \((a, b)\) sweeps the nodal cubic \( N \) at the beginning, we have local intersection multiplicities at the node \((C \cdot B_1) = a, (C \cdot B_2) = b\). Thus, by Lemma 3.4.2, first bullet point, after the first blow-up, \( C \) has a cusp type \((a, b - a)\) (recall that we can assume \(2a \leq b\)), \((C \cdot E_i) = a\) and \((C \cdot B_2) = b - a\). The next 6 blow-ups do not affect the cusp type and these local intersection multiplicities, as everything happens at the other branch \( B_1 \). Now when blowing down \( N \), by Lemma 3.4.2, we get a singularity of type \((b - a, b)\), and local intersections \((C \cdot E_i) = a + b - a = b, (C \cdot E_7) = b - a\). Then, after blowing down \( E_i, i = 1 \ldots 6 \), we have a cusp type \((b, (i + 1)b - a)\) and local intersections \((C \cdot E_{i+1}) = b, (C \cdot E_7) = (i + 1)b - a\). In particular, after the last blow-down, the strict transform of \( E_7 \) is a nodal cubic with node at the singular point of \( C \) such that \( C \) has intersection multiplicity \( b \) with one branch and \( 7b - a \) with the other. We will call the first branch \( B_1 \), the second \( B_2 \) and (the strict transform of) the divisor \( E_7 \) we denote by \( N^1 \). We see that (the strict transform of) \( C \) now sweeps the nodal cubic \( N^1 \) as claimed.

The proof of the remaining two cases is a similar step-by-step check of cusp types and local intersection multiplicities.

For elements \( \zeta = x + y\sqrt{5} \) of \( \mathbb{Z}[\sqrt{5}] \) introduce the notation \( \zeta^+ = x + |y|\sqrt{5} \). We have the following corollary of the proposition above.

**Lemma 3.4.4.** Let \( C \) be an \((a, b)\)-unicuspidal curve of genus \( g \) sweeping a nodal cubic \( N \), and suppose that \((a, b)\) corresponds to \( \zeta \in \mathbb{Z}[\sqrt{5}] \). Then there exists \( m \) such that for all \( i \neq m - 1, m \) there exists a 1-unicuspidal curve of genus \( g \) with the Puiseux pair corresponding to \((\zeta \phi^{4i})^+\). Finally, if \( b > 7a \), we can choose \( m = -1 \).

**Proof.** Suppose first for simplicity that \( b > 7a \); we will see later that this assumption is not very restrictive. For each \( i \geq 0 \) we are going to construct an \((a_i, b_i)\)-unicuspidal curve \( C_i \) and an \((a'_i, b'_i)\)-unicuspidal curve \( \overline{C}_i \), where \((a_i, b_i)\) corresponds to \((\zeta \phi^{4i})^+\) and \((a'_i, b'_i)\) corresponds to \((\zeta \phi^{-4(i+3)})^+)\).

The first family of curves is constructed as follows. Let \( C_0 = C, N_0 = N \). For every \( i \geq 0 \), define inductively \( C_{i+1} = f_1^{N_i}(C_i) \) and let \( N_{i+1} = N_i^1 \) be the nodal cubic associated to \( f_1^{N_i} \). Then by Proposition 3.4.3 (1), \( C_i \) is an \((a_i, b_i)\)-unicuspidal curve of genus \( g \) with \((a_i, b_i)\) corresponding to \( \phi^i \zeta \).

The second family is constructed in an analogous way. Define \( \overline{C}_0 = f_2^N(C), \overline{N}_0 = N^2 \). By Proposition 3.4.3 (3), \( \overline{C}_0 \) is a \((b - 7a, 7b - 48a)\)-unicuspidal curve. Call \((a'_0, b'_0)\) the Puiseux pair \((b - 7a, 7b - 48a)\), which corresponds to \( \overline{\zeta} \phi^{-12} \).
As above, for \( i \geq 0 \), define inductively \( \mathcal{C}_{i+1} = \mathcal{f}^N_i(\mathcal{C}_i) \) and let \( \mathcal{N}_{i+1} = \mathcal{N}_i^1 \) be the nodal curve associated to \( \mathcal{N}_i \). We remark here that we use the labelling of the branches of \( \mathcal{N}_i \) induced by the curve \( \mathcal{C}_i \).

Then, by Proposition 3.4.3 (1), \( \mathcal{C}_i \) is an \((a_i', b_i')\)-unicuspidal curve of genus \( g \), where \((a_i', b_i')\) corresponds to \( \phi^{4i} \cdot \zeta^{-12} \); however, observe that

\[
\phi^{-12} \cdot \phi^{4i} = \zeta^{-12} \cdot \phi^{-4i} = \zeta^{-4(i+3)} = (\zeta^{-4(i+3)})^+.
\]

This concludes the proof under the assumption \( b > 7a \); we prove now that this assumption is not needed. To this end, assume that \( b < 7a \), and consider the curve \( f_2^N(C) \); by Proposition 3.4.3 (2), this is an \((a^*, b^*)\)-unicuspidal curve of genus \( g \), and \((a^*, b^*) = (7a - b, a)\) corresponds to \( \phi^{-4}\zeta \).

As remarked in the proof of Proposition 3.4.3, the assumption \( a + b = 3d \) implies that \( b/a > \phi^4 > 41/6 \), hence it is easy to see that \( b^* - 7a^* > b - 7a \). An inductive argument shows that there exists an integer \( m \leq 0 \) such that there exists an \((a^*, b^*)\)-unicuspidal curve \( C^* \) with \( b^* > 7a^* \), with \((a^*, b^*)\) corresponding to \( \phi^{4(m+1)}\zeta \), that sweeps a nodal curve \( N^* \). That is, the assumption \( b > 7a \) was not restrictive, and this concludes the proof. \( \square \)

Before turning to the proof of Theorem 3.1.7, we prove the following:

**Lemma 3.4.5.** Let \( N \subset \mathbb{CP}^2 \) be the nodal cubic defined by the equation \( x^3 + y^3 - xyz = 0 \). Then for any positive integer \( d \), there exists a reduced smooth projective curve \( C \) of degree \( d \) having local intersection multiplicity \( 3d \) with \( N \) at its node.

**Proof.** Set \( h(x, y, z) = x^3 + y^3 - xyz \) for the polynomial defining \( N \). The node of \( N \) is at the point \([x : y : z] = [0 : 0 : 1] \).

First, we prove that there exists a smooth local curve branch of degree at most \( d \) having local intersection multiplicity \( 3d \) with \( N \) at its node. To show this, we will work in the affine chart \( z = 1 \) of \( \mathbb{CP}^2 \). \( N \subset \mathbb{CP}^2 \) has a parametrization with \([t : s] \in \mathbb{CP}^1 \) as follows:

\[
[x : y : z] = [ts^2 : t^2s : t^3 + s^3].
\]

Therefore, in the affine chart with coordinates \( x \) and \( y \) around \((0, 0)\),

\[
(x(t), y(t)) = \left( \sum_{k=0}^{\infty} (-1)^{k}t^{3k+1}, \sum_{k=0}^{\infty} (-1)^{k}t^{3k+2} \right), \quad |t| < \frac{1}{2}
\]

is a parametrization of the branch of \( N \) tangent to the \( x \)-axis, mapping \( t = 0 \) to the node. For simplicity, while working in this chart, we denote by \( h(x, y) \) the polynomial \( x^3 + y^3 - xy \) as well.

We claim that for each \( d \) there exists a polynomial \( f_d(x, y) \) of degree \( d \) such that it has local intersection multiplicity \( 3d \) with \( N \) at its node and it is smooth at that point. For
$d = 1, 2$ we set
\[ f_1(x, y) = y, \quad f_2(x, y) = y - x^2: \]
the order of \( t \) in the expansion of \( f_1(x(t), y(t)) \) is 2, and the coefficient of \( t^2 \) is 1; and the order of \( t \) in the expansion of \( f_2(x(t), y(t)) \) is 5, and the coefficient of \( t^5 \) is 1.

Let \( c_1 = c_2 = 1 \). For \( n \geq 3 \), define recursively the pair \((f_n(x, y), c_n)\) as follows:
\[ f_n(x, y) = c_{n-2}f_{n-1}(x, y) - c_{n-1}xyf_{n-2}(x, y) \]
and let \( c_n \) denote the coefficient of \( t^{3n-1} \) in the expansion of \( f_n(x(t), y(t)) \).

**Claim 3.4.6**. For any integer \( n \geq 1 \), the following hold:

1. \( f_n(x, y) \) is a polynomial of degree \( n \). If the coefficient of \( x^iy^j \) in \( f_n(x, y) \) is nonzero, then \( i + 2j \equiv 2 \pmod{3} \). The coefficient of the monomial \( y \) in \( f_n(x, y) \) is nonzero.
2. The order of \( t \) in the expansion of \( f_n(x(t), y(t)) \) is \( 3n - 1 \); in particular, \( c_n \neq 0 \).

**Proof.** We prove this by induction on \( n \). Both properties can be easily checked for \( n = 1, 2 \). Assume we know the claim for \( f_{n-1}(x, y) \) and \( f_{n-2}(x, y) \), and let us prove it for \( f_n(x, y) \). For an integer \( m \), let \( Q_m(t) \) be defined by \( f_m(x(t), y(t)) / t^{3m-1} \); the inductive hypothesis tells us that \( Q_{n-1}(t) \) and \( Q_{n-2}(t) \) are power series satisfying \( Q_{n-1}(0) = c_{n-1} \neq 0 \) and \( Q_{n-2}(0) = c_{n-2} \neq 0 \).

All the properties in (1) are obvious from the definition; for the non-vanishing of the coefficient of \( y \), we observe that it is obtained from the coefficient of \( f_{n-1}(x, y) \) by multiplication with \( c_{n-2} \neq 0 \), and therefore it is nonzero.

For (2), write
\[ f_n(x(t), y(t)) = \sum_{m=0}^{\infty} a_mt^m. \]
First, we prove that \( a_m = 0 \) for \( m \leq 3n - 2 \). As all powers of \( t \) in the expansion of \( x(t) \) are congruent to 1 (mod 3) and all powers of \( t \) in the expansion of \( y(t) \) are congruent to 2 (mod 3), using part (1) we immediately get that \( a_m = 0 \) if \( m \neq 2 \) (mod 3). Therefore, it is enough to show that \( a_m = 0 \) for \( m \equiv 2 \pmod{3} \) with \( m \leq 3n - 4 \). From the definition of \( f_n(x, y), Q_{n-1}(t), Q_{n-2}(t) \), and the inductive hypothesis, we have
\[ f_n(x(t), y(t)) = c_{n-2}t^{3(n-1)-1}Q_{n-1}(t) - c_{n-1}t^3P(t) \cdot t^{3(n-2)-1}Q_{n-2}(t), \]
where \( P(t) \) is the power series associated with \((1 + t^3)^{-2}\). Therefore, there is indeed no power \( t^m \) in the above expansion for \( m \leq 3n - 4 \).

Assume now that \( t^{3n-1} \) vanishes as well, i.e. \( c_n = 0 \). This would mean that \( f_n(x, y) \) has local intersection multiplicity at least \( 3n \) with the parametrized branch of \( N \), therefore (since it also intersects the other branch at the node), it has intersection multiplicity at
least \(3n + 1\) with the cubic \(N\) altogether. As the degree of \(f_n(x, y)\) is \(n\), from Bézout’s theorem, it follows that \(h(x, y)\) divides \(f_n(x, y)\). But this is impossible, as \(y\) has non-zero coefficient in \(f_n(x, y)\); a contradiction. \(\Box\)

**Remark 3.4.7.** Numerical evidence indicates that \(c_n\) is in fact 1 for every \(n \geq 1\). However, this is not needed in the argument.

The curve \(C_d\) defined by the equation \(f_d(x, y) = 0\) has local intersection multiplicity \(3d - 1\) with one branch of the node of \(N\), and at least one with the other branch. Since it does not contain the curve \(N\), its intersection multiplicity with \(N\) at the node is at most \(3d\), by Bézout’s theorem. This also shows that the intersection multiplicity of \(C_d\) with one of the two branches of \(N\) is 1, hence \(C_d\) is reduced and smooth at the node of \(N\).

In this way, we have proven the existence of a local germ \(f_d(x, y)\) with the above properties. We will also denote by \(f_d(x, y, z)\) the homogenization of the germ, and with \(C_d\) the associated, degree-\(d\) projective curve. As indicated above, \(C_d\) is reduced and it intersects \(N\) only at the node with multiplicity \(3d\). However, it may have singular points away from \([0 : 0 : 1]\) if \(d \geq 3\).

Recall that we have defined two polynomials \(f_1(x, y)\) and \(f_2(x, y)\) of degrees 1 and 2 respectively, the zero sets of their homogenizations are curves that sweep \(N\) at the node. Suppose now that \(d > 2\). For a non-negative integer \(m\) let \(g_m(x, y, z)\) be the Fermat polynomial \(g_m(x, y, z) = x^m + y^m + z^m\), defining the Fermat curve \(\{g_m(x, y, z) = 0\}\).

Consider the linear pencil of curves
\[
\{\lambda f_d(x, y, z) + \mu h(x, y, z)g_{d-3}(x, y, z) = 0\}
\]
as \([\lambda : \mu]\) varies in \(\mathbb{CP}^1\).

The basepoints of the linear system are the intersection points of \(C_d\) and \(\{h(x, y, z)g_{d-3}(x, y, z) = 0\}\).

These are the point \([0 : 0 : 1]\) and the intersections of \(\{g_{d-3}(x, y, z) = 0\}\) with \(C_d\).

We claim that at each basepoint one of the two generating curves of the pencil is smooth. In fact, at \([0 : 0 : 1]\) the curve \(C_d\) is smooth by construction; on the other hand, since \(C_d\) and \(N\) intersect only at \([0 : 0 : 1]\), at every other basepoint the curve \(\{h(x, y, z)g_{d-3}(x, y, z) = 0\}\) coincides with the Fermat curve, and hence it is smooth. Therefore, the generic member of the pencil is smooth and reduced by Bertini’s theorem [26, Theorem 17.16].

Notice also that the intersection multiplicity of both generators of the pencil with \(N\) is at least \(3d\): it is at least \(3d - 1\) with one branch \(B_2\) and at least 1 with the other branch \(B_1\) of \(N\) at the node. In this way, for all members of the pencil the intersection
multiplicity is at least $3d$ (at least $3d - 1$ with $B_2$ and at least 1 with $B_1$). Now again by Bézout’s theorem, it can not be higher. In particular, the generic member of the pencil has intersection multiplicity exactly $3d - 1$ with $B_2$ and 1 with $B_1$; altogether $3d$ with $N$ at its node. So the generic member of the pencil is the curve we are looking for. □

**Proof of Theorem 3.1.7.** Recall from Proposition 3.4.3 that a curve $C$ sweeping a cubic $N$ induces a labelling of the branches of $N$ at its node, hence the pair $(C, N)$ determines two Cremona transformations $f_1^N$ and $f_2^N$ which in turn define the two associated nodal cubics $N_1$ and $N_2$. Moreover, $f_j^N(C)$ sweeps $N_j$ for $j = 1, 2$.

By Lemma 3.4.4 we can find a smooth curve $C$ of degree $d = k + 1 = L_k^5$ and genus $g = k(k - 1)/2$ that sweeps a nodal cubic $N$. Notice that, since $C$ is smooth, its ‘Puiseux pair’ at the node is $(L_k^1, L_k^5) = (1, 3k + 2)$; also, observe that $(1, 3k + 2)$ corresponds to $\zeta = 7k + 4 + (3k + 2)\sqrt{5} \in \mathbb{Z}[\sqrt{5}]$. Notice that $25 \nmid 2g - 1 = k^2 - k - 1$, cf. Example 3.3.14.

Also, since $k > 1$, $L_k^5 > 7L_k^1$, Lemma 3.4.4 gives, for every $i \neq -1, -2$, a 1-unicuspidal curve whose Puiseux pair corresponds to $(\phi^4i\zeta)^+$. Now, simply observe that $(L_k^4, L_k^5)$ corresponds to $\phi^4\zeta$.

As for the second part of the statement, by Theorem 3.1.2, almost all Puiseux pairs $(a, b)$ such that there exists an $(a, b)$-unicuspidal curve of genus $g$ satisfy (3.1.5). As noted in Remark 3.3.13 if $g \equiv 1 \pmod{3}$ and $25 \nmid 2g - 1$ is a power of a prime, then all solutions of the generalized Pell equation (3.1.5) correspond to elements of form $(\zeta \phi^n)^+, n \in \mathbb{Z}$ for some $\zeta \in \mathcal{O}$. On the other hand, since we realized above all but two elements in the orbit given by the action of multiplication by $\pm \phi^4$, for these genera we have constructed examples corresponding to almost all singularities of 1-unicuspidal curves. □

**Example 3.4.8.** Let us consider the case $g = 6$. According to Theorem 3.1.7 for every $n \neq -1, -2$ we can construct a singular genus-6 curve with a cusp of type corresponding to $(\phi^n(32 + 14\sqrt{5}))^+ \in \mathcal{O}$ (see the proof above with $k = 4$).

Setting $\zeta = 32 + 14\sqrt{5}$, we list explicitly a few solutions of the corresponding generalized Pell equation:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\zeta \phi^n$</th>
<th>$(\zeta \phi^n)^+$</th>
<th>Corresponding cusp type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4$</td>
<td>$767 - 343\sqrt{5}$</td>
<td>$767 + 343\sqrt{5}$</td>
<td>(50, 343)</td>
</tr>
<tr>
<td>$-3$</td>
<td>$112 - 50\sqrt{5}$</td>
<td>$112 + 50\sqrt{5}$</td>
<td>(7, 50)</td>
</tr>
<tr>
<td>$-2$</td>
<td>$17 - 7\sqrt{5}$</td>
<td>$17 + 7\sqrt{5}$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td>$7 + \sqrt{5}$</td>
<td>$7 + \sqrt{5}$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$32 + 14\sqrt{5}$</td>
<td>$32 + 14\sqrt{5}$</td>
<td>(1, 14)</td>
</tr>
<tr>
<td>$1$</td>
<td>$217 + 97\sqrt{5}$</td>
<td>$217 + 97\sqrt{5}$</td>
<td>(14, 97)</td>
</tr>
<tr>
<td>$2$</td>
<td>$1487 + 665\sqrt{5}$</td>
<td>$1487 + 665\sqrt{5}$</td>
<td>(97, 665)</td>
</tr>
</tbody>
</table>
Notice that, since $g$ is divisible by 3, by Proposition 3.3.12, the elements $(\phi^{4n+2}(32 + 14\sqrt{5}))^+$, too, correspond to solutions of Equation (3.1.5).

Setting $\zeta' = \zeta \phi^2 = 83 + 37\sqrt{5}$, we get some solutions as follows:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\zeta' \phi^{4n}$</th>
<th>$(\zeta' \phi^{4n})^+$</th>
<th>Corresponding cusp type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4$</td>
<td>$293 - 13\sqrt{5}$</td>
<td>$293 + 13\sqrt{5}$</td>
<td>$(19, 131)$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$43 - 19\sqrt{5}$</td>
<td>$43 + 19\sqrt{5}$</td>
<td>$(2, 19)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$8 - 2\sqrt{5}$</td>
<td>$8 + 2\sqrt{5}$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td>$13 + 5\sqrt{5}$</td>
<td>$13 + 5\sqrt{5}$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$83 + 37\sqrt{5}$</td>
<td>$83 + 37\sqrt{5}$</td>
<td>$(5, 37)$</td>
</tr>
<tr>
<td>$1$</td>
<td>$568 + 254\sqrt{5}$</td>
<td>$568 + 254\sqrt{5}$</td>
<td>$(37, 254)$</td>
</tr>
<tr>
<td>$2$</td>
<td>$3893 + 1741\sqrt{5}$</td>
<td>$3893 + 1741\sqrt{5}$</td>
<td>$(254, 1741)$</td>
</tr>
</tbody>
</table>

**Question 3.4.9.** Is it the case that for almost all $n$ the elements $(\phi^{4n+2}(32 + 14\sqrt{5}))^+ \in \mathcal{O}$ correspond to the Puiseux pair of a 1-unicuspidal curve $C_n$?

In light of Lemma 3.4.4, the answer is positive if we can find an integer $n^*$ and an $(a^*, b^*)$-unicuspidal curve $C^*$ that sweeps a nodal cubic $N^*$, where $(a^*, b^*)$ corresponds to $(\phi^{4n^*+2}(32 + 14\sqrt{5}))^+$.

Notice also that, since $2g - 1 = 11$ is a prime, together with the previous construction, this would realize almost all possible Puiseux pairs for this genus (see Remark 3.3.13).

**Example 3.4.10.** Let us consider the case $g = 10$. Theorem 3.1.7 (with $k = 5$) gives us a family of curves, each with a singularity corresponding to $(\phi^{4n}(39 + 17\sqrt{5}))^+ \in \mathcal{O}$ for any $n \neq -1, -2$.

Setting $\zeta = 39 + 17\sqrt{5}$, some solutions are as follows:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\zeta \phi^{4n}$</th>
<th>$(\zeta \phi^{4n})^+$</th>
<th>Corresponding cusp type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4$</td>
<td>$1089 - 487\sqrt{5}$</td>
<td>$1089 + 487\sqrt{5}$</td>
<td>$(71, 487)$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$159 - 7\sqrt{5}$</td>
<td>$159 + 7\sqrt{5}$</td>
<td>$(10, 71)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$24 - 10\sqrt{5}$</td>
<td>$24 + 10\sqrt{5}$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td>$9 + \sqrt{5}$</td>
<td>$9 + \sqrt{5}$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>$39 + 17\sqrt{5}$</td>
<td>$39 + 17\sqrt{5}$</td>
<td>$(1, 17)$</td>
</tr>
<tr>
<td>$1$</td>
<td>$264 + 118\sqrt{5}$</td>
<td>$264 + 118\sqrt{5}$</td>
<td>$(17, 188)$</td>
</tr>
<tr>
<td>$2$</td>
<td>$1809 + 809\sqrt{5}$</td>
<td>$1487 + 665\sqrt{5}$</td>
<td>$(118, 809)$</td>
</tr>
</tbody>
</table>
Since \( g \equiv 1 \pmod{3} \) and \( 2g - 1 = 19 \) is a prime, by Remark 3.3.13, all the solutions of the Pell equation which can correspond to possible Puiseux pairs are in the set \( \{ (\zeta \phi^n)^+ : n \in \mathbb{Z} \} \).

So we constructed all but finitely many 1-unicuspidal genus-10 curves (up to equisingularity).

Similar arguments apply as soon as \( g \) is a triangular number such that \( 2g - 1 \) is a power of a prime greater than 5. That is, if \( g \equiv 1 \pmod{3} \), then Theorem 3.1.7 produces almost all 1-unicuspidal curves of genus \( g \), up to equisingularity. The first triangular number congruent to 1 modulo 3 for which we are unable to produce almost all 1-unicuspidal curves is \( g = 325 \), for which \( 2g - 1 = 649 = 11 \cdot 59 \).

If, on the other hand, \( 2g - 1 \) is congruent to \(-1\) modulo 3 and larger than 5, Theorem 3.1.7 only provides explicit examples of 1-unicuspidal curves corresponding to half of the solutions of Equation (3.1.5). We have a natural generalisation of Question 3.4.9 above:

**Question 3.4.11.** Let \( g > 3 \) be a triangular number such that \( g \equiv 0 \pmod{3} \) and \( 2g - 1 \) is a power of a prime. Is it the case that for almost all coprime solutions \((a, b)\) of Equation (3.1.5) there exists an \((a, b)\)-unicuspidal curve of genus \( g \)?

**Example 3.4.12.** For \( g = 1 \) and \( g = 3 \) some unusual things happen, as \( 2g - 1 \) is 1 in the first case, and 5 in the second.

If \( g = 1 \), we can start with \((a, b) = (1, 8)\) corresponding to \( \zeta = 18 + 8\sqrt{5} \). We get that there exist curves corresponding to \((\zeta \phi^n)^+\) for any \( n \in \mathbb{Z} \), except \( n = -1 \) (yielding \( \zeta \phi^{-4} = 3 + \sqrt{5} \)) and \( n = -2 \) (yielding \( \zeta \phi^{-8} = 3 - \sqrt{5} \)). However, now \((\zeta \phi^{4n})^+ = (\zeta \phi^{-4(n+3)})^+\), so the set of possible Puiseux pairs is indexed by \( \mathbb{N} \) rather than \( \mathbb{Z} \).

If \( g = 3 \), we can start with \((a, b) = (1, 11)\) corresponding to \( \zeta = 25 + 11\sqrt{5} \). We get that there exist curves corresponding to \((\zeta \phi^n)^+\) for any \( n \in \mathbb{Z} \), except \( n = -1 \) (yielding \( \zeta \phi^{-4} = 5 + \sqrt{5} \)) and \( n = -2 \) (yielding \( \zeta \phi^{-8} = 10 - 4\sqrt{5} \)). Now although \( 5 \equiv -1 \pmod{3} \) and \( \zeta \phi^2 = 65 + 29\sqrt{5} \) (so 3 does not divide \( y \) here), we do not get another family of solutions, as \( \zeta \phi^2 = 65 + 29\sqrt{5} = \zeta \phi^{-12} \).

**Example 3.4.13.** The smallest non-triangular \( g \) for which we have infinitely many coprime solutions of the Pell equation is \( g = 16 \). By Proposition 3.3.12, all coprime solutions of Equation (3.1.5) correspond to \((\phi^{4n}(57 + 25\sqrt{5}))^+, n \in \mathbb{Z} \).

The natural question to ask in this setting is the following:

**Question 3.4.14.** Is it the case that for every \( g \), almost all coprime solutions \((a, b)\) of Equation (3.1.5) are realized by an \((a, b)\)-unicuspidal curve of genus \( g \)?
3.4.2. More examples. In the next two examples, we exhibit 1-unicuspidal genus-\(g\) curves for some \(g \geq 4\). These curves are of minimal degree \(d_{\text{min}}\) among all curves with the given singularity, but \(d_{\text{min}}\) is not the minimal degree for which the degree-genus formula \((3.1.3)\) has a solution.

Also, in both cases, \(a + b\) is not \(3d\), thus yielding examples of curves (for infinitely many \(g\)) not covered by the identity of Theorem \(3.1.2\).

Recall that for short we write \(R_n = H(n)\) for the values of the semigroup counting function of the singularity.

**Example 3.4.15.** Consider the projective curve \(C_p\) defined by the equation

\[
C_p = \{x^{p+3} + y^{p+3} + x^pz^3 = 0\}
\]

for \(p \geq 2\), and \(p\) not divisible by 3. This is a unicuspidal curve of degree \(d = p + 3\) and genus \(g = p + 2\). The cusp is of type \((p,p + 3)\). This local type is always a candidate with \(d = p + 2\) as well; in this case \(g = 1\), but it is not an admissible candidate since Equation \((3.1.2)\) obstructs the existence of such curves: set \(j = 1\) and \(k = 0\) and notice that \(R_{d+1} = R_{p+3} = \#(\Gamma \cap [0, p + 3]) = 2 < 3\), as the semigroup is generated by two elements \(p\) and \(p + 3\).

Notice that in this case \(a + b = 2p + 3\), while \(3d = 3(p + 3)\).

In fact, for certain local types, Theorem \(3.1.1\) can exclude arbitrarily many candidate degrees as well, as the next example shows.

**Example 3.4.16.** Consider the projective curve \(D_p\) defined by

\[
D_p = \{x^p z^{p-1} + x^{2p-1} + y^{2p-1} = 0\}
\]

for \(p \geq 2\). This is a unicuspidal curve of degree \(d = 2p - 1\) and genus \(g = (p - 1)(p - 2)\). The cusp has a torus knot of type \((p,2p - 1)\). This local type is always a candidate with all possible smaller \(d\)’s (such that \(\left(\frac{d-1}{2}\right) \geq \delta = (p - 1)^2\)), but the inequality \((3.1.2)\) obstructs the existence of such curves (set \(j = 1\) and \(k = 0\) and notice that in order to have \(R_{d+1} \geq 3\) we need \(d \geq 2p - 1\)). In this way, this example shows that for the topological type \((p,2p - 1)\) approximately \(2 - \sqrt{2})p\) candidate degrees can be obstructed.

In this case, \(a + b = 3p - 1\), while \(3d = 6p - 3\).

**Remark 3.4.17.** In fact, the above applications of \((3.1.2)\) (and, surprisingly, the proof of Proposition \(3.2.3\) too) use only the lower bound for \(R_{jd+1-2k}\) and only with \(k = 0\). This special case can be obtained by applying Bézout’s theorem. Indeed, one can repeat the
argument of the proof of [20, Proposition 2] without any change, and get
\[
\frac{(j + 1)(j + 2)}{2} \leq R_{jd+1}
\]
for \(j = 0, \ldots, d - 3\).

Notice that in all the examples above, \(a + b < 3d\). The next example shows that there are infinitely many 1-unicuspidal curves with \(a + b > 3d\) as well.

**Example 3.4.18.** For any positive integer \(n > 2\), there exists a \((3n, 21n+1)\)-unicuspidal curve \(C'_n\) of degree \(d = 8n\) and genus \((n - 1)(n - 2)/2\). To construct such curves, consider a smooth curve \(C_n\) of degree \(n\) touching a nodal cubic \(N\) in one single point different from its node (with local intersection multiplicity \(3n\)). By Lemma 3.4.19 stated below, such a pair \((N, C_n)\) exists. Choose a branch of \(N\) at the node, and let \(f^N\) denote the associated Orevkov’s Cremona transformation (we do not have a specified order of branches at the node here, so there is no point in distinguishing \(f^N_1\) and \(f^N_2\), as described in Subsection 3.4.1). Then one can check that the \(C'_n = f^N(C_n)\) has degree and cusp type as described above, therefore, \(a + b = 24n + 1 > 24n = 3d\).

Recall that the birational map \(f^N\) automatically produces a new nodal cubic \(N^1\) as the image of \(E_7\) (the exceptional divisor obtained at the last blow-up) after the series of blow-downs. Notice that we now can distinguish the two branches of \(N^1\): call \(B_1\) the branch having intersection multiplicity \(3n\) with \(C'_n\) and \(B_2\) the one having intersection multiplicity \(21n\) with \(C'_n\). Now applying the birational map \(f^{N^1}_2\) leads back to the original smooth curve \(C_n\). On the other hand, applying \(f^{N^1}_1\) leads to another cuspidal curve; however, this has more than one Puiseux pair.

**Lemma 3.4.19.** Let \(N \subset \mathbb{CP}^2\) be the nodal cubic defined by the equation \(x^3 + x^2y - yz^2 = 0\). Then for any positive integer \(d > 2\), there exists a reduced smooth projective curve \(C\) of degree \(d\) having local intersection multiplicity \(3d\) with \(N\) at the point \([x : y : z] = [0 : 0 : 1]\).

**Proof.** The proof is similar to the proof of Lemma 3.4.5. Set \(h(x, y, z) = x^3 + x^2y - yz^2\) for the equation of \(N\) itself. In the affine chart of \(\mathbb{CP}^2\) given by \(z = 1\) with coordinates \(x\) and \(y\) around \((0, 0)\),
\[
(x(t), y(t)) = \left(t, \frac{t^3}{1 - t^2}\right), \quad |t| < 1/2
\]
is a parametrization of the local branch of \(N\), mapping \(t = 0\) to the inflection point \((0, 0)\).

Set \(f_d(x, y) = y^d + x^3 + x^2y - y\). It is easy to check that \(f_d(x, y)\) is smooth at \((0, 0)\) and has local intersection multiplicity \(3d\) with \(N\). We will also denote by \(f_d(x, y, z)\) the
homogenization of this germ. Consider the linear pencil
\[
\{ \lambda f_d(x, y, z) + \mu h(x, y, z)g_{d-3}(x, y, z) = 0 \}
\]
where \([\lambda : \mu] \in \mathbb{CP}^1\) and \(g_{d-3}(x, y, z)\) is the Fermat polynomial defined above. Similarly as in the proof of Lemma 3.4.5, by Bertini’s theorem it follows that the generic member of the pencil determines a smooth curve with the desired properties. \(\square\)

**Remark 3.4.20.** Call an \((a, b)\)-unicuspidal degree-\(d\) curve *exceptional* if \(a + b \neq 3d\). Theorem 3.1.2 says that there are at most finitely many exceptional curves for each fixed genus \(g\). More precisely, analysing the proof of Theorem 3.1.2, one can obtain an upper bound quadratic in \(g\) for the degree of exceptional curves, that is, \(d \leq O(g^2)\) for all exceptional curves of degree \(d\) and genus \(g\).

Of course, there can be infinitely many exceptional curves of varying genus \(g\). Indeed, the above examples provide us infinitely many such curves: in Example 3.4.15 we have \(a + b < 3d\) and \(d = O(g)\), in Example 3.4.16 we have \(a + b < 3d\) again but \(d = O(\sqrt{g})\); in Example 3.4.18 we have \(a + b > 3d\) and \(d = O(\sqrt{g})\) again.

The following natural question arises.

**Question 3.4.21.** Let \(d_g\) be the largest degree of an exceptional curve of genus \(g\). How fast does \(d_g\) grow with \(g\)? What is the smallest power \(g^\mu\) of \(g\) such that \(d_g = O(g^{\mu+\varepsilon})\) for all \(\varepsilon > 0\)?

The same question can be asked separately for exceptional curves with \(a + b > 3d\) and \(a + b < 3d\).

From the proof of Theorem 3.1.2 and Example 3.4.15 it follows that the exponent we are looking for is between 1 and 2.
CHAPTER 4

Topological invariants of Dehn surgeries

4.1. Introduction

This chapter contains a result from a joint work with András Némethi and can be found in our joint manuscript [8]. I am grateful to András Némethi for his work and several useful suggestions, especially for the proof of Claim 4.4.5, further, for his help with the proofs of Lemma 4.3.1, Lemma 4.3.2 and Lemma 4.3.3.

4.1.1. Motivation. In this chapter we prove an additivity property of the 3-dimensional (normalized) Seiberg–Witten invariant with respect to taking the universal abelian cover, valid for some surgery 3-manifolds. Namely, assume that $M$ is obtained as a negative rational surgery along connected sum of algebraic knots in the three-sphere $S^3$. Let $\Sigma$ be its universal abelian cover. Theorem 4.4.1 states that the sum over all spin$^c$ structures of the Seiberg–Witten invariants of $M$ (after normalization) equals to the canonical Seiberg–Witten invariant of $\Sigma$.

Both covers of manifolds and manifolds of form $S^3 - \frac{p}{q}(K)$ are extensively studied. The stability of certain properties and invariants with respect to the coverings is a key classical strategy in topology, it is even more motivated by the recent proof of Thurston’s virtually fibered conjecture [1, 81]. Manifolds of form $S^3_{-\frac{p}{q}}(K)$ can be particularly interesting due to the theorem of Lickorish and Wallace [34, 80] stating that every closed oriented three-manifold can be expressed as surgery on a link in $S^3$. Based on this result, one can ask which manifolds have surgery representations with some restrictions. For example, using Heegaard Floer homology, [27] provides necessary conditions on manifolds having surgery representation along a knot. In this context, Theorem 4.4.1 can be viewed also as a criterion for a manifold having surgery representation of form $S^3_{-\frac{p}{q}}(K)$ with $K$ a connected sum of algebraic knots.

In fact, Seiberg–Witten invariants (SW) and Heegaard Floer homologies are closely related. The SW invariants were originally introduced by Witten in [82], but they also arise as Euler characteristics of Heegaard Floer homologies, cf. [67, 66]. Here we will involve another cohomology theory with similar property. Since $S^3_{-\frac{p}{q}}(K)$ is representable
by a negative definite plumbing graph, via \cite{53} we can view the SW invariants as Euler characteristics of lattice cohomologies introduced in \cite{52}. The big advantage of the lattice cohomology over the classical definition of Heegaard Floer homology is that it is computable algorithmically from the plumbing graph in an elementary way. In the last section of applications and examples the above ‘covering additivity property’ will be combined with results involving lattice cohomology.

Another strong motivation to study the above property is provided by the theory of complex normal surface singularities: the geometric genus of the analytic germ is conjecturally connected with the SW invariant of the link of the germ (see \cite{55, 56, 57, 58}). Since the geometric genus satisfies the ‘covering additivity property’ (cf. \S \ref{4.2.1}), it is natural to ask for the validity of similar property at purely topological level.

Furthermore, from the point of view of singularity theory, the motivation for the surgery manifolds $S^3_{p/q}(K)$ is also strong: the link of the so called superisolated singularities (introduced in \cite{37}) are of this form. These singularities are key test-examples for several properties and provide counterexamples for several conjectures. They ‘embed’ the theory of projective plane curves to the theory of surface singularities. For their brief introduction see Section 2.3 earlier or Example 4.5.1 below, for a detailed presentation see \cite{37, 38}.

All these connections with the analytic theory will be used deeply in several points of the proof. For consequences of the main result regarding analytic invariants see the end of Subsection 4.5.3.

\subsection{Notations.} We recall some facts about negative definite plumbed 3- and 4-manifolds, their spin$^c$ structures and Seiberg–Witten invariants. For more see \cite{56, 58}.

Let $M$ be a 3–manifold which is a rational homology sphere ($\mathbb{Q}HS^3$). Assume that it has a negative definite plumbing representation (see e.g. \cite{61}) with a decorated connected graph $G$ with vertex set $\mathcal{V}$. In particular, $M$ is the boundary of a plumbed 4-manifold $P$, which is obtained by plumbing disk bundles over oriented surfaces $E_v \simeq S^2, v \in \mathcal{V}$ (according to $G$), and which has a negative definite intersection form. A vertex $v \in \mathcal{V} = \mathcal{V}(G)$ is decorated by the self–intersection $e_v \in \mathbb{Z}$ of $E_v$ in $P$. One can think of $e_v$ also as the Euler number of the disk bundle over $E_v \cong S^2$ used in the plumbing construction. Since $M$ is a $\mathbb{Q}HS^3$, the graph $G$ is a tree. We set $\# \mathcal{V}(G)$ for the number of vertices of $G$.

Below all the (co)homologies are considered with $\mathbb{Z}$–coefficients.

Denote by $L = L_G = \mathbb{Z}\langle E_v \rangle_{v \in \mathcal{V}}$ the free abelian group generated by basis elements $\{E_v\}_{v \in \mathcal{V}}$, indexed by $\mathcal{V}$. It can be identified with $H_2(P)$, where $\{E_v\}_{v \in \mathcal{V}}$ represent the zero
4.1. INTRODUCTION

sections of the disk bundles used in the plumbing construction. It carries the negative definite intersection form \((.,.) = (.,.)_G\) (of \(P\); readable from \(G\) too). This form naturally extends to \(L \otimes \mathbb{Q}\). Denoting by \(L' = L'_G = \text{Hom}_\mathbb{Z}(L, \mathbb{Z})\) the dual lattice, one gets a natural embedding \(L \to L'\) by \(l \mapsto (.,l)\). Furthermore, we can regard \(L'\) as a subgroup of \(L \otimes \mathbb{Q}\), therefore \((\cdot,\cdot)\) extends to \(L'\) as well. We introduce the anti-dual basis elements \(E_v^*\) in \(L'\) defined by \((E_{v'},E_v^*)\) being \(-1\) if \(v = v'\) and \(0\) otherwise. Notice that \(L' \cong H^2(P) \cong H_2(P,M)\). The short exact sequence \(0 \to H_2(P) \to H_2(P,M) \to H_1(M) \to 0\) identifies \(L'/L\) with \(H_1(M)\), which will be denoted by \(H\). We denote the class of \(l' \in L'\) by \([l'] \in H\), and we call \(l' \in L'\) a representative of \([l']\).

Assume that the intersection form in the basis \(\{E_v\}_{v \in \mathcal{V}}\) has matrix \(I\); then we define \(\text{det}(G) := \text{det}(-I)\). It also equals to the order of \(H\). Since \(I\) is negative definite, \(\text{det}(G) > 0\).

For any negative definite plumbing graph \(G\), which is a tree, and for any two vertices \(u, v\), the following holds (see [15] §10 in the integral homology case and [56] in general):

\[
\begin{align*}
\text{(4.1.1)} & \quad - \text{det}(G) \cdot (E_u^*, E_v^*) = \text{the product of the determinants of the} \\
& \quad \text{connected components of that graph, which is obtained from \(G\) by} \\
& \quad \text{deleting the shortest path connecting \(u\) and \(v\) and the adjacent edges.}
\end{align*}
\]

For any \(h \in H\), we denote by \(r_h = \sum_{v \in \mathcal{V}} c_v E_v\) ‘the smallest effective representative’ of its class in \(L'\), determined by the property \(0 \leq c_v < 1\) for all \(v\).

Finally, we define the canonical characteristic element in \(L'\). It is the unique element \(k_G \in L'\) such that \((k_G,E_v) = -(E_v,E_v) - 2\) for every \(v \in \mathcal{V}\). (In fact, \(P\) carries the structure of a smooth complex surface – in the case of singularities, \(P\) is a resolution, cf. Subsection 4.2.1 –, and \(k_G\) is the first Chern class of its complex cotangent bundle.)

The Seiberg–Witten invariants of \(M\) associate a rational number to each spin\(^c\) structure of \(M\). There is a ‘canonical’ spin\(^c\) structure \(\sigma_{\text{can}} \in \text{Spin}^c(M)\), the restriction of that spin\(^c\) structure of \(P\), which has first Chern class \(k_G \in H^2(P)\). As we assumed \(M\) to be a \(\mathbb{Q}HS^3\), \(\text{Spin}^c(M)\) is finite. It is an \(H\) torsor: for \(h \in H\), we denote this action by \(\sigma \mapsto h \ast \sigma\).

We denote by \(\text{sw}_\sigma(M) \in \mathbb{Q}\) the Seiberg–Witten invariant of \(M\) corresponding to the spin\(^c\) structure \(\sigma\). This is the classical monopole counting Seiberg–Witten invariant of \(M\) corrected by the Kreck–Stolz invariant to make it dependent only on the manifold \(M\). Nevertheless, we will adopt the approach from [56], and we regard \(\text{sw}_\sigma(M)\) as the sum of the sign–refined Reidemeister–Turaev torsion and the Casson–Walker invariant. By [56] it can be computed from the plumbing graph \(G\), and in this chapter this combinatorial approach (and its consequences and related formulae from succeeding articles) will be used. Here we will use the sign convention of \(\text{sw}\) which is opposite to the one used in [56]:
we adopt the sign convention from later articles [13, 53], consistently with Chapter 2 (cf. the discussion after Formula (2.3.1) in Section 2.3).

Next, we consider the following normalization term: for each \( h \in H \) we set

\[
(4.1.2) \quad i_h(M) := \frac{(k_G + 2r_h, k_G + 2r_h)}{8} + \#V.
\]

It does not depend on the particular plumbing representation of the manifold \( M \) (or \( P \)), hence it is an invariant of the manifold \( M \). Then, the normalized Seiberg–Witten invariant is defined as follows: for any \( h \in H \), we set

\[
(4.1.3) \quad s_h(M) = -\mathfrak{sw}_{h,\sigma_{can}}(M) - i_h(M).
\]

Sometimes we will also use the notations \( s_h(G) = s_h(M) \), or \( \mathfrak{sw}_h(G) = \mathfrak{sw}_{h,\sigma_{can}}(M) \).

In fact, \( s_h(M) \in \mathbb{Z} \). This can be seen through the identity (4.5.1), where \( s_h(M) \) appears as the Euler characteristic of the lattice cohomology. We also refer to Section 4.5.1 for the fact that \( s_h(M) \) and \( i_h(M) \) are indeed independent of the plumbing representation of the manifold.

Let \( \Sigma \) be the universal abelian cover (UAC) of the \( \mathbb{Q}HS^3 \) manifold \( M \): it is associated with the abelianization \( \pi_1(M) \to H_1(M) \). Usually the UAC of a rational homology sphere is not a rational homology sphere. However, here we will consider only those situations when this happens.

**Definition 4.1.1.** Assume that \( \Sigma \), the UAC of \( M \) is also a rational homology sphere. We say that for the manifold \( M \) the ‘covering additivity property’ of the invariant \( s \) holds with respect to the universal abelian cover (shortly, ‘\( CAP \) of \( s \) holds’) if

\[
(4.1.4) \quad s_0(\Sigma) = \sum_{h \in H_1(M)} s_h(M).
\]

The index 0 in the left hand side is the unit element in \( H_1(\Sigma) \).

The main result of this chapter is the following.

**Theorem 4.1.2.** Let \( M = S^3_{-p/q}(K) \) be a manifold obtained by a negative rational Dehn surgery of \( S^3 \) along a connected sum of algebraic knots \( K = K_1 \# \ldots \# K_\nu \) \( (p,q > 0, \gcd(p,q) = 1) \). Assume that \( \Sigma \), the UAC of \( M \), is a \( \mathbb{Q}HS^3 \). Then \( CAP \) of \( s \) holds.

Though the statement is topological, in the proof we use several analytic steps based on the theory of singularities. These steps not only emphasize the role of the algebraic knots and of the negative definite plumbing construction, but they also provide the possibility to use certain deep results valid for singularities (which are transported into the proof).
We emphasize that the above covering additivity property is not true for general negative definite plumbed 3–manifolds (hence for general 3-manifolds either), cf. Example 4.5.3. In particular, we cannot expect a proof of the main theorem by a general topological machinery unless some restrictions are made.

It is convenient to extend the definitions (4.1.2) and (4.1.3) for any representative \( l' \in L' \):

\[
\iota_{l'}(G) := \frac{(k_G + 2l', k_G + 2l')_G + \# V(G)}{8} \quad \text{and} \quad s_{l'}(M) = -\mathfrak{sw}_{l'}(G) - \iota_{l'}(G).
\]

By a computation, for two representatives \([l'_1] = [l'_2] = h \in H\) one has:

\[
(4.1.4) \quad s_{l'_2}(G) - s_{l'_1}(G) = \chi(l'_2) - \chi(l'_1),
\]

where

\[
(4.1.5) \quad \chi(l') := -(l', l' + k_G)/2.
\]

In particular,

\[
(4.1.6) \quad s_{l'}(G) = s_{[l']}(G) + \chi(l') - \chi(r[l']).
\]

Note that for \( l \in L \) one has \( \chi(l) = -(l, l + k_G)/2 \in \mathbb{Z} \). In fact, By Riemann–Roch theorem, \( \chi(l) \) is the topological description of the analytic Euler characteristic \( \chi(\mathcal{O}_l) \) of the structure sheaf \( \mathcal{O}_l \) of any non–zero effective cycle \( l \in L \).

### 4.2. Motivation: normal surface singularities

#### 4.2.1. Connection with singularity theory.

Theorem 4.1.2 is motivated by a geometric genus formula valid for normal surface singularities. Next we present two pieces of this connection, namely, the definition and covering properties of the equivariant geometric genera of normal surface singularities and the Seiberg–Witten Invariant Conjecture of Némethi and Nicolaescu [56]. For details we refer to [51, 55, 53, 56, 58].

Let \((X, 0)\) be a complex normal surface singularity germ with link \( M \). Let \( \pi : \widetilde{X} \rightarrow X \) be a good resolution with negative definite dual resolution graph \( G \), which can be regarded also as a plumbing graph for the 4-manifold \( \widetilde{X} \) and its boundary \( M \). (Hence, the \( E_v \)'s in this context are the irreducible exceptional curves.) The geometric genus of the singularity is defined as \( p_g(X) = \dim_C H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \), where \( \mathcal{O}_{\widetilde{X}} \) is the structure sheaf of \( \widetilde{X} \). Although \( p_g(X) \) is defined via \( \widetilde{X} \), it does not depend on the particular choice of the resolution. In [56] the following conjecture was formulated for certain (analytic types of) singularities, as a topological characterization of \( p_g(X) \):

\[
(4.2.1) \quad p_g(X) = s_0(M).
\]
We say that the *Seiberg–Witten Invariant Conjecture* (SWIC) holds for \((X, 0)\) if (4.2.1) is valid.

It is natural to ask whether there is any similar connection involving the other Seiberg–Witten invariants? The answer is given in [51, 55]. Let \((Y, 0)\) be the universal abelian cover of the singularity \((X, 0)\) (that is, its link \(\Sigma\) is the regular UAC of \(M\), \((Y, 0)\) is normal, and \((Y, 0) \to (X, 0)\) is analytic). The covering action of \(H = H_1(M)\) on \(Y\) extends to an action on the resolution \(\tilde{Y}\) of \(Y\). Hence \(H\) acts on \(H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})\), and it provides an eigenspace decomposition \(\bigoplus_{\xi \in \hat{H}} H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})\xi\), indexed by the characters \(\xi \in \hat{H} := \text{Hom}(H, \mathbb{C}^*)\) of \(H\). Set

\[ p_g(X)_h = \dim_{\mathbb{C}} H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})\xi, \]

where \(\xi_h \in \hat{H}\) is the character given by \(h'' \mapsto e^{2\pi i (l', l'')}\), \([l'] = h, [l''] = h''\). The numbers \(p_g(X)_h\) are called the *equivariant geometric genera* of \((X, 0)\). Note that \(p_g(X)_0 = p_g(X)\).

We say that the *Equivariant Seiberg–Witten Invariant Conjecture* (EqSWIC) holds for \((X, 0)\) if the next identity (4.2.2) is satisfied for every \(h\in H\):

\[(4.2.2)\]

\[ p_g(X)_h = s_h(M). \]

The EqSWIC holds for several families of singularities: rational, minimally elliptic, weighted homogeneous and splice quotient singularities [51, 57, 55].

Observe that by the definition, \(p_g(Y) = \sum_{h \in H} p_g(X)_h\). Hence, the next claim is obvious.

**Claim 4.2.1.** If for a singularity \((X, 0)\) with \(\mathbb{Q}H^3\) link the EqSWIC holds, and for its (analytic) universal abelian cover \((Y, 0)\) with \(\mathbb{Q}H^3\) link the SWIC holds too, then for the link \(M\) of \((X, 0)\) the (purely topological) covering additivity property of \(s\) also holds.

**Example 4.2.2.** As we already mentioned above, by [51, 57] the assumptions of Claim 4.2.1 are satisfied e.g. by cyclic quotient and weighted homogeneous singularities, hence the CAP of \(s\) holds e.g. for all lens–spaces and Seifert rational homology sphere 3–manifolds.

**Theorem 4.1.2** of the present note proves CAP for surgery manifolds. Furthermore, Example 4.5.3 shows that CAP does not hold for arbitrary plumbed 3–manifolds.

The invariants \(\{s_h(G)\}_h\) for many 3–manifolds (graphs) are computed. The next statement basically follows from Example 4.2.2 combined with the fact that the UAC of a lens space is \(S^3\) (that is, the UAC of a cyclic quotient is \((\mathbb{C}^2, 0)\) with \(p_g(\mathbb{C}^2) = 0\), hence \(s_h(G) = p_g(X)_h = 0\)).
Proposition 4.2.3 ([51, 52, 55]). If $G$ is a (not necessarily minimal) graph of $S^3$ or of a lens–space, then $\mathfrak{s}_h(G) = 0$ for every $h \in H$.

4.3. Dehn surgeries and their universal abelian covers

4.3.1. The structure of the plumbing graph $G$ of $S^3_{-p/q}(K)$. In this subsection we describe the plumbing graph of $S^3_{-p/q}(K)$ and we also fix some additional notations.

For $j = 1, \ldots, \nu$, let $K_j \subset S^3$ be the embedded knot of an irreducible plane curve singularity $\{ f_j(x,y) = 0 \} \subset (\mathbb{C}^2,0)$, where $f_j$ is a local holomorphic germ $(\mathbb{C}^2,0) \to (\mathbb{C},0)$, which is locally irreducible. Let $G_j$ be the minimal embedded resolution graph of $\{ f_j(x,y) = 0 \} \subset (\mathbb{C}^2,0)$, which is a plumbing graph (of $S^3$) with several additional decorations. It has an arrowhead $a_j \notin \mathcal{V}(G_j)$ supported on a vertex $u_j \in \mathcal{V}(G_j)$, this arrowhead represents $K_j$ (or, in a different language, it represents the strict transform $S(f_j)$ of $\{ f_j = 0 \}$ intersecting the exceptional $(-1)$–curve $E_{u_j}$). Furthermore, $G_j$ has a set of multiplicity decorations, the vanishing orders $\{ m_v \}_{v \in \mathcal{V}(G_j)}$ of the pullback of $f_j$ along the irreducible exceptional divisors. We collect them in the total transform $\text{div}(f_j) = S(f_j) + \sum_{v \in \mathcal{V}(G_j)} m_v E_v = S(f_j) + (f_j)$ of $f_j$; here $S(f_j)$ is supported on the arrowhead $a_j$ and represents the strict transform, while $(f_j)$ is the part of the total transform supported on $\cup_{v \in \mathcal{V}(G_j)} E_v$ and is characterized by $(f_j), E_v)_{G_j} + \delta_{v,u_j} = 0$ for any $v \in \mathcal{V}(G_j)$ (with the Kronecker delta notation, that is, $\delta_{v,u_j}$ being 1 if $v = u_j$, and 0 otherwise). For more on the graphs of plane curve singularities see [14, 15].

Define $K := K_1 \# K_2 \# \ldots \# K_{\nu}$.

Next, we write the surgery coefficient in Hirzebruch–Jung continued fraction

\begin{equation}
\frac{p}{q} = k_0 - \frac{1}{k_1 - \frac{1}{k_2 - \frac{1}{\ldots - \frac{1}{k_s}}}} =: [k_0, k_1, \ldots, k_s],
\end{equation}

where $k_i \in \mathbb{Z}$, $k_0 \geq 1$, $k_1, \ldots, k_s \geq 2$.

Then $M = S^3_{-p/q}(K)$ can be represented by a negative definite plumbing graph $G$, which is constructed as follows, cf. [51, 59]. $G$ consists of $\nu$ blocks, isomorphic to $G_1, \ldots, G_{\nu}$ (with their Euler decorations but without the multiplicity decorations and arrowheads), a chain $G_0$ of length $s$ consisting of vertices $\overline{u}_1, \ldots, \overline{u}_s = u'$ with decorations $e_{\overline{u}_1} = -k_1, \ldots, e_{\overline{u}_s} = -k_s$, respectively, and one ‘central vertex’ $u$. Moreover, we add $\nu + 1$ new edges: $u$ and the vertex $u_j$ from each block $G_j$ is connected by an edge, and $u$ and the
first vertex $\overline{\pi}_1$ with decoration $-k_1$ of the chain $G_0$ is connected by an edge. The vertex $u$ gets decoration $e_u = -k_0 - \sum_{j=1}^\nu m_{uj}$.

Note that if $q = 1$ then $G_0$ is empty. In this case, we have $s = 0$ and $u = u'$.

We use the notation $E_v$, $v \in \mathcal{V}(G)$, for the basis of the lattice $L_G$ associated with $G$. We simply write $(\ldots)$ for the intersection form $(\ldots)_G$, and $E^*_v$ for the anti-dual elements in $G$; that is, $(E^*_{v'}, E^*_v) = -\delta_{v,v'}$ with the Kronecker delta notation.

Similarly, we write $(\ldots)_{G_j}$ for the intersection form of $G_j$ ($j = 0, \ldots, \nu$). For any $v \in \mathcal{V}(G_j)$, we set $E^*_v \in L'(G_j)$ for the anti-dual of $E_v$ in the graph $G_j$; that is, $(E^*_{v'}, E^*_v)_{G_j} = -\delta_{v,v'}$, $v', v' \in \mathcal{V}(G_j)$.

We denote the canonical class of $G$ by $k_G$ and the canonical class of $G_j$ by $k_{G_j}$.

By a general fact from the theory of surgeries, $H_1(M) = \mathbb{Z}_p$. In fact, $[E^*_u]$ is a generator of this group. For the convenience of the reader we provide a proof of these two statements (which will serve as a model for the corresponding statements valid for the UAC).

**Lemma 4.3.1.** $H = \{[hE^*_u]\}_h$, where $h \in \{0, 1, \ldots, p - 1\}$.

**Proof.** Consider the following element $D$ of $L_G$. On each $G_j$ it is $(f_j)$, we put multiplicity 1 on $u$, multiplicity $k_0$ on $\overline{\pi}_1$, and in general, the numerator of the fraction having Hirzebruch–Jung form $[k_0, \ldots, k_{i-1}]$ on $\overline{\pi}_i$, $1 \leq i \leq s$. Furthermore, put an arrowhead on $\overline{\pi}_s$ with multiplicity $p$. If this arrowhead represents a cut $S$ supported by $E_{\pi_s}$ (that is, a disk in $P = P(G)$, which intersects $E_{\pi_s}$ transversely), then $pS + D$, as an element of $H_2(P, \partial P, \mathbb{Z})$, has the property that $(pS + D, E_v) = 0$ for all $v \in \mathcal{V}(G)$.

This shows that, in fact, $E^*_v = D/p$. Note also that the $E_u$–coefficient of $D/p$ is $1/p$, hence the class of $E^*_u$ (or of $D/p$) in $H = L'(G)/L(G)$ has order $p$.

Hence, if we show that $H$ itself has order $p$ then we are done. For this we verify that $\det(G) = p$. In determinant computations of decorated trees $\mathfrak{G}$, the following formula is useful; see e.g. [13] 4.0.1(d)].

\begin{equation}
\text{det}(\mathfrak{G}) = \det(\mathfrak{G} \setminus e) - \det(\mathfrak{G} \setminus \{a, b, \text{and their adjacent edges}\}).
\end{equation}

We proceed by induction over $\nu$. In order to run the induction properly we introduce slightly more general graphs. Let $G^\ell(x)$ be the graph constructed similarly as $G$ above, but now we glue only $G_0$ and $G_1, \ldots, G_\ell$, where $1 \leq \ell \leq \nu$, and we put for the Euler decoration $e_u$ of the central vertex $u$ the general integer $e_u := -k_0 - x$ for some $x \in \mathbb{Z}$.

With this notation, we have $G = G^\nu(\sum_{j=1}^\nu m_{uj})$.

Note that $\det(G_j) = 1$ (since the graph represents $S^3$), and $\det(G_j \setminus u_j) = m_{uj}$. Indeed, using (4.1.1), $m_{uj} = -((f_j), E^*_u)_{G_j} = -E^*_u, E^*_u)_{G_j} = \det(G_j \setminus u_j)$. 

\begin{equation}
\text{det}(G_j) = 1 \text{ (since the graph represents } S^3), \text{ and } \det(G_j \setminus u_j) = m_{uj} \text{. Indeed, using (4.1.1), } m_{uj} = -((f_j), E^*_u)_{G_j} = -E^*_u, E^*_u)_{G_j} = \det(G_j \setminus u_j).
\end{equation}
Additionally, let us define also $G_0(x)$, the string with $s + 1$ vertices with decorations $-k_0 - x, -k_1, \ldots, -k_s$. Note that from the definitions $\text{det}(G_0(0)) = p = k_0q - r$, and by (4.3.2) (used for the first edge) $\text{det}(G_0(x)) = (x + k_0)q - r = p + xq$.

Now, let us compute $\text{det}(G^e(x))$. If $e = 1$, and if we take for $e$ in (4.3.2) the edge $(u_1, u)$, then $\text{det}(G^1(x)) = \text{det}(G_1) \cdot \text{det}(G_0(x)) - \text{det}(G_1 \setminus u_1) \cdot \text{det}(G_0(x) \setminus u) = p + xq - m_{u_1}q$. Hence for $x = m_{u_1}$ we get $p$.

If $e = 2$, and if we take $e = (u, u_2)$, then $\text{det}(G^2(x)) = \text{det}(G^1(x)) - m_{u_2}q = p + xq - m_{u_1}q - m_{u_2}q$, which equals $p$ for $x = m_{u_1} + m_{u_2}$. For arbitrary $e$ we run induction. □

### 4.3.2. The structure of plumbing graph $\Gamma$ of the UAC $\Sigma$ of $M = S^3_{p/q}(K)$

We construct a plumbing graph $\Gamma$ as follows. $\Gamma$ consists of $\nu$ blocks $\Gamma_1, \ldots, \Gamma_\nu$ with distinguished vertices $w_1, \ldots, w_\nu$ (which will be described later), a chain $\Gamma_0$ of length $q - 1$ consisting of vertices $\overline{w}_1, \ldots, \overline{w}_{q-1} = w'$, all with decoration $-2$, and a ‘central vertex’ $w'$, and some additional edges. These edges are: each vertex $w_j$ is connected by an edge to $w$, and $\overline{w}_1$ is also connected by an edge to $w$. If $q = 1$ then $\Gamma_0$ is empty and $w = w'$.

$\Gamma_j, j \geq 1$ is a plumbing graph of the link of the suspension hypersurface singularity $\{(g_j = 0), 0\} \subset (\mathbb{C}^3, 0)$, where $g_j(x, y, z_j) = f_j(x, y) + z_j^p$ (for the shape of $\Gamma_j$ see [47]). The vertex $w_j$ of $\Gamma_j$ is that vertex which supports the arrowhead $\alpha_j$ (representing the strict transform $S(z_j)$ of $\{z_j = 0\}$), if we regard $\Gamma_j$ as the embedded resolution graph of $(\{z_j = 0\}, 0) \subset (\{g_j = 0\}, 0)$ (that is, $w_j$ supports the strict transform of $\{z_j = 0\}$).

The self–intersection of $w$ is determined as follows.

Let $F_v, v \in \mathcal{V}(\Gamma)$, denote the basis elements of the lattice $L(\Gamma)$ associated with $\Gamma$. We regard $\Gamma_j$ as a subgraph of $\Gamma$, hence $\{F_v\}_{v \in \mathcal{V}(\Gamma_j)}$ are regarded as generators of $L(\Gamma_j)$.

We write $\text{div}(z_j) = S(z_j) + \sum_{v \in \mathcal{V}(\Gamma_j)} n_v F_v = S(z_j) + (z_j)$ for the total transform of $\{z_j = 0\}$ in the embedded resolution of $(\{z_j = 0\}, 0) \subset (\{g_j = 0\}, 0)$ with resolution graph $\Gamma_j$. This means that $(z_j)$ topologically is characterized by $((z_j), F_v)_{\Gamma_j} + \delta_{v}\alpha_j = 0$ for any $v \in \mathcal{V}(\Gamma_j)$ (with the Kronecker delta notation); the strict transform $S(z_j)$ can be represented as an arrowhead $\alpha_j$ on $w_j$.

Then we define the decoration of the central vertex $w$ in $\Gamma$ by $e_w = -1 - \sum_{j=1}^{\nu} n_{w_j}$.

**Lemma 4.3.2.** $\Gamma$ is a (possible) plumbing graph of the UAC $\Sigma$.

**Proof.** This follows basically from the topological interpretation of the cyclic covering algorithms from [47, 48]. Although we will not review the whole algorithm, we will explain how it applies (in particular, some familiarity of the reader with this algorithm is needed), and we also determine the main blocks of $\Gamma$ provided by the algorithm.
Consider the element $D$ of $L_G$ constructed in the proof of Lemma 4.3.1. Its multiplicities can be completed with an arrowhead with multiplicity $p$ supported on $\overline{a}_s$. Equivalently, we can put a cut $S$ on $E_{\overline{a}_s}$, and then $pS + D \in H_2(P, \partial P, \mathbb{Z})$ has the property that $(pS + D, E_v) = 0$ for all $v \in \mathcal{V}(G)$, cf. the proof of Lemma 4.3.1. This means that $pS + D$ is a topological analogue of the divisor of a function. The algorithm which provides the (topological) cyclic $\mathbb{Z}_p$-covering of the plumbed 4–manifold $P$ with branch locus $pS + D$ is identical with the algorithm from [47, 48] (which provides branched cyclic covers associated with an analytic function with, say, divisor $pS + D$ in $\tilde{X}$).

The point is that $S$ has multiplicity $p$, hence the $\mathbb{Z}_p$-covering will have no branching along it. This can be proved as follows (we prefer again an analytic language, but the interested reader might rewrite it in a purely topological language). Let $U$ be a local neighborhood of the intersection point $S \cap E_{\overline{a}_s}$, let $(\zeta_1, \zeta_2)$ be local analytic coordinates in $U$ such that $\{\zeta_1 = 0\} = S \cap U$, $\{\zeta_2 = 0\} = U \cap E_{\overline{a}_s}$. Then $pS + D$ in $U$ is the divisor of $\zeta_1^p \zeta_2^m$ for some integer $m$ with $(p, m) = 1$. Then the cyclic $\mathbb{Z}_p$ covering above $U$ is the normalization of $\{\zeta_1^p \zeta_2^m = \xi^p\}$. This is smooth with coordinates $(\zeta_1, \sigma)$, and the normalization map is $\zeta_1 = \zeta_1$, $\zeta_2 = \sigma^p$, $\xi = \zeta_1 \sigma^m$, which projects to $U$ as $\zeta_1 = \zeta_1$ and $\zeta_2 = \sigma^p$. Hence above the generic point of $S$ there are $p$ points.

In particular, the branch locus is supported in $\cup_v E_v$, that is, we get an unbranched covering of $M$. Note also that the $E_u$–coefficient of $D/p$ is $1/p$, hence the class of $D/p$ (which in fact equals $E_u^*$) has order $p$ in $H$, and it generates $H$, see also the proof of Lemma 4.3.1. This implies that this algorithm provides exactly the UAC of $M$.

Since the algorithm is ‘local’, and the multiplicity of $u$ is 1, it follows that over the subgraphs $G_j$ it is identical with that one which provides the graph of the suspension singularity $f_j(x, y) + z_j^p$. Moreover, over $u$ we will have exactly one vertex in the covering, namely $w$, and this will get multiplicity 1 (see again [47, 48]).

Next, we verify its behaviour over the graph $G_0$. This graph is the graph of a cyclic quotient singularity of type $(q, r)$, where $q/r = [k_1, \ldots, k_s]$. This is the normalization of $xy^{q-r} = z^q$. For details regarding cyclic quotient singularities see [3].

Using this coordinate choice, the strict transform of $y$ is exactly $qS$, the strict transform of $x$ is a disk $S'$ in $E_u$ (a disk neighborhood of $E_u \cap E_{\overline{a}_s}$ in $E_u$) with multiplicity $q$; and finally, the strict transform of $z$ is $S' + (q − r)S$. In particular, the cyclic covering we consider over $G_0$ is exactly the cyclic $\mathbb{Z}_p$–covering of the normalization of $xy^{q-r} = z^q$ along the divisor of $zy^{k_0−1}$ (here for the $S$–multiplicity use $q−r+(k_0−1)q = k_0q−r = p$). This is a new cyclic quotient singularity, the normalization of $\{(x, y, z, w) : xy^{q-r} = z^q, zy^{k_0−1} = w^p\}$. 
The $q$-power of the second equation combined with the first one gives $xy^p = w^{pq}$, hence $t := w^q/y$ is in the integral closure with $x = t^p$. Hence, after eliminating $x$, the new equations are $ty = w^q$, $t^py^{-r} = z^q$ and $zy^{k_0-1} = w^p$. A computation shows that the integral closure of this ring is given merely by $ty = w^q$. (For this use $zy^{k_0-1} = w^p = w^{k_0q-r} = w^{(k_0-1)q}w^{q-r} = t^{k_0-1}y^{k_0-1}w^{q-r}$, that is $y^{k_0-1}(z - t^{k_0-1}w^{q-r}) = 0$.)

This is an $A_{q-1}$ singularity, whose minimal resolution graph is $\Gamma_0$.

Finally, notice that the above algorithm provides a system of multiplicities, which can be identified with a homologically trivial divisor, hence, similarly as in \cite{47,48}, from the multiplicities we get (via intersection with $F_w$) the last ‘missing Euler number’ $e_w$ too. □

The intersection form of $\Gamma$ will be denoted by $\langle \ldots \rangle = \langle \ldots \rangle_\Gamma$. Similarly, $\langle \ldots \rangle_j = \langle \ldots \rangle_{\Gamma_j}$ will denote the intersection form of $\Gamma_j$. The canonical class of $\Gamma$ is $k_\Gamma$, the canonical class of $\Gamma_j$ is $k_{\Gamma_j}$. For any $v \in \mathcal{V}(\Gamma)$, $F^*_v$ will denote the anti-dual of the corresponding divisor $F_v$ in $\Gamma$. Similarly, for a vertex $v \in \mathcal{V}(\Gamma_j)$, $F^*_v$ is the anti-dual of $F_v$ in $\Gamma_j$. Set $J = H_1(\Sigma)$ and let $J_j$ be the first homology group of the 3–manifolds determined by $\Gamma_j$.

**Lemma 4.3.3.**

\[ J \cong J_1 \times \cdots \times J_\nu. \]

**Proof.** Let $\rho_j \in \hat{\Gamma}_j$ be a character of $\Gamma_j$, $j \geq 1$. In \cite{58}, §6.3] is proved that $\rho_j$ takes value 1 on $F^*_w$ (recall that the vertex $w_j$ of $\Gamma_j$ is connected with the central vertex $w$). Hence, for $j \neq i$, $j, i \geq 1$, there is no edge $(v_j, v_i)$ of $\Gamma$, such that $v_j$ is in the support of $\rho_j$ and $v_i$ is in the support of $\rho_i$. This means that each $\rho_j \in \hat{\Gamma}_j$ can be extended to a character of $J$, by setting $\rho_j(F^*_v) = 1$ whenever $v \notin \mathcal{V}(\Gamma_j)$; in this way providing a monomorphism $\hat{\Gamma}_j \hookrightarrow \hat{J}$. Moreover, the same property also guarantees that in fact one has a simultaneous embedding $\prod_{j \geq 1} \hat{\Gamma}_j \hookrightarrow \hat{J}$. Therefore, if we prove that $\prod_{j \geq 1} \det(\Gamma_j) = \det(\Gamma)$, then the above embedding becomes an isomorphism, hence the statement follows.

For the determinant identity we use similar method as in the proof of Lemma 4.3.1. Let $\Gamma^t(x)$ be the graph obtained by connecting (the distinguished vertices of) $\Gamma_0$ and $\Gamma_1, \ldots, \Gamma_\ell$ to $w$, where $1 \leq \ell \leq \nu$, and we put for the Euler decoration $e_w$ the integer $e_w := -1 - x$. Furthermore, let $\Gamma_0(x)$ be the string with $q$ vertices and decorations $-1 - x, -2, -2, \ldots$. Hence $\det(\Gamma_0(x)) = 1 + xq$.

We write $d_j$ for $\det(\Gamma_j) = |J_j|$, and note that $\det(\Gamma_j \setminus w_j) = d_j n_{w_j}$ (use (4.1.1)).

Now, we are ready to compute $\det(\Gamma^t(x))$. If $\ell = 1$, and in (4.3.2) we take $e = (w_1, w)$, then $\det(\Gamma^1(x)) = d_1(1 + xq) - d_1n_{w_1}q$, which equals $d_1$ whenever $x = n_{u_1}$.

If $\ell = 2$ and $e = (u, w_2)$ one has $\det(\Gamma^2(x)) = \det(\Gamma^1(x))d_2 - d_2n_{w_2}d_1q = d_1d_2(1 + xq - n_{w_1}q - n_{w_2}q)$, which equals $d_1d_2$ whenever $x = m_{u_1} + m_{u_2}$. For arbitrary $\ell$ run induction. □
Lemma 4.3.4.
\[
\begin{align*}
(a) \quad -p \cdot (E_u^*, E_u^*) &= q \quad \text{and} \quad -p \cdot (E_{u'}^*, E_{u'}^*) = 1; \\
(b) \quad q \cdot (E_{u_j}^*, E_{v_j}^*) &= p \cdot (E_u^*, E_v^*) \quad \text{for any } v \in V(G_j), \ j \geq 1; \\
(c) \quad -\langle F_w^*, F_w^* \rangle &= q \quad \text{and} \quad -\langle F_{w'}^*, F_{w'}^* \rangle = 1; \\
(d) \quad q \cdot \langle F_{u_j}^*, F_{v_j}^* \rangle &= \langle F_w^*, F_v^* \rangle \quad \text{for any } v \in V(\Gamma_j), \ j \geq 1.
\end{align*}
\] (4.3.3)

Proof. Identity (4.1.1) applied for $G$ and the pair of vertices $u, u'$ (and $u, u'$) gives $(a)$, since $\det(G) = p$, $\det(G_j) = 1$ for $j \geq 1$ and $\det(G_0) = q$. $(b)$ follows similarly. $(c)$ and $(d)$ follows from this property combined with Lemma 4.3.3.

4.4. The Covering Additivity Property

4.4.1. Proof of the main theorem. Now we are ready to prove Theorem 4.1.2. To adjust it to its proof, we recall it in a more explicit form, in the language of plumbing graphs.

Theorem 4.4.1. Let $K$ be a connected sum of algebraic knots, $p, q$ coprime positive integers. Let $G$ be the plumbing graph of $S^3_{-p/q}(K)$ and $\Gamma$ the plumbing graph of its universal abelian cover $\Sigma$. Assume that both $S^3_{-p/q}(K)$ and its universal abelian cover $\Sigma$ are rational homology spheres. Then the following additivity holds:
\[
-s_{w_0}(\Gamma) - \frac{\langle k_\Gamma, k_\Gamma \rangle + \#V(\Gamma)}{8} = \sum_{h=0}^{p-1} \left( -s_{w_h}(G) - \frac{\langle k_G + 2r_h, k_G + 2r_h \rangle + \#V(G)}{8} \right).
\]

On the left hand side the index 0 of $s_{w_0}(\Gamma)$ is the unit element of $J = H_1(\Sigma)$ and on the right hand side we identified elements of $H \cong \mathbb{Z}_p$ with elements of $\{0, 1, \ldots, p-1\}$, 0 being the unit element and 1 being the generator $[E_u^*]_1$, i.e., $r_h = r[E_u^*]_1$.

In fact, the condition whether $\Sigma$ is a $\mathbb{Q}HS^3$ or not is readable already from $p$ and the plane curve singularity invariants describing the knots $K_j$; cf. [58] §6.2 (c)].

Proof. Notice that deleting from $G$ the ‘central vertex’ $u$ (and all its adjacent edges), one gets $G_0, G_1, \ldots, G_\nu$ as connected components of the remaining graph. Also, deleting from $\Gamma$ the ‘central vertex’ $u$ (and all its adjacent edges), one gets $\Gamma_0, \Gamma_1, \ldots, \Gamma_\nu$ as connected components of the remaining graph. $\Gamma_0$ and $G_0$ are present only if $q > 1$.

We use the notations $R_j$, resp. $\tilde{R}_j$ for the ‘restriction’ homomorphisms $L'_G \to L'_{G_j}$, resp. $L'_\Gamma \to L'_{\Gamma_j}$, dual to the natural inclusions $L_{G_j} \to L_G$, resp. $L_{\Gamma_j} \to L_\Gamma$. They are characterized by $R_j(E_u^*) = E_{u_j}^*$, if $v \in V(G_j)$ and 0 otherwise, resp. $\tilde{R}_j(F_v^*) = F_{v_j}^*$, if $v \in V(\Gamma_j)$ and 0 otherwise. E.g., $R_j(k_G) = k_{G_j}$ and $\tilde{R}_j(k_\Gamma) = k_{\Gamma_j}$ (cf. [13] Def. 3.6.1 (2))].
We can apply the surgery formula of [13] Theorem 1.0.1 and get the following two formulae. The new symbols $\mathcal{H}_{u,h}^\text{pol}(1)$ and $\mathcal{F}_{w,0}^\text{pol}(1)$ are values at $t = 1$ of certain polynomials as in [13] §3.5; their definitions will be recalled later in (4.4.3) and (4.4.4).

\[(4.4.1) \quad \mathcal{S}_h(G) = \mathcal{H}_{u,h}^\text{pol}(1) + \mathcal{S}_{R_0}(r_h)(G_0) + \sum_{j=1}^\nu \mathcal{S}_{R_j}(r_h)(G_j),\]

\[(4.4.2) \quad \mathcal{S}_0(\Gamma) = \mathcal{F}_{w,0}^\text{pol}(1) + \mathcal{S}_0(\Gamma_0) + \sum_{j=1}^\nu \mathcal{S}_0(\Gamma_j).\]

In (4.4.1), for $j \geq 1$, $[R_j(r_h)] = 0 \in L'_G / L_G$, as the latter one is the trivial group $H_1(S^3)$.

Hence, by (4.1.6), $s_{R_j}(r_h)(G_j) = s_0(G_j) + \chi_j(R_j(r_h))$, where $\chi_j(x) := -\frac{1}{2}(x, x + k_{G_j})_j$.

Furthermore, by Proposition 4.2.3, $s_0(\Gamma_0) = 0$, and $s_0(G_j) = 0$ for $j \geq 1$ (as $G_j$ is a plumbing graph for $S^3$). Therefore, the desired equality $s_0(\Gamma) = \sum_{h=0}^{p-1} \mathcal{S}_h(G)$ reduces to the proof of the following three lemmas.

\[\text{Lemma 4.4.2.} \quad \sum_{h=0}^{p-1} \mathcal{H}_{u,h}^\text{pol}(1) = \mathcal{F}_{w,0}^\text{pol}(1).\]

\[\text{Lemma 4.4.3.} \quad \sum_{h=0}^{p-1} \chi_j(R_j(r_h)) = s_0(\Gamma_j) \quad (\text{for any } j \geq 1).\]

\[\text{Lemma 4.4.4.} \quad \sum_{h=0}^{p-1} \mathcal{S}_{R_0}(r_h)(G_0) = 0.\]

In the next paragraphs we recall the definition of $\mathcal{H}_{u,h}^\text{pol}$ and $\mathcal{F}_{w,0}^\text{pol}$ (following [13] §3.5) adapted to the present case and notations, and then we provide the proofs of the lemmas.

First, given a rational function $\mathcal{R}(t)$ of $t$, one defines its polynomial part $\mathcal{R}_\text{pol}(t)$ as the unique polynomial in $t$ such that $\mathcal{R}(t) - \mathcal{R}_\text{pol}(t)$ is either 0 or it can be written as a quotient of two polynomials of $t$ such that the numerator has degree strictly less than the denominator. Now $\mathcal{F}_{w,0}^\text{pol}$ and $\mathcal{H}_{u,h}^\text{pol}$ are polynomial parts of rational functions defined as follows.

\[(4.4.3) \quad \mathcal{H}_{u,h}(t) = \frac{1}{p} \cdot \sum_{\varrho \in \hat{H}} \varrho^{-1}(h) \prod_{v \in \mathcal{V}(G)} (1 - \varrho([E^*_v])) t^{-p(E^*_v, E^*_v)} \delta_v^{-2},\]
where \( \delta_v \) denotes the degree (number of adjacent edges) of a vertex \( v \in \mathcal{V}(G) \).

\[
F_{w,0}(t) = \frac{1}{|J|} \cdot \sum_{\varrho \in \hat{J}} \prod_{v \in \mathcal{V}(\Gamma)} (1 - \varrho([F_v^*]) t^{-|J|(F_w^*,F_v^*)})^{\delta_v - 2},
\]

where \( \tilde{\delta}_v \) denotes the degree of a vertex \( v \in \mathcal{V}(\Gamma) \).

**Proof of Lemma 4.4.2.** By Fourier summation

\[
H_u(t) := \sum_{h=0}^{p-1} H_{u,h}(t) = \prod_{v \in \mathcal{V}(G)} (1 - t^{-p(E_u^*,E_v^*)})^{\delta_v - 2}.
\]

As taking polynomial parts of rational functions is additive, Lemma 4.4.2 follows if we prove

\[
H_u(t^{[J]}) = F_{w,0}(t).
\]

Let \( \Delta_j = \Delta_{s^{d_j}}(K_j) \) be the Alexander polynomial of the knot \( K_j \) (defined as in [58 §2.6, (8)], or [15]). Then, since \( E_{v_j}^* = (f_j) \in L_{G_j} \),

\[
\frac{\Delta_j(t)}{1 - t} = \prod_{v \in \mathcal{V}(G_j)} (1 - t^{-p(E_{v_j}^*,E_{v_j}^*)})^{\delta_v - 2}.
\]

Comparing (4.4.5) with the above formula for the Alexander polynomials and using identities (a), (b) of Lemma 4.3.4 we get that

\[
H_u(t) = \frac{\prod_{j=1}^{J} \Delta_j(t^q)}{(1 - t)(1 - t^q)}.
\]

Recall that \( J_j = L_{G_j}/L_{\Gamma_j} \) is the first homology group of the manifold determined by \( \Gamma_j \) and that \( F_{v_j}^* = (z_j) \in L_{G_j} \). Let \( \Delta_{j,G_j} \) be the Alexander polynomial of the knot \( K_{j,G_j} \) in the manifold of \( \Gamma_j \) determined by \( z_j = 0 \) (see [58 §2.6, (8)]). That is,

\[
\frac{\Delta_{j,G_j}(t)}{1 - t} = \frac{1}{|J_j|} \cdot \sum_{\varrho_j \in \hat{J}_j} \prod_{v \in \mathcal{V}(\Gamma_j)} (1 - \varrho_j([F_v^*]) t^{-p(F_{v_j}^*,F_v^*)})^{\delta_v - 2}.
\]

Recall that \( J = J_1 \times \cdots \times J_\nu \). Consequently, any character \( \varrho \in \hat{J} \) can be written as a \( \nu \)-tuple of characters, \( \varrho = (\varrho_1, \ldots, \varrho_\nu) \) with \( \varrho_j \in \hat{J}_j = \text{Hom}(J_j, \mathbb{C}^*) \). Furthermore, for any \( v \in \mathcal{V}(\Gamma_j) \), \( \varrho([F_v^*]) = \varrho_j([F_{v_j}^*]) \) and \( \varrho([F_v^*]) = 1 \) if \( v = w \) or \( v \in \Gamma_0 \) as in that case \( F_v^* \) represents the trivial element in \( L_{G_j}/L_{\Gamma_j} \) (see also the proof of Lemma 4.3.3).

Comparing (4.4.4) with the above formula for the Alexander polynomials and using identities (c), (d) of Lemma 4.3.4 we get that, setting \( s = t^{[J]} \),

\[
F_{w,0}(t) = \frac{\prod_{j=1}^{J} \Delta_{j,G_j}(s^q)}{(1 - s)(1 - s^q)}.
\]
4.4. THE COVERING ADDITIVITY PROPERTY

By [58, Proposition 6.6] \( \Delta_j = \Delta_{j, \Gamma_j} \), so via (4.4.7) and (4.4.8) we obtain (4.4.6).

**Proof of Lemma 4.4.3.** For any element \( l' = \sum_{v \in \mathcal{V}(G)} c_v E_v \in L'_G \), let

\[
[l'] := \sum_{v \in \mathcal{V}(G)} [c_v] E_v, \text{ resp. } \{l'\} := l' - [l']
\]
denote the coordinatewise integer, resp. fractional part of \( l' \) in the basis \( \{E_v\}_{v \in \mathcal{V}(G)} \). We use this notation for other graphs as well.

Using the description of \( E^*_u \) in the proof of Lemma 4.3.2 we have

\[
(4.4.9) \quad hE^*_u = \sum_{j \geq 1} h \cdot (f_j)/p + hE_u/p + D_0 \quad (0 \leq h < p),
\]
where \( D_0 \) is supported on \( G_0 \). Since \( r_h = \{hE^*_u\} \), we obtain

\[
r_h = hE^*_u - \sum_{j \geq 1} [h \cdot (f_j)/p] - [D_0].
\]

Since \( R_j(E^*_u) = 0, R_j(E_v) = E_v \) for \( v \in \mathcal{V}(\Gamma_j) \), and \( R_j(E_v) = 0 \) for \( v \notin (\mathcal{V}(\Gamma_j) \cup u) \), we get

\[
(4.4.10) \quad R_j(r_h) = -[h \cdot (f_j)/p].
\]

As \( \Gamma_j \) is the plumbing graph of a suspension hypersurface singularity \( \{g_j = 0\} \), where \( g_j(x, y, z_j) = f_j(x, y) + z_j^p \) and, as it is proved in [58], for such suspension singularities the SWIC holds (see §4.2.1), we have \( \mathfrak{g}_0(\Gamma_j) = p_g(\{g_j = 0\}) \). Hence, the statement of the Lemma is equivalent with the following geometric genus formula valid for suspension singularities:

\[
p_g(\{g_j = 0\}) = \sum_{h=0}^{p-1} \chi_j(-[h \cdot (f_j)/p]).
\]

This formula has major importance even independently of the present application. We separate the statement in the following claim.

**Claim 4.4.5.** Let \( f(x, y) \in \mathbb{C}\{x, y\} \) be the equation of an irreducible plane curve singularity. Let \( G_f \) be the dual resolution graph of a good embedded resolution of \( f \), from which we delete the arrowhead (strict transform) of \( f \) and all the multiplicities. Let \( (f) \) be the part of the divisor of \( f \) supported on the exceptional curves. Then for any positive integer \( p \) the geometric genus of the suspension singularity \( \{g(x, y, z) = 0\} \) with \( g(x, y, z) = f(x, y) + z^p \) is

\[
p_g(\{g = 0\}) = \sum_{h=0}^{p-1} \chi(\mathbb{R}(-[h \cdot (f)/p])).
\]

**Remark 4.4.6.** A combinatorial formula (involving Dedekind sums) for the signature of (the Milnor fibre of) suspension singularities was presented in [47]. Recall that Durfee
and Laufer type formulae imply that the geometric genus and the signature determine each other modulo the link (see e.g. [49] Theorem 6.5 and the references therein). In particular, the mentioned signature formulae provide expressions for the geometric genus as well. Nevertheless, the above formula is of different type.

Proof of Claim 4.4.5 by András Némethi. Let $\phi : Z \to (\mathbb{C}^2, 0)$ be the embedded resolution of $f$. Consider the $\mathbb{Z}_p$ branched covering $c : \{(g = 0), 0 \to (\mathbb{C}^2, 0)$, the restriction of $(x, y, z) \mapsto (x, y)$. Let $c_\phi : W \to Z$ be the pullback of $c$ via $\phi$ and let $\hat{c}_\phi : \hat{W} \to Z$ be the composition of the normalization $n : \hat{W} \to W$ with $c_\phi$. Then $\hat{W} \to W \to \{g = 0\}$ is a partial resolution of $\{g = 0\}$: although it might have some cyclic quotient singularities, since these are rational, one has $p_g(\{g = 0\}) = h^1(\mathcal{O}_{\hat{W}})$. On the other hand, we claim that

\[(4.4.11) \quad (\hat{c}_\phi)_* (\mathcal{O}_{\hat{W}}) = \bigoplus_{h=0}^{p-1} \mathcal{O}_Z([h \cdot (f)/p])\]

This follows basically from [30] §9.8. For the convenience of the reader we sketch the proof.

We describe the sheaves $(c_\phi)_* (\mathcal{O}_W)$ and $(\hat{c}_\phi)_* (\mathcal{O}_{\hat{W}})$ in the neighborhood $U$ of a generic point of the exceptional set $E$ of $\phi$. Consider such a point with local coordinates $(u, v)$, $\{u = 0\} = E \cap U, (f) in U is given by $u^n = 0$. Consider the covering, a local neighborhood of type $\{(u, v, z) : z^p = u^m\}$ in $W$. Then $\mathcal{O}_{W,0}$ as $\mathbb{C}\{u, v\}$–module is $\bigoplus_{h=0}^{p-1} z^h \cdot \mathbb{C}\{u, v\}$. For simplicity we assume $\text{gcd}(m, p) = 1$. The $\mathbb{Z}_p$–action is induced by the monodromy on the regular part, namely by the permutation of the $z$–pages, induced over the loop $u(s) = \{e^{2\pi is}\}_{0 \leq s \leq 1}$. This is the multiplication by $\xi := e^{2\pi im/p}$. Hence, $z^h \mathbb{C}\{u, v\}$ is the $\xi^h$–eigensheaf of $( c_\phi)_* (\mathcal{O}_W)$.

If we globalize $z\mathbb{C}\{u, v\}$, we get a line bundle on $Z$, say $\mathcal{L}$. Then the local representative of $\mathcal{L}^p$ is $z^p \mathbb{C}\{u, v\} = u^m \mathbb{C}\{u, v\} = \mathbb{C}\{u, v\}(-f)$. Hence $\mathcal{L}^p$ is trivialized by $f \circ \phi$. Since $\text{Pic}(Z) = 0$, $\mathcal{L}$ itself is a trivial line bundle on $Z$.

Next, we consider the normalization $\hat{W}$. Above $U$ it is $(\mathbb{C}^2, 0)$ with local coordinates $(t, v)$, and the normalization is $z = t^m, u = tv$. In particular,

\[(\hat{c}_\phi)_* (\mathcal{O}_{\hat{W},0}) = \bigoplus_{h=0}^{p-1} t^h \cdot \mathbb{C}\{u, v\},\]

where $\mathcal{F}^{(h)} := t^h \cdot \mathbb{C}\{u, v\}$ is the $e^{2\pi ih/p}$–eigensheaf. Set the integer $m'$ with $0 \leq m' < p$ and $mm' = 1 + kp$ for certain $k \in \mathbb{Z}$. Then one has the following eigensheaf inclusions: $t^h \mathbb{C}\{u, v\} \supset z^{(hm')} \cdot \mathbb{C}\{u, v\} = \mathcal{L}^{(hm')/p}|_U$. Hence, for some effective cycle $D$ we must have $t^h \mathbb{C}\{u, v\} = \mathcal{L}^{(hm')/p}(D)|_U$. This, by taking $m$–power reads as $z^h \mathbb{C}\{u, v\} = z^{(hm')/p_m} \mathbb{C}\{u, v\}(mD)$. This means that if $\{hm'/p\} = mh/p$ and $mm_h = k_h p + h$ for certain integers $m_h$ and $k_h, 0 \leq m_h < p$, then the local equation of $mD$ is $z^{(hm')/p_m-h} = z^{kh_p}$.
Hence $D$ locally is given by $t^{kh} = u^{kh}$. Since $kh = \lfloor mm/p \rfloor$, the global reading of this fact is $D = \lfloor m_h \cdot (f)/p \rfloor$. Hence

$$(\hat{c}_\phi)_*(\mathcal{O}_{\widetilde{\mathcal{R}}}) = \mathcal{L}^{(kh')}_{h>0}p(\lfloor m_h \cdot (f)/p \rfloor).$$

Since $\mathcal{L}$ is a trivial bundle, and $h \mapsto m_h$ is a permutation of $\{0, \ldots, p-1\}$, (4.4.11) follows.

Next, from (4.4.11) we obtain $p_g(\{g = 0\}) = \sum_h h^1(\mathcal{O}_Z(\lfloor h \cdot (f)/p \rfloor))$. Therefore

$$h^1(\mathcal{O}_Z(D')) = h^1(\mathcal{O}_D'(D')), \quad h^1(\mathcal{O}_D'(D')) = 0,$$

we have

$$h^1(\mathcal{O}_Z(D')) = -\chi(\mathcal{O}_D'(D')) = -(D', D') + (D', D' + K)/2 = \chi(-D').$$

This ends the proof of the claim. $\square$

By Claim 4.4.5, the proof of Lemma 4.4.3 is completed. $\square$

Proof of Lemma 4.4.4. We observe two facts. First, from the proof of Lemma 4.3.2 we obtain that $R_0(r_h)$ only depends on the value $p/q$ (and not on the blocks $G_j$, $j \geq 1$). (Equivalently, from (4.4.9), we have that $D_0$ is the unique rational cycle on $G_0$ such that, when completed with an arrowhead supported on $u' = \overline{u}_s$ with multiplicity one and with an arrowhead supported on $\overline{u}_1$ with multiplicity $h/p$, it has the property that intersected by any $E_{\overline{\pi}_1}$ the result is zero.) It has the same expression even if we replace all the graph $G_j$ by the empty graph.

Second, from Equations (4.4.1) and (4.4.2) and the discussion following them we get that under the validity of Lemmas 4.4.2 and 4.4.3 (notice that we already proved these for any situation) the main Theorem 4.4.1 (property CAP) is equivalent with Lemma 4.4.4. Putting these facts together, we get that the validity of Lemma 4.4.4 is equivalent with the validity of CAP in the case when $G_j = \emptyset$ for all $j \geq 1$. But CAP for $G_0 \cup \{u\}$ is true by Claim 4.2.1 and Example 4.2.2.

Of course, there is also a direct argument by a combinatorial computation of the involved invariants on the string $G_0$. $\square$

4.5. Examples and applications

4.5.1. The invariant $s_h$ and lattice cohomology. The normalized SW invariant $s_h(G)$ can also be expressed as the Euler characteristic of the lattice cohomology, cf. Remark 2.7.17 (a) in Chapter 2. The advantage of this approach is that it provides an
alternative, completely elementary way to define $s$, as the definition of the lattice cohomology is purely combinatorial from the plumbing graph $G$. This description is rather different than the one used in [56, 57, 58], or the one used in the above proofs.

An another advantage is that for integral surgeries there are several computations and formulae for the lattice cohomology in the literature, which provide additional information on the main theorem, or on the different surgery pieces used in its proof.

Recall the definition of the lattice cohomology associated with a weight function from Subsection 2.7.3, Chapter 2.

For details on lattice cohomologies of negative definite plumbing graphs, see [52, 59].

We just recall here that given a negative definite plumbing graph $G$ of a $\mathbb{Q}HS^3$ 3-manifold $M$ and a representative $l' \in L'$ of an element $[l'] = h \in H$, one works with the lattice $L = L_G = \mathbb{Z}\langle E_v \rangle_{v \in \mathcal{V}(G)}$ and weight function $L \ni l \mapsto -\frac{1}{2}(l, l + k_G + 2l')$. The cohomology theory corresponding to this weight function in this chapter will be denoted by $H^*(G; k_G + 2l')$.

If for $h \in H$ we choose the minimal representative $r_h \in L'$, then the cohomology theory $H^*(G; k_G + 2r_h)$ is in fact an invariant of the pair $(M, h)$ (i.e. it does not depend on the plumbing representation) and thus can be denoted by $H^*_h(M)$. It is proved in [53] (cf. Remark 2.7.17 in Chapter 2) that for any $l' \in L'$ and $h = [l'] \in H$

\begin{equation}
(4.5.1) 
\ s'_v(G) = eu \ H^*(G; k_G + 2l') \quad \text{and} \quad \ s_h(M) = eu \ H^*_h(M).
\end{equation}

Notice that in [53], both the definition of $s$ and the lattice cohomology are slightly different: our invariant $s$ as defined in (4.1.3) is the whole expression [53] (1.0.2)] with the minimal representative $r_h \in L'$ (the same symbol $s$ was denoting a different quantity in [53], being equal to $sw$).

4.5.2. Lattice cohomology of integral surgeries. The lattice cohomology of integral surgeries ($q = 1$) along $K = K_1 \# \ldots \# K_r$ was treated in [7, 50, 51, 59]. We use the notation $p/q = d \in \mathbb{Z}$. Clearly $u = u'$.

Let $\Delta_j(t)$ be the Alexander polynomial of the algebraic knot $K_j$ (recall that in our convention, $\Delta_j(t) \in \mathbb{Z}[t]$ and $\Delta_j(1) = 1$). Let $\delta_j$ be the Seifert genus of $K_j \subset S^3$ (or, the delta invariant of the corresponding plane curve singularity), and write $\delta := \sum_{j=1}^r \delta_j$. Set also $\Delta(t) = \prod_{j=1}^r \Delta_j(t)$ and write it in the form (cf. (2.2.4) in Subsection 2.2.1)

\begin{equation}
\Delta(t) = 1 + \delta(t - 1) + (t - 1)^2 Q(t) \quad \text{with} \quad Q(t) = \sum_{i=0}^{2\delta - 2} q_i t^i.
\end{equation}

Note that $Q(1) = \Delta''(1)/2$.

Example 4.5.1. The case of integral surgeries is especially important in singularity theory, since the links of superisolated singularities (see [37, 38]) are of this type. Recall
that they appear as follows. Let \( f \in \mathbb{C}[x, y, z] \) be an irreducible homogeneous polynomial of degree \( d \) such that its zero set in \( \mathbb{CP}^2 \) is a rational cuspidal curve; i.e. \( C = \{ f = 0 \} \) is homeomorphic to \( S^2 \) and all the singularities of \( C \) are locally irreducible. Let their number be \( \nu \). Assume that there is no singular point on the projective line given by \( z = 0 \). Then the equation \( f(x, y, z) + z^{d+1} = 0 \) in \( (\mathbb{C}^3, 0) \) determines an isolated complex surface singularity with link homeomorphic to \( S^3_{-d}(K) \), where \( K \) is the connected sum of algebraic knots given by the local topological types of the singularities on \( C \). In this case, by genus formula, \( (d-1)(d-2) = 2\delta \), a relation which connects \( K \) with \( d \). However, we can take the surgery (and plumbed) manifold \( M = S^3_{-d}(K) \) for any \( K \) being a connected sum of \( \nu \) algebraic knots \( K_j, j = 1, \ldots, \nu \) and with arbitrary \( d > 0 \), even without the ‘compatibility condition’ \( (d-1)(d-2) = 2\delta \) (imposed in the case of superisolated singularities).

Next we recall some results on lattice cohomology of \( M = S^3_{-d}(K) \), which will be combined with the above proved CAP. We emphasize that in the next general discussion the identity \( (d-1)(d-2) = 2\delta \) will not be assumed.

Use the notations of §4.3.1 and the proof of Lemma 4.4.3 and write \( s_h := hE_u^* \), then \( r_h = \{ hE_u^* \} = \{ s_h \} \), and set also \( c_h := \chi(r_h) - \chi(s_h) \).

From [59, Theorem 7.1.1] (see also [7] Theorem 3.1.3], cf. (2.7.5) from Theorem 2.7.11) we know that

\[
\mathfrak{g}_{s_h}(G) = \text{eu } \mathbb{H}^+ \left( S^3_{-d}(K); k_G + 2s_h \right) = \sum_{n \equiv h \mod d, 0 \leq n \leq 2\delta - 2} q_n.
\]

By the surgery formula [13, Theorem 1.0.1] one has

\[
\mathfrak{g}_{s_h}(G) = \mathcal{H}_{u,h}^\text{pol}(1) + \sum_{j=1}^{\nu} \mathcal{G}_{R_j(s_h)}(G_j).
\]

Since \( R_j(s_h) = 0 \) and \( \mathcal{G}_0(G_j) = 0 \) (see Proposition 4.2.3) we get \( \mathfrak{g}_{s_h}(G) = \mathcal{H}_{u,h}^\text{pol}(1) \), hence

\[
\mathcal{H}_{u}^\text{pol}(1) = \sum_{h=0}^{p-1} \mathfrak{g}_{s_h}(G) = q_n = Q(1).
\]

This is related to the invariants \( \mathfrak{g}_{h}(G) \) as follows. From (4.1.4) and (4.5.1) we have

\[
\sum_{h=0}^{d-1} \mathfrak{g}_{h}(G) = \sum_{h=0}^{d-1} \mathfrak{g}_{s_h}(G) + \sum_{h=0}^{d-1} (\chi(r_h) - \chi(s_h)) = Q(1) + \sum_{h=0}^{d-1} c_h.
\]

Furthermore, the identity \( p_y(\{ g_j = 0 \}) = \sum \chi_j(-[h \cdot (f_j)/d]) \) from Claim 4.4.5 has also the following addendum:

\[
\sum_{j=1}^{\nu} \chi_j(-[h \cdot (f_j)/d]) = \chi(r_h) - \chi(s_h) = c_h.
\]
Indeed, by (4.4.9),
\[ s_h = hE_u^* = hE_u/d + \sum_j h \cdot (f_j)/d, \]
hence \( r_h = hE_u/d + \sum_j \{h(f_j)/d\} \), and \( s_h - r_h = \sum_j [h(f_j)/d] \). This shows that \((s_h, s_h - r_h) = (hE_u^*, s_h - r_h) = 0\). Therefore \( \sum_j \chi_j(-[[h \cdot (f_j)/d]]) = \chi(r_h - s_h) = \chi(r_h) - \chi(s_h) \).

**Figure 4.1.** The plumbing graph of \( S^3_{-8}(K_{(6,7)} \# K_{(2,9)} \# K_{(2,5)}) \), the Dehn surgery with coefficient \(-8\) on the connected sum of the torus knots \((6, 7), (2, 9)\) and \((2, 5)\), and the plumbing graph of its universal abelian cover.

### 4.5.3. Examples and applications.

**Example 4.5.2.** Consider the plumbing graph of a superisolated singularity corresponding to a curve of degree \( d = 8 \) with three singular points whose knots \( K_1, K_2, K_3 \)
are the torus knots of type \((6, 7), (2, 9), (2, 5)\), respectively (see Figure 4.1). Such a curve exists, see \[22\] Theorem 3.5] (cf. Proposition \[2.6.3\] in Chapter 2).

One computes that
\[
\mathcal{H}_u(t) = (1 - t) \cdot \frac{1 - t^{42}}{(1 - t^6)(1 - t^7)} \cdot \frac{1 - t^{10}}{(1 - t^2)(1 - t^5)} \cdot \frac{1 - t^{18}}{(1 - t^2)(1 - t^9)} = \frac{\Delta_1(t)\Delta_2(t)\Delta_3(t)}{(1 - t)^2}
\]
and \(\mathcal{H}_{u}^{\text{pol}}(1) = 293\). Correspondingly, \(\sum_{h=0}^{7} s_{sh}(G) = Q(1) = 293\). One also computes that \(\sum_{h=0}^{7} c_{h} = 34\). Therefore, \(\sum_{h=0}^{7} s_{h}(G) = 293 + 34 = 327\).

After computing the graph \(\Gamma\) of the UAC (see Figure 4.1), we have \(J = \mathbb{Z}_7 \times \mathbb{Z}_9 \times \mathbb{Z}_5\). Setting \(s = t^{7 \cdot 9 \cdot 5}\), after summation \(\Sigma_{s}\) over \(\zeta_1 \in \mathbb{Z}_7\), \(\zeta_2 \in \mathbb{Z}_9\), \(\zeta_3 \in \mathbb{Z}_5\) (where \(\mathbb{Z}_i = \{e^{2\pi i m}\}_{m \in \mathbb{Z}}\) are cyclic groups), the rational function \(\mathcal{F}_{w,0}(t)\) equals
\[
\frac{1 - s}{7 \cdot 9 \cdot 5} \sum_{\Gamma} \frac{(1 - s^{21})^2}{(1 - s^7)(1 - \zeta_1^2 s^3)(1 - \zeta_1^{-2} s^3)} \cdot \frac{(1 - s^9)}{(1 - \zeta_2^5 s)(1 - \zeta_2^{-5} s)} \cdot \frac{(1 - s^5)}{(1 - \zeta_3 s)(1 - \zeta_3^{-1} s)} = \frac{\Delta_{1,\Gamma}(s)\Delta_{2,\Gamma}(s)\Delta_{3,\Gamma}(s)}{(1 - s)^2}.
\]
Then \(\mathcal{F}_{w,0}(t) = \mathcal{H}_u(s)\) holds indeed with \(s = t^{7 \cdot 9 \cdot 5}\). Correspondingly,
\[
s_{0}(\Gamma) = \mathcal{F}_{w,0}^{\text{pol}}(1) + \sum_{j=1}^{3} p_{g}(\{g_{j} = 0\}) = 293 + 34 = 327.
\]

**Example 4.5.3.** We wish to emphasize that the covering additivity property of \(s\) is not true in general, not even when restricting ourselves to integral surgeries along algebraic knots in integral homology spheres (instead of \(S^3\)). This is shown by the next example (motivated by \[58\] Remark 6.8.(2)); the arrowhead of that graph is replaced by the \(-8\) vertex below.

![Graph](image)

If we replace the \((-8)\)-vertex of the graph \(G\) above by an arrowhead (representing a knot \(K\)), we get the graph of an integral homology sphere \(\mathcal{S}^3\) with a knot it having \(m_{u_1} = 6\), hence \(M(G) = \mathcal{S}_{-2}^3(K)\), and \(G\) has determinant 2. One computes that \(s_{0}(G) + s_{1}(G) = 15 + 14 = 29\), while \(s_{0}(\Gamma) = 21\). In fact, when trying to copy the proof of Theorem 4.4.1 one finds that neither the polynomial identity of Lemma 4.4.2 holds (this is why the example was present in \[58\] Remark 6.8.(2))), nor is \(\Gamma\) \(w\) of suspension type (so we can not use the SWIC as in the proof of Lemma 4.4.3).
On the other hand, there are facts suggesting that the CAP of $s$ can hold in more general settings. Indeed, as we indicated in Claim 4.2.1 if for a given $M$ one can find a surface singularity with link $M$ such that the EqSWIC holds for the singularity $(X, 0)$ and the SWIC holds for its UAC, then the additivity of $s$ holds automatically. (Eq)SWIC was verified for many analytic structures, whose links are not of surgery type. On the other hand, the family of superisolated singularities is the main source of counterexamples for SWIC (and this was one of the motivations to test CAP for them).

Remark 4.5.4. Independently of any analytic argument, one can also find purely topological examples for which CAP still works (and in which cases not only that we cannot verify the presence of EqSWIC/SWIC, but we cannot even identify any specific analytic structure on the topological type, or on certain special subgraphs). Here is one, for which the assumptions of Theorem 4.4.1 do not hold either.

One verifies that $\det(G) = 2$, and $s_0(G) + s_1(G) = 147 + 132 = 279 = s_0(\Gamma)$.

This raises the interesting question that what are the precise limits of the CAP.

Remark 4.5.5. The lattice cohomology plays an intermediate role connecting the analytic invariants of a normal surface singularity $X$ with the topology of its link $M = M(G)$. E.g., one proves using [52] Proposition 6.2.2, Example 6.2.3, Theorem 7.1.3, 7.2.4 that for any $h \in H_1(M)$ one has

$$p_g(X)_h \leq \text{eu} \mathbb{H}^0(M(G); k_G + 2r_h).$$

Furthermore, for surgery manifolds $M(G) = S^3_{-d}(K)$, $K$ being the connected sum of $\nu$ algebraic knots, one has the vanishing $\mathbb{H}^q(M(G), k_G + 2r_h) = 0$ for $q \geq \nu$ ([59 Corollary 4.2.11]). In particular, for superisolated singularities corresponding to unicuspidal rational plane curves ($\nu = 1$) one has

$$(4.5.2) \quad p_g(X)_h \leq \text{eu} \mathbb{H}^+(M(G); k_G + 2r_h) = s_h(M).$$

Therefore, for $M = S^3_{-d}(K)$ with $\nu = 1$, if the SWIC holds for the UAC $(Y, 0)$, that is, if $p_g(Y) = s_0(\Sigma)$, then this identity, the CAP and (4.5.2) implies $p_g(X)_h = s_h(M)$ for any $h$, that is, the EqSWIC for $(X, 0)$. 
This is important for the following reason: for superisolated singularities we do not know (even at conjectural level) any candidate (either topological or analytic!) for their equivariant geometric genera. It is not hard to verify that 
\[ p_g(X) = d(d - 1)(d - 2)/6 \] (see e.g. [38 §2.3]), but no formulas exist for \( p_g(X)_h \), and no (topological or analytic) prediction exists for \( p_g(Y) \) either. For some interesting computations involving the universal abelian cover of superisolated singularities, see [71].

**Example 4.5.6.** Set \( \nu = 1, d = 4 \), and let \( K_1 \) be the (3, 4) torus knot. This can be realized by the superisolated singularity \( zx^3 + y^4 + z^5 = 0 \). In this case \( M = S^3_{-4}(K_1) \).

One verifies that \( \sum_{h=0}^3 s_h(M) = 9 = s_0(\Sigma) \), cf. Theorem 4.4.1.

On the other hand, the UAC \((Y, 0)\) of the singularity is the Brieskorn singularity \( x^3 + y^4 + z^{16} = 0 \), whose geometric genus is \( p_g(Y) = 9 \) too. Hence, by the above remark, \( p_g(X)_h = s_h(M) \) for any \( h \).

**Remark 4.5.7.** (Continuation of Remark 4.5.5.) It is interesting that we have two sets of invariants, an analytic package \( \{p_g(X)_h\}_h, p_g(Y) \) and a topological one \( \{s_h(M)\}_h, s_0(\Sigma) \) and both of them satisfy the additivity property. Nevertheless, in some cases, they do not agree. For example, if \( p_g(Y) < s_0(\Sigma) \), then by (4.5.2) necessarily at least one of the inequalities in (4.5.2) is strict. In particular, for both sets of invariants (topological and analytical), the additivity property always holds, but the equality of the two packages in certain cases fails.

**Example 4.5.8.** Set again \( \nu = 1 \) and \( d = 4 \), but this time let \( K_1 \) be the (2, 7) torus knot. As usual \( M = S^3_{-4}(K_1) \). By a computation \( \sum_{h=0}^3 s_h(M) = 10 = s_0(\Sigma) \).

A suitable superisolated singularity is given by \( (zy - x^2)^2 - xy^3 + z^5 = 0 \). By [38, the end of Section 4.5] the universal abelian cover \( Y \) satisfies the strict inequality \( p_g(Y) < 10 \). Therefore, \( p_g(X)_h < s_h(M) \) for at least one \( h \) (in fact, not for \( h = 0 \)).
List of computer programs

- Maple\textsuperscript{TM} of Maplesoft\textsuperscript{TM} was used to produce Figure 1.2 and to help calculations about algebraic curves and their intersections, especially in Section 2.6 of Chapter 2 and Section 3.4 of Chapter 3.
http://www.maplesoft.com/

- SINGULAR available from https://www.singular.uni-kl.de/ and developed under the GNU General Public License was used in Section 2.6 of Chapter 2 and Section 3.4 of Chapter 3 to help with computations of cusp types and parametrizations.

- A program written by Baldur Sigurðsson was used to help with computations of lattice cohomologies in Chapter 4.
https://bitbucket.org/baldursigurds/lattice

- Figures 1.1, 2.1, 2.2 and 2.3 were made by KolourPaint, Figure 4.1 was made with the help of GeoGebra.
https://www.kde.org/applications/graphics/kolourpaint/
http://www.geogebra.org/
Bibliography


[18] T. Fenske, *Rational cuspidal plane curves of type $(d, d-4)$ with $\chi(\Theta_V(D)) \leq 0$*, Manuscripta Math. **98** (1999), no. 4, 511–527.


Summary

The thesis deals with singularities of complex projective plane curves and the topology of some related complex normal surface singularities and surgery manifolds.

In the first chapter, after an overview of invariants of local complex plane curve singularities, we prove a formula describing the behaviour of the semigroup counting function of a plane curve singularity under blow-up. The identity simplifies the computation of the Upsilon function of algebraic knots and it can be used to compute the zeroth lattice cohomology of surgery manifolds obtained by a negative Dehn surgery along an algebraic knot.

In the next chapter, we study rational cuspidal curves in the complex projective plane, that is, curves homeomorphic to the 2-sphere and having locally irreducible singularities only. The main tool is the semigroup distribution property, conjectured for unicuspidal curves by Fernández de Bobadilla, Luengo, Melle-Hernández and Némethi and proved by Borodzik and Livingston in general. We prove that it implies a conjecture of Fernández de Bobadilla, Luengo, Melle-Hernández and Némethi on the product of Alexander polynomials of the singularity links provided the number of singular points is at most two. On the other hand, we show that the original conjecture is false in general for curves with at least three cusps. Nevertheless, a weakened form of this conjecture can be formulated, which turns out to be true for all currently known rational cuspidal curves. We relate these conjectures to lattice and Heegaard Floer homologies of links of superisolated singularities corresponding to rational cuspidal curves.

The new results of the first two chapters are from a joint work with András Némethi. A generalization of the semigroup distribution property for cuspidal curves of arbitrary genus was proved in a joint work with Daniele Celoria and Marco Golla, and independently by Borodzik, Hedden and Livingston. Using this result, one can obtain very strong obstructions on the possible cusp types of projective curves. In the third chapter, we restrict ourselves to unicuspidal curves of arbitrary genus with singularity whose link is a torus knot. For such curves, we give an almost complete classification of possible cusp types for certain genera. Explicit constructions using birational maps of the projective plane are given to produce the realizable types.

These results are from a joint work with Daniele Celoria and Marco Golla. The links of superisolated singularities obtained from rational cuspidal curves can be represented as Dehn surgeries along a connected sum of algebraic knots. This motivates the result of the last chapter, where we prove an identity for certain topological invariants of such 3-manifolds.

Namely, we prove that the sum of the normalized Seiberg–Witten invariants over all spin$^c$ structures of such a manifold equals to the canonical Seiberg–Witten invariant of its universal abelian cover, provided it is a rational homology sphere. This fact is rather surprising, as it is motivated by the Seiberg–Witten invariant conjecture of Némethi and Nicolaescu (and a similar additivity property valid for the equivariant geometric genera of singularities), which conjecture fails in the very case of superisolated singularities.

This covering additivity property is proved in a joint work with András Némethi.
Összefoglalás (Summary in Hungarian)

Az értekezésben komplex projektív síkgörbék szingularitásaival és bizonyos, hozzájuk rendelhető normális felületszingularitások és 3-sokaságok topológiájával foglalkozunk.

Az első fejezetben a síkgörbe-szingularitások lokális típusait jellemző invariánsok áttekintése után bizonyítunk egy formulát, mely leírja a gőrbeszingularitások felcsoportháromláló függvényének viselkedését az algebrai felfújás hatására. Az azonosság nagyban egyszerűsíti az algebrai csomók Üpszilon-függvényének kiszámítását és algebrai csomók menti negatív együtthatós Dehn-műtéttel kapható sokaságok nulladik rácsponthomológiájának meghatározását is könnyebbe teszi.


Az első két fejezet új eredményei egy Némethi Andrásal közös munkából származnak.

A felcsoportharangozás tulajdonság egy, tetszőleges génuszú gőrbékre vonatkozó általánosítása megtalálható egy Daniele Celoriával és Marco Gollával közösen írt munkában. Az általánosítást ezzel egyidőben, de tőlünk függetlenül Borodzik, Hedden és Livingston is jelentett. Ezt az általánosítást használva nagyon erős szükséges feltételek nyerhetők egy csúcsos projektív gőrbe szingularitásainak lehetséges lokális típusaira vonatkozóan. Ha olyan, egysúcsú gőrbékre szorítkozunk, melynek a szingularitása egy tóruszarcos linkkel rendelkezik, akkor bizonyos génuszok esetén belátunk egy majdnem teljes osztályozási tételt a lehetséges tóruszarcos-típusokra vonatkozóan. A lehetséges típusok megvalósításaúhoz a projektív sík biracionális leképezéseit használó explicit konstrukciókat adunk.

A fenti eredmények egy Daniele Celoriával és Marco Gollával közös munkából származnak.


Ez a fedési additivitást egy, Némethi Andrással közösen írt munkában bizonyítottuk.