Hilbert schemes of points on some classes of surface singularities

Ph.D. theses by

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1. Introduction

The punctual Hilbert scheme parametrizing the zero-dimensional subschemes of a quasi-projective variety contains a large amount of information about the geometry and topology of the base variety. As a set, it consists of ideal sheaves of the sheaf of regular functions on the variety, such that the quotient by the ideal has finite length. The Hilbert schemes of points on smooth curves and surfaces have been investigated for a long time by several people, including Hartshorne, Fogarty, Macdonald, Iarrobino, Briançon. Due to the work of Nakajima, Grojnowski, and many others, it has turned out that the surface case has an especially rich geometrical structure, see e.g. [12].

In the recent years a new direction has emerged, which also allows singularities on the base variety. A breakthrough result was obtained by Maulik [10], who proved the conjecture of Oblomkov and Shende relating an integral of the length function with respect to the Euler characteristic over the Hilbert scheme of points of a curve with planar singularities to the HOMFLY polynomial of its link. The result shows that this polynomial invariant of the link of a singularity contains information about the invariants of the Hilbert scheme of points on the singularity.

Motivated by these results, it is natural to consider the Hilbert scheme of points on singular surfaces. The aim of this thesis is to describe the Euler characteristics of the Hilbert schemes parametrizing the zero-dimensional subschemes of some basic classes of surface singularities.

The well-know simple singularities are the simplest type of normal surface singularities, and it is known that they have an orbifold structure. There are at least two natural versions of the punctual Hilbert scheme in the case of quotient singularities. We show that the generating series of the Euler characteristics of the coarse Hilbert schemes of points on the singularities of type $A_n$ and $D_n$ can be computed from the multivariable generating series of the corresponding equivariant Hilbert schemes. The remaining cases $E_6$, $E_7$, and $E_8$ are not treated here, but computer calculations lead us to an analogous conjecture. The proofs might be similar as well, once the representation theoretic tools become available.

The moduli space of torsion free sheaves on surfaces are higher rank analogs of the Hilbert schemes. In type $A$ our results reveal their Euler characteristic generating function as well. Another very interesting class of normal surface singularities is the so-called cyclic quotient singularities of type $(p,1)$. As an outlook we also obtain some results about the associated generating functions.

2. Preliminaries

For a general reference on Hilbert scheme of points on surfaces, we refer the reader to [12]. Let us fix an arbitrary quasi-projective variety $X$. A central invariant in this thesis will be the generating series of the Euler characteristics
of its Hilbert schemes:

\begin{equation}
Z_X(q) = \sum_{m=0}^{\infty} \chi(\text{Hilb}^m(X)) q^m.
\end{equation}

The higher rank analog of the Hilbert scheme of points on $\mathbb{C}^2$ is the moduli space of framed torsion free sheaves on $\mathbb{P}^2$. Torsion free sheaves are generalizations of vector bundles, essentially they can be viewed as vector bundles which are allowed to have some singularities: the dimensions of the fibers do not all have to be equal. Their moduli space is denoted as $\mathcal{M}^{r,m}(\mathbb{C}^2)$. We put $\mathbb{C}^2$ instead of $\mathbb{P}^2$ in the argument to keep the analogy with the Hilbert scheme, but this should not eventuate any confusion.

Let $G < \text{GL}(2, \mathbb{C})$ be a small finite subgroup and denote by $\mathbb{C}^2/G$ the corresponding quotient variety. There are two different types of Hilbert schemes attached to this data. First, there is the classical Hilbert scheme $\text{Hilb}(\mathbb{C}^2/G)$ of the quotient space. This is the moduli space of ideal sheaves in $\mathcal{O}_{\mathbb{C}^2/G}(\mathbb{C}^2/G)$ of finite colength. We call this the \textit{coarse Hilbert scheme of points}. It decomposes

$$
\text{Hilb}(\mathbb{C}^2/G) = \bigsqcup_{m \in \mathbb{N}} \text{Hilb}^m(\mathbb{C}^2/G)
$$

into components which are quasiprojective but singular varieties indexed by “the number of points”, the codimension $m$ of the ideal. Second, there is the moduli space of $G$-invariant finite colength subschemes of $\mathbb{C}^2$, the invariant part of $\text{Hilb}(\mathbb{C}^2)$ under the lifted action of $G$. This Hilbert scheme is also well known and is variously called the \textit{orbifold Hilbert scheme} [16] or \textit{equivariant Hilbert scheme} [9]. We denote it by $\text{Hilb}([\mathbb{C}^2/G])$. This space also decomposes as

$$
\text{Hilb}([\mathbb{C}^2/G]) = \bigsqcup_{\rho \in \text{Rep}(G)} \text{Hilb}^\rho([\mathbb{C}^2/G]),
$$

where

$$
\text{Hilb}^\rho([\mathbb{C}^2/G]) = \{ I \in \text{Hilb}(\mathbb{C}^2)^G : H^0(\mathcal{O}_{\mathbb{C}^2/I}) \simeq_G \rho \}
$$

for any finite-dimensional representation $\rho \in \text{Rep}(G)$ of $G$; here $\text{Hilb}(\mathbb{C}^2)^G$ is the set of $G$-invariant ideals of $\mathbb{C}[x, y]$, and $\simeq_G$ means $G$-equivariant isomorphism. Being components of the fixed point set of a finite group acting on smooth quasiprojective varieties, the orbifold Hilbert schemes themselves are smooth and quasiprojective.

We collect the topological Euler characteristics of the two versions of the Hilbert scheme into two generating functions. Let $\rho_0, \ldots, \rho_n \in \text{Rep}(G)$ denote the (isomorphism classes of) irreducible representations of $G$, with $\rho_0$ the trivial representation.
Definition 2.1. (a) The orbifold generating series of the orbifold $[\mathbb{C}^2/G]$ is
\[ Z_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n) = \sum_{m_0, \ldots, m_n = 0}^{\infty} \chi \left( \operatorname{Hilb}^{m_0 \rho_0 + \cdots + m_n \rho_n ([\mathbb{C}^2/G])} \right) q_0^{m_0} \cdots q_n^{m_n}. \]

(b) The coarse generating series of the singularity $\mathbb{C}^2/G$ is just the series defined in (2.1):
\[ Z_{\mathbb{C}^2/G}(q) = \sum_{m=0}^{\infty} \chi \left( \operatorname{Hilb}^m (\mathbb{C}^2/G) \right) q^m. \]

Assume that $G < \text{SL}(2, \mathbb{C})$. Then $G$ fixes the line $l_\infty \subset \mathbb{P}^2$. Let us take and fix a lift of the $G$-action to $O \oplus r l_\infty$. This can be written as $W \otimes \mathbb{C} O l_\infty$. Then the $G$-action on $\mathbb{C}^2$ lifts naturally to a $G$-action on $M_{r,m}(\mathbb{C}^2)$ [13]. We define the moduli space of $G$-equivariant torsion free sheaves on $\mathbb{P}^2$ with a framing as the $G$-invariant part of the space $M_{r,m}(\mathbb{C}^2)$ with respect to the induced $G$-action. It will be denoted by $M_{r,m}(\mathbb{C}^2/G)$. The dependence on the choice of $W$ is suppressed in the notation. In fact, it is easy to see that the isomorphism type of $M_{r,m}(\mathbb{C}^2/G)$ only depends on the isomorphism class of $W$. We denote
\[ M_r(\mathbb{C}^2) = \bigsqcup_m M_{r,m}(\mathbb{C}^2). \]

Using Beilinson’s spectral sequence, it can be shown that the analog of the quotient $\mathbb{C}[x,y]/I$ for a higher rank framed sheaf $(E, \phi)$ is $H^1(E(-1))$, and also that $H^0(E(-1)) = H^2(E(-1)) = 0$ [12, Chapter 2]. If $E$ is $G$-invariant, then $H^1(E(-1))$ carries naturally a $G$-representation, and there is a decomposition
\[ M^r(\mathbb{C}^2) = \bigsqcup_{\rho \in \operatorname{Rep}(G)} M^{r,\rho}(\mathbb{C}^2), \]
where
\[ M^{r,\rho}(\mathbb{C}^2/G) = \{(E, \Phi) \in M^r(\mathbb{C}^2)^G : H^1(E(-1)) \simeq_G \rho \}. \]

The Euler characteristics of the higher rank equivariant moduli spaces for a fixed isomorphism class of $W$ are collected again into generating series:

Definition 2.2.
\[ Z^W_{[\mathbb{C}^2/G]}(q_0, \ldots, q_n) = \sum_{m_0, \ldots, m_n = 0}^{\infty} \chi \left( M^{r,m_0 \rho_0 + \cdots + m_n \rho_n ([\mathbb{C}^2/G])} \right) q_0^{m_0} \cdots q_n^{m_n}. \]

In this thesis almost always we are only concerned with finite subgroups $G < \text{SL}(2, \mathbb{C})$. As it is well known, these are classified into three types: type $A_n$ for $n \geq 1$, type $D_n$ for $n \geq 4$ and type $E_n$ for $n = 6, 7, 8$. The type of the singularity can be parametrized by a simply laced irreducible Dynkin diagram with $n$ nodes, arising from an irreducible simply laced root system $\Delta$. 
Additionally, we will also make some investigation with another class of singularities. Fix a positive integer \( p \). Let \( \mathbb{Z}_p \) be the cyclic group of order \( p \) with generator \( g \) and let it act on \( \mathbb{C}^2 \) as: \( g.x = e^{\frac{2\pi i}{p}} x \), \( g.y = e^{\frac{2\pi i q}{p}} y \) where \( q \) is coprime to \( p \). Then we get an action of \( \mathbb{Z}_p \) on \( \mathbb{C}^2 \) which is free away from the origin. Let \( X(p,q) \) denote the quotient variety. It is called the cyclic quotient singularity of type \((p,q)\).

The Hilbert scheme \( \text{Hilb}(X(p,q)) \) of points on \( X(p,q) \) is the moduli space of ideals sheaves in \( \mathcal{O}_{X(p,q)} \) of finite colength. The case when \( q = p - 1 \) is just the type \( A_p \) singularity introduced above. We will present results about the other extreme case when \( q = 1 \), that is, about

\[
Z_{X(p,1)}(q) = \sum_{m=0}^{\infty} \chi(\text{Hilb}^m(X(p,1))) q^m.
\]

### 3. The main results

It can be shown that the Hilbert schemes on \([\mathbb{C}^2/G_\Delta]\) where \( \Delta \) is an irreducible simply-laced root system have a description as Nakajima quiver varieties \([11]\). Moreover, their homologies carry a representation on an affine Lie algebra. This implies the following theorem.

**Theorem 3.1 (13).** Let \([\mathbb{C}^2/G_\Delta]\) be a simple singularity orbifold, where \( \Delta \) is any of the types \( A, D, \) and \( E \). Then its orbifold generating series can be expressed as

\[
Z_{[\mathbb{C}^2/G_\Delta]}(q_0, \ldots, q_n) = \left( \prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \sum_{\underline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n} q_1^{m_1} \cdots q_n^{m_n} (q_1^{1/2})^{m_\top} C_\Delta \cdot \underline{m},
\]

where \( q = \prod_{i=0}^{n} q_i^{d_i} \) with \( d_i = \dim \rho_i \), and \( C_\Delta \) is the finite type Cartan matrix corresponding to \( \Delta \).

Our first main result is a strengthening of this theorem. Given a root system \( \Delta \) of type \( A \) or \( D \) there is a certain combinatorial set, the set of Young walls \( Z_\Delta \) of type \( \Delta \).

**Theorem 3.2 (4331).** Let \([\mathbb{C}^2/G_\Delta]\) be a simple singularity orbifold, where \( \Delta \) is of type \( A_n \) for \( n \geq 1 \) or \( D_n \) for \( n \geq 4 \). Then there exists a decomposition

\[
\text{Hilb}([\mathbb{C}^2/G_\Delta]) = \bigsqcup_{Y \in Z_\Delta} \text{Hilb}([\mathbb{C}^2/G_\Delta])_Y
\]

into locally closed strata indexed by the set of Young walls \( Z_\Delta \) of the appropriate type. Each stratum is isomorphic to an affine space of a certain dimension, and in particular has Euler characteristic \( \chi(\text{Hilb}([\mathbb{C}^2/G_\Delta])_Y) = 1 \).
For type $A$, the set of Young walls is simply the set of finite partitions, represented as Young diagrams, equipped with a diagonal labelling. In this case, Theorem 3.2 is well known; the decomposition in type $A$ is not unique, but depends on a choice of a one-dimensional subtorus of the full torus $(\mathbb{C}^*)^2$ acting on the affine plane $\mathbb{C}^2$. On the other hand, the type $D$ case appears to be new; in this case, our decomposition is unique, there is no further choice to make.

We explain combinatorially that the right hand side of (3.1) enumerates the set of Young walls $Z_{\Delta}$ of the appropriate type \cite{[3]}. In particular, Theorem 3.2 implies Theorem 3.1.

The second main result of the thesis is the following formula, which says that the coarse generating series is a very particular specialization of the orbifold one.

**Theorem 3.3** (\cite{[4],[3]}). Let $\mathbb{C}^2/G_{\Delta}$ be a simple singularity, where $\Delta$ is of type $A_n$ for $n \geq 1$ or $D_n$ for $n \geq 4$. Let $h^\vee$ be the (dual) Coxeter number of the corresponding finite root system (one less than the dimension of the corresponding simple Lie algebra divided by $n$). Then

$$Z_{\mathbb{C}^2/G_{\Delta}}(q) = \left( \prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\overline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n} \zeta^{m_1 + m_2 + \cdots + m_n} (q^{1/2})^m \cdot C_{\Delta} \cdot m,$$

where $\zeta = \exp\left(\frac{2\pi i}{1 + h^\vee}\right)$ and $C_{\Delta}$ is the finite type Cartan matrix corresponding to $\Delta$.

Thus $Z_{\mathbb{C}^2/G_{\Delta}}(q)$ is obtained from $Z_{[\mathbb{C}^2/G_{\Delta}]}(q_0, \ldots, q_n)$ by the substitutions

$$q_1 = \cdots = q_n = \exp\left(\frac{2\pi i}{1 + h^\vee}\right), \quad q_0 = q \exp\left(-\frac{2\pi i}{1 + h^\vee} \sum_{i \neq 0} \dim \rho_i\right).$$

In type $A$, the formula in Theorem 3.3 is not new: it was proved directly (in a slight disguise) by Dijkgraaf and Sulkowski in \cite{[7]} and also recently, using completely different methods, by Toda in \cite{[14]}. Our main contribution is the general Lie-theoretic formulation, as well as a proof in type $D$; we also provide a direct combinatorial proof in type $A$, which appears to be new.

One can check directly that the generating series in Theorem 3.3 has also integer coefficients for $E_6$, $E_7$ and $E_8$ to a high power in $q$. This motivates the following.

**Conjecture 3.4** (\cite{[4]}). Let $\mathbb{C}^2/G_{\Delta}$ be a simple singularity of type $E_n$ for $n = 6, 7, 8$. Let $h^\vee$ be the (dual) Coxeter number of the corresponding finite
root system. Then, as for other types,

$$Z_{C^2/G_\Delta}(q) = \left( \prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\bar{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n} \zeta^{m_1+m_2+\cdots+m_n} (q^{1/2})^{\bar{m}^\top \cdot C_{\Delta} \cdot \bar{m}},$$

where $\zeta = \exp\left(\frac{2\pi i}{1 + h^\top}\right)$ and $C_{\Delta}$ is the finite type Cartan matrix corresponding to $\Delta$.

The key tool in our proof of Theorem 3.3 for types $A$ and $D$ is the combinatorics of Young walls, in particular their abacus representation.

In the type $A$ case we can go further to the higher rank case. The higher rank orbifold generating series can again be derived from the results of [13] or [8], but our methods give a new proof in this case as well. Let $W$ be the isomorphism class of the framing and let $a = (0, \ldots, 0, \ldots, n-1, \ldots, n-1)$, where for each $c \in C$ the number of $c$’s in $a$ is $w_c$.

**Theorem 3.5 (1).** Let $[C^2/G_\Delta]$ be a simple singularity orbifold, where $\Delta$ is of type $A_n$ for $n \geq 1$. Then

$$Z_{[C^2/G_\Delta]}^W(q) = \prod_{m=1}^{l} Z_{\Delta(a_m)}(q),$$

where $l$ is the length of $a$,

$$Z_{\Delta(a)}(q) = \left( \prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\bar{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n} q_1^{m_1} \cdots q_{n+a}^{m_n} (q^{1/2})^{\bar{m}^\top \cdot C \cdot \bar{m}},$$

$q = \prod_{i=0}^{n} q_i$, and $C$ is the Cartan matrix of finite type $A_n$.

Let us consider now the cyclic quotient singularities. Our main result in this direction is a representation of $Z_{X(p,1)}(q)$ as coefficient of a two variable generating function. In this two variable generating function continued fractions appear. We introduce the notation $[z^0] \sum_n A_n z^n = A_0$.

Let

$$F(q, z) = \frac{(qz)^{p-1}}{1 - qz \left( 1 - q^2 z \left( \frac{(q^2 z)^{p-1}}{1 - q^2 z} + \frac{1 - (q^2 z)^{p-1}}{1 - q^2 z} \right) + \frac{1 - (qz)^{p-1}}{1 - qz} \right)} + \frac{1 - (qz)^{p-1}}{1 - qz},$$

(3.2)
and
\[ T(q, z) = (qz) \prod_{n=1}^{\infty} (1 + z^p q^{np+1})(1 + z^{-p} q^{(n-1)p-1})(1 - q^{np}). \]

**Theorem 3.6** ([2]).
\[ Z_{X(p,1)}(q) = [z^0]T(q, z) \left( F(q^{-1}, z^{-1}) - (qz)^{-p} F(q^{-1}, (qz)^{-1}) \right) . \]

**Remark 3.7.** (1) For \( p = 1 \), \( X(1,1) = \mathbb{C}^2 \). Then, by Theorem 3.3, \( Z_{X(1,1)}(q) \) is also equal to
\[ \prod_{m=1}^{\infty} \frac{1}{1 - q^m}. \]

(2) For \( p = 2 \), \( X(2,1) \) is the \( A_1 \) singularity. By Theorem 3.3, \( Z_{X(2,1)}(q) \) is also equal to
\[ \left( \prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^2 \cdot \sum_{m \in \mathbb{Z}} \xi^m q^m, \]
where \( \xi = \exp \left( \frac{2\pi i}{3} \right) \).

We are not aware of a direct proof for these equalities.

Finally, we are able to globalize the statements so far. Let \( X = S \) be a quasi-projective surface which is non-singular outside a finite number of surface singularities \( \{ P_1, \ldots, P_k \} \), with \( (P_i \in S) \) a singularity locally analytically isomorphic to \( (0 \in \mathbb{C}^2/G_{\Delta_i}) \) for \( G_{\Delta_i} < \text{SL}(2, \mathbb{C}) \) a small finite subgroup, or to \( (0 \in (X(p_i,1))) \) for a positive integer \( p_i \). Let \( S^0 \subset S \) be the nonsingular part of \( S \).

**Theorem 3.8** ([4]). The generating function \( Z_S(q) \) of the Euler characteristics of Hilbert schemes of points of \( S \) has a product decomposition
\[ Z_S(q) = \left( \prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{\chi(S^0)} \cdot \prod_{j=1}^{k} Z^{(P_j,S)}(q). \]

The local terms can be expressed
- either as
\[ Z^{(P_i,S)}(q) = Z_{\mathbb{C}^2/G_{\Delta_i}}(q) \text{ if } (P_i \in S) \cong (0 \in \mathbb{C}^2/G_{\Delta_i}), \text{ and they are given by Theorem 3.3 for type } A \text{ and } D; \]
- or as
\[ Z^{(P_i,S)}(q) = Z_{X(p_i,1)}(q) \text{ if } (P_i \in S) \cong (0 \in (X(p_i,1))), \text{ and they are given by Theorem 3.6}. \]
Formulas (3.4)-(3.5)-(3.6) are our analogue for the case of surfaces with simple singularities of the Oblomkov–Shende–Maulik formula. Note that each $C^2/G_{\Delta_i}$ is in particular a hypersurface singularity, as are planar singularities in the curve case. The main difference is the fact that (conjecturally, for type $E$) our local terms $Z^{(P,S)}(q)$ are expressed in terms of Lie-theoretic and not topological data.

The $S$-duality conjecture [15] predicts that for a smooth surface $S$, up to a fractional power of $q$, the Euler characteristic generating function of certain modular spaces of torsion free sheaves on $S$ has to be a meromorphic modular form for a finite index subgroup of $SL(2, \mathbb{Z})$. Our formulae lead to the following new modularity results, extending the results of [14] for type $A$.

**Corollary 3.9.** (S-duality for simple singularities, [4]) For type $A$ and type $D$, and, assuming Conjecture 3.4 for all types, the partition function $Z_{C^2/G_{\Delta}}(q)$ is, up to a suitable fractional power of $q$, the $q$-expansion of a meromorphic modular form of weight $-\frac{1}{2}$ for some congruence subgroup of $SL(2, \mathbb{Z})$.

**Corollary 3.10.** (S-duality for surfaces with simple singularities, [4]) Let $S$ be a quasiprojective surface with simple singularities of type $A$ and $D$, or, assuming Conjecture 3.4 of arbitrary type. Then the generating function $Z_S(q)$ is, up to a suitable fractional power of $q$, the $q$-expansion of a meromorphic modular form of weight $-\frac{\chi(S)}{2}$ for some congruence subgroup of $SL(2, \mathbb{Z})$.

**Publications concerning this thesis**


**Other publications**


References


