MATHEMATICAL ANALYSIS – EXERCISES I

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# MATHEMATICAL ANALYSIS – EXERCISES I

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KEY WORDS: Analysis, calculus, derivate, integral, multivariable, complex.

SUMMARY: This problem book is for students learning mathematical calculus and analysis. The main task of it to introduce the derivate and integral calculus and their applications.

# Contents

# Chapter 1

# Basic Notions, Real Numbers

> It is encouraging that the refutation of the unfounded rumor claiming that it is not a lie to deny that there will be at least one student who will not fail the exam without knowing the proof of any of the theorems in analysis proved to be wrong.
>
> (Baranyai Zsolt)

**1.1** A set $A \subset \mathbb{R}$ is called **bounded** if there is a real number $K \in \mathbb{R}$ such that for all $a \in A$ $|a| \leq K$.
A set $A \subset \mathbb{R}$ is **bounded from above** if there is a real number $M \in \mathbb{R}$ (**upper bound**) such that for all $a \in A$ implies $a \leq M$.
A set $A \subset \mathbb{R}$ is **bounded from below** if there is a real number $m \in \mathbb{R}$ (**lower bound**) such that for all $a \in A$ implies $a \geq m$.

**1.2** **Cantor's Axiom:** The intersection of a nested sequence of closed bounded intervals is not empty.

**1.3** **supremum**: If a set $A$ has a least upper bound, and this number is $M$, then $M$ is the **supremum** of the set $A$ denoted by the expression $M = \sup A$.

**1.4** If a nonempty set $A \subset \mathbb{R}$ is bounded from above, then $A$ has a least upper bound.

**1.5** **Bernoulli Inequality:** If $n \in \mathbb{N}$ and $x > -1$, then

$$(1 + x)^n \geq 1 + n \cdot x.$$

The equality is true if and only if $n = 0$ or $n = 1$ or $x = 0$.

## 1.1 Elementary Exercises

**Plot the solutions of the following inequalities on the number line.**

**1.1.** $|x-5|<3$

**1.2.** $|5-x|<3$

**1.3.** $|x-5|<1$

**1.4.** $|5-x|<0.1$

**Find the solutions of the following inequalities.**

**1.5.** $\dfrac{1}{5x+6} \geq -1$

**1.6.** $6x^2+7x-20>0$

**1.7.** $10x^2+17x+3 \leq 0$

**1.8.** $-6x^2+8x-2>0$

**1.9.** $8x^2-30x+25 \geq 0$

**1.10.** $-4x^2+4x-2 \geq 0$

**1.11.** $9x^2-24x+17 \geq 0$

**1.12.** $-16x^2+24x-11<0$

**1.13.** Find the mistake.

$$\log_2 \frac{1}{2} \leq \log_2 \frac{1}{2} \quad \text{and} \quad 2<4$$

Multiplying the two inequalities:

$$2\log_2 \frac{1}{2} < 4\log_2 \frac{1}{2}.$$

Using the logarithmic identities:

$$\log_2 \left(\frac{1}{2}\right)^2 < \log_2 \left(\frac{1}{2}\right)^4.$$

Since the function $\log_2 x$ is strictly monotonically increasing:

$$\frac{1}{4} < \frac{1}{16}.$$

Multiplying with the denominators: $16<4$.

**Find the solutions of the following equations and inequalities.**

**1.14.** $|x+1|+|x-2| \leq 12$

**1.15.** $\sqrt{x+3}+\sqrt{x-5}=0$

**1.16.** $\left|\dfrac{x+1}{2x+1}\right| > \dfrac{1}{2}$

**1.17.** $|2x-1| < |x-1|$

**1.18.** $\sqrt{x+3}+|x-2|=0$

**1.19.** $\sqrt{x+3}+|x-2| \leq 0.$

## 1.2 Basic Logical Concepts

**1.20.** State the negation of each of the following statements as simple as you can.

**(a)** All mice like cheese.

**(b)** He who brings trouble on his family will inherit only wind.

**(c)** There is an $a$, such that for all $b$ there is a unique $x$ such that $a+x=b$.

**(d)** 3 is not greater than 2, or 5 is a divisor of 10.

**(e)** If my aunt had wheels, she would be the express train.

**1.21.** There are 5 goats and 20 fleas in a court. Does the fact that there is a goat bitten by all fleas imply that there is a flea which bit all the goats?

**1.22.** Let us assume that the following statements are true.

**(a)** If an animal is a mammal, then it has a tail or a gill.

**(b)** No animal has a tail.

**(c)** All animals are either mammals or have a tail or have a gill.

Is it implied by the previous statements that all animals have a gill?

**1.23.** Left-handed Barney, who is really left-handed, can write with his left hand only true statements, and with his right hand only false statements. With which hand can he write down the following statements?

**(a)** I am left-handed.

**(b)** I am right-handed.

**(c)** I am left-handed and my name is Barney.

**(d)** I am right-handed and my name is Barney.

**(e)** I am left-handed or my name is Barney.

**(f)** I am right-handed or my name is Barney.

**(g)** The number 0 is not even, and not odd.

**1.24.** Seeing a black cat is considered bad luck. Which of the following statements is the negation of the previous statement?

**(a)** Seeing a black cat is considered as good luck.

**(b)** Not the seeing a black cat is considered as bad luck.

**(c)** Seeing a white cat is considered as bad luck.

**(d)** Seeing a black cat is not considered as bad luck.

**1.25.** : -) "All of the Mohicans are liar" - said the last of the Mohicans. Did he tell the truth?

**1.26.** : -)

| |
|---|
| 1) 3 is a prime number.<br>2) 4 is divisible by 3.<br>3) There is exactly 1 true statement in this frame. |

How many true statements are there in the frame?

**1.27.** If it's Tuesday, this must be Belgium. Which of the following statements are implied by this?

**(a)** If it's Wednesday, this must not be Belgium.

**(b)** If it is Belgium, this must be Tuesday.

**(c)** If it is not Belgium, this must not be Tuesday.

**How many subsets of the set $H = \{1, 2, 3, \dots 100\}$ are there for which the following statement is true and for how many of them it is false?**

**1.28.** 1 is an element of the subset;

**1.29.** 1 and 2 are elements of the subset;

**1.30.** 1 or 2 are elements of the subset;

**1.31.** 1 is an element of the subset or 2 is not an element of the subset;

**1.32.** if 1 is an element of a subset, then 2 is an element of the subset.

**1.33.** There is a bag of candies on the table, and there are some students. Which of the following statements implies the other?

**(a)** All of the students licked a candy (from the bag).

**(b)** There is a candy (from the bag), such that it is licked by all of the students.

**(c)** There is a student, who licked all of the candies (from the bag).

**(d)** All candies (from the bag) are licked by some students.

## 1.3 Methods of Proof

**Prove that**

**1.34.** $\sqrt{3}$ is irrational;

**1.35.** $\dfrac{\sqrt{2}}{\sqrt{3}}$ is irrational;

**1.36.** $\dfrac{\dfrac{\sqrt{2}+1}{2}+3}{4}+5$ is irrational!

**1.37.** We know that $x$ and $y$ are rational numbers. Prove that

**(a)** $x+y$

**(b)** $x-y$

**(c)** $xy$

**(d)** $\dfrac{x}{y}$, if $y \neq 0$

are rational.

**1.38.** We know that $x$ is a rational number, and $y$ is an irrational number.

**(a)** Can $x+y$ be rational? **(b)** Can $x-y$ be rational?

**(c)** Can $xy$ be rational? **(d)** Can $\dfrac{x}{y}$ be rational?

**1.39.** We know that $x$ and $y$ are irrational numbers.

**(a)** Can $x+y$ be rational? **(b)** Can $xy$ be rational?

**1.40.** Is it true that

**(a)** if $a$ and $b$ are rational numbers, then $a+b$ is rational?
**(b)** if $a$ and $b$ are irrational numbers, then $a+b$ is irrational?
**(c)** if $a$ is a rational number, $b$ is an irrational number, then $a+b$ is rational?
**(d)** if $a$ is a rational number, $b$ is an irrational number, then $a+b$ is irrational?

**1.41.** Let $A_1, A_2, \dots$ be a sequence of statements. What can we conclude from the fact that

**(a)** $A_1$ is true. If $A_1, A_2, \dots, A_n$ are all true, then $A_{n+1}$ is also true.
**(b)** $A_1$ is true. If $A_n$ and $A_{n+1}$ are true, then $A_{n+2}$ is true.
**(c)** If $A_n$ is true, then $A_{n+1}$ is also true. $A_{2^n}$ is false for all $n$.
**(d)** $A_{100}$ is true. If $A_n$ is true, then $A_{n+1}$ is also true.
**(e)** $A_{100}$ is true. If $A_n$ is false, then $A_{n+1}$ is also false.
**(f)** $A_1$ is false. If $A_n$ is true, then $A_{n+1}$ is also true.
**(g)** $A_1$ is true. If $A_n$ is false, then $A_{n-1}$ is also false.

**1.42.** Prove that $16|5^{n+1}-4n-5$ for all $n \in \mathbb{N}$.

**1.43.** Prove that $\tan 1°$ is irrational.

**1.44.** Prove that $n! \le \left(\dfrac{n+1}{2}\right)^n$.

**1.45.** Let $a_1 = 0.9,\ a_{n+1} = a_n - a_n^2$.
Is it true that there is such an $n$ that $a_n < 10^{-6}$ ?

**1.46.** Write down the following expressions for $n = 1, 2, 3, 6, 7, k$ and $k+1$.

**(a)** $\sqrt{n}$

**(b)** $\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}$

**(c)** $1^2+2^2+3^2+\cdots+n^2$

**(d)** $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)\cdot n}$

**(e)** $1\cdot 4+2\cdot 7+3\cdot 10+\cdots+n(3n+1)$

**(f)** $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)$

**1.47.** After calculating the first terms, find simple expressions for the following sums, then prove this by induction.

**(a)** $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)\cdot n}$

**(b)** $1+3+\ldots+(2n-1)$

**Prove that for all positive integers $n$ the following equations hold:**

**1.48.** $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1})$

**1.49.** $1+2+\cdots+n=\frac{n(n+1)}{2}$

**1.50.** $1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$

**1.51.** $1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$

**1.52.** $1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$

**Write down a more simple form for the following expressions:**

**1.53.** $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)\cdot n}$

**1.54.** $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)\cdot(n+2)}$

**1.55.** $1\cdot 2+2\cdot 3+\cdots+n\cdot(n+1)$

**1.56.** $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n \cdot (n+1) \cdot (n+2)$

**1.57.** A newly born pair of rabbits, one male and one female, is put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Rabbits never die, and a mating pair always produces one new pair (one male, one female) every month from the second month on. How many pairs of rabbits will be there at the end of the 2nd, 3rd, 4th, 5th and 6th month?

**Let $(u_n)$ be the Fibonacci sequence, that is, $u_0 = 0$, $u_1 = 1$, and if $n > 1$, then $u_{n+1} = u_n + u_{n-1}$.**

**1.58.** Prove that $u_n$ and $u_{n+1}$ are relatively primes.

**1.59.** Prove that $\dfrac{1.6^n}{3} < u_n < 1.7^n \quad (n > 0)$.

**1.60.** Prove the following equations:

**(a)** $u_1 + u_2 + \cdots + u_n = u_{n+2} - 1$ **(b)** $u_n^2 - u_{n-1}u_{n+1} = (-1)^{n+1}$

**(c)** $u_1^2 + u_2^2 + \cdots + u_n^2 = u_n u_{n+1}$

**1.61.** Simplify the following expressions:

**(a)** $s_n = u_0 + u_2 + \cdots + u_{2n}$ **(b)** $s_n = u_1 + u_3 + \cdots + u_{2n+1}$

**(c)** $s_n = u_0 + u_3 + \cdots + u_{3n}$ **(d)** $s_n = u_1u_2 + u_2u_3 + \cdots + u_{2n-1}u_{2n}$

**1.62.** **Theorem:** All of the horses have the same color.

**Proof:** We prove by induction that any $n$ horses are same colored. For $n = 1$ the statement is obvious. Let assume that it is true for $n$, and from this we prove it for $n+1$: By the induction hypothesis from the given $n+1$ horses the $1st, 2nd, \ldots, nth$ horses are same colored, and also the $2nd, \ldots, nth, (n+1)th$ horses are same colored, therefore all of the $n+1$ horses are same colored.

Is this proof correct? If not, where is the mistake?

**1.63.** **Theorem:** There is no sober sailor.

**Proof:** By induction. Let's assume that the statement is true for $n$ sailors, and we prove the statement for $n+1$ sailors. By the induction hypothesis from the given $n+1$ sailors $1th, 2nd, \ldots, nth$ sailors are not sober, and also the $2nd, \ldots, nth, (n+1)th$ sailors are not sober, therefore all of the $n+1$ are drunk.

Is this proof correct? If not, where is the mistake?

**1.64.** Prove the arithmetic and geometric means inequality for 2 terms.

**1.65.** Show that the arithmetic, geometric and harmonic means of some positive real numbers $a_1, a_2, \ldots, a_n$ are between the biggest and the smallest numbers.

**We know that $a, b, c > 0$ and $a+b+c = 18$. Find the values of $a, b$ and $c$ such that the following expressions are maximal:**

**1.66.** $abc$

**1.67.** $a^2bc$

**1.68.** $a^3b^2c$

**1.69.** $\dfrac{abc}{ab+bc+ac}$

**We know that $a, b, c > 0$ and $abc = 18$. Find the values of $a, b$ and $c$ such that the following expression are minimal:**

**1.70.** $a+b+c$

**1.71.** $2a+b+c$

**1.72.** $3a+2b+c$

**1.73.** $a^2+b^2+c^2$

**1.74.** We know that the product of three positive numbers is 1.

**(a)** At least how much can be their sum?

**(b)** At most how much can be their sum?

**(c)** At least how much can be the sum of their reciprocal?

**(d)** At most how much can be the sum of their reciprocal?

**1.75.** Prove that if $a > 0$, then $a + \dfrac{1}{a} \geq 2$.

**1.76.** Prove that if $a, b$ and $c$ are positive numbers, then $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} \geq 3$.

**1.77.** Prove that if $n$ positive, then $\left(1 + \dfrac{1}{n}\right)^{2n} \geq 4$.

**1.78.** A storekeeper has a pair of scales, but the arms of the scale have different length. The storekeeper knows this, so if a customer buys some goods, he puts the half of the goods in the left container and the known weight in the right container, and he puts the other half of the goods in the right container, and the known weight in the left container. The storekeeper thinks that in this way he can compensate the inaccuracy of the scale. Is he right?

**1.79.** Find the maximum of the function $f(x) = x(1-x)$ in the closed interval $[0, 1]$.

**What is the minimum of the following function if $x > 0$, and at which point is it attained?**

**1.80.** $f(x) = x + \dfrac{4}{x}$

**1.81.** $g(x) = \dfrac{x^2 - 3x + 5}{x}$

**1.82.** Find the maximum of the function $x^2(1 - x)$ in the closed interval $[0, 1]$.

**1.83.** What is the maximum of the function $g(x) = x(1 - x)^3$ in the closed interval $[0, 1]$?

**1.84.** What is the minimum of the function $f(x) = 2x^2 + \dfrac{3}{x^2 + 1} + 5$?

**1.85.** Which point of the $y = \dfrac{1}{4}x^2$ parabola is closest to $(0, 5)$?

**1.86.** Which rectangle has maximal area that we can write in the circle of radius 1?

## 1.4 Sets

**Which statements are true, and which statements are false? If a statement is true, then prove it, if a statement is false, then give a counterexample.**

**1.87.** $A \setminus B = A \cap \overline{B}$

**1.88.** $(A \cup B) \setminus B = A$

**1.89.** $(A \setminus B) \cup (A \cap B) = A$

**1.90.** $\overline{A} \setminus B = A \setminus \overline{B}$?

**1.91.** $(A \cup B) \setminus A = B$

**1.92.** $(A \cup B) \setminus C = A \cup (B \setminus C)$

**1.93.** $(A \setminus B) \cap C = (A \cap C) \setminus B$

**1.94.** $A \setminus B = A \setminus (A \cap B)$

**1.95.** Which statement is **not** true?

**(a)** $A \setminus B = \{x : x \in A \vee x \notin B\}$ **(b)** $A \setminus B = A \cap \overline{B}$

**(c)** $A \setminus B = (A \cup B) \setminus B$ **(d)** $A \setminus B = A \setminus (A \cap B)$

**1.96.** Which of the following sets is equal to $\overline{A \cup B}$?

**(a)** $\{x : x \notin A \vee x \notin B\}$ **(b)** $\{x : x \notin A \wedge x \notin B\}$

**(c)** $\{x : x \in A \vee x \in B\}$ **(d)** $\{x : x \in A \wedge x \in B\}$

**1.97.** Which of the following sets is equal to $A \cap (B \cup C)$?

**(a)** $A \cup (B \cap C)$ **(b)** $(A \cap B) \cup C$

**(c)** $(A \cup B) \cap C$ **(d)** $(A \cap B) \cup (A \cap C)$

**Let $A, B, C$ be sets. Write down the following sets with $A, B, C$ and with the help of the set operations, for example: $(A \setminus B) \cup C$.**

**1.98.** Elements which are in $A$, but are not in $B$ nor in $C$.

**1.99.** Elements which are exactly in one of $A$, $B$ and $C$.

**1.100.** Elements which are exactly in two of $A$, $B$ and $C$.

**1.101.** Elements which are exactly in three of $A$, $B$ and $C$.

**1.102.** Prove that for arbitrary sets $A$ and $B$ it is true that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

**1.103.** Prove the De Morgan's laws:

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i} \quad \text{and} \quad \overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$$

## 1.5 Axioms of the Real Numbers

**1.104.** Prove that for all real numbers $a, b$

**(a)** $|a| + |b| \geq |a + b|$ **(b)** $|a| - |b| \leq |a - b| \leq |a| + |b|$

**1.105.** Prove that for all real numbers $a_1, a_2, \ldots, a_n$

$$|a_1| + |a_2| + \ldots |a_n| \geq |a_1 + a_2 + \cdots + a_n|\,.$$

**1.106.** Is it true that

**(a)** if $x < A$, then $|x| < |A|$, **(b)** if $|x| < A$, then $|x^2| < A^2$?

**1.107.** Is it true for all real numbers $a_1, a_2, \ldots, a_n$ that

**(a)** $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$,
**(b)** $|a_1 + a_2 + \cdots + a_n| \geq |a_1| + |a_2| + \cdots + |a_n|$,
**(c)** $|a_1 + a_2 + \cdots + a_n| < |a_1| + |a_2| + \cdots + |a_n|$,
**(d)** $|a_1 + a_2 + \cdots + a_n| > |a_1| + |a_2| + \cdots + |a_n|$?

**1.108.** Is it true for all real numbers $a, b$ that

**(a)** $|a+b| \geq |a| - |b|$,
**(b)** $|a+b| \leq |a| - |b|$,

**(c)** $|a-b| < ||a| - |b||$,
**(d)** $|a-b| \leq |a| - |b|$?

**1.109.** Let $H$ be a nonempty subset of the real numbers. What do the following statements mean?

**(a)** $\forall x \in H \; \exists y \in H \; (y < x)$
**(b)** $\forall y \in H \; \exists x \in H \; (y < x)$

**(c)** $\exists x \in H \; \forall y \in H \; (y \leq x)$
**(d)** $\exists y \in H \; \forall x \in H \; (y \leq x)$

**1.110.** Let $H_1 = \{h \in \mathbb{R} : -3 < h \leq 1\}$ and $H_2 = \{h \in \mathbb{R} : -3 \leq h < 1\}$. Which statements are true, if $H = H_1$ or $H = H_2$?

**(a)** $\forall x \in H \; \exists y \in H \; (y < x)$
**(b)** $\forall y \in H \; \exists x \in H \; (y < x)$

**(c)** $\exists x \in H \; \forall y \in H \; (y \leq x)$
**(d)** $\exists y \in H \; \forall x \in H \; (y \leq x)$

**1.111.** Let $A = \{a \in \mathbb{R} : -3 < a \leq 1\}$ and $B = \{b \in \mathbb{R} : -3 < b < 1\}$. Which statements are true?

**(a)** $\forall a \in A \; \exists b \in B \; b < a$
**(b)** $\exists b \in B \; \forall a \in A \; b < a$

**(c)** $\forall b \in B \; \exists a \in A \; b < a$
**(d)** $\exists a \in A \; \forall b \in B \; b < a$

**Determine the intersection of the following sequences of sets.**

**1.112.** $A_n = \{a \in \mathbb{Q} : -\frac{1}{n} < a < \frac{1}{n}\}$

**1.113.** $B_n = \{b \in \mathbb{R} \backslash \mathbb{Q} : -\frac{1}{n} < b < \frac{1}{n}\}$

**1.114.** $C_n = \{c \in \mathbb{Q} : \sqrt{2} - \frac{1}{n} < c < \sqrt{2} + \frac{1}{n}\}$

**1.115.** $D_n = \{d \in \mathbb{N} : -n < d < n\}$

**1.116.** $E_n = \{e \in \mathbb{R} : -n < e < n\}$

**1.117.** Let $H \subset \mathbb{R}$. Write down the negation of the following statement:

$$\forall x \in H\ \exists y \in H\ (x > 2 \implies y < x^2).$$

**Determine the intersection of the following sequences of intervals. (For example, find the intersection $M$ with the help of a figure, then prove that $\forall x \in M$ implies that $\forall n\ x \in I_n$, and if $y \notin M$, then $\exists k\ y \notin I_k$. (We note that $k$ and $n$ are positive integers.)**

**1.118.** $I_n = [-1/n, 1/n]$

**1.119.** $I_n = (-1/n, 1/n)$

**1.120.** $I_n = [2 - 1/n, 3 + 1/n]$

**1.121.** $I_n = (2 - 1/n, 3 + 1/n)$

**1.122.** $I_n = [0, 1/n]$

**1.123.** $I_n = (0, 1/n)$

**1.124.** $I_n = [0, 1/n)$

**1.125.** $I_n = (0, 1/n]$

**1.126.** Which statements are true? (Give the reasoning for the answer!)

**(a)** If the intersection of a nested sequence of intervals is not empty, then the intervals are closed.

**(b)** If the intersection of a nested sequence of intervals is empty, then the intervals are open.

**(c)** The intersection of a nested sequence of closed intervals is one point.

**(d)** If the intersection of a nested sequence of intervals is empty, then there is an open interval among the intervals.

**(e)** If the intersection of a nested sequence of intervals is empty, then there is a not closed interval among the intervals.

**(f)** If the intersection of intervals is not empty, then the intervals are nested.

**Satisfy your answers.**

**1.127.** Can the intersection of a nested sequence of intervals be empty?

**1.128.** Can the intersection of nested sequence of closed intervals be empty?

**1.129.** Can the intersection of nested sequence of closed intervals be a single point?

**1.130.** Can the intersection of nested sequence of open intervals be not empty?

**1.131.** Can the intersection of nested sequence of open intervals be empty?

**1.132.** Can the intersection of nested sequence of closed intervals be a proper interval (not a single point)?

**1.133.** Can the intersection of nested sequence of open intervals be a proper interval?

**1.134.** Can the intersection of nested sequence of closed intervals be a proper open interval?

**1.135.** Can the intersection of nested sequence of open intervals be a proper open interval?

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**1.136.** Which of the axioms of the real numbers are fulfilled by the rational numbers?

**1.137.** Prove from the Archimedes' axiom that $(\forall b, c < 0)\ (\exists n \in \mathbb{N})\ nb < c$.

**1.138.** Prove that there is a finite decimal number between any two real numbers.

**1.139.** Prove that there is a rational number between any two real numbers.

**1.140.** What is the connection between the finite decimal numbers and the rational numbers?

**1.141.** Prove that a decimal form of a real number is repeating decimal if and only if the number is rational.

**1.142.** Prove that Cantor's axiom doesn't remain true, if we omit any of its assumption.

**1.143.** Prove from the field axioms the following identities:

**(a)** $-a = (-1) \cdot a$

**(b)** $(a-b) - c = a - (b+c)$

**(c)** $(-a) \cdot b = -(a \cdot b)$

**(d)** $\dfrac{1}{a/b} = \dfrac{b}{a}$

**(e)** $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{a \cdot c}{b \cdot d}$

## 1.6 The Number Line

**Draw the following sets on the number line. Decide which one is an interval, and which one is not. Decide which intervals are closed, which ones are open, and which ones are nor open, neither closed.**

**1.144.** $A = \{1, 2, 3\}$

**1.145.** $B = \{x \in \mathbb{R} : 2 < x < 6\}$

**1.146.** $C = \{5.6\}$

**1.147.** $D = \{x \in \mathbb{N} : 2 \le x \le 6\}$

**1.148.** $E = \{x \in \mathbb{R} : 2 \le x \le 6\}$

**1.149.** $F = \{x \in \mathbb{R} : 2 < x \le 6\}$

**1.150.** $G = \{x \in \mathbb{R} : 2 \le x < 6\}$

**1.151.** $H = \{x \in \mathbb{Q} : 2 \le x \le 6\}$

**Which ones of the following sets are bounded, bounded from above, bounded from below? Do they have minimal or maximal elements?**

**1.152.** set of the prime numbers

**1.153.** set of the positive numbers

**1.154.** $[-5, -2)$

**1.155.** $\left\{\dfrac{1}{n} : n \in \mathbb{N}^+\right\}$

**1.156.** $\{x \in \mathbb{R} : x \le 73\}$

**1.157.** $\{x \in \mathbb{Q} : x \le 73\}$

**1.158.** $\{x \in \mathbb{R} : x \le \sqrt{2}\}$

**1.159.** $\{x \in \mathbb{Q} : x \le \sqrt{2}\}$

**1.160.** $\{n \in \mathbb{N} : n \text{ is prime } \wedge \ n+2 \text{ is prime}\}$

**1.161.** Which of the following statements implies the other one?

**P:** The set $A$ is finite (that is, the number of the elements of $A$ is finite).

**Q:** The set $A$ is bounded.

**1.162.** Is there any sequence of numbers $a_1, a_2, \dots$ such that the set $\{a_1, a_2, \dots\}$ is bounded, but the sequence has no maximal and no minimal elements?

**Write down with logical symbols the following statements.**

**1.163.** The set $A$ is bounded.

**1.164.** The set $A$ is not bounded from below.

**1.165.** The set $A$ has no minimal element.

**1.166.** How many maxima, or upper bounds can a set have?

**1.167.** Which of the following statements implies the other one?

**P:** The set $A$ has a minimal element.

**Q:** The set $A$ is bounded from below.

**1.168.** Let $A \cap B \neq \emptyset$. What can we say about the connection of $\sup A$, $\sup B$, $\sup(A \cup B)$, $\sup(A \cap B)$ and $\sup(A \setminus B)$?

**1.169.** Let $A = (0,1)$, $B = [-\sqrt{2}, \sqrt{2}]$ and $C = \left\{ \frac{1}{2^n} + \frac{1}{2^m} : n, m \in \mathbb{N}^+ \right\}$. Find, if there exist, the supremum, the infimum, the maximum and the minimum of the previous sets.

**1.170.** Let $A$ be an arbitrary set of numbers, and

$$B = \{-a : a \in A\}, C = \left\{ \frac{1}{a} : a \in A, a \neq 0 \right\}.$$

What is the connection between the supremum and the infimum of the sets?

**Find, if there exist, the supremum, the infimum, the maximum and the minimum of the following sets.**

**1.171.** $[1,2]$

**1.172.** $(1,2)$

**1.173.** $\left\{\dfrac{1}{2n-1} : n \in \mathbb{N}^+\right\}$

**1.174.** $\mathbb{Q}$

**1.175.** $\left\{\dfrac{1}{n} + \dfrac{1}{\sqrt{n}} : n \in \mathbb{N}^+\right\}$

**1.176.** $\{\sqrt[n]{3} : n \in \mathbb{N}^+\}$

**1.177.** $\{x : x \in (0,1) \cap \mathbb{Q}\}$

**1.178.** $\left\{\dfrac{1}{n} + \dfrac{1}{k} : n, k \in \mathbb{N}^+\right\}$

**1.179.** $\{\sqrt{n+1} - \sqrt{n} : n, k \in \mathbb{N}^+\}$

**1.180.** $\left\{n + \dfrac{1}{n} : n \in \mathbb{N}^+\right\}$

**1.181.** $\left\{\sqrt[n]{2} : n \in \mathbb{N}^+\right\}$

**1.182.** $\{\sqrt[n]{2^n - n} : n \in \mathbb{N}\}$

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**1.183.** Let $H$ be a nonempty subset of the real numbers. Which of the following statements implies an other one?

**(a)** $H$ is not bounded from below. **(b)** $H$ has no minimal element.

**(c)** $\forall x \in H\ \exists y \in H\ (y < x)$. **(d)** $\forall y \in H\ \exists x \in H\ (y < x)$.

**1.184.** We know that $c$ is an upper bound of $H$. Does it imply that $\sup H = c$?

**1.185.** We know that there is no less upper bound of $H$, than $c$. Does it imply that $\sup H = c$?

**1.186.** Let $A$ and $B$ be not empty subsets of the real numbers. Prove that if

$$\forall a \in A\ \exists b \in B (a \leq b),$$

then $\sup A \leq \sup B$.

**1.187.** Prove that any nonempty set, which is bounded from below, has an infimum.

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**Let $x, y, A, B$ be arbitrary real numbers, and $\varepsilon$ be a positive real number. Which of the following statements (P and Q) implies the other one?**

**1.188.** **P:** $|x - A| < \varepsilon$ **Q:** $A - \varepsilon < x < A + \varepsilon$

**1.189.** **P:** $|x - y| < 2\varepsilon$ **Q:** $|x - A| < \varepsilon$ and $|y - A| < \varepsilon$

**1.190.** **P:** $|x| < A$ and $|y| < B$ **Q:** $|x| - |y| < A - B$

**1.191.** **P:** $|x| < A$ and $|y| < B$ **Q:** $|x| + |y| < A + B$

**1.192.** **P:** $|x| < A$ and $|y| < B$ **Q:** $|x| - |y| < A + B$

**1.193.** Show an example of a nonempty set of real numbers, which is bounded, but has no minimum.

**1.194.** Let us assume that the set $H \subset \mathbb{R}$ is nonempty. Which of the following statements implies the other one?

**P:** $H$ has no minimum. **Q:** $\forall a \in \mathbb{R}^+ \; \exists b \in H \quad b < a$

# Chapter 2

# Convergence of a Sequence

**2.1** The sequence $(a_n)$ **converges** to the number $b \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \; \exists n_0 \; \forall n \geq n_0 (|a_n - b| < \varepsilon).$$

We call the natural number $n_0$ the **threshold** for the given $\varepsilon$.
If the sequence $(a_n)$ converges to the number $b$, we can use the following notations:

$$\lim_{n\to\infty} a_n = b \text{ or } \lim a_n = b \text{ or } a_n \to b, \text{ if } n \to \infty \text{ or } a_n \to b.$$

If the sequence $(a_n)$ is not convergent, we say that the sequence $(a_n)$ is divergent.

**2.2** We say that the **limit** of the sequence $(a_n)$ is **infinity**, or $(a_n)$ diverges to $\infty$, if

$$\forall P \in \mathbb{R} \; \exists n_0 \; \forall n \geq n_0 (a_n > P).$$

The notations:

$$\lim_{n\to\infty} a_n = \infty \text{ or } \lim a_n = \infty \text{ or } a_n \to \infty, \text{ if } n \to \infty \text{ or } a_n \to \infty.$$

**2.3** We say, that the **limit** of the sequence $(a_n)$ is **-infinity**, or $(a_n)$ diverges to $-\infty$, if

$$\forall P \in \mathbb{R} \; \exists n_0 \; \forall n \geq n_0 (a_n < P).$$

The notation:

$$\lim_{n\to\infty} a_n = -\infty \text{ or } \lim a_n = -\infty \text{ or } a_n \to -\infty, \text{ or } n \to \infty \text{ or } a_n \to -\infty.$$

## 2.1 Limit of a Sequence

**Let the sequence $(a_n)$ be: $a_n = 1+\dfrac{1}{\sqrt{n}}$. In the exercises the letters $n$ and $n_0$ denote positive integers.**

**2.1.** Find a number $n_0$ such that $\forall n > n_0$ implies that

**(a)** $|a_n - 1| < 0,1$ **(b)** $|a_n - 1| < 0,01$

**2.2.** Is there any $n_0$ number such that $\forall n > n_0$ implies $|a_n - 2| < 0,001$?

**2.3.** Is it true that

**(a)** $\forall \varepsilon > 0\ \exists n_0\ \forall n > n_0\ (|a_n - 1| < \varepsilon)$
**(b)** $\exists n_0\ \forall \varepsilon > 0\ \forall n > n_0\ (|a_n - 1| < \varepsilon)$
**(c)** $\exists \varepsilon > 0\ \exists n_0\ \forall n > n_0\ (|a_n - 1| < \varepsilon)$
**(d)** $\exists \varepsilon > 0\ \exists n_0\ \forall n > n_0\ (|a_n - 1| > \varepsilon)$
**(e)** $\forall \varepsilon > 0\ \exists n_0\ \forall n \le n_0\ (|a_n - 1| < \varepsilon)$
**(f)** $\forall \varepsilon > 0\ \exists n_0\ \forall n \le n_0\ (|a_n - 1| > \varepsilon)$

**Find a threshold $N$ from which all of the terms of one of the sequences is greater than the terms of the other one.**

**2.4.** $a_n = 10n^2 + 25$
$b_n = n^3$

**2.5.** $a_n = 4n^5 - 3n^2 - 7$
$b_n = 10n + 30$

**2.6.** $a_n = 3^n - n^2$
$b_n = 2^n + n$

**2.7.** $a_n = 2^n + 3^n$
$b_n = 4^n$

**2.8.** $a_n = 2^n$
$b_n = n!$

**2.9.** $a_n = n!$
$b_n = n^n$

**2.10.** $a_n = \sqrt{n+1} - \sqrt{n}$
$b_n = \dfrac{1}{n}$

**2.11.** $a_n = 2^n$
$b_n = n^3$

**2.12.** $a_n = 0.999^n$
$b_n = \frac{1}{n^2}$

**2.13.** $a_n = 10^n$
$b_n = n!$

**Find a number $N$ such that $\forall n > N$ implies that**

**2.14.** $1.01^n > 1000;$

**2.15.** $0.9^n < \frac{1}{100};$

**2.16.** $\sqrt[n]{2} < 1.01.$

**2.17.** $\sqrt[n]{n} < 1.0001.$

**2.18.** $n^2 > 6n + 15$

**2.19.** $n^3 > 6n^2 + 15n + 37$

**2.20.** $n^3 - 4n + 2 > 6n^2 - 15n + 37$

**2.21.** $n^5 - 4n^2 + 2 > 6n^3 - 15n + 37$

**Show that there exists a number $n_0$ such that for all $n > n_0$ implies**

**2.22.** $\sqrt{n+1} - \sqrt{n} < 0.01$

**2.23.** $\sqrt{n+3} - \sqrt{n} < 0.01$

**2.24.** $\sqrt{n+5} - \sqrt{n+1} < 0.01$

**2.25.** $\sqrt{n^2+5} - n < 0.01$

**Prove the following inequalities.**

**2.26.** $\forall n > 10 \quad 2^n > n^3;$

**2.27.** $\sqrt{n} \leq 1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$

**2.28.** Which statement implies the other?
**P:** In the sequence $(a_n)$ there is a smallest and a greatest term.
**Q:** The sequence $(a_n)$ is bounded.

**2.29.** Is it true that $b$ is the limit of the sequence $(a_n)$ if and only if

**(a)** for any $\varepsilon > 0$ the sequence $a_n$ has infinitely many terms closer to $b$ than $\varepsilon$?

**(b)** for any $\varepsilon > 0$ the sequence $a_n$ has only finitely many terms at least $\varepsilon$ distance to $b$?

**(c)** there exists $\varepsilon > 0$ such that the sequence $a_n$ has infinitely many terms closer to $b$ than $\varepsilon$?

**(d)** there exists $\varepsilon > 0$ such that the sequence $a_n$ has infinitely many terms at least distance $\varepsilon$ to $b$?

**What can we say about the limit of the sequence $(-a_n)$ if**

**2.30.** $\lim\limits_{n\to\infty} a_n = a$ $(a \in \mathbb{R})$;

**2.31.** $\lim\limits_{n\to\infty} a_n = \infty$;

**2.32.** $\lim\limits_{n\to\infty} a_n = -\infty$?

**2.33.** $a_n$ is oscillating divergent?

**2.34.** Which statement implies the other?

**P:** $\lim\limits_{n\to\infty} a_n = \infty$

**Q:** $(a_n)$ is bounded below, but isn't bounded above.

**Find the limits of the following sequences, and give a threshold depending on $\varepsilon$:**

**2.35.** $\dfrac{(-1)^n}{n}$

**2.36.** $\dfrac{1}{\sqrt{n}}$

**2.37.** $\dfrac{1+\sqrt{n}}{n}$

**2.38.** $\dfrac{n}{n+1}$

**2.39.** $\dfrac{5n-1}{7n+2}$

**2.40.** $\dfrac{2n^6+3n^5}{7n^6-2}$

**2.41.** $\dfrac{n+\frac{1}{n}}{n+1}$

**2.42.** $\sqrt{n+1}-\sqrt{n}$

**2.43.** $\sqrt{n^2+1}-n$

**2.44.** $\dfrac{1}{n-\sqrt{n}}$

**2.45.** $\dfrac{1+\cdots+n}{n^2}$

**2.46.** $n\left(\sqrt{1+\dfrac{1}{n}}-1\right)$

**2.47.** $\sqrt{n^2+1}+\sqrt{n^2-1}-2n$

**2.48.** $\sqrt[3]{n+2}-\sqrt[3]{n-2}$

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**2.49.** Are the following sequences convergent or divergent? Find the limits if they exist.

**(a)** $a_n = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$

**(b)** $a_n = \begin{cases} 3 & \text{if } n \leq 100 \\ 4 & \text{if } n > 100 \end{cases}$

**(c)** $a_n = \begin{cases} 3n & \text{if } n \text{ is even} \\ 4n^2 & \text{if } n \text{ is odd} \end{cases}$

**(d)** $a_n = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

**2.50.** Prove that the sequence $\dfrac{1}{n}$ does not converge to 7.

**2.51.** Prove that the sequence $(-1)^n\dfrac{1}{n}$ does not converge to 7.

**2.52.** Prove that the sequence $(-1)^n$ does not converge to 7.

**2.53.** Prove that the sequence $(-1)^n$ is divergent.

**2.54.** Prove that a convergent sequence always has a minimal or maximal term.

**2.55.** Show an example such that $a_n - b_n \to 0$ but $\dfrac{a_n}{b_n} \not\to 1$.

**2.56.** Prove that if $(a_n)$ is convergent, then also $(|a_n|)$ is convergent. Is the reverse of the statement true?

**2.57.** Does $a_n^2 \to a^2$ imply that $a_n \to a$?
And does $a_n^3 \to a^3$ imply that $a_n \to a$?

**2.58.** Prove that if $a_n \to a > 0$, then $\sqrt{a_n} \to \sqrt{a}$.

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**Which statement implies that $a_n \to \infty$?**

**2.59.** $\forall K$ it is true that outside the interval $(K, \infty)$ the sequence $a_n$ has only finitely many terms.

**2.60.** $\forall K$ it is true that inside the interval $(K, \infty)$ the sequence $a_n$ has infinitely many terms.

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**2.61.** Let's assume that $\lim_{n\to\infty} a_n = \infty$. Which statements are true for this sequence? Which statements imply that $\lim_{n\to\infty} a_n = \infty$?

**(a)** The sequence $a_n$ has no maximal term.

**(b)** The sequence $a_n$ has a minimal term.

**(c)** Outside the interval $(3, \infty)$ the sequence $a_n$ has only finitely many terms.

**(d)** $\forall K$ it is true that outside the interval $(K, \infty)$ the sequence $a_n$ has only finitely many terms.

**(e)** Inside the interval $(3, \infty)$ the sequence $a_n$ has infinitely many terms.

**(f)** $\forall K$ it is true that inside the interval $(K, \infty)$ the sequence $a_n$ has infinitely many terms.

**2.62.** Is it true that if a sequence has a (finite or infinite) limit, then the sequence is bounded from below or above?

**2.63.** Which statement implies the other?

**P:** The sequence $(a_n)$ is strictly monotonically increasing.

**Q:** The limit of $(a_n)$ is infinity.

---

**Can the limit of the sequence $a_n$ be $-\infty$, $\infty$ or a finite number, if**

**2.64.** the sequence has infinitely many terms greater than 3?

**2.65.** the sequence has infinitely many terms smaller than 3?

**2.66.** the sequence has a maximal term?

**2.67.** the sequence has a minimal term?

**2.68.** the sequence has no minimal term?

**2.69.** the sequence has no maximal term?

**2.70.** Is there any oscillating divergent sequence, which is

**(a)** bounded? **(b)** not bounded?

**2.71.** A sequence has infinitely many positive and infinitely many negative terms. Can the sequence be convergent?

**Find a threshold for the sequences with limit infinity:**

**2.72.** $n - \sqrt{n}$

**2.73.** $\dfrac{1+2+\cdots+n}{n}$

**2.74.** $\dfrac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n}$

**2.75.** $\dfrac{n^2-10n}{10n+100}$

**2.76.** $\dfrac{2^n}{n}$

**2.77.** $\dfrac{n!}{2^n}$

**2.78.** Find the limit of $\dfrac{n^2+1}{n+1} - an$ if $a$ is an arbitrary real number.

**2.79.** Find the limit of $\sqrt{n^2-n+1} - an$ if $a$ is an arbitrary real number.

**2.80.** Find the limit of $\sqrt{(n+a)(n+b)} - n$ if $a, b$ are arbitrary real numbers.

**2.81.** Prove that if $a_{n+1} - a_n \to c > 0$ , then $a_n \to \infty$.

**2.82.** Prove that if $a_n > 0$, $\dfrac{a_{n+1}}{a_n} \to c > 1$ , then $a_n \to \infty$.

**2.83.** For which real numbers is it true that the sequence of its decimal numbers is oscillating divergent?

## 2.2 Properties of the Limit

**Can we decide from the given inequalities, whether the sequence $b_n$ has a limit or has not, and if there is a limit, can we determine the value of the limit? If the answer is "yes", then find the limit of $b_n$.**

**2.84.** $\frac{1}{n} < b_n < \frac{2}{n}$

**2.85.** $-\frac{1}{n} \le b_n \le \frac{1}{\sqrt{n}}$

**2.86.** $\frac{1}{n} < b_n < \sqrt{n}$

**2.87.** $n \le b_n$

**2.88.** $b_n < -1.01^n$

**2.89.** $b_n < n^2$

**2.90.** Prove that if the sequence $(a_n)$ has no subsequence, which goes to infinity, then the sequence $(a_n)$ is bounded from above.

**2.91.** Prove that if the sequences $(a_{2n}), (a_{2n+1}), (a_{3n})$ are convergent, then $(a_n)$ is convergent, too.

**2.92.** Is there any sequence $(a_n)$ which has no convergent subsequence, but $(|a_n|)$ is convergent?

**Let $a$ be a real number, and $a_n \to a$. Prove that**

**2.93.** if $a > 1$, then $a_n^n \to \infty$.

**2.94.** if $|a| < 1$, then $a_n^n \to 0$.

**2.95.** if $a > 0$, then $\sqrt[n]{a_n} \to 1$.

**2.96.** if $a < -1$, then $a_n^n$ is divergent.

**2.97.** Prove that if $(a_n + b_n)$ is convergent, and $(b_n)$ is divergent, then $(a_n)$ is divergent.

**2.98.** Is it true that if $(a_n \cdot b_n)$ is convergent, and $(b_n)$ is divergent, then $(a_n)$ is divergent?

**2.99.** Is it true that if $(a_n/b_n)$ is convergent, and $(b_n)$ is divergent, then $(a_n)$ is divergent?

**2.100.** Prove that if $\lim \dfrac{a_n - 1}{a_n + 1} = 0$, then $(a_n)$ is convergent, and $\lim a_n = 1$.

**2.101.** Let's assume that $(a_n)$ satisfies that $\dfrac{a_n - 5}{a_n + 3} \to \dfrac{5}{13}$. Prove that $a_n \to 10$.

**2.102.** Let's assume that $\sqrt[n]{a_n} \to 0,3$. Prove that $a_n \to 0$.

**2.103.** Let $p(x)$ be a polynomial. Prove that $\dfrac{p(n+1)}{p(n)} \to 1$.

**Let's assume that the sequence $a_n$ has a limit. Which statement implies the other?**

**2.104.** **P:** For all large enough $n$ $\quad \dfrac{1}{n} < a_n$ $\qquad$ **Q:** $\lim\limits_{n\to\infty} a_n > 0$

**2.105.** **P:** For all large enough $n$ $\quad \dfrac{1}{n} \leq a_n$ $\qquad$ **Q:** $\lim\limits_{n\to\infty} a_n \geq 0$

**2.106.** **P:** For all large enough $n$ $\quad \dfrac{1}{n} < a_n$ $\qquad$ **Q:** $\lim\limits_{n\to\infty} a_n \geq 0$

**2.107.** **P:** For all large enough $n$ $\quad \dfrac{1}{n} \leq a_n$ $\qquad$ **Q:** $\lim\limits_{n\to\infty} a_n > 0$

**Let's assume that the sequences $a_n$ and $b_n$ have limits. Which statement implies the other?**

**2.108.** **P:** For all large enough $n$ $\quad a_n < b_n$ $\qquad$ **Q:** $\lim\limits_{n\to\infty} a_n < \lim\limits_{n\to\infty} b_n$

**2.109.** **P:** For all large enough $n$ $\quad a_n \leq b_n$ $\qquad$ **Q:** $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$

**Which statement implies that the sequence $a_n$ has a limit? Which statement implies that the sequence $a_n$ is convergent? Which statement implies that the sequence $a_n$ is divergent?**

**2.110.** $b_n$ is convergent and $a_n > b_n$ for all large enough $n$.

**2.111.** $\lim_{n\to\infty} b_n = \infty$ and $a_n > b_n$ for all large enough $n$.

**2.112.** $\lim_{n\to\infty} b_n = -\infty$ and $a_n > b_n$ for all large enough $n$.

**2.113.** $b_n$ and $c_n$ are convergent and $b_n \le a_n \le c_n$ for all large enough $n$.

**2.114.** $\lim_{n\to\infty} b_n = \infty$ and $a_n < b_n$ for all large enough $n$.

**Are the following sequences bounded from above? Find the limits if they exist.**

**2.115.** $\dfrac{1+2+\cdots+n}{n}$

**2.116.** $\dfrac{1+2+\cdots+n}{n^2}$

**2.117.** $\dfrac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n}$

**2.118.** $\dfrac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n^2}$

**Are the following sequences convergent or divergent? Find the limits if they exist.**

**2.119.** $\sqrt[n]{2^n+3^n}$

**2.120.** $\sqrt[n]{3^n-2^n}$

**2.121.** $\sqrt[n]{7+(-1)^n}$

**2.122.** $\sqrt[n]{2^n-n}$

**2.123.** $\sqrt[n]{2^n+n^2}$

**2.124.** $\sqrt[n]{2^n-n^2}$

**2.125.** $\dfrac{1-2+3-\cdots-2n}{\sqrt{n^2+1}}$

**2.126.** $\left(\dfrac{n-1}{3n}\right)^n$

**2.127.** $\dfrac{n^3-n^2+1}{\sqrt{n^6+1}+100n^2+n+1}$

**2.128.** $\sqrt[n]{\dfrac{n^3-n^2+1}{n^6+100n^2+n+1}}$

**2.129.** $\sqrt[n]{\dfrac{2^n+n^2+1}{3^n+n^3+1}}$

**2.130.** $\dfrac{n^2+(-1)^n}{3n^2+1}$

**2.131.** $\left(1+\dfrac{1}{n}\right)^{n^2}$

**2.132.** $\dfrac{n^2-1}{n^2+1}$

**2.133.** $\dfrac{1}{n^3}$

**2.134.** $\dfrac{5n-1}{7n+2}$

**2.135.** $\dfrac{n}{n+1}$

**2.136.** $\dfrac{2n^6+3n^5}{7n^6-2}$

**2.137.** $\dfrac{n+1/n}{n+1}$

**2.138.** $\dfrac{7n^5+2}{5n-1}$

**2.139.** $\dfrac{3n^7+4}{-5n^2+2}$

**2.140.** $\dfrac{2^n+3^n}{4^n+(-7)^n}$

**2.141.** $\dfrac{3n^{5/3}+n\sqrt{n}}{n^{1/4}+\sqrt[5]{n}}$

**2.142.** $\dfrac{7n-2n^3}{3n^3+18n^2-9}$

## Which statement implies the other?

**2.143.** **P:** $a_n$ is convergent and $b_n$ is convergent **Q:** $a_n+b_n$ is convergent

**2.144.** **P:** $a_n+b_n\to\infty$ **Q:** $a_n\to\infty$ and $b_n\to\infty$

**2.145.** **P:** $a_n+b_n\to\infty$ **Q:** $a_n\to\infty$ or $b_n\to\infty$

**2.146.** **P:** $a_n\cdot b_n\to 0$ **Q:** $a_n\to 0$ or $b_n\to 0$

**2.147.** **P:** $a_n$ and $b_n$ are bounded **Q:** $a_n+b_n$ is bounded

**2.148.** **P:** $a_n$ and $b_n$ are bounded **Q:** $a_n\cdot b_n$ is bounded

**2.149.** Show examples of the possible behavior of the sequence $a_n+b_n$ if $\lim\limits_{n\to\infty} a_n=\infty$ and $\lim\limits_{n\to\infty} b_n=-\infty$.

**2.150.** Show examples of the possible behavior of the sequence $a_n\cdot b_n$ if $\lim\limits_{n\to\infty} a_n=0$ and $\lim\limits_{n\to\infty} b_n=\infty$.

**2.151.** Show examples of the possible behavior of the sequence $\dfrac{a_n}{b_n}$ if $\lim\limits_{n\to\infty} a_n=0$ and $\lim\limits_{n\to\infty} b_n=0$.

**2.152.** Show examples of the possible behavior of the sequence $\frac{a_n}{b_n}$ if $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = \infty$.

**2.153.** Let's assume that none of the terms of the sequence $b_n$ is 0. Which statement implies the other?

**P:** $b_n \to \infty$ **Q:** $\frac{1}{b_n} \to 0$

**2.154.** Which statement implies the other?

**P:** $\frac{a_n}{b_n} \to 1$ **Q:** $a_n - b_n \to 0$

**2.155.** Let's assume that $a_n \to \infty$ and $b_n \to \infty$. Which statement implies the other?

**P:** $\frac{a_n}{b_n} \to 1$ **Q:** $a_n - b_n \to 0$

**2.156.** Let's assume that $a_n \to 0$ and $b_n \to 0$. Which statement implies the other?

**P:** $\frac{a_n}{b_n} \to 1$ **Q:** $a_n - b_n \to 0$

## 2.3 Monotonic Sequences

**Let $(a_n)$ and $(b_n)$ be two monotonic sequences. What can we say about the monotonity of the following sequences? What additional conditions are required for monotonity?**

**2.157.** $(a_n + b_n)$

**2.158.** $(a_n - b_n)$

**2.159.** $(a_n \cdot b_n)$

**2.160.** $\left(\frac{a_n}{b_n}\right)$

**2.161.** Let $a_1 = 1$, and $a_{n+1} = \sqrt{2a_n}$, if $n \geq 1$. Prove that the sequence $a_n$ is monotonically increasing.

**2.162.** Let $a_1 = \frac{1}{2}$, and $a_{n+1} = 1 - \sqrt{1 - a_n}$, if $n \geq 1$. Prove that all terms of the sequence are positive, and the sequence is monotonically decreasing.

**2.163.** Let $a_1 = 0.9$, and $a_{n+1} = a_n - a_n^2$, if $n \geq 1$. Prove that all terms of the sequence are positive, and the sequence is monotonically decreasing. Prove that there is $n \in \mathbb{N}^+$ such that $a_n < 10^{-6}$, and find such an $n$ number.

**2.164.** Let $a_1 > 0$, and $\frac{a_{n+1}}{a_n} > 1.1$ for all $n \in \mathbb{N}^+$. Prove that there is $n \in \mathbb{N}^+$ such that $a_n > 10^6$, and find such an $n$ number.

### Which statement implies the other?

**2.165.** **P:** The sequence $a_n$ is monotonically increasing **Q:** The sequence $a_n$ goes to infinity.

**2.166.** **P:** The sequence $a_n$ is monotonically decreasing. **Q:** The sequence $a_n$ goes to minus infinity.

**2.167.** Let's assume that the terms of the sequence satisfy the inequality $a_n \leq \frac{a_{n-1} + a_{n+1}}{2}$ if $n > 1$. Prove that the sequence $(a_n)$ cannot be oscillating divergent.

**2.168.** Let $a_1 = a > 0$ be arbitrary, and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$. Prove that $a_n \to \sqrt{a}$.

### Find the limits of the following recursive sequences if the limits exist. In the recurrence formulas $n \geq 1$.

**2.169.** $a_1 = 2, \quad a_{n+1} = \frac{2a_n}{1 + a_n^2}$

**2.170.** $a_1 = 1,5, \quad a_{n+1} = -a_n + 1$

**2.171.** $a_1 = 3, \quad a_{n+1} = \frac{a_n + \frac{5}{a_n}}{2}$

**2.172.** $a_1 = 6, \quad a_{n+1} = \dfrac{a_n + \dfrac{5}{a_n}}{2}$

**2.173.** $a_1 = 0, \quad a_{n+1} = \sqrt{2 + a_n}$

**2.174.** $a_1 = 0, \quad a_{n+1} = \dfrac{1}{2 - a_n}$

**2.175.** $a_1 = 0, \quad a_{n+1} = \dfrac{1}{4 - a_n}$

**2.176.** $a_1 = 0, \quad a_{n+1} = \dfrac{1}{1 + a_n}$

**2.177.** $a_1 = 1, \quad a_{n+1} = a_n + \dfrac{1}{a_n}$

**2.178.** $a_1 = 0,9, \quad a_{n+1} = a_n - a_n^2$

**2.179.** $a_1 = 1, \quad a_{n+1} = \sqrt{2a_n}$

**2.180.** $a_1 = 1, \quad a_{n+1} = a_n + \dfrac{1}{a_n^3 + 1}$

**Are the following sequences bounded or monotonic? Find the limits if they exist.**

**2.181.** $\left(1 + \dfrac{1}{n}\right)^n$

**2.182.** $\left(1 + \dfrac{1}{n}\right)^{n+1}$

**2.183.** $\left(1 - \dfrac{1}{n}\right)^n$

**2.184.** $\left(1 + \dfrac{1}{2n}\right)^n$

## 2.4 The Bolzano–Weierstrass theorem and the Cauchy Criterion

**2.185.** Write down the negation of Cauchy's criterion for a sequence $(a_n)$. What is the logical connection between the negation of Cauchy's criterion and the statement "$(a_n)$ is divergent", that is, which statement implies the other?

## Which statement implies the other?

**2.186.** **P:** $a_{2n}$ and $a_{2n+1}$ are convergent **Q:** $a_n$ is convergent

**2.187.** **P:** $a_{2n}$, $a_{2n+1}$ and $a_{3n}$ is convergent **Q:** $a_n$ is convergent

**2.188.** **P:** $a_{2n} \to 5$ **Q:** $a_n \to 5$

## Which statement implies that the sequence is convergent?

**2.189.** $a_{n+1} - a_n \to 0$, if $n \to \infty$

**2.190.** $|a_n - a_m| < \dfrac{1}{n+m}$ for all $n, m$

## Which sequence has a convergent subsequence?

**2.191.** $(-1)^n$

**2.192.** $\dfrac{1}{n}$

**2.193.** $\sqrt{n}$

**2.194.** $(-1)^n \dfrac{1}{n}$

**2.195.** Prove that if the sequence $(a_n)$ has no convergent subsequence, then $|a_n| \to \infty$.

**2.196.** Prove that if $(a_n)$ is bounded, and all of its convergent subsequences go to $a$, then $a_n \to a$.

**2.197.** Prove that if the sequence $(a_n)$ has no two subsequences going to two different limits, then the sequence has a limit.

**2.198.** Prove that if $|a_{n+1} - a_n| \leq 2^{-n}$ for all $n$, then the sequence $(a_n)$ is convergent.

**2.199.** Let's assume that $a_{n+1} - a_n \to 0$. Does it imply that $a_{2n} - a_n \to 0$?

## 2.5 Order of Growth of the Sequences

**2.200.** Prove that $n! \prec n^n$ is true.

**2.201.** Give the order of growth of the following sequences.

$$(n^7), \qquad (n^2+2^n), \qquad (100\sqrt{n}), \qquad \left(\frac{n!}{10}\right)$$

**2.202.** Insert into the order of growth $n \prec n^2 \prec n^3 \prec \cdots \prec 2^n \prec 3^n \prec \cdots \prec n! \prec n^n$ into the right places the sequences $\sqrt{n}$, $\sqrt[3]{n}$, ..., $\sqrt[k]{n}$.

**2.203.** Find all of the asymptotically equal pairs among the following sequences.

$(n!)$, $(n^n)$, $(n!+n^n)$, $(\sqrt{n})$, $(\sqrt[n]{n})$, $(\sqrt{n+1})$, $(\sqrt[n]{2})$

**Are the following sequences convergent or divergent? Find the limits if they exist.**

**2.204.** $\dfrac{2^n}{3^n}$

**2.205.** $\dfrac{3^n}{2^n}$

**2.206.** $(1.1)^n$

**2.207.** $\left(-\dfrac{4}{5}\right)^n$

**2.208.** $\dfrac{1}{(1.2)^n+1}$

**2.209.** $\dfrac{n+2}{\sqrt{n}-3^{-n}}$

**2.210.** $\dfrac{3.01^n}{2^n+3^n}$

**2.211.** $\dfrac{3^n}{(-3)^n}$

**2.212.** $\dfrac{3^n-\sqrt{n}+n^{10}}{2^n-\sqrt[n]{n}+n!}$

**2.213.** $\dfrac{n^{100}}{100^n}$

**2.214.** $\dfrac{10^n}{n!}$

**2.215.** $0.99^n n^2$

**2.216.** $\dfrac{n!-3^n}{n^{10}-2^n}$

**2.217.** $\dfrac{1.01^n}{n^2}$

**2.218.** $\dfrac{n^3}{1.2^n}$

**2.219.** $\sqrt[n]{2^n + n - 1}$

**2.220.** $\dfrac{3^{n+6} + n^2}{2^{n+3}}$

**2.221.** $\dfrac{4^n + 5^n}{6^n + (-7)^n}$

## 2.6 Miscellaneous Exercises

**2.222.** Let $a_n = \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$ (the sum has $n$ terms). Since the sequences in all terms go to 0, so the sequence $a_n$ goes to 0. On the other hand $a_n = n \cdot \frac{1}{n} = 1$ for all $n$, therefore $a_n \to 1$. Which of the reasonings contains any error, and what is the error?

**2.223.** We know that $1 + \frac{1}{n} \to 1$, and $1^n = 1$, therefore $\left(1 + \frac{1}{n}\right)^n \to 1$.

On the other hand, applying Bernoulli's inequality, we can prove that $\left(1 + \frac{1}{n}\right)^n \geq 2$,

therefore the limit of $\left(1 + \frac{1}{n}\right)^n$ cannot be smaller than 2.

Which of the reasonings contains any error, and what is the error?

**2.224.** Let's assume that $\sqrt[n]{a_n} \to 2$. What can we say about $\lim\limits_{n\to\infty} a_n$?

**2.225.** Let's assume that $\sqrt[n]{a_n} \to \frac{1}{2}$. What can we say about $\lim\limits_{n\to\infty} a_n$?

**2.226.** Let's assume that $\sqrt[n]{a_n} \to 1$. What can we say about $\lim\limits_{n\to\infty} a_n$?

**2.227.** Let's assume that $a_n \to 2$. What can we say about $\lim\limits_{n\to\infty} a_n^n$?

**2.228.** Let's assume that $a_n \to \frac{1}{2}$. What can we say about $\lim\limits_{n\to\infty} a_n^n$?

**2.229.** Let's assume that $a_n \to 1$. What can we say about $\lim\limits_{n\to\infty} a_n^n$?

**Show an example for a sequence $a_n$, for which is true that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1$, and**

**2.230.** $\lim_{n\to\infty} a_n = 1$

**2.231.** $\lim_{n\to\infty} a_n = \infty$

**2.232.** $\lim_{n\to\infty} a_n = 0$

**2.233.** $\lim_{n\to\infty} a_n = 7$

# Chapter 3

# Limit and Continuity of Real Functions

**3.1 Jensen's inequality.** The function $f$ is convex on the interval $(a, b)$ if and only if for arbitrarily chosen finitely many $x_1, x_2, \cdots, x_n \in (a, b)$ numbers and $t_1, t_2, \ldots, t_n \geq 0$ weights, where $\sum_{i=1}^{n} t_i = 1$

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i)$$

holds.

**3.2 Limits and inequalities.**

— If there is some neighborhood of $a$ such that $f(x) \leq g(x)$, and the limits of $f$ and $g$ exist at $a$, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

— If the limits of $f$ and $g$ exist at $a$, and

$$\lim_{x \to a} f(x) < \lim_{x \to a} g(x),$$

then in some neighborhood of $a$ $f(x) < g(x)$.

— **Squeeze theorem.** If in some neighborhood of $a$ $f(x) \leq g(x) \leq h(x)$, and the limits of $f$ and $h$ exist at $a$, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x),$$

then the limit of $g$ also exists at $a$, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x).$$

— **"0 times bounded is 0".** If $\lim_{x \to a} f(x) = 0$, and $g(x)$ is bounded, then

$$\lim_{x \to a} f(x) g(x) = 0.$$

### 3.3 Continuous functions and their limits.

— The function $f$ is continuous at $a$ if and only if there exists the limit of the function at $a$ and the limit is $f(a)$.

— The function $f$ is continuous from right at $a$ if and only if there exists its right-hand side limit at $a$ and it is $f(a)$.

— The function $f$ is continuous from left at $a$ if and only if there exists its left-hand side limit at $a$ and it is $f(a)$.

### 3.4 Continuous functions in closed interval.

— **Weierstrass theorem:** If a function is continuous in a bounded and closed interval, then the function has maximum and minimum value.

— **Intermediate value theorem (Bolzano's theorem):** If the function $f(x)$ is continuous in the bounded and closed $[a, b]$ interval, then every value between $f(a)$ and $f(b)$ is attained in $[a, b]$.

— **Inverse of a continuous function:** If a function is continuous and invertible in a bounded and closed interval, then the range of the function is a closed interval, and in this interval the inverse function is continuous.

### 3.5 Uniform continuity.

— **Heine-Borel theorem:** If a function is continuous in a bounded and closed interval, then the function is uniformly continuous.

— The function $f(x)$ is uniformly continuous in a **bounded and open** $(a, b)$ interval if and only if the function is continuous in $(a, b)$ and the $\lim\limits_{x\to a^+} f(x)$, $\lim\limits_{x\to b^-} f(x)$ finite limits exist.

— If $f(x)$ is continuous in $[a, \infty)$, differentiable in $(a, \infty)$, and its derivative is bounded, then $f(x)$ is uniformly continuous in $[a, \infty)$.

## 3.1 Global Properties of Functions

**3.1.** Let $[x]$ be the floor of $x$, that is, the maximal integer which is not greater than $x$. Plot the following functions!

**(a)** $[x]$ **(b)** $[-x]$

**(c)** $[x+0,5]$ **(d)** $[2x]$

**3.2.** Let $\{x\}$ be the fractional part of $x$, that is, $\{x\} = x - [x]$. Plot the following functions!

**(a)** $\{x\}$ **(b)** $\{-x\}$

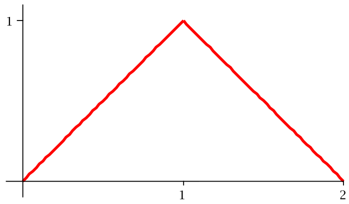
**(c)** $\{x+0,5\}$ **(d)** $\{2x\}$

**3.3.** Is this a formula for a function?

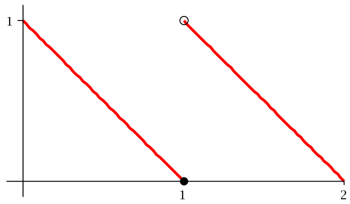
$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

**3.4.** Find the formulas for the following graphs!

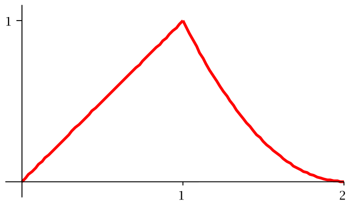
**(a)**



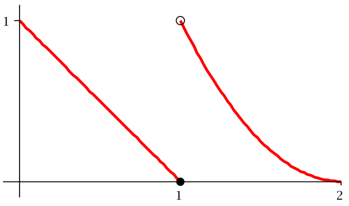

**(b)**



**(c)**




**(d)**




**Determine the maximal domain of the real numbers for the following functions!**

**3.5.** $\log_2 x^2$

**3.6.** $\sqrt{x^2-16}$

**3.7.** $\sqrt{\sin x}$

**3.8.** $\dfrac{\log_2(-x)}{\sqrt{-x}}$

**3.9.** Match the formulas and the graphs!

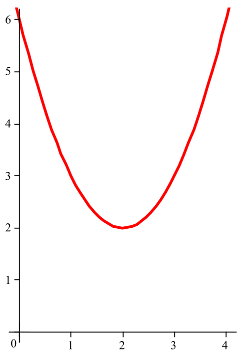
**(a)** $(x-1)^2-4$

**(b)** $(x-2)^2+2$

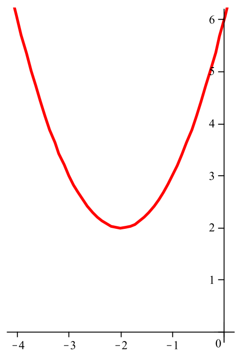
**(c)** $(x+2)^2+2$

**(d)** $(x+3)^2-2$

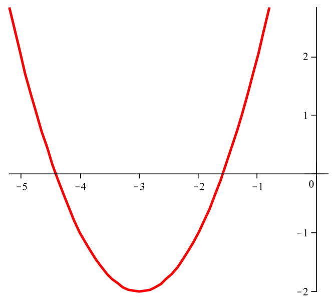
(A)



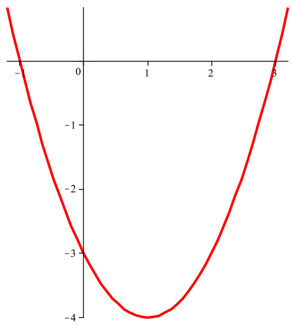

(B)



(C)

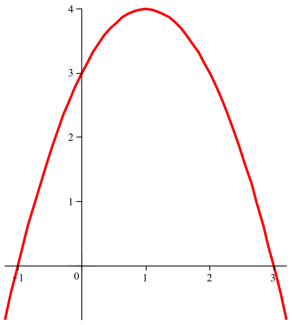


(D)

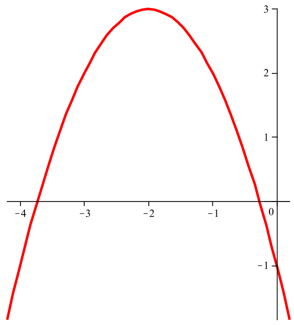


**3.10.** We plotted four transforms of the function $y = -x^2$. Find the formulas for the graphs!

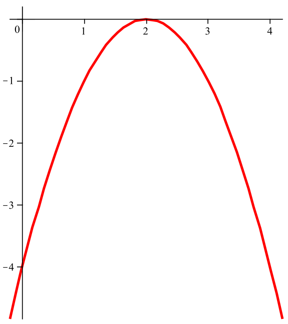
**(a)**



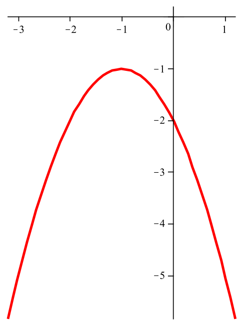
**(b)**



**(c)**



**(d)**



**3.11.** Are there some equivalent among the following functions?

**(a)** $f_1(x) = x$ **(b)** $f_2(x) = \sqrt{x^2}$

**(c)** $f_3(x) = \left(\sqrt{x}\right)^2$ **(d)** $f_4(x) = \ln e^x$

**(e)** $f_5(x) = e^{\ln x}$ **(f)** $f_6(x) = \left(\sqrt{-x}\right)^2$

**3.12.** Find the values of the following functions if $f(x) = x + 5$ and $g(x) = x^2 - 3$.

**(a)** $f(g(0))$ **(b)** $g(f(0))$

**(c)** $f(g(x))$ **(d)** $g(f(x))$

**(e)** $f(f(-5))$ **(f)** $g(g(2))$

**(g)** $f(f(x))$ **(h)** $g(g(x))$

**3.13.** Find the values of the following functions if $f(x) = x - 1$ and $g(x) = \dfrac{1}{x+1}$

**(a)** $f(g(1/2))$ **(b)** $g(f(1/2))$

**(c)** $f(g(x))$ **(d)** $g(f(x))$

**(e)** $f(f(2))$ **(f)** $g(g(2))$

**(g)** $f(f(x))$ **(h)** $g(g(x))$

**Which function is even, which one is odd, which one is both, and which one is neither even, nor odd?**

**3.14.** $x^3$

**3.15.** $x^4$

**3.16.** $\sin x$

**3.17.** $\cos x$

**3.18.** $2 + \sin x$

**3.19.** $2 + \cos x$

**3.20.** 3

**3.21.** $(x+1)^2$

**3.22.** 0

**3.23.** $|x^3|$

**3.24.** $[x]$

**3.25.** $\{x\}$

**Let's assume that the domains of $f$ and $g$ are $\mathbb{R}$. Which statements are true? Explain your answers!**

**3.26.** If $f$ is odd, then $f(0) = 0$.

**3.27.** If $f(0) = 0$, then $f$ is odd.

**3.28.** If $f$ even, then $f(-5) = f(5)$.

**3.29.** If $f(-5) = f(5)$, then $f$ is even.

**3.30.** If $f$ and $g$ even, then $fg$ is even.

**3.31.** If $f(-5) \neq -f(5)$, then $f$ is not odd.

**3.32.** If $f$ and $g$ odd, then $fg$ is even.

**3.33.** If $f$ and $g$ odd, then $fg$ is odd.

**3.34.** Plot the graphs of the following functions! Color the intervals on the $x$-axis red, where the function is monotonically decreasing. Is any of the following functions monotonically decreasing on its whole domain?

**(a)** $\sin x$

**(b)** $\cos x$

**(c)** $x^2$

**(d)** $\dfrac{1}{x}$

**(e)** $|x|$

**(f)** $|x^2 - 2|$

**(g)** $\tan x$ **(h)** $\cot x$

**3.35.** Is there any function in $\mathbb{R}$, which is monotonically decreasing and monotonically increasing? If there is such a function, find all of them!

**Answer the following questions! Reason the answers!**

**3.36.** Can the sum of two strictly monotonically increasing functions be strictly monotonically decreasing?

**3.37.** Can the product of two strictly monotonically increasing functions be strictly monotonically decreasing?

**3.38.** Is it true that the sum of two strictly monotonically decreasing functions is strictly monotonically decreasing?

**3.39.** Is it true that the product of two strictly monotonically decreasing functions is strictly monotonically decreasing?

**Let $D(f)$ be the domain and $R(f)$ be the range of the $f$ function. Is there a monotonically increasing function such that**

**3.40.** $D(f) = (0, 1)$ and $R(f) = [0, 1]$

**3.41.** $D(f) = [0, 1]$ and $R(f) = (0, 1)$?

**3.42.** Write down with logic symbols that $f$ is bounded!

**Find lower and upper bounds for the following functions if there exist. Which functions are bounded?**

**3.43.** $x^2$ **3.44.** $\sin x$

**3.45.** $\{x\}$ **3.46.** $\dfrac{[x]}{x}$

**3.47.** $\sin^2 x$

**3.48.** $2^{-x}$

**3.49.** $\log_2 x$

**3.50.** $\dfrac{1}{1+x^2}$

**Let's assume that the domain of $f$ is $\mathbb{R}$. Write down with logic symbols, and give examples for such an $f$ function which**

**3.51.** has maximum at 3!

**3.52.** has a maximum 3.

**3.53.** has a maximum!

**3.54.** has no maximum!

**Which statement implies the other?**

**3.55.** **P:** $f$ has a maximum. **Q:** $f$ is bounded from above.

**3.56.** **P:** $f$ has no minimum. **Q:** $f$ is not bounded from below.

**Find the $M$ maximum and $m$ minimum of the following functions, if there exist.**

**3.57.** $x^2$ $(-\infty, \infty)$

**3.58.** $|x|$ $[-1, 3]$

**3.59.** $x^3$ $[-1, 1)$

**3.60.** $\sin x$ $(-\pi, \pi)$

**3.61.** $\cos x$ $(-\pi, \pi)$

**3.62.** $[x]$ $[-1, 1]$

**3.63.** $[x]$ $(-1, 1)$

**3.64.** $\{x\}$ $[-1, 1]$

**Give an example of functions with domain $\mathbb{R}$ such that**

**3.65.** the function is not bounded from above and not bounded from below.

**3.66.** bounded, but has no maximum and no minimum.

**Give an example of such a function whose domain is $[-1,1]$, and which**

**3.67.** is not bounded from above, and not bounded from below.

**3.68.** is bounded, but has no maximum and no minimum.

**Is there any function such that it is**

**3.69.** strictly monotonically decreasing in $(-\infty,0)$, strictly monotonically increasing in $(0,\infty)$, and has no minimum at 0?

**3.70.** monotonically decreasing in $(-\infty,0]$, monotonically increasing in $[0,\infty)$, and has no minimum at 0?

**3.71.** not bounded in $[0,1]$?

**3.72.** bounded in $[0,1]$, but has no minimum, and no maximum in $[0,1]$?

**3.73.** positive in $\mathbb{R}$, but has no minimum?

**Find the least positive period for the following functions!**

**3.74.** $\sin x$

**3.75.** $\sin(2x)$

**3.76.** $\sin\dfrac{x}{2}$

**3.77.** $\tan x$

**3.78.** $\sin x+\tan x$

**3.79.** $\sin 2x+\tan\dfrac{x}{2}$

**3.80.** Prove that if $p$ is a period for a function, then any positive integer times $p$ is also a period.

**3.81.** Is the function $f(x)=3$ periodic? If yes, then find all of its periods!

**3.82.** Do all non-constant periodic functions have a least positive period?

**3.83.** Is the Dirichlet-function

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

periodic? If the answer is "yes", then give all periods!

## Is the given function convex or concave in $(0, \infty)$?

**3.84.** $x$

**3.85.** $x^2$

**3.86.** $\sqrt{x}$

**3.87.** $-x^3$

**3.88.** $\sin x$

**3.89.** $[x]$

**3.90.** Let the domain of the real function $f$ be $(0, 10)$. Which statement implies the other?

**P:** $f$ is convex in $(3, 8)$ **Q:** $f$ is convex in $(5, 7)$.

**3.91.** Give all of the functions that are both convex and concave in $(1, 2)$! Is there among the functions a strictly convex or a strictly concave function?

**3.92.** Let's assume that the domain of $f$ is $(-1, 3)$. Which statement implies the other?

**P:** $f(1) \le \dfrac{f(0) + f(2)}{2}$ **Q:** $f$ is convex in $(-1, 3)$

**3.93.** Is the function $\sqrt{x}$ convex, concave, both or neither in the interval $[0, \infty)$? Write down the Jensen-inequality with the weights $t_1 = \ldots = t_n = \dfrac{1}{n}$!

**3.94.** Plot the graph of $x^{10}$, and the chord in the interval $[1, 2]$! Write down the equation of the chord of $x^{10}$ in $[1, 2]$! Prove that $x^{10} \le 1023x - 1022$ is true for all $x \in [1, 2]$.

**3.95.** Write down the equation of the chord of the function $\sin x$ in $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.
Which number is greater: $\dfrac{\sin(\pi/6)+\sin(\pi/2)}{2}$ or $\sin\dfrac{\pi/6+\pi/2}{2}$?

**3.96.** Write down the equation of the chord of the function $\log_7 x$ in $[2,4]$.
Which number is greater: $\log_7 3$ or $\dfrac{\log_7 2+\log_7 4}{2}$?

### Plot the graphs of some functions so that the function is

**3.97.** monotonically increasing in $[1,2]$ and monotonically decreasing in $[3,4]$,

**3.98.** monotonically increasing in $[1,4]$ and monotonically decreasing in $[3,5]$,

**3.99.** convex in $[1,4]$, and concave in $[4,5]$,

**3.100.** convex in $[1,4]$, and concave in $[2,5]$,

**3.101.** strictly monotonically increasing in $[1,2]$, strictly monotonically decreasing in $[2,4]$, and has a maximum at 2,

**3.102.** strictly monotonically increasing in $[1,2]$, strictly monotonically decreasing in $[2,4]$, and has a minimum at 2.

### Plot the graphs of some functions so that

**3.103.** $\forall x_1 \in [1,2] \wedge \forall x_2 \in [1,2] \quad f(x_1) = f(x_2)$,

**3.104.** $\forall x_1 \in [1,2] \wedge \forall x_2 \in [1,2] \quad (x_1 > x_2 \implies f(x_1) > f(x_2))$,

**3.105.** $\forall x_1 \in [1,2] \wedge \forall x_2 \in [1,2] \quad (x_1 > x_2 \implies f(x_1) \leq f(x_2))$,

**3.106.** $\forall x_1 \in [1,2] \wedge \forall x_2 \in [1,2] \qquad \exists c \in [x_1, x_2] \quad f(c) = \dfrac{f(x_1)+f(x_2)}{2}$,

**3.107.** $\exists x_1 \in [1,2] \wedge \exists x_2 \in [1,2] \qquad \forall x \in [1,2] \quad f(x) \neq \dfrac{f(x_1)+f(x_2)}{2}$,

**3.108.** $\forall x_1 \in [1,2] \wedge \forall x_2 \in [1,2] \quad f\left(\dfrac{x_1+x_2}{2}\right) > \dfrac{f(x_1)+f(x_2)}{2}$,

**3.109.** $\forall x_1 \in [1,2] \wedge \forall x_2 \in [1,2] \quad f\left(\frac{1}{4}x_1 + \frac{3}{4}x_2\right) < \frac{1}{4}f(x_1) + \frac{3}{4}f(x_2)$,

**3.110.** $\exists x_0 \in [1,2] \quad \forall x \in [1,2] \quad f(x) \le f(x_0)$,

**3.111.** $(\forall x_1 \in [1,2]\ \exists x_2 \in [1,2]\ f(x_1) < f(x_2)) \wedge (\forall x_1 \in [1,2]\ \exists x_2 \in [1,2]\ f(x_1) > f(x_2))$.

**3.112.** Which of the following functions are bijective on the whole number-line?

**(a)** $x$

**(b)** $x^2$

**(c)** $x^3$

**(d)** $\sqrt{x}$

**(e)** $\sqrt[3]{x}$

**(f)** $\sqrt{|x|}$

**(g)** $\frac{1}{x}$

**(h)** $f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

**3.113.** Give the inverses of the following functions! Plot in the same coordinate system the inverse pairs!

**(a)** $x^3$

**(b)** $x^3 + 1$

**(c)** $2^x$

**(d)** $2^x - 1$

**Find intervals such that the function is injective in those intervals! Find the inverses of the function in these intervals!**

**3.114.** $x^2$

**3.115.** $\sqrt{x}$

**3.116.** $\sin x$

**3.117.** $2^x$

**3.118.** Find some functions that are equal to their inverses!

**3.119.** Which statement implies the other?

**P:** $f$ is strictly monotonic **Q:** $f$ has an inverse function

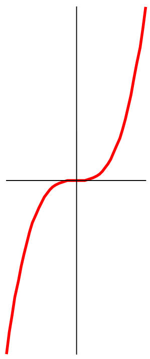
**3.120.** Show that the function

$$f(x) = \begin{cases} x \text{ if } x \in \mathbb{Q} \\ -x \text{ if } x \notin \mathbb{Q} \end{cases}$$

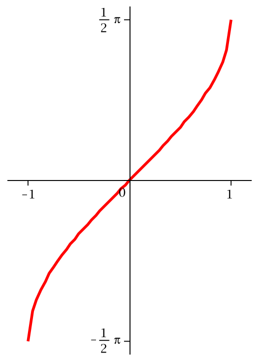
is not monotonic in any interval, but the function has an inverse!

**3.121.** Find the inverse pairs among the graphs!

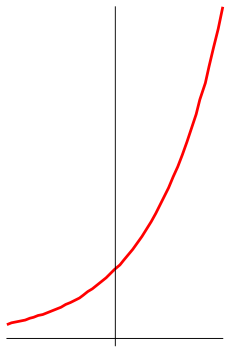
**(a)**



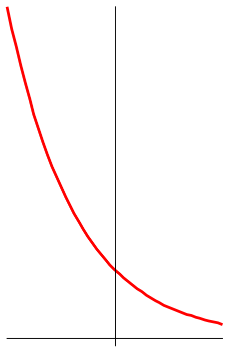
**(b)**



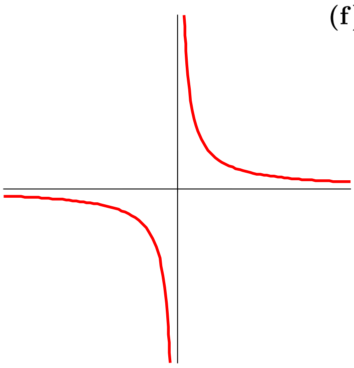

**(c)**



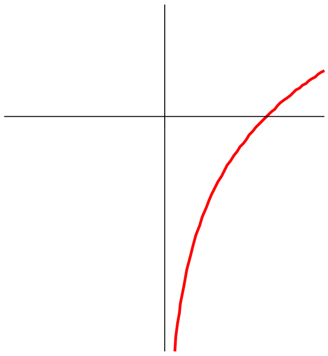
**(d)**



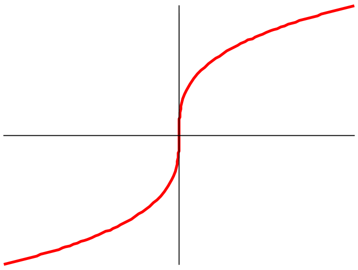
**(e)**



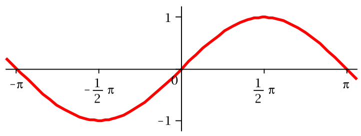
**(f)**



**(g)**



**(h)**




**3.122.** Is there a function with domain $\mathbb{R}$ whose graph is symmetric, and the line of symmetry is the

**(a)** axis $x$?

**(b)** axis $y$?

**3.123.** Which statement implies the other?

**P:** $f$ is monotonically increasing in $\mathbb{R}$.
**Q:** $f(x+1) \geq f(x)$ for all $x \in \mathbb{R}$

**3.124.** Prove that the function $f(x) = \dfrac{1}{x} + \dfrac{1}{x-1}$ attains each value exactly once in $(0,1)$!

**3.125.** Prove that if for all $x \in \mathbb{R}$ $f(x+1) = \dfrac{1+f(x)}{1-f(x)}$, then $f$ is periodic!

**3.126.** Let's assume that $f$ is an even function. Can $f$ have an inverse?

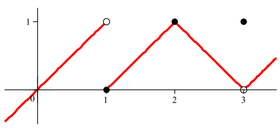
**3.127.** Let's assume that $f$ is an odd function. Does that imply that $f$ has an inverse?

**3.128.** Plot the functions $f$ and $g$. Give the function $g \circ f$. Is it true that $g$ is the inverse of $f$?

$$f(x) = \begin{cases} x, & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ x+1 & \text{if } x > 0 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x, & \text{if } x < 0 \\ 0 & \text{if } 0 \le x < 1 \\ x-1 & \text{if } x \ge 1 \end{cases}$$

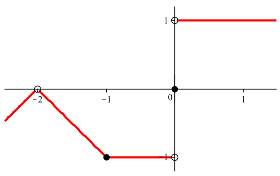
## 3.2 Limit

**3.129.** Do the given limits exist according to the graph? If the answer is "yes", find the limits!



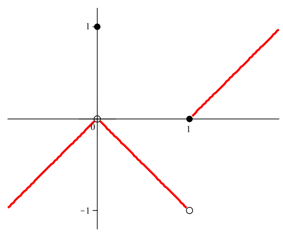

**(a)** $\lim\limits_{x\to 1} f(x)$ **(b)** $\lim\limits_{x\to 2} f(x)$ **(c)** $\lim\limits_{x\to 3} f(x)$

**3.130.** Do the given limits exist according to the graph? If the answer is "yes", find the limits!



**(a)** $\lim_{x\to -2} f(x)$ **(b)** $\lim_{x\to -1} f(x)$ **(c)** $\lim_{x\to 0} f(x)$

**3.131.** Which statements are true according to the graph?

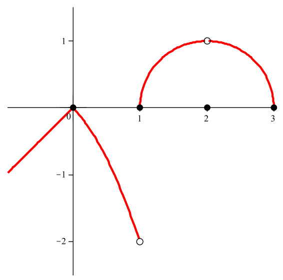


**(a)** $\lim_{x\to 0} f(x)$ exists. **(b)** $\lim_{x\to 0} f(x) = 0$ **(c)** $\lim_{x\to 0} f(x) = 1$

**(d)** $\lim_{x\to 1} f(x) = 1$ **(e)** $\lim_{x\to 1} f(x) = 0$

**(f)** The function has a limit at each point of $(-1, 1)$.

**3.132.** Which statements are true according to the graph of the function?



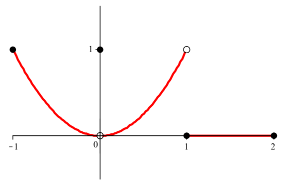
**(a)** $\lim_{x\to 2} f(x)$ does not exist. **(b)** $\lim_{x\to 2} f(x) = 2$

**(c)** $\lim_{x\to 1} f(x)$ does not exist.

**(d)** $f(x)$ has a limit at each point of $(-1, 1)$.

**(e)** $f(x)$ has a limit at each point of $(1, 3)$.

**3.133.** Which statements are true according the graph of the function?




**(a)** $\lim\limits_{x\to 1^+} f(x) = 1$ **(b)** $\lim\limits_{x\to 0^-} f(x) = 0$

**(c)** $\lim\limits_{x\to 0^-} f(x) = 1$ **(d)** $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$

**(e)** $\lim\limits_{x\to 0} f(x)$ exists. **(f)** $\lim\limits_{x\to 0} f(x) = 0$

**(g)** $\lim\limits_{x\to 0} f(x) = 1$ **(h)** $\lim\limits_{x\to 1} f(x) = 1$

**(i)** $\lim\limits_{x\to 1} f(x) = 0$ **(j)** $\lim\limits_{x\to 2^-} f(x) = 2$

**(k)** $\lim\limits_{x\to 1^-} f(x)$ does not exist. **(l)** $\lim\limits_{x\to 2^+} f(x) = 0$

**3.134.** Write down the following statements with logic symbols! Find functions of which the statements are true!

**(a)** $\lim\limits_{x\to 3} f(x) = 4$ **(b)** $\lim\limits_{x\to 4} f(x) = \infty$ **(c)** $\lim\limits_{x\to 5} f(x) = -\infty$

**(d)** $\lim\limits_{x\to 3^+} f(x) = 4$ **(e)** $\lim\limits_{x\to 3^+} f(x) = \infty$ **(f)** $\lim\limits_{x\to 3^+} f(x) = -\infty$

**(g)** $\lim\limits_{x\to 3^-} f(x) = 4$ **(h)** $\lim\limits_{x\to 3^-} f(x) = \infty$ **(i)** $\lim\limits_{x\to 3^-} f(x) = -\infty$

**(j)** $\lim\limits_{x\to \infty} f(x) = 4$ **(k)** $\lim\limits_{x\to \infty} f(x) = \infty$ **(l)** $\lim\limits_{x\to \infty} f(x) = -\infty$

**(m)** $\lim\limits_{x\to -\infty} f(x) = 4$ **(n)** $\lim\limits_{x\to -\infty} f(x) = \infty$ **(o)** $\lim\limits_{x\to -\infty} f(x) = -\infty$

**3.135.** Find the functions which have limit at 3. Which functions have the same limit?

**(a)** $5$

**(b)** $6$

**(c)** $\begin{cases} 5, & \text{if } x \neq 3 \\ 6, & \text{if } x = 3 \end{cases}$

**(d)** $\begin{cases} 5, & \text{if } x \in \mathbb{Q} \\ 6, & \text{if } x \notin \mathbb{Q} \end{cases}$

**(e)** $\dfrac{1}{(x-3)^2}$

**(f)** $\dfrac{1}{\cos(x-3)}$

**(g)** $\dfrac{1}{\sin(x-3)}$

**(h)** $\dfrac{1}{x-3}$

## Find the following limits with substitution!

**3.136.** $\lim\limits_{x\to 3} 5x$

**3.137.** $\lim\limits_{x\to 0} 5x$

**3.138.** $\lim\limits_{x\to 1/7} (7x-3)$

**3.139.** $\lim\limits_{x\to 1} \dfrac{-2}{7x-3}$

**3.140.** $\lim\limits_{x\to -1} 3x^2(7x-3)$

**3.141.** $\lim\limits_{x\to 1} \dfrac{3x^2}{7x-3}$

**3.142.** $\lim\limits_{x\to \pi/2} x\sin x$

**3.143.** $\lim\limits_{x\to \pi} \dfrac{\cos x}{1-\pi}$

## Find the following limits after simplifying the fractions!

**3.144.** $\lim\limits_{x\to 5} \dfrac{x-5}{x^2-25}$

**3.145.** $\lim\limits_{x\to -3} \dfrac{x+3}{x^2+4x+3}$

**3.146.** $\lim\limits_{x\to -5} \dfrac{x^2+3x-10}{x+5}$

**3.147.** $\lim\limits_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

**3.148.** $\lim\limits_{t\to 1} \dfrac{t^2+t-2}{t^2-1}$

**3.149.** $\lim\limits_{t\to -1} \dfrac{t^2+3t+2}{t^2-t-2}$

**3.150.** $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

**3.151.** $\lim\limits_{x\to 4} \dfrac{4x-x^2}{2-\sqrt{x}}$

**3.152.** $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$

**3.153.** $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

**3.154.** $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

**3.155.** $\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x^2}$

**Find the following trigonometric limits!**

**3.156.** $\lim_{x\to 0} \frac{\sin x}{x}$

**3.157.** $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

**3.158.** $\lim_{\vartheta\to 0} \frac{\sin(\vartheta\sqrt{2})}{\vartheta\sqrt{2}}$

**3.159.** $\lim_{t\to 0} \frac{\sin kt}{t}$

**3.160.** $\lim_{y\to 0} \frac{\sin 3y}{4y}$

**3.161.** $\lim_{h\to 0} \frac{h}{\sin 3h}$

**3.162.** $\lim_{x\to 0} \frac{\tan 2x}{x}$

**3.163.** $\lim_{t\to 0} \frac{2t}{\tan t}$

**Find the following limits if exist!**

**3.164.** $\lim_{x\to 0} x \sin x$

**3.165.** $\lim_{x\to 0} \sin \frac{1}{x}$

**Find the limits of the following functions at $\infty$ and at $-\infty$.**

**3.166.** $\frac{2x+3}{5x+7}$

**3.167.** $\frac{2x^2-7x+1}{\sqrt{x^2+1}+1}$

**3.168.** $\frac{2x^3-7x}{x^3+1}$

**3.169.** $\frac{2x+3}{5x^2+7}$

**3.170.** $\frac{2x^2-7x}{x^3+1}$

**3.171.** $\frac{x^{-1}+x^{-5}}{x^{-2}-x^{-3}}$

**Find the (finite or infinite) limits of the following functions at $\infty$.**

**3.172.** $\frac{2\sqrt{x}+x^{-1}}{3x-7}$

**3.173.** $\frac{2+\sqrt{x}}{2-\sqrt{x}}$

**3.174.** $\dfrac{2x^3 - 7x}{x^2 + 1}$

**3.175.** $\dfrac{2x^2 - 7x + 1}{\sqrt{x^3 + 3} + 7}$

**3.176.** $\dfrac{2x^2 - 7x + 1}{\sqrt{x^4 + 1} + 1}$

**3.177.** $\dfrac{\sqrt[3]{2x^2 + 1} + 1}{\sqrt{x^2 + 1} + 1}$

**Find both the right-hand side and the left-hand side limits in each problem!**

**3.178.** $\displaystyle\lim_{x \to 0^+} \frac{1}{3x}$

**3.179.** $\displaystyle\lim_{x \to 0^-} \frac{5}{2x}$

**3.180.** $\displaystyle\lim_{x \to 2^-} \frac{3}{x - 2}$

**3.181.** $\displaystyle\lim_{x \to 3^+} \frac{1}{x - 3}$

**3.182.** $\displaystyle\lim_{x \to -8^+} \frac{2x}{x + 8}$

**3.183.** $\displaystyle\lim_{x \to -5^-} \frac{3x}{2x + 10}$

**3.184.** $\displaystyle\lim_{x \to 7^+} \frac{4}{(x - 7)^2}$

**3.185.** $\displaystyle\lim_{x \to 0^-} \frac{-1}{x^2(x + 1)}$

**Find the following limits!**

**3.186.** $\displaystyle\lim_{x \to \infty} \frac{\sin x}{x}$

**3.187.** $\displaystyle\lim_{x \to \infty} \frac{e^x}{x}$

**3.188.** $\displaystyle\lim_{x \to \infty} \frac{\ln x}{x}$

**3.189.** $\displaystyle\lim_{x \to \infty} \frac{x^2}{e^x}$

**Let $k$ be a fixed positive number. Find the following limits:**

**3.190.** $\displaystyle\lim_{x \to \infty} \frac{x^k}{e^x}$

**3.191.** $\displaystyle\lim_{x \to \infty} \frac{\ln x}{\sqrt[k]{x}}$

**Do the limits of the following functions exist at 0? Do the right-hand side or the left-hand side limits exist at 0?**

**3.192.** $[x]$

**3.193.** $\{x\}$

**3.194.** $\begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$

**3.195.** $\begin{cases} x, & \text{if } x \in \mathbb{Q} \\ -x, & \text{if } x \notin \mathbb{Q} \end{cases}$

**3.196.** Find a function with domain $\mathbb{R}$, which has limits at exactly 2 points!

**3.197.** Is there any function with domain $\mathbb{R}$, whose limit is infinity at infinitely many points?

**3.198.** Prove that if $f$ is not a constant, periodic function, then $f$ has no limit at infinity.

### Have the following functions limits at infinity?

**3.199.** $[x]$

**3.200.** $\{x\}$

**3.201.** $\sin x$

**3.202.** $\tan x$

### Which statement implies the other?

**3.203.** **P:** $\lim\limits_{x\to\infty} f(x) = 5$ **Q:** $\lim\limits_{x\to\infty} f^2(x) = 25$

**3.204.** **P:** $\lim\limits_{x\to\infty} f(x) = -5$ **Q:** $\lim\limits_{x\to\infty} |f(x)| = 5$

**3.205.** **P:** $\lim\limits_{x\to\infty} f(x) = \infty$ **Q:** $\lim\limits_{x\to\infty} \dfrac{1}{f(x)} = 0$

**3.206.** Are there any limit of

**(a)** the sequence $a_n = \sin(n\pi)$?

**(b)** the function $f(x) = \sin x$ in infinity?

**(c)** the sequence $a_n = \left[\dfrac{1}{n}\right]$?

**(d)** the function $f(x) = [x]$ at 0?

**Which statement implies the other?**

**3.207.** **P:** The limit of the sequence $f(n)$ is 5. **Q:** $\lim_{x\to\infty} f(x) = 5$.

**3.208.** **P:** The limit of the sequence $f\left(\frac{1}{n}\right)$ is 5. **Q:** $\lim_{x\to 0} f(x) = 5$.

**3.209.** **P:** $\lim_{x\to\infty} (f(x) + g(x)) = \infty$ **Q:** $\lim_{x\to\infty} f(x)g(x) = \infty$

**3.210.** **P:** $\lim_{x\to\infty} (f(x) + g(x)) = \infty$ **Q:** $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} g(x) = \infty$

**3.211.** **P:** $\lim_{x\to\infty} f(x)g(x) = \infty$ **Q:** $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} g(x) = \infty$

**3.212.** Let the domain of $f$ be $\mathbb{R}$. Which statement implies the other:

**P:** $\lim_{x\to\infty} f(x) = 0$ **Q:** The limit of $f(n)$ is 0

if

**(a)** $f$ can be any arbitrary function?

**(b)** $f$ is continuous?

**(c)** $f$ is monotonic?

**(d)** $f$ is bounded?

## 3.3 Continuous Functions

**3.213.** Write down with logic symbols that $f$ is continuous at 3!

**Which statement implies the other?**

**3.214.** **P:** $f$ has a limit at 3. **Q:** $f$ is continuous at 3.

**3.215.** **P:** $f$ has no limit at 3. **Q:** $f$ is not continuous at 3.

**3.216.** Are the following functions continuous at 0?

**(a)** $D(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$ **(b)** $f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ -x, & \text{if } x \notin \mathbb{Q} \end{cases}$

**3.217.** Find a function that is continuous at exactly 2 points!

**3.218.** The functions $f$ and $g : \mathbb{R} \to \mathbb{R}$ are different at a point, but equal to each other at all other points. Can be both functions continuous at every point?

**3.219.** Let's assume that $f$ and $g : \mathbb{R} \to \mathbb{R}$ have finite limits at every point, and their limits are equal. Does it imply that $f = g$ in every point? Does it imply that $f = g$ in every point if both $f$ and $g$ are continuous?

**3.220.** Which statement implies the other?

**P:** $f$ and $g$ are continuous at 3. **Q:** $f + g$ is continuous at 3.

**3.221.** Let's assume that $f$ is continuous, and $g$ is not continuous at 3. Can

**(a)** $f + g$ **(b)** $fg$

be continuous at 3?

**3.222.** Let's assume that nor $f$, neither $g$ is continuous at 3. Does it imply that

**(a)** $f + g$ **(b)** $fg$

is not continuous at 3?

**3.223.** Let's assume that $f$ and $g$ are continuous at 3. Does it imply that $\dfrac{f}{g}$ is continuous at 3?

## At which points are the following functions continuous?

**3.224.** $\dfrac{x^2 - 4}{x + 2}$

**3.225.** $\dfrac{x^3 - 1}{x - 1}$

**3.226.** $\sqrt{x}$

**3.227.** $\sqrt[3]{x}$

**3.228.** Find a function $f : \mathbb{R} \to \mathbb{R}$ which is not continuous at some points, but $|f|$ is continuous at every point.

**For what number $c$ are the following functions continuous at 0?**

**3.229.** $f(x) = \begin{cases} x^2 + 2 & \text{if } x \geq 0 \\ mx + c & \text{if } x < 0 \end{cases}$

**3.230.** $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ c & \text{if } x = 0 \end{cases}$

**3.231.** $f(x) = \begin{cases} x^3 + x + 1 & \text{if } x > 0 \\ ax^2 + bx + c & \text{if } x \leq 0 \end{cases}$

**3.232.** $f(x) = \begin{cases} \sqrt{x+2} & \text{if } x \geq 0 \\ (x+c)^2 & \text{if } x < 0 \end{cases}$

**3.233.** Prove that all polynomials with degree 3 have a real root.

**3.234.** Let's assume that $f$ is continuous in $[a, b]$. Prove that there is a $c \in [a, b]$ such that

**(a)** $f(c) = \dfrac{f(a) + f(b)}{2}$ **(b)** $f(c) = \sqrt{f(a)f(b)}$

**3.235.** Let's assume that $f$ is continuous in $[a, b]$, and $f(a) \geq a$ and $f(b) \leq b$. Prove that there is a $c \in [a, b]$ such that $f(c) = c$.

**3.236.** Let's assume that both $f$ and $g$ are continuous in $[a, b]$, and $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that there is a $c \in [a, b]$ such that $f(c) = g(c)$.

**3.237.** Let's assume that $f$ and $g$ are continuous in $[a, b]$, and for all $x \in [a, b]$ $f(x) < g(x)$. Prove that there is an $m > 0$ such that for all $x \in [a, b]$ $g(x) - f(x) \geq m$.

**3.238.** Find a function $f : [0, 1] \to \mathbb{R}$ which is continuous except at one point, and

**(a)** not bounded. **(b)** bounded, but has no maximum.

### Which statement implies the other?

**3.239.** **P:** $f$ is continuous in $[1,2]$ **Q:** $f$ has a maximum and a minimum in $[1,2]$

**3.240.** **P:** $f$ is continuous in $(1,2)$ **Q:** $f$ has a maximum and a minimum in $(1,2)$

**3.241.** **P:** $f$ is bounded in $(1,2)$ **Q:** $f$ has a maximum and a minimum in $(1,2)$

**3.242.** **P:** $f$ is bounded in $[1,2]$ **Q:** $f$ has a maximum and a minimum in $[1,2]$

### Is there a function which is

**3.243.** not continuous in $[0,1]$, but has both a maximum and a minimum in $[0,1]$?

**3.244.** continuous in $(0,1)$, and has both a maximum and a minimum in $(0,1)$?

**3.245.** continuous in $(0,1)$, but has neither a maximum, nor a minimum in $(0,1)$?

**3.246.** continuous in $[0,1]$, but has neither a maximum, nor a minimum in $[0,1]$?

### Have the following functions got a maximum in $[77, 888]$?

**3.247.** $3^{x+5}\sin x + \sqrt{x}$

**3.248.** $\sin(2x) + \cos(3x)$

**3.249.** $[x]$

**3.250.** $\{x\}$

### $D(f)$ is the domain, and $R(f)$ is the range of the function $f$. Is there any function such that

**3.251.** $D(f) = (0, 1)$ and $R(f) = [0, 1]$

**3.252.** $D(f) = [0, 1]$ and $R(f) = (0, 1)$

**3.253.** $D(f) = [0, 1]$ and $R(f) = [3, 4] \cup [5, 6]$

**Is there any monotonically increasing function such that**

**3.254.** $D(f) = (0, 1)$ and $R(f) = [0, 1]$

**3.255.** $D(f) = [0, 1]$ and $R(f) = (0, 1)$

**3.256.** $D(f) = [0, 1]$ and $R(f) = [3, 4] \cup [5, 6]$

**Is there any continuous function such that**

**3.257.** $D(f) = (0, 1)$ and $R(f) = [0, 1]$

**3.258.** $D(f) = [0, 1]$ and $R(f) = (0, 1)$

**3.259.** $D(f) = [0, 1]$ and $R(f) = [3, 4] \cup [5, 6]$

**3.260.** Prove that if a function is continuous in a (bounded) closed interval, then the range of the function is a (bounded) closed interval.

**3.261.** Prove that if $f$ is a continuous function in $\mathbb{R}$, and its limit is 0 both at infinity and minus infinity, then $f$ is bounded!

**3.262.** Prove that if $f$ is a continuous function in $\mathbb{R}$, and its limit is infinity both in infinity and minus infinity, then $f$ has a minimum.

**3.263.** Prove that the equation $x \sin x = 100$ has infinitely many roots!

**At which points are the following functions continuous, or continuous from left or right?**

**3.264.** $[x]$

**3.265.** $[-x]$

**3.266.** $[x]+[-x]$

**3.267.** $[x]-[-x]$

**At which points are the following functions continuous?**

**3.268.** $f(x)=\begin{cases}\cos\dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$

**3.269.** $f(x)=\begin{cases}x\sin\dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$

**Are the following functions uniformly continuous in the given intervals?**

**3.270.** $f(x)=x^2 \qquad (-\infty,\infty),\quad [-2,2],\quad (-2,2)$

**3.271.** $f(x)=\dfrac{1}{x} \qquad (0,\infty),\quad [1,2],\quad (1,2),\quad [1,\infty)$

## Chapter 4

# Differential Calculus and its Applications

**4.1** The function $f$ has a tangent line at point $a$ if and only if $f$ is differentiable at $a$. The equation of the tangent line is

$$y = f'(a)(x - a) + f(a).$$

**4.2** If $f(x)$ is differentiable at $a$, then the function is continuous at $a$.
The converse of the theorem is not true: for example, $f(x) = |x|$ is continuous at 0, but not differentiable at 0!

**4.3** **Derivative rules.** If $f$ and $g$ are differentiable at $a$, then

— for any $c \in \mathbb{R}$ $c \cdot f$ is differentiable at $a$, and

$$(c \cdot f)'(a) = c \cdot f'(a)$$

— $f + g$ is differentiable at $a$, and

$$(f + g)'(a) = f'(a) + g'(a)$$

— $f \cdot g$ is differentiable at $a$, and

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

— if $g(a) \neq 0$, then $\dfrac{f}{g}$ is differentiable at $a$, and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{g^2(a)}$$

**4.4 Chain rule.** If $g$ is differentiable at $a$, and $f$ is differentiable at $g(a)$, then $f \circ g$ is differentiable at $a$, and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

**4.5 Derivative of the inverse function.** If $f$ is continuous and has an inverse in a neighbourhood of the point $a$, and it is differentiable at $a$, and $f'(a) \neq 0$, then $f^{-1}$ is differentiable at $f(a)$, and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

**4.6 Mean value theorems.**

— **Rolle's theorem.** If $f$ is continuous on a closed interval $[a, b]$, and differentiable on the open interval $(a, b)$, and $f(a) = f(b)$, then there exists a $c \in (a, b)$ such that $f'(c) = 0$.

— **Mean value theorem.** If $f$ is continuous on the closed interval $[a, b]$, and differentiable on the open interval $(a, b)$, then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, for any function that is continuous on $[a, b]$, and differentiable on $(a, b)$ there exists a $c \in (a, b)$ such that the secant joining the endpoints of the interval $[a, b]$ is parallel to the tangent at $c$.

— **Cauchy's theorem.** If $f$ and $g$ are continuous on the closed interval $[a, b]$, differentiable on the open interval $(a, b)$, and for any $x \in (a, b)$ $g'(x) \neq 0$, then there exists a $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

— **Basic theorem of antiderivatives.** If $f$ and $g$ are continuous on the closed interval $[a, b]$, differentiable on the open interval $(a, b)$, and if for $\forall x \in (a, b)$ $f'(x) = g'(x)$, then $f - g$ is constant.

**4.7 Darboux's theorem.** If $f$ is differentiable on $(a, b)$, differentiable from the right-hand side at $a$ and from the left hand-side at $b$, then the range of the derivative function $f'(x)$ contains each value between $f'_+(a)$ and $f'_-(b)$.

**4.8 Relationship between monotonicity and derivative.** Let $f(x)$ be continuous on $[a, b]$, and differentiable on $(a, b)$.

— $f(x)$ is monotonically increasing on $[a, b]$ if and only if for all $x \in (a, b)$ $f'(x) \geq 0$.

— If for all $x \in (a, b)$ $f'(x) > 0$, then $f(x)$ is strictly monotonically increasing on $[a, b]$.

The converse of the statement is not true, for example $f(x) = x^3$ is strictly monotonically increasing, but $f'(0) = 0$.

— $f(x)$ is strictly monotonically increasing on $[a, b]$ if and only if for all $x \in (a, b)$ $f'(x) \geq 0$ and for all $a < c < d < b$ $f'(x)$ has only finitely many roots on $(c, d)$.

**4.9 Relationship between local extrema and derivative.** Let's assume that $f(x)$ is differentiable at $a$.

— If $f(x)$ has a local extremum (maximum or minimum) at $a$, then $f'(a) = 0$.

— If $f(x)$ is differentiable in a neighbourhood of $a$, $f'(a) = 0$ and $f'(x)$ changes sign at $a$, then $f(x)$ has a local extremum at $a$, namely
  – (strict) local maximum if $x < a$ implies $(f'(x) > 0)$ $f'(x) \geq 0$ and $x > a$ implies $(f'(x) < 0)$ $f'(x) \leq 0$,
  – (strict) local minimum if $x < a$ implies $(f'(x) < 0)$ $f'(x) \leq 0$ and $x > a$ implies $(f'(x) > 0)$ $f'(x) \geq 0$.

— If $f(x)$ is differentiable two times at $a$, $f'(a) = 0$ and $f''(a) \neq 0$, then $f(x)$ has a local extremum at $a$, namely
  – strict local maximum if $f"(a) < 0$,
  – strict local minimum if $f"(a) > 0$.

**4.10 Relationship between convexity and derivative**. Let's assume that $f(x)$ is differentiable on $(a, b)$.

— $f(x)$ is (strictly) convex on $(a, b)$ if and only if $f'(x)$ is (strictly) monotonically increasing on $(a, b)$.

— $f(x)$ is (strictly) concave on $(a, b)$ if and only if $f'(x)$ is (strictly) monotonically decreasing on $(a, b)$.

— $f(x)$ has an **inflection point** at $c \in (a, b)$ if and only if $f'(x)$ has local extremum at $c$.

**4.11 L'Hospital's rule.** Let's assume that $f$ and $g$ are differentiable in a punctured neighbourhood of $a$, $f$ and $g$ have limits at $a$, and either both limits are 0 or both limits are $\infty$, that is, the limit of the quotient of the two function is critical. In this case **if there exists** the limit $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$, **then also exists** the limit $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$, and

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

This theorem is also valid for one-sided limits or limits at infinity or minus infinity.

## 4.1 The Concept of Derivative

**4.1.** Find the derivative of $\sqrt{x}$ and $\sqrt[3]{x}$ at point $x=a$ using the definition! What is the domain, where are the functions $\sqrt{x}$ and $\sqrt[3]{x}$ continuous, and where are they differentiable? Give the derivatives!

**4.2.** Let's assume that

$$\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}=4.$$

Does it imply that $f$ is continuous at 3?

**4.3.** Let's assume that $f$ is continuous at 3. Does it imply that the limit

$$\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}$$

exists and it is finite?

**Find the following limits!**

**4.4.** $\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

**4.5.** $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

**4.6.** $\lim\limits_{h\to 0}\dfrac{1/(x+h)-1/x}{h}$

**4.7.** $\lim\limits_{h\to 0}\dfrac{(x+h)^3-x^3}{h}$

**4.8.** $\lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0}$

**4.9.** $\lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0}$

**4.10.** $\lim_{x \to x_0} \frac{1/x - 1/x_0}{x - x_0}$

**4.11.** $\lim_{x \to x_0} \frac{x^3 - x_0^3}{x - x_0}$

**Where are the following functions continuous and differentiable?**

**4.12.** $|x|$

**4.13.** $|x^3|$

**4.14.** $|x^2 - 1|$

**4.15.** $|\sqrt[3]{x}|$

**For which values of $b$ and $c$ are the following functions differentiable at 3? Find the derivatives!**

**4.16.** $f(x) = \begin{cases} x & \text{if } x \geq 3 \\ bx^2 - c & \text{if } x < 3 \end{cases}$

**4.17.** $g(x) = \begin{cases} x^2 & \text{if } x \leq 3 \\ b - cx & \text{if } x > 3 \end{cases}$

**4.18.** $h(x) = \begin{cases} (1-x)(2-x) & \text{if } x \geq -3 \\ bx + c & \text{if } x < -3 \end{cases}$

**At which points are the following functions differentiable? At which points are the derivatives continuous?**

**4.19.** $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

**4.20.** $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

**4.21.** $f(x) = |x^3|$

**4.22.** $f(x) = \{x\} \sin \pi x$

**4.23.** $f(x) = [x] \sin^2 \pi x$

**4.24.** $f(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$

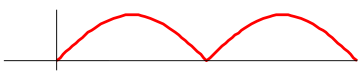
**4.25.** $f(x) = \begin{cases} 1-x & \text{if } x < 1 \\ (1-x)(2-x) & \text{if } 1 \leq x \leq 2 \\ -(2-x) & \text{if } 2 < x \end{cases}$

**4.26.** $f(x) = \left(\{x\} - \dfrac{1}{2}\right)^2$, where $\{x\}$ denotes the fraction part of $x$.

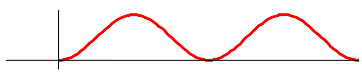
**4.27.** $f(x) = [x]\sin \pi x$, where $[x]$ denotes the integer part of $x$.

**4.28.** Which of the following graph belongs to $f(x) = \sin^2 x$ and which one to $g(x) = |\sin x|$?

**(a)**



**(b)**



**Find the first, second, . . . $n$th derivatives of the following functions!**

**4.29.** $x^6$ **4.30.** $\dfrac{1}{x}$ **4.31.** $\sin x$ **4.32.** $\cos x$

## 4.2 The Rules of the Derivative

**Which statement implies the other?**

**4.33.** **P:** $f$ is even. **Q:** $f'$ is odd.

**4.34.** **P:** $f$ is odd. **Q:** $f'$ is even.

**Let's assume that the function $f$ is differentiable. Which statement implies the other?**

**4.35.** **P:** $f$ is periodic. **Q:** $f'$ is periodic.

**4.36.** **P:** $\lim\limits_{x\to\infty} f(x) = 0$ **Q:** $\lim\limits_{x\to\infty} f'(x) = 0$

**4.37.** **P:** $\lim\limits_{x\to\infty} f(x) = \infty$ **Q:** $\lim\limits_{x\to\infty} f'(x) = \infty$

**4.38.** **P:** $f$ is differentiable at $a$ **Q:** $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a-h)}{2h}$ exists.

**Find the points where the tangent line of $\sin x$ is parallel to the**

**4.39.** $x$-axis;

**4.40.** line $y = x$.

**4.41.** Give the equation of the tangent line of the graph of $\cos x$ at point $x = \dfrac{\pi}{3}$.

**4.42.** Give the equation of the tangent line of the graph of $f(x) = x^3 - 2x^2 + 3x + 4$ at point $(1; 6)$.

**4.43.** Where is the tangent line of the graph of $2x^3 - 6x^2 + 8$ horizontal?

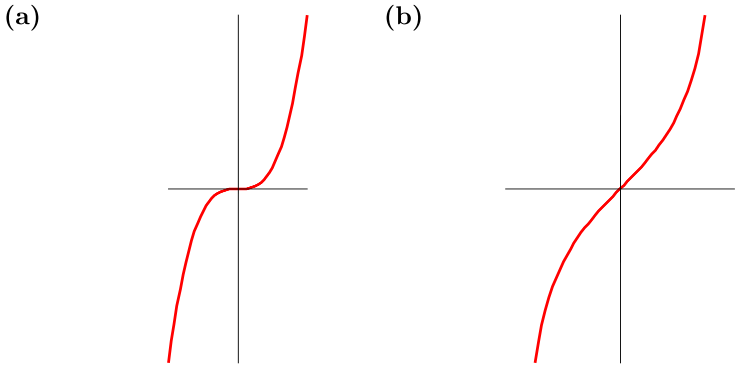
**4.44.** Find those tangent lines of the graph of $2x^3 - 6x^2 + 8$ whose angle with the $x$-axis is 45 degree and 30 degree!

**4.45.** What is the angle between the graph of $x^2$ and the line $y = 2x$, that is, what is the angle between the tangent line and the line $y = 2x$ at the intersection point?

**4.46.** Prove that the curves $x^2 - y^2 = a$ and $xy = b$ are perpendicular, that is, their tangent lines are perpendicular to each other at the intersection point!

**4.47.** Where is the tangent line of the graph of $\sqrt[3]{\sin x}$ vertical?

**4.48.** In the following figure one of the graphs is the graph of $\tan x$, and the other is the graph of $x^3$. Which graph belongs to $\tan x$, and which to $x^3$?




**4.49.** The position-time function of a car is $s(t) = 3t^2 + 5t + 8$. Find the instantaneous velocity at $t = 3$. Find the velocity-time function!

**4.50.** The velocity-time function of a car is $v(t) = 5t + 3$. Find the instantaneous velocity of the car at $t = 7$. Give the acceleration-time function!

**4.51.** The displacement-velocity of vibrating mass point is $y(t) = 5\sin\ t$. Give the velocity-time and acceleration-time functions!

**Find the domain of the following functions! Find the derivatives and the domains of the derivatives as well!**

**4.52.** $3x^8 - \dfrac{3}{4}x^6 + 2$

**4.53.** $\dfrac{5x+3}{2x-1}$

**4.54.** $x + \dfrac{1}{x} + \sqrt{x}$

**4.55.** $3x^2 - \dfrac{\sqrt[3]{x}}{5} + 7$

**4.56.** $\sqrt{x\sqrt{x\sqrt{x}}}$

**4.57.** $\sqrt{x + \sqrt{x + \sqrt{x}}}$

**4.58.** $4\sin x$

**4.59.** $\dfrac{\sin x + \cos x}{3}$

**4.60.** $\sin\left(x^{22}\right)$

**4.61.** $(\sin x)^{22}$

**4.62.** $\dfrac{\sin x - x\cos x}{\cos x + x\sin x}$

**4.63.** $4x^3\tan(x^2+1)$

**4.64.** $\dfrac{\sin x}{\cos x}$

**4.65.** $\sin \dfrac{1}{x}$

**4.66.** $x \sin x$

**4.67.** $(3x^5 + 1)\cos\ x$

**4.68.** $x^x$

**4.69.** $\sqrt[x]{x}$

**4.70.** $x^{-x}$

**4.71.** $e^{\sqrt{x}}$

**4.72.** $\sqrt[x]{x^2+1}$

**4.73.** $e^{-3x^2}$

**4.74.** $\log_4 x$

**4.75.** $\log_x 4$

**4.76.** $\ln(\ln x)$

**4.77.** $\dfrac{1}{2}\ln\dfrac{x+1}{x-1}$

**4.78.** $\ln\left(x + \sqrt{x^2+1}\right), \quad (x > 1)$

**4.79.** $\ln\left(e^x + \sqrt{1+e^{2x^2}}\right)$

**4.80.** $4\sinh x$

**4.81.** $\dfrac{\sinh x + \cosh x}{3}$

**4.82.** $\sinh\left(x^{22}\right)$

**4.83.** $(\sinh x)^{22}$

**4.84.** $\dfrac{\sinh x - x\cosh x}{\cosh x + x\sinh x}$

**4.85.** $4x^3\tanh(x^2+1)$

**4.86.** $\dfrac{\sinh x}{\cosh x}$

**4.87.** $\sinh\dfrac{1}{x}$

**4.88.** $x\sinh x$

**4.89.** $(3x^5+1)\cosh x$

**4.90.** $\log_3 x \cdot \cos x$

**4.91.** $\dfrac{\sin x + 2\ln x}{\sqrt{x}+1}$

**4.92.** $\dfrac{x^2e^x - 3^x\ln x + \cos\pi}{x^2+1}$

**4.93.** $\ln\left(\sin x + \cos^2 x\right)$

**4.94.** $\dfrac{\cos(3^x)+5}{\ln(\sin x)+x^2}$

**4.95.** $(\sin x)^{\cos x}$

**4.96.** $\ln(\sin x)$

**4.97.** $x^{\tan x}$

**Show that the following functions have inverses on an adequate interval, and find the derivatives of the inverses at the given points!**

**4.98.** $x + \sin x, \quad a = 1 + \pi/2$

**4.99.** $3x^3 + x, \quad a = 4$

**4.100.** $f(x) = x^5 + x^2, \quad a = 2$

**4.101.** $-2x^3 + \sqrt{x}, \quad a = -1$

**Find the derivatives of the inverses of the following trigonometric functions!**

**4.102.** $\arcsin x$

**4.103.** $\arccos x$

**4.104.** $\arctan x$

**4.105.** $\operatorname{arccot} x$

**Where are the following functions differentiable, and what are the derivatives?**

**4.106.** $\arcsin(\cos x)$

**4.107.** $\sin(\arccos x)$

**4.108.** $\arctan(\sin x)$

**4.109.** $\tan(\arcsin x)$

**4.110.** $\operatorname{arsinh}(\cosh x)$

**4.111.** $\sinh(\operatorname{arcosh} x)$

**4.112.** $\operatorname{artanh}(\sinh x)$

**4.113.** $\tanh(\operatorname{arsinh} x)$

**Find the following limits if they exist!**

**4.114.** $\lim\limits_{x \to \pi/6} \dfrac{2\sin x - 1}{6x - \pi}$

**4.115.** $\lim\limits_{x \to \pi/2} \dfrac{\sin x - 2/(\pi x)}{\cos x}$

**4.116.** $\lim\limits_{x \to \pi/2} \dfrac{\cos x}{x - \pi/2}$

**4.117.** $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$

**Find the following limits if they exist!**

**4.118.** $\lim_{n\to\infty} n\left(\sqrt{\frac{n+1}{n}}-1\right)$

**4.119.** $\lim_{n\to\infty} n\left(\cos\frac{1}{n}-1\right)$

**4.120.** $\lim_{n\to\infty} ne^{1/n}-1$

**4.121.** $\lim_{n\to\infty} n^2\sin\frac{1}{n}\left(\sqrt[n]{e}-1\right)$

**Find the second derivatives of the following functions!**

**4.122.** $x^3+2x^2+x+1$

**4.123.** $e^{\sin x}$

**4.124.** $\ln\cos x$

**4.125.** $\arctan\frac{1}{x}$

## 4.3 Mean Value Theorems, L'Hospital's Rule

**4.126.** Is there any function whose derivative is $[x]$, that is, the integer part of $x$ on the interval $[-3,5]$?

**4.127.** Is there any function whose derivative is not continuous?

**4.128.** Let's $f(x)=\arctan x$, $\quad g(x)=\arctan\frac{1+x}{1-x}$ and $h(x)=f(x)-g(x)$. Prove that $h'(x)=0$. Does it imply that $h(x)$ is a constant function? Calculate $h(0)$, and the limit $\lim_{x\to\infty} h(x)$. Explain the results!

**4.129.** Let's assume that $f$ and $g$ are differentiable functions, and $f(0)\geq g(0)$, and for all $x\in\mathbb{R}$ $f'(x)>g'(x)$. Prove that $f(x)>g(x)$, if $x>0$.

**How many roots do the following equations have?**

**4.130.** $x^3+2x+4=0$

**4.131.** $x^5-5x+2=0$

**4.132.** $e^x=2x+2$

**4.133.** $\sin x=\frac{x}{2}$

**Find the following limits!**

**4.134.** $\lim_{x\to\infty} x\left(\frac{\pi}{2}-\arctan x\right)$

**4.135.** $\lim_{x\to 0} \frac{x}{\ln(1+x)}$

**4.136.** $\lim_{x\to\pi/2} \frac{1-\sin x}{1+\cos 2x}$

**4.137.** $\lim_{x\to 0^+} \frac{x}{\sin\sqrt{x}}$

**4.138.** $\lim_{x\to 0^+} \frac{\ln x}{\cot x}$

**4.139.** $\lim_{x\to\infty} \frac{x+\ln x}{x+1}$

**4.140.** $\lim_{x\to-\infty} \frac{x+1}{e^{-x}}$

**4.141.** $\lim_{x\to 0} \frac{\sin x - x}{\tan x - x}$

**4.142.** $\lim_{x\to 0} \frac{x\cot x - 1}{x^2}$

**4.143.** $\lim_{x\to 0} \left(\cot x - \frac{1}{x}\right)$

**4.144.** $\lim_{x\to 0} (1+x)^{1/x}$

**4.145.** $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

**4.146.** $\lim_{x\to 0} \left(\frac{1+e^x}{2}\right)^{\cot x}$

**4.147.** $\lim_{x\to\infty} \frac{x^3+x^2+1}{e^x}$

**4.148.** $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

**4.149.** $\lim_{x\to\infty} x\sin\frac{1}{x+1}$

**4.150.** $\lim_{x\to 0} \frac{\cosh x - \cos x}{x^2}$

**4.151.** $\lim_{x\to\infty} \frac{1+\arctan x}{\sinh x + \cosh x}$

**4.152.** Calculate the limit $\lim_{x\to 0} \frac{\sin x}{x+1}$.

**Solution:** Applying L'Hospital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x+1} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

Find the error!

**4.153.** Calculate the limit a $\lim_{x\to 0} \frac{x+1}{2x+2}$.

**Solution:** Applying L'Hospital's rule:

$$\lim_{x\to 0} \frac{x+1}{2x+2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Find the error!

**4.154.** Calculate the limit $\lim\limits_{x\to\infty} \frac{\sin x}{x}$.

**Solution:** Applying L'Hospital's rule:

$$\lim_{x\to\infty} \frac{\sin x}{x} = \lim_{x\to\infty} \frac{\cos x}{1} = \lim_{x\to\infty} \cos x. \text{ The limit does not exist.}$$

Find the error!

## 4.4 Finding Extrema

**Find the extrema of the following functions on the given intervals!**

**4.155.** $x^3 - 12x \qquad [-10;3], \quad [0;3]$

**4.156.** $x^3 + 2x^5 \qquad [-1,4], \quad [2,5], \quad [-7,3]$

**4.157.** For a thin lens the object ($t$) and image ($k$) distances are related by the equation

$$\frac{1}{f} = \frac{1}{t} + \frac{1}{k},$$

where $f$ is the focal length. Find the value of $t$ for the given $f$ such that $t + k$ is maximal or minimal.

**4.158.** At a projectile motion the particle is thrown obliquely from the earth surface. The angle between the initial velocity $v_0$ and the horizontal line is $\alpha$. The length of the motion is

$$\frac{2v_0^2}{g} \sin\alpha\cos\alpha.$$

Find $\alpha$ for the given $v_0$ such that the length of the motion is maximal!

**4.159.** Which of the rectangles has the maximal area in a right-angled, isosceles triangle? And which of them has the maximal perimeter?

We examine only those rectangles, which have two vertices on the hypotenuse of the triangle, and the other vertices are on the legs.

**4.160.** Find the radius $R$ and the height $m$ of a right circular cone with given generatrix $a$ such that the volume of the cone is maximal!

**4.161.** Which of the rectangles with area 16 $cm^2$ has the minimal perimeter? Find the sides of this rectangle!

**4.162.** Why has the square the maximal area among the rectangles with perimeter 8?

**4.163.** What can the area of a right-angled triangle be at most, if the sum of its one leg and its hypotenuse is 10 cm?

**4.164.** One side of a rectangle is on the x-axis, and the two upper vertices are on the parabola $y = 12 - x^2$. Find the maximum of the area of these kinds of rectangles!

**4.165.** We want to make an upper open box from a 8 x 15 dm cardboard, so we cut out congruent squares from the cardboard's corners, and turn up the sides. What are the sides of the box with maximal volume? What is this maximal volume?

**4.166.** The two vertices of a triangle in a coordinate system's first quarter are $(a, 0)$ and $(0, b)$, and the length of the side connecting these two vertices is 20 length unit. Prove that the area of the triangle is maximal, if $a = b$.

**4.167.** There is a farm alongside a river. We want to enclose a rectangle for the animals. One side of the rectangle is the river, and we want to build an electric fence along the other three sides, for which we have 800m length of electric wire. Find the side of the rectangle for the maximal area! What is the maximal area?

**4.168.** Find the $R$ radius and the $m$ height of a cylinder with given volume $V$ such that the surface area of the cylinder is maximal!

**4.169.** We should fence a rectangle plantation, which has 216 $m^2$ area, then we should divide the rectangle to two equal parts with a fence parallel to one of the sides of the rectangle. Find the side of the rectangle such that the length of the fence is minimal. What is the length of the fence in this case?

## 4.5 Examination of Functions

**Let's assume that $f$ is differentiable on $\mathbb{R}$. Which statement implies the other?**

**4.170.** **P:** $f'(a) = 0$ **Q:** $f$ has a local extremum at $a$

**4.171.** **P:** $f'(a) \neq 0$ **Q:** $f$ has no local extremum in $a$

**4.172.** **P:** $f'(x) \geq 0$ on $(3,5)$ **Q:** $f$ is monotonically increasing on $(3,5)$

**4.173.** **P:** $f'(x) > 0$ on $(3,5)$ **Q:** $f$ is strictly increasing on $(3,5)$

**Let's assume that $f$ is two times differentiable on $\mathbb{R}$. Which statement implies the other?**

**4.174.** **P:** $f''(a) > 0$ **Q:** $f$ has a local minimum at $a$

**4.175.** **P:** $f'(a) = 0$ and $f''(a) > 0$ **Q:** $f$ has a local minimum at $a$

**4.176.** **P:** $f''(a) = 0$ **Q:** $f$ has an inflection point at $a$

**4.177.** **P:** $f''(a) = 0$ **Q:** $f$ has an inflection point or local extremum at $a$

**4.178.** **P:** $f''(x) \geq 0$ on $(3,5)$ **Q:** $f$ is convex on $(3,5)$

**4.179.** **P:** $f''(x) > 0$ on $(3,5)$ **Q:** $f$ is strictly convex on $(3,5)$

**On which intervals are the following functions monotonically increasing or decreasing? Find the local extremum as well!**

**4.180.** $f(x) = -x^2 - 3x + 23$

**4.181.** $f(x) = 2x^3 - 18x + 23$

**4.182.** $f(x) = x^4 - 4x^3 + 4x^2 + 23$

**4.183.** $f(x) = x\sqrt{9 - x^2}$

**4.184.** $f(x) = x^2\sqrt{5-x}$

**4.185.** $f(x) = \dfrac{x^2-9}{x-2}$

## Find the local extrema of the following functions and the types of the extrema!

**4.186.** $y = xe^{-x}$

**4.187.** $y = 2 + x - x^2$

**4.188.** $y = x^3 - 6x^2 + 9x - 4$

**4.189.** $y = x + \sin x$

## Analyze the following functions!

**4.190.** $f(x) = x + \dfrac{1}{x}$

**4.191.** $f(x) = x - \dfrac{1}{x^2}$

**4.192.** $f(x) = \dfrac{1}{1+x^2}$

**4.193.** $f(x) = \dfrac{x}{1+x^2}$

**4.194.** $f(x) = \dfrac{x+1}{1+x^2}$

**4.195.** $f(x) = \dfrac{x^3}{x^2+1}$

**4.196.** $f(x) = \dfrac{x^3}{x^2-1}$

**4.197.** $f(x) = x + \dfrac{2x}{x^2-1}$

**4.198.** $f(x) = \dfrac{1}{x^2} - \dfrac{1}{(x-1^2)}$

**4.199.** $f(x) = \dfrac{x\sqrt{1-x}}{1+x}$

## Plot the graphs of the following functions!

**4.200.** $f(x) = 6 - 2x - x^2$

**4.201.** $f(x) = x^3 - 3x + 3$

**4.202.** $f(x) = x(6-2x)^2$

**4.203.** $f(x) = 1 - 9x - 6x^2 - x^3$

**4.204.** $f(x) = (x-2)^3 + 1$

**4.205.** $f(x) = 1 - (x+1)^3$

## 4.6 Elementary Functions

**Analyze the following functions!**

**4.206.** $x^{2n}, \quad n \in \mathbb{N}^+$

**4.207.** $x^{2n+1}, \quad n \in \mathbb{N}^+$

**4.208.** $x^{-2n}, \quad n \in \mathbb{N}^+$

**4.209.** $x^{-(2n+1)}, \quad n \in \mathbb{N}^+$

**4.210.** $\sqrt[2n]{x}, \quad n \in \mathbb{N}^+$

**4.211.** $\sqrt[2n+1]{x}, \quad n \in \mathbb{N}^+$

**4.212.** $a^x, \quad (a > 1)$

**4.213.** $a^x, \quad (0 < a < 1)$

**4.214.** $\log_a x, \quad (a > 1)$

**4.215.** $\log_a x, \quad (0 < a < 1)$

**4.216.** $\sin x$

**4.217.** $\tan x$

**4.218.** $\arcsin x$

**4.219.** $\arctan x$

**4.220.** $\sinh x = \dfrac{e^x - e^{-x}}{2}$

**4.221.** $\cosh x = \dfrac{e^x + e^{-x}}{2}$

**4.222.** $\operatorname{arsinh} x$

**4.223.** $\operatorname{arcosh} x$

**Find the following values!**

**4.224.** $2^{\frac{\ln 100}{\ln 2}}$

**4.225.** $\left(\dfrac{1}{9}\right)^{-\log_3 7}$

**4.226.** $\arcsin \dfrac{\sqrt{3}}{2}$

**4.227.** $\arctan(-1)$

**4.228.** $\arccos(\cos(9\pi))$

**4.229.** $\sin\left(\arcsin \dfrac{1}{3}\right)$

**4.230.** $\tan(\arctan 100)$

**4.231.** $\arcsin(\sin 3)$

**Are the following functions periodic? If "yes", find a period!**

**4.232.** $\tan(10x)$

**4.233.** $\cot(\pi x)$

**4.234.** $\sin\dfrac{x}{5}$

**4.235.** $\cos\dfrac{x}{2} + \tan\dfrac{x}{3}$

**4.236.** There are the graphs of some functions below. Find the formulas belonging to the graphs!

$\sinh x$, $\cosh x$, $e^x$, $e^{-x}$, $\log_3 x$, $\log_{0,5} x$, $\ln(-x)$, $x^3$, $x^{-3}$

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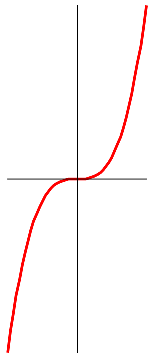

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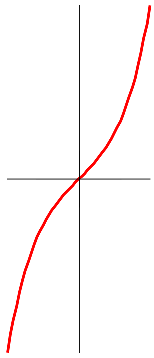

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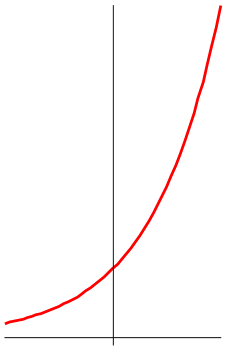

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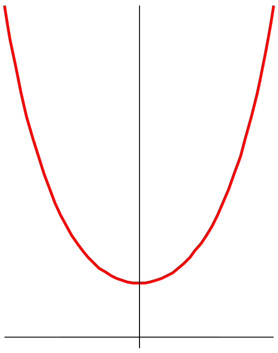

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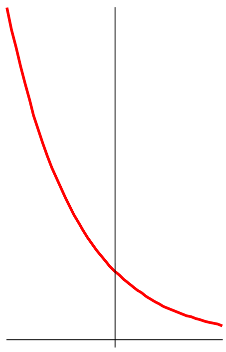

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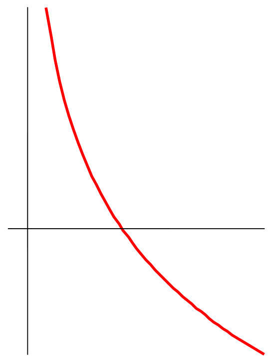
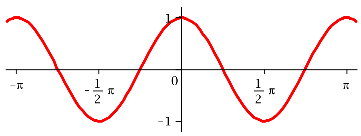


**4.237.** There are the graphs of some functions below. Find the formulas belonging to the graphs!

$$\sin x,\quad \cos 2x,\quad 2\sin x,\quad \sin(x-2),\quad -\cos x,\quad \tan x,\quad \cot x,$$

$$\sin^2 x,\quad |\cos x|$$

**(a)**

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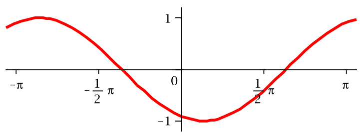


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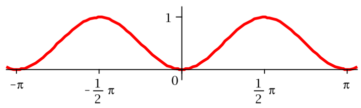


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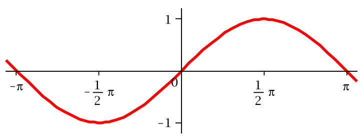

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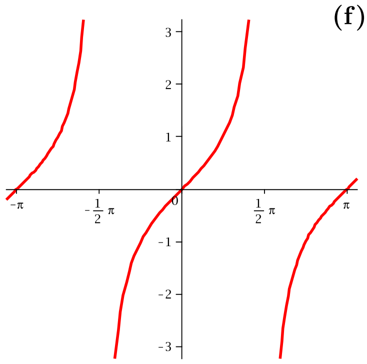


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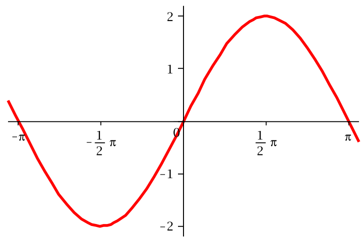
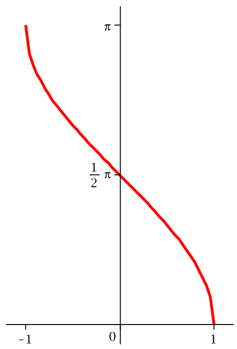



**4.238.** There are the graphs of some functions below. Find the formulas belonging to the graphs!

$$\arcsin x, \quad \arccos x, \quad \arctan x, \quad \operatorname{arccot} x$$

$$\sin(\arcsin x), \quad \arcsin(\sin x), \quad \tan(\arctan x), \quad \arctan(\tan x)$$

**(a)**

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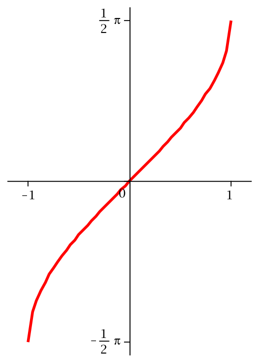


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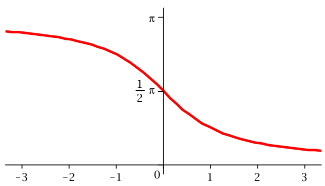


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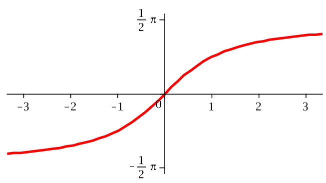

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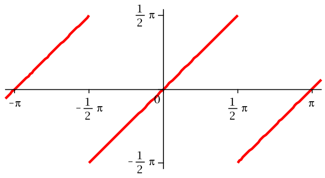

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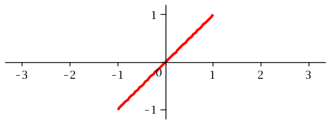

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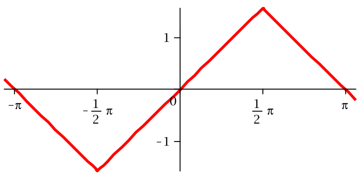

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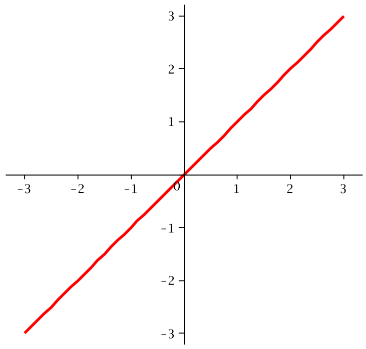
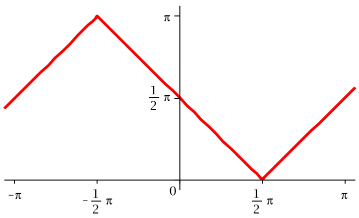


**4.239.** There are the graphs of some functions below. Find the formulas belonging to the graphs!

$$\sin(\arccos x),\quad \arccos(\sin x),\quad \cos(\arcsin x),\quad \arcsin(\cos x)$$

$$\tan(\operatorname{arccot} x),\quad \operatorname{arccot}(\tan x),\quad \cot(\arctan x),\quad \arctan(\cot x)$$

**(a)**



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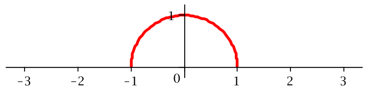

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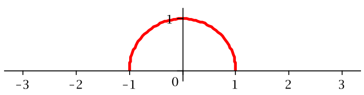
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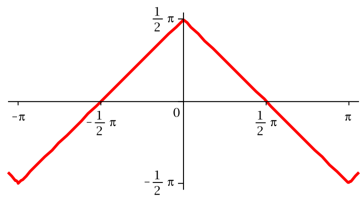
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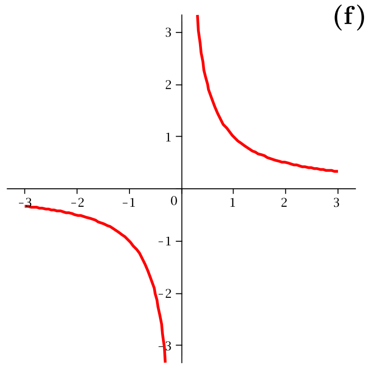
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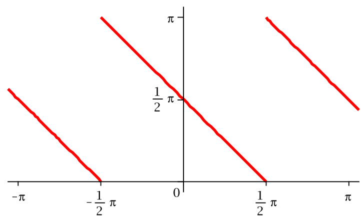
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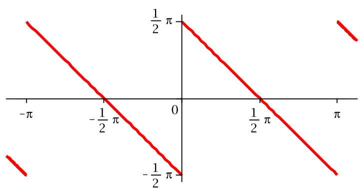
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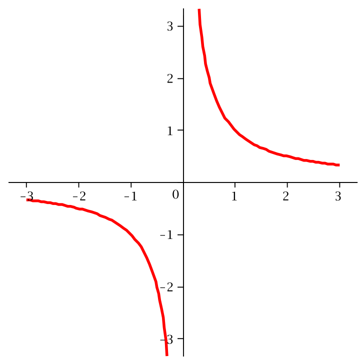
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# Chapter 5

# Riemann Integral

## 5.1 Basic antiderivatives.

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1} + C \quad (\alpha \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C \quad (a \neq 1) \qquad \int e^x\,dx = e^x + C$$

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + C \qquad \int \frac{1}{\sin^2 x}\,dx = -\cot x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \cosh x\,dx = \sinh x + C \qquad \int \sinh x\,dx = \cosh x + C$$

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \operatorname{arsinh} x + C \qquad \int \frac{1}{\sqrt{x^2-1}}\,dx = \operatorname{arcosh} x + C$$

## 5.2 Rules of integration

— If $f$ and $g$ have primitive functions, then $f+g$ and $c \cdot f$ have one, too, namely

$$\int (f+g) = \int f + \int g, \qquad \int c \cdot f = c \int f.$$

— If $F$ is a primitive function of $f$, then for all $a, b \in \mathbb{R},\ a \neq 0$

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C.$$

— If $f$ is differentiable and positive everywhere, then

$$\int f^{\alpha}(x)f'(x)\,dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + C \quad (\alpha \neq -1).$$

— If $f$ is differentiable and not equal to zero at any point, then

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

— **Integration by parts:**

If $f$ and $g$ are differentiable and $fg'$ has a primitive function, then $f'g$ has a primitive function, too, and

$$\int f'g = fg - \int fg'.$$

— **Integration by substitution:**

If $g(x)$ is differentiable, and the range of $g$ is part of the domain of $f$, and $f$ has primitive function there, then the composite function $f(g(x)) \cdot g'(x)$ has a primitive function, and

$$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C,$$

where $F(y)$ is one of the primitive functions of $f(y)$.

**5.3 The fundamental theorem of calculus (Newton-Leibniz formula).** If $f$ is integrable on the closed interval $[a,b]$, has a primitive function $F$ on the open interval $(a,b)$, and continuous on the closed interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

.

**5.4 Integral transform.** If $f$ is continuous on $[a,b]$, $g : [c,d] \to [a,b]$ is continuously differentiable, and $g(c) = a$ and $g(d) = b$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(t))g'(t)\,dt.$$

The advantage of the formula above is that we don't have to know the inverse of $g$, it is enough to determine the points $c$ and $d$.

## 5.5 Applications of integration.

— **The area below of the graph of a function.** If $f : [a, b] \to \mathbb{R}$ is integrable and not negative at any points, then the domain

$$A = \{(x; y) : x \in [a, b], 0 \le y \le f(x)\}$$

has an area and

$$t(A) = \int_a^b f(x)\,dx.$$

— **Area between two functions.** If $f$ and $g$ are integrable functions on $[a, b]$, and for all $x \in [a, b]$ $g(x) \ge f(x)$, then the domain between the two graphs

$$N = \{(x; y) : x \in [a, b], f(x) \le y \le g(x)\}$$

has an area, and

$$t(N) = \int_a^b (g(x) - f(x))\,dx.$$

— **Polar integration: area of a sector-like domain.** If the domain $S$ is bounded by the curve given by the polar equation $r = r(\varphi)$, $\varphi \in [\alpha, \beta]$ and by two lines connecting the origin and the endpoints of the curve, and $r(\varphi)$ is integrable, then $S$ has an area, and

$$t(S) = \frac{1}{2}\int_\alpha^\beta r^2(\varphi)\,d\varphi.$$

— **Length of the function graph.** If the function $f$ is continuously differentiable on $[a, b]$, then the graph of the function has a length, and

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

— **Volume of a solid of revolution.** If $f : [a, b] \to \mathbb{R}$ is integrable and not negative at any points, then the solid of revolution

$$A = \left\{(x; y; z) : a \le x \le b,\ y^2 + z^2 \le f^2(x)\right\}$$

has a volume, and

$$V = \pi \int\limits_a^b f^2(x)\,dx$$

**5.6 Comparison test.** If $f$ and $g$ are integrable on every closed and bounded subinterval of $[a,\infty)$, $x \in [a,\infty)$ implies $|f(x)| \le g(x)$, and $\int\limits_a^\infty g(x)\,dx$ is convergent, then $\int\limits_a^\infty f(x)\,dx$ is (absolute) convergent.

**5.7 Limit comparison test.** If $f$ and $g$ are positive and integrable functions on every bounded and closed subinterval of $[a,\infty)$, the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

exists and $0 < L < \infty$, then the improper integrals

$$\int\limits_a^\infty f(x)\,dx \qquad \text{and} \qquad \int\limits_a^\infty g(x)\,dx$$

are both convergent or both divergent.

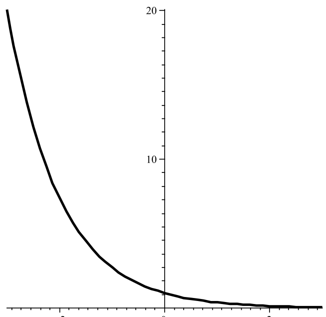
The criteria of convergence above can be worded for improper integrals on bounded intervals, too.

## 5.1 Indefinite Integral

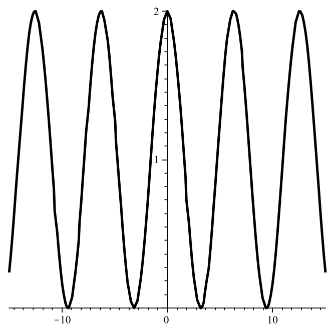
**5.1.** There are functions on the left-hand side column, and their derivatives on the right-hand side column. Find the corresponding pairs!

$$x^3 - 4x, \quad x^3, \quad x + \sin x, \quad \tan x, \quad e^{-x}$$

(1)




(A)

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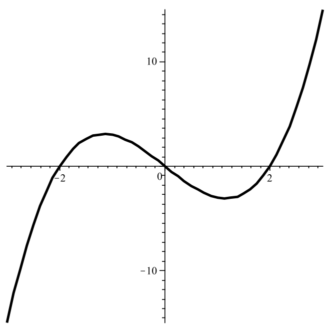
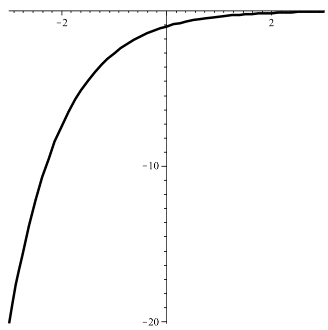
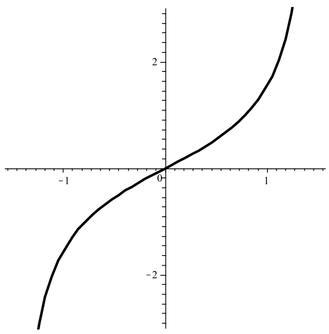



(B)




(3)

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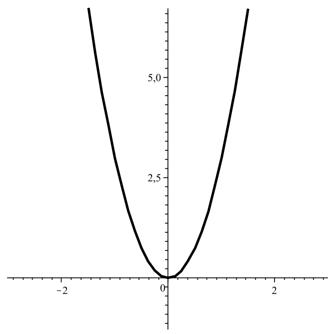

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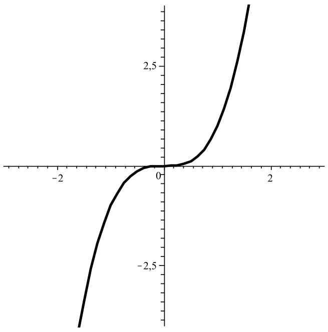


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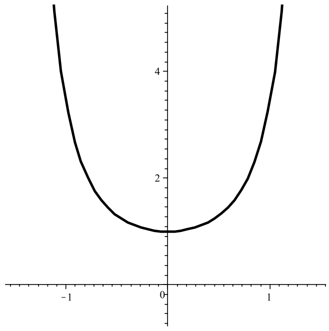
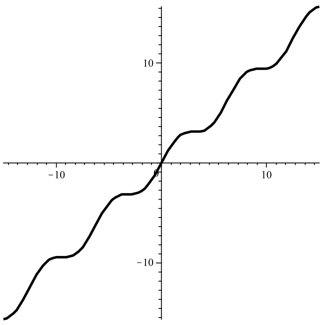
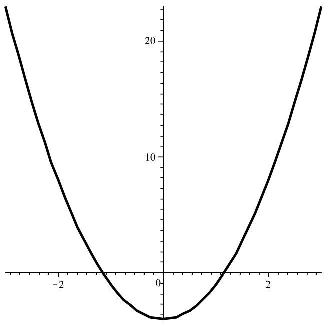



(5)




(E)




**Find the following indefinite integrals!**

**5.2.** $\displaystyle\int \sin(x+3)\ dx$

**5.3.** $\displaystyle\int (1+x+5x^2)\ dx$

**5.4.** $\displaystyle\int \sqrt{x+2}\ dx$

**5.5.** $\displaystyle\int (\sin x+\cos x)^2\ dx$

**5.6.** $\displaystyle\int \frac{1}{(x+2)^3}\ dx$

**5.7.** $\displaystyle\int \mathrm{e}^{2x+3}\ dx$

**Find the following indefinite integrals by using the basic antiderivatives, or linear substitution! Always verify the result by derivation!**

**5.8.** $\displaystyle\int x^{3/2}\,dx$

**5.9.** $\displaystyle\int \left(\sqrt{x}+\frac{1}{x}\right)\,dx$

**5.10.** $\displaystyle\int \frac{-5}{x-7}\,dx$

**5.11.** $\displaystyle\int \frac{x^5-3x^3+x-2}{x^2}\,dx$

**5.12.** $\displaystyle\int \sin 2x + 3\cos x\,dx$

**5.13.** $\displaystyle\int 2^x\,dx$

**5.14.** $\displaystyle\int e^{2x-3}\,dx$

**5.15.** $\displaystyle\int e^{-x}+3\cos\frac{x}{2}\,dx$

**Find the following integrals applying the formulas for the special forms $\displaystyle\int \frac{f'}{f}$ or $\displaystyle\int f^{\alpha} f'$. Always verify the result by derivation!**

**5.16.** $\displaystyle\int \frac{x^2}{x^3+1}\,dx$

**5.17.** $\displaystyle\int \ln^2 x\cdot\frac{1}{x}\,dx$

**5.18.** $\displaystyle\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx$

**5.19.** $\displaystyle\int x\sqrt{x^2+1}\,dx$

**5.20.** $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$

**5.21.** $\displaystyle\int \frac{1}{x\ln x}\,dx$

**5.22.** $\displaystyle\int \frac{1}{(1+x^2)\arctan x}\,dx$

**5.23.** $\displaystyle\int \frac{\sin x}{\cos^{2013} x}\,dx$

**There are quadratic expressions in the following denominators. Solve the integrals by partial fraction decomposition or the form $\dfrac{f'}{f}$, or by using the basic antiderivative formula for $\displaystyle\int\frac{1}{1+x^2}\,dx$. Always verify the result by derivation!**

**5.24.** $\displaystyle\int \frac{1}{2+x^2}\,dx$

**5.25.** $\displaystyle\int \frac{4}{5+6x^2}\,dx$

**5.26.** $\displaystyle\int \frac{1}{4-x^2}\,dx$

**5.27.** $\displaystyle\int \frac{1}{4-9x^2}\,dx$

**5.28.** $\displaystyle\int \frac{3}{(x+3)(x+2)}dx$

**5.29.** $\displaystyle\int \frac{3}{(x+3)(2-x)}dx$

**5.30.** $\displaystyle\int \frac{1}{10-x^2}dx$

**5.31.** $\displaystyle\int \frac{1}{10+x^2}dx$

**5.32.** $\displaystyle\int \frac{x-2}{x^2-2x+6}dx$

**5.33.** $\displaystyle\int \frac{x}{x^2-2x+6}dx$

**5.34.** $\displaystyle\int \frac{x^2+1}{x^2-1}dx$

**5.35.** $\displaystyle\int \frac{x^2-1}{x^2+1}dx$

**5.36.** $\displaystyle\int \frac{3x^3+2x-1}{x^2-x-6}dx$

**5.37.** $\displaystyle\int \frac{3x^2+2x-1}{x^2-2x+6}dx$

**5.38.** $\displaystyle\int \frac{x^4+2}{x^2+1}dx$

**5.39.** $\displaystyle\int \frac{x^4-2}{x^2-1}dx$

**Integrate the following rational fraction functions!**

**5.40.** $\displaystyle\int \frac{1}{x^3+x^2}\,dx$

**5.41.** $\displaystyle\int \frac{1}{x^3+x}\,dx$

**5.42.** $\displaystyle\int \frac{x}{x^2-2x+5}\,dx$

**5.43.** $\displaystyle\int \frac{x}{x^2-2x-3}\,dx$

**5.44.** $\displaystyle\int \frac{x+2}{x-1}\,dx$

**5.45.** $\displaystyle\int \frac{x^2+2}{x-1}\,dx$

**5.46.** $\displaystyle\int \frac{x^2-2}{x+1}\,dx$

**5.47.** $\displaystyle\int \frac{3x^2+2}{x^3+2x}\,dx$

**5.48.** $\displaystyle\int \frac{x+2}{x^2+2x+2}\,dx$

**5.49.** $\displaystyle\int \frac{2}{(x-1)^2}\,dx$

**Find the following indefinite integrals using the basic antiderivatives recognizing the special forms $\displaystyle\int \frac{f'}{f}$ or $\displaystyle\int f'f^a$, or using trigonometric formulas! Always verify the results by derivation!**

**5.50.** $\displaystyle\int \sin^2 x\,dx$

**5.51.** $\displaystyle\int \cos^2 2x\,dx$

**5.52.** $\displaystyle\int \frac{3}{\cos^2 x}\,dx$

**5.53.** $\displaystyle\int \frac{4}{\sin^2(3x-5)}\,dx$

**5.54.** $\displaystyle\int \frac{5}{\cos^2(1-x)}\,dx$

**5.55.** $\displaystyle\int \tan x\,dx$

**5.56.** $\displaystyle\int \frac{(\sin x+\cos x)^2}{\sin^2 x}\,dx$

**5.57.** $\displaystyle\int \frac{(\sin x-\cos x)^2}{\cos^2 x}\,dx$

**5.58.** $\displaystyle\int \frac{\sin^2 2x+1}{\cos^2 x}\,dx$

**5.59.** $\displaystyle\int \sqrt{1+\sin x}\,dx$

**Find the following indefinite integrals by using integration by parts!**

**5.60.** $\displaystyle\int x\cos x\,dx$

**5.61.** $\displaystyle\int x^2\sin x\,\,dx$

**5.62.** $\displaystyle\int e^x\sin x\,\,dx$

**5.63.** $\displaystyle\int x^2e^{-x}\,dx$

**5.64.** $\displaystyle\int \arctan x\,dx$

**5.65.** $\displaystyle\int x\arctan x\,dx$

**5.66.** $\displaystyle\int x\sinh x\,dx$

**5.67.** $\displaystyle\int x\ln^2 x\,dx$

**5.68.** $\displaystyle\int x\ln\frac{1+x}{1-x}\,dx$

**5.69.** $\displaystyle\int \sin 3x\cdot\cos 4x\,dx$

**5.70.** Is the following integration by parts correct? If yes, then $0 = 1$?

$$\int \frac{1}{x}\cdot\frac{1}{\ln x}\,dx = \ln x\cdot\frac{1}{\ln x} - \int \ln x\cdot\frac{-\frac{1}{x}}{\ln^2 x}\,dx = 1 + \int \frac{1}{x}\cdot\frac{1}{\ln x}\,dx$$

**Find the following indefinite integrals by the given substitutions!**

**5.71.** $\displaystyle\int xe^{x^2}\,dx,\qquad t = x^2$

**5.72.** $\displaystyle\int x^2\sqrt{x^3+1}\,dx,$ $\quad t=\sqrt{x^3+1}$

**5.73.** $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx,$ $\quad x=\sin t$

**5.74.** $\displaystyle\int \frac{1}{\sqrt{x(1-x)}}\,dx,$ $\quad x=\sin^2 t$

**5.75.** $\displaystyle\int \frac{1}{\sin x}\,dx,$ $\quad t=\cos x$

**5.76.** $\displaystyle\int \frac{1}{1+\cos^2 x}\,dx,$ $\quad t=\tan x$

**5.77.** $\displaystyle\int \frac{1}{2^x+4^x}\,dx,$ $\quad t=2^x$

**5.78.** $\displaystyle\int \frac{1}{1+\sin^2 x}\,dx,$ $\quad t=\tan x$

**5.79.** $\displaystyle\int \frac{1}{(1-x^2)\sqrt{1-x^2}}\,dx,$ $\quad x=\sin t$

**5.80.** $\displaystyle\int \frac{1}{(1+x^2)\sqrt{1+x^2}}\,dx,$ $\quad x=\tan t$

**5.81.** $\displaystyle\int \frac{1}{(1+x^2)\sqrt{1+x^2}}\,dx,$ $\quad x=\sinh t$

**5.82.** $\displaystyle\int \frac{1}{(x+1)^2(x-2)^3}\,dx,$ $\quad t=\frac{x+1}{x-2}$

**Find the following indefinite integrals by substitutions!**

**5.83.** $\displaystyle\int (2x+1)\mathrm{e}^{x^2+x+1}\,dx$

**5.84.** $\displaystyle\int \cos x\,\mathrm{e}^{\sin x}\,dx$

**5.85.** $\displaystyle\int \frac{dx}{\sqrt{x(1-x)}}$

**5.86.** $\displaystyle\int \frac{x}{(x^2+1)^2}\,dx$

**5.87.** $\displaystyle\int \frac{e^x+2}{e^x+e^{2x}}\,dx$

**5.88.** $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx$

**5.89.** $\displaystyle\int \frac{2^x+3}{2^x+2^{2x}}\,dx$

**5.90.** $\displaystyle\int \frac{1}{\cos x}\,dx$

**Find the following indefinite integrals!**

**5.91.** $\displaystyle\int \frac{1}{(x+2)(x-1)}\,dx$

**5.92.** $\displaystyle\int \frac{x+1}{x^2+x+1}\,dx$

**5.93.** $\displaystyle\int \cos^3 x \sin^2 x\ dx$

**5.94.** $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

**5.95.** $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx$

**5.96.** $\displaystyle\int x\sin(x^2+1)\ dx$

**5.97.** $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

**5.98.** $\displaystyle\int \ln x\ dx$

**5.99.** $\displaystyle\int \ln(x^2+1)\ dx$

**5.100.** $\displaystyle\int \frac{1}{(1+x^2)\arctan x}\,dx$

**5.101.** $\displaystyle\int \frac{1}{1+\cos x}\,dx$

**5.102.** $\displaystyle\int \frac{\arctan x}{1+x^2}\,dx$

**5.103.** $\displaystyle\int 2x\sin(x^2+1)\,dx$

**5.104.** $\displaystyle\int e^{-3x}-2\sin(3x-2)\,dx$

**5.105.** $\displaystyle\int (2x+2)(x^2+2x)^{222}\,dx$

**5.106.** $\displaystyle\int \cot x\,dx$

**5.107.** $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx$

**5.108.** $\displaystyle\int \sqrt{2-x}\,dx$

**5.109.** $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

**5.110.** $\displaystyle\int x^2\ln x\,dx$

**5.111.** $\displaystyle\int (x^2+x)\ln x\,dx$

**5.112.** $\displaystyle\int \left(3\sqrt[4]{x}+2\right)\ dx$

**5.113.** $\displaystyle\int \sin x\cdot\cos^{3000} x\,dx$

**5.114.** $\displaystyle\int \frac{1}{x\ln x}\,dx$

**5.115.** $\displaystyle\int 3^{-2x}\,dx$

**5.116.** $\displaystyle\int \sqrt{2-x^2}\,dx$

**5.117.** $\int \cos x \cdot e^{\sin x}\, dx$

**5.118.** $\int \frac{\cos^3 x}{\sin^4 x}\, dx$

**5.119.** $\int \frac{1}{\sin^2 x \cos^4 x}\, dx$

**5.120.** $\int \frac{\cos^5 x}{\sin^3 x}\, dx$

**5.121.** $\int \frac{1}{\cos^4 x}\, dx$

**5.122.** $\int \cos x \sin^2 x\, dx$

## 5.2 Definite Integral

**5.123.** Decide whether the given points are a partition of the interval $[-2, 4]$ according to the definition of Riemann integral? If yes, find the norm of the partition!

**(a)** $x_0 = -2, x_1 = -1, x_2 = 0, x_3 = \frac{1}{2}, x_4 = 4$

**(b)** $x_0 = -1, x_1 = 2, x_2 = 4$

**(c)** $x_0 = -2, x_1 = 4$

**(d)** $x_0 = -2, x_1 = x_0 + 1, x_2 = x_1 + \frac{1}{2}, \ldots, x_n = x_{n-1} + \frac{1}{n}$

**(e)** $x_0 = -2, x_1 = -1, 5, x_2 = 3, x_3 = 4$

**5.124.** Are the following partitions refinements of each other in the interval $[-2, 4]$?

**(a)** F={-2; -1; 0; 4} $\Phi = \{-2; -1; 0; \frac{1}{2}; 3; 4\}$

**(b)** F={-2; -1.5; 3; 4} $\Phi = \{-2; -1; 0; 3; 4\}$

**Find the lower and the upper Riemann sums of the following function with the given partition of the interval $[-2, 4]$!**

**5.125.** $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ , $\Phi = \{-2, 1, 5, 4\}$

**5.126.** $x^2$, $\Phi = \{-2, -1, 0, 4\}$

**5.127.** $x^2$, $\Phi = \{-2, 4\}$

**5.128.** $[x]$, $\Phi = \{-2, -1, 0, 4\}$ **5.129.** $[x]$, $\Phi = \{-2, 1, 5, 4\}$

**5.130.** Find the sets of the lower and upper Riemann sums of the function

$$f(x) = \begin{cases} 0 \text{ if } 0 \le x < \dfrac{1}{2} \\ 1 \text{ if } \dfrac{1}{2} \le x \le 1 \end{cases}$$

in the interval $[0, 1]$. Give the supremum of the lower sums, and the infimum of the upper sums!

**Are the following functions Riemann integrable on the given $I$ intervals?**

**5.131.** $f(x) = 5$ $I = [0, 1]$ **5.132.** $f(x) = -4$ $I = (-1, 2)$

**5.133.** $f(x) = [x]$ $I = [2, 4]$ **5.134.** $f(x) = |x|$ $I = [-2, 1]$

**5.135.** $f(x) = \begin{cases} \dfrac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ $I = [0, 1]$

**5.136.** $D(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$ $I = [3, 5]$

**5.137.** $g(x) = \begin{cases} x \text{ if } x \in \mathbb{Q} \\ -x \text{ if } x \notin \mathbb{Q} \end{cases}$ $I = [3, 5]$

**5.138.** Prove that if for all $x \in [a, b]$ imply that $m \le f(x) \le M$, and $f$ is integrable on $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

**5.139.** Prove that if for all $x \in [a, b]$ imply that $f(x) \leq g(x)$, and $f$ and $g$ are integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

**Prove the following inequalities!**

**5.140.** $0 \leq \int_1^2 \frac{1}{x^2 + e^x}\,dx \leq 1$ **5.141.** $1 \leq \int_4^5 \sqrt{\ln x + 0,2}\,dx \leq 2$

**5.142.** Let

$$f(t) = \operatorname{sgn} t \text{ and } G(x) = \int_{-6}^{x} f(t)\,dt \quad (x > -6).$$

Find the values of $G(-4), G(0), G(1), G(6)$! Find the derivative of $G(x)$.

**5.143.** Let

$$f(t) = \begin{cases} 1 \text{ if } t = \frac{1}{n} (n \in \mathbb{N}^+) \\ 0 \text{ otherwise} \end{cases} \quad \text{and } G(x) = \int_0^x f(t)\,dt \quad (x > 0).$$

Find the functions $G(x)$ and $G'(x)$.

**5.144.** Can the function $\operatorname{sgn} x$ be the integral function of any functions on $[-1, 1]$? Has the function $\operatorname{sgn} x$ got a primitive function on $(-1, 1)$?

**5.145.** Let

$$F(x) = \begin{cases} x^2 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases} \quad \text{and } g(x) = \begin{cases} F'(x) \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}.$$

Does the function $g$ have a primitive function? Is $g$ integrable? Is $g$ differentiable?

**Find the limits of the following sequences!**

**5.146.** $\lim\limits_{n\to\infty} \dfrac{\sin\frac{1}{n} + \sin\frac{2}{n} + \cdots + \sin\frac{n}{n}}{n}$

**5.147.** $\lim\limits_{n\to\infty} \dfrac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n\sqrt{n}}$

**5.148.** $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{1} + \sqrt[3]{2} + \cdots + \sqrt[3]{n}}{n\sqrt[3]{n}}$

**5.149.** $\lim\limits_{n\to\infty} n \sum\limits_{i=1}^{n} \left(\dfrac{i}{n}\right)^2$

**5.150.** $\lim\limits_{n\to\infty} n \sum\limits_{i=1}^{n} \dfrac{i}{n^2 + i^2}$

**5.151.** $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(\ln \sqrt[n]{n+i} - \ln \sqrt[n]{n}\right)$

**5.152.** Let $f$ be a bounded function on $[0,1]$, and assume that

$$\lim_{n\to\infty} \frac{f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)}{n} = 5.$$

Does it imply that $f$ is integrable on $[0,1]$, and $\int\limits_0^1 f(x)\,dx = 5$?

**5.153.** Which statement implies the other?

**P:** $f$ is integrable on $[a,b]$. **Q:** $|f|$ is integrable on $[a,b]$.

**5.154.** Find the derivative of

$$G(x) = \begin{cases} x^2 \sin\frac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Using this result prove that the function

$$f(x) = \begin{cases} \cos \frac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

has a primitive function!

**5.155.** We know that the function

$$f(x) = \begin{cases} \cos \frac{1}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

has a primitive function. Can the function

$$g(x) = \begin{cases} \cos \frac{1}{x} \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}$$

have a primitive function?

**Find the derivatives of the following functions!**

**5.156.** $H(x) = \int_2^x \frac{1}{\ln t}\, dt.$

**5.157.** $L(x) = \int_2^{\cosh x} \frac{1}{\ln t}\, dt.$

**Find the following limits!**

**5.158.** $\lim_{x\to\infty} \frac{\ln x}{x} \int_2^x \frac{1}{\ln t}\, dt$

**5.159.** $\lim_{x\to\infty} \frac{\ln x}{x} \int_2^{\cosh x} \frac{1}{\ln t}\, dt$

**Let $f$ be integrable on $[-a, a]$. Prove that if $f$ is**

**5.160.** even, then $\int_{-a}^{a} f(x)\, dx = 2 \int_0^a f(x)\, dx,$

**5.161.** odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

**Calculate the following definite integrals!**

**5.162.** $\displaystyle\int_{2}^{3} x^2\,dx$

**5.163.** $\displaystyle\int_{4}^{6} 4^{5x+6}\,dx$

**5.164.** $\displaystyle\int_{0}^{\pi} \sin x\,dx$

**5.165.** $\displaystyle\int_{3}^{4} x^2 \ln x\,dx$

**5.166.** $\displaystyle\int_{-2\pi}^{0} \sin^2 x\,dx$

**5.167.** $\displaystyle\int_{-2}^{3} \frac{3x^4+4x^3-2x+1}{x^2+1}\,dx$

**Calculate the following definite integrals by applying the given substitutions!**

**5.168.** $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx, \quad x = \sin t$

**5.169.** $\displaystyle\int_{2}^{5} x^2\sqrt{x^3+1}\,dx, \quad t = \sqrt{x^3+1}$

**Calculate the following definite integrals by substitutions!**

**5.170.** $\displaystyle\int_{1}^{2} \frac{3^x+2}{3^x+3^{2x}}\,dx$

**5.171.** $\displaystyle\int_{0}^{1} \arctan x \cdot \sqrt{x}\,dx$

**5.172.** $\displaystyle\int_{\pi/6}^{\pi/2} \frac{1}{1+\tan x}\,dx$

**5.173.** $\displaystyle\int_{1}^{2} \frac{5^{2x}}{1+5^x}\,dx$

**5.174.** Calculate the integral $\int\limits_{-2}^{2} \sin^9 x \cdot e^{x^4}\, dx$.

## 5.3 Applications of the Integration

**5.175.** Find the area below the graph of the function $-x^2+3$.

**5.176.** Find the area below one "hump" of the graph of the function $\sin^2 x$.

**Find the area between the graphs of $f$ and $g$.**

**5.177.** $f(x) = x^2, \qquad g(x) = -x+2$

**5.178.** $f(x) = -x^2+2x, \qquad g(x) = -x$

**5.179.** $f(x) = \sqrt{1-x^2}, \qquad g(x) = 0$

**5.180.** $f(x) = \sqrt{1-x^2}, \qquad g(x) = -x$

**Find the area between the curves!**

**5.181.** the $x$ axis, the graph of $\ln x$ and the line $x = e$

**5.182.** the $x$ axis, the graph of $\tan x$ and the line $x = \pi/4$

**5.183.** the $y$ axis, the graph of $x^2$ and the line $y = 3$

**5.184.** the $y$ axis, the graph of $e^x$ and the line $y = 5$

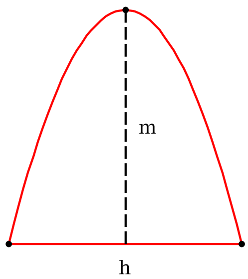
**5.185.** the graph of $\dfrac{1}{1+x^2}$ and the graph of $\dfrac{x^2}{2}$

**5.186.** the graph of $x^2$, the line $y = x$ and the line $y = -x+1$

**5.187.** the graph of $x^2$, the graph of $2x^2$ and the line $y = x$

**5.188.** the graph of $\dfrac{1}{x}$, the line $y = x$ and the line $y = -2x + 4.5$

**5.189.** Prove that the segment of the parabola on the figure given below, which has height $m$, and chord $h$, has the area $T = \dfrac{2}{3}mh$.




**Find the area bounded by the following curves given with polar coordinates!**

**5.190.** $r = \cos\varphi$

**5.191.** $r = \cos 2\varphi$

**5.192.** $r = \cos 3\varphi$

**5.193.** $r = \dfrac{1}{1 + \frac{1}{2}\cos\varphi}, \quad 0 \le \varphi \le \dfrac{\pi}{2}$

**Rotate the graphs of the following functions around the $x$-axis! Find the volume of the solids of revolution!**

**5.194.** $e^{-x} \quad I = [0, 1]$

**5.195.** $\sqrt{x} \quad I = [0, 1]$

**5.196.** $\sin x \quad I = [0, \pi]$

**5.197.** $\dfrac{1}{x} \quad I = [1, 4]$

**Rotate the graphs of the following functions around the $y$-axis! Find the volume of the solids of revolution!**

**5.198.** $e^{-x} \quad x \in [0, 1]$

**5.199.** $\sqrt{x} \quad x \in [0, 1]$

**5.200.** $\sin x \quad x \in [0, \pi/2]$

**5.201.** $\frac{1}{x} \quad x \in [1, 4]$

**Rotate the graphs of $f$ and $g$ around the $x$-axis on the intervals $[x_1, x_2]$, then calculate the volume of the solids of revolution bounded by the rotated graphs and the planes $x = x_1$ and $x = x_2$.**

**5.202.** $f(x) = -x^2 + 4 \quad g(x) = -2x^2 + 8 \quad x_1 = -2 \quad x_2 = 2$

**5.203.** $f(x) = \sin x \quad g(x) = -4x^4 + 4 \quad x_1 = 0 \quad x_2 = 1$

**5.204.** $f(x) = e^x \quad g(x) = \frac{1}{x} \quad x_1 = 1 \quad x_2 = 2$

**5.205.** $f(x) = \ln x \quad g(x) = \cosh x \quad x_1 = 1 \quad x_2 = 2$

**Find the arc length of the graph of the following functions on the given intervals!**

**5.206.** $\sqrt{x}, \quad I = [0, 1]$

**5.207.** $\ln x, \quad I = [\sqrt{3}, \sqrt{8}]$

**5.208.** $\cosh x, \quad I = [-1, 1]$

**5.209.** $x^{3/2} \quad I = [0, 4]$

**Find the displacement of the mass-point moving along a line in the given time intervals, if the velocity-time function of the point is**

**5.210.** $v(t) = 5t^2 \quad t \in [0, 2]$

**5.211.** $v(t) = 3\sin 2t \quad t \in [0, 2\pi]$

**A mass-point is moved on the $x$-axes from the point $x_1$ to the point $x_2$ by the force $F(x)$ codirectional with the $x$-axis. What is the work of the force?**

**5.212.** $F(x) = 2x, \quad [x_1, x_2] = [0, 2]$

**5.213.** $F(x) = 3\sin x, \quad [x_1, x_2] = [0, \pi/4]$

**There is a 1 m length rod on the $x$-axis with its left endpoint at the origin. What is the mass of the rod, if its density at distance $x$ from the origin is $\varrho(x)$?**

**5.214.** $\varrho(x) = 2 + x$

**5.215.** $\varrho(x) = 2 + \dfrac{x^2}{1000}$

**5.216.** Find the centers of mass of the rods in the previous problem!

**5.217.** Find the moment of inertia around the $y$-axis of the rods in the previous problems!

**5.218.** How much is needed to move an electric charge $q$ on the $x$-axis from the point $x_0$ to the point $2x_0$, if there is a fixed electric charge $Q$ at the origin?

**5.219.** There is a rod with length $l$, mass $m$ on the $x$-axis with left endpoint at the origin. What is the force between the rod and a mass point $M$ at the point $(0, y_0)$?

## 5.4 Improper integral

**5.220.** For which value of $c$ is the improper integral $\displaystyle\int_1^\infty \frac{1}{x^c}\,dx$ convergent?

**5.221.** For what $c$ is the improper integral $\displaystyle\int_0^1 \frac{1}{x^c}\,dx$ convergent?

**5.222.** Which statement implies that the improper integral $\int_1^\infty f(x)\,dx$ is convergent? Which statement implies that the improper integral $\int_1^\infty f(x)\,dx$ is divergent?

**(a)** $\forall x \in [1, \infty) \quad |f(x)| < \dfrac{1}{x^2}$ **(b)** $f(x) > \dfrac{1}{x^3}$

**(c)** $\forall x \in [1, \infty) \quad |f(x)| > \dfrac{1}{x}$ **(d)** $\forall x \in [1, \infty) \quad |f(x)| < \dfrac{1}{x}$

**Are the following improper integrals convergent? If yes, find the value of the integrals!**

**5.223.** $\displaystyle\int_3^{\infty} 2^{-x}\,dx$

**5.224.** $\displaystyle\int_0^1 \frac{dx}{x-1}$

**5.225.** $\displaystyle\int_2^{\infty} \frac{dx}{x^3}$

**5.226.** $\displaystyle\int_0^7 \frac{dx}{x^3}$

**5.227.** $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x}}$

**5.228.** $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

**5.229.** $\displaystyle\int_1^2 \frac{dx}{x\ln x}$

**5.230.** $\displaystyle\int_2^{\infty} \frac{dx}{x\ln x}$

**5.231.** $\displaystyle\int_{\frac{1}{2}}^1 \frac{dx}{x\ln x}$

**5.232.** $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{x\ln x}$

**5.233.** $\displaystyle\int_1^{\infty} \frac{dx}{x+\sqrt{x}}$

**5.234.** $\displaystyle\int_0^1 \frac{dx}{x+\sqrt{x}}$

**5.235.** $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

**5.236.** $\displaystyle\int_0^1 \ln x\,dx$

**5.237.** $\displaystyle\int_0^{\pi/2} \tan x\,dx$

**5.238.** $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

**Which of the following integrals are convergent, absolute convergent or divergent?**

**5.239.** $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x}+x^2}$

**5.240.** $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}+x^2}$

**5.241.** $\displaystyle\int_2^{\infty} \frac{dx}{x\ln^2 x}\,dx$

**5.242.** $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x+x^3}}$

**5.243.** $\displaystyle\int_{0}^{+\infty} \frac{x^2}{x^4 - x^2 + 1}\, dx$

**5.244.** $\displaystyle\int_{1}^{+\infty} \frac{x+1}{\sqrt{x^4+1}}\, dx$

**5.245.** $\displaystyle\int_{1}^{\infty} \frac{x+1}{x^3+x}\, dx$

**5.246.** $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{x+x^3}}$

**5.247.** $\displaystyle\int_{0}^{\pi/2} \frac{dx}{\sin x}$

**5.248.** $\displaystyle\int_{0}^{\infty} e^{-x^2}\, dx$

**5.249.** $\displaystyle\int_{1}^{\infty} \frac{\cos x}{x^2}\, dx$

**5.250.** $\displaystyle\int_{1}^{\infty} \frac{\ln x}{x^2}\, dx$

**5.251.** $\displaystyle\int_{0}^{\infty} x e^{-x^2}\, dx$

**5.252.** $\displaystyle\int_{0}^{\infty} x^2 e^{-x^2}\, dx$

**5.253.** $\displaystyle\int_{0}^{+\infty} \frac{dx}{x^2}$

**5.254.** $\displaystyle\int_{1}^{+\infty} \frac{\cos x}{x^2}\, dx$

# Chapter 6

# Numerical Series

## 6.1 Convergence criteria.

— If the sum $\sum_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$.

— **Cauchy convergence test for series.** The sum $\sum_{n=1}^{\infty} a_n$ is convergent if and only if

$$\forall \varepsilon > 0 \; \exists n_0 \; \forall n, m \; \left( n_0 \leq n \leq m \implies \left| \sum_{k=n}^{m} a_k \right| < \varepsilon \right).$$

— **Direct comparison test.** If $n \in \mathbb{N}^+$ implies $|a_n| \leq b_n$ except for finitely many $n$ and the sum $\sum_{n=1}^{\infty} b_n$ is convergent, then the sum $\sum_{n=1}^{\infty} a_n$ is absolute convergent.

— **Limit comparison test**. If the terms of the sums $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive, and there exists the limit $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $0 < c < \infty$, then the two series are both convergent or both divergent.

— **Ratio test.** If the limit $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists and equals to $q$, then $q < 1$ implies that $\sum_{n=1}^{\infty} a_n$ is absolute convergent, $q > 1$ implies that the it is divergent. In case of $q = 1$ the ratio test is inconclusive, and the series may converge or diverge.

— **Root test.** If the limit $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ exists and equal to $q$, then $q<1$ implies that $\sum\limits_{n=1}^{\infty} a_n$ is absolute convergent, $q>1$ implies that the series is divergent. If $q=1$, the root test is inconclusive, and the series may converge or diverge.

— **Integral test.** If $f$ is a monotonically decreasing function on the half-line $[1,\infty)$, then the infinite series $\sum\limits_{n=1}^{\infty} f(n)$ and the improper integral $\int\limits_{1}^{\infty} f(x)\,dx$ are both convergent or both divergent.

— **Leibniz's test or alternating series test.** The sum $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ is a *Leibniz-sum*, if the sequence $(a_n)$ **decreases monotonically** to zero. The Leibniz-sums are convergent.

## 6.1 Convergence of Numerical Series

**Write down the $n$th partial sum of the following series! Find the limits of the partial sums! Find the sums of the series!**

**6.1.** $1+\dfrac{1}{2}+\dfrac{1}{4}+\cdots+\dfrac{1}{2^k}+\cdots$

**6.2.** $1+\dfrac{1}{10}+\dfrac{1}{100}+\cdots+\dfrac{1}{10^k}+\cdots$

**6.3.** $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{k\cdot(k+1)}+\cdots$

**6.4.** $\dfrac{1}{1\cdot 4}+\dfrac{1}{4\cdot 7}+\cdots+\dfrac{1}{(3k-2)\cdot(3k+1)}+\cdots$

**6.5.** $\dfrac{1}{1\cdot 4}+\dfrac{1}{2\cdot 5}+\cdots+\dfrac{1}{k\cdot(k+3)}+\cdots$

**Find the sum of the following series, if they are convergent!**

**6.6.** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{9^n}$

**6.7.** $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{9^n}$

**6.8.** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n+5^n}{9^n}$

**6.9.** $\displaystyle\sum_{n=1}^{\infty} \frac{2\cdot 4^n}{(-9)^n}$

**6.10.** $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{4^n+5^n}$

**6.11.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{5^n}$

**6.12.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{10^n}$

**6.13.** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(-3)^n}$

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**6.14.** Is the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n$ convergent?

**6.15.** Prove that if $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$ (see the criteria of convergence about the terms converge to 0).

**6.16.** Prove that the **harmonic series** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent!

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**Can the convergence or the sum of an infinite series change, if we**

**6.17.** insert some new parentheses into the series?

**6.18.** eliminate some parentheses from the series?

**6.19.** insert finitely many new terms into the series?

**6.20.** delete finitely many terms from the series?

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**Are the following statements true?**

**6.21.** If $a_n \to 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

**6.22.** If $\sum_{n=1}^{\infty} a_n$ is convergent, then $a_n \to 0$.

**6.23.** If $a_n \to 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

**6.24.** If $\sum_{n=1}^{\infty} a_n$ is divergent, then $a_n \to 1$.

**What are the limits of the sequences of the terms of the following series? Are these series convergent?**

**6.25.** $\sum_{n=1}^{\infty} \frac{1}{n+1}$

**6.26.** $\sum_{n=1}^{\infty} \frac{1}{n^2}$

**6.27.** $\sum_{n=1}^{\infty} \frac{1}{n+n^2}$

**6.28.** $\sum_{n=1}^{\infty} \frac{2}{n^2}$

**6.29.** $\sum_{n=1}^{\infty} \frac{1}{3n}$

**6.30.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

**6.31.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

**6.32.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{3}}$

**6.33.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{0,01}}$

**6.34.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$

**6.35.** $\sum_{n=1}^{\infty} \sin(n\pi)$

**6.36.** $\sum_{n=1}^{\infty} \cos(n\pi)$

**6.37.** Prove that if for all positive integers $n$ $a_n > 0$, $a_n \leq b_n \leq c_n$, $\sum_{n=1}^{\infty} a_n = 7$ and $\sum_{n=1}^{\infty} c_n = 10$, then $\sum_{n=1}^{\infty} b_n$ is convergent!

**6.38.** Prove that if for all positive integers $n$ $a_n > 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent or equals to infinity.

**6.39.** Let's assume that $\sum_{n=1}^{\infty} a_n$ is convergent, and that for all positive integers $n$ $b_n < a_n$. Does it imply that $\sum_{n=1}^{\infty} b_n$ is convergent?

**6.40.** Let's assume that $\sum_{n=1}^{\infty} a_n$ is divergent, and that for all integers $n$ $b_n > a_n$. Does it imply that $\sum_{n=1}^{\infty} b_n$ is divergent?

## 6.2 Convergence Tests for Series with Positive Terms

**Decide whether the following series are convergent by applying the ratio test!**

**6.41.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+4}$

**6.42.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$

**6.43.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

**6.44.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

**6.45.** Let's assume that for all positive integers $n$ $a_n > 0$. What can we say about the convergence of $\sum_{n=1}^{\infty} a_n$ for sure, if

**(a)** $\displaystyle\lim \frac{a_{n+1}}{a_n} > 1$, **(b)** $\displaystyle\lim \frac{a_{n+1}}{a_n} < 1$, **(c)** $\displaystyle\lim \frac{a_{n+1}}{a_n} = 1$?

**Decide whether the following series are convergent by applying the ratio test!**

**6.46.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

**6.47.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$

**6.48.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$

**6.49.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

**6.50.** Let's assume that for all positive integers $n$ $a_n > 0$. What can we say about the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ for sure, if

**(a)** $\lim \sqrt[n]{a_n} > 1$ **(b)** $\lim \sqrt[n]{a_n} < 1$ **(c)** $\lim \sqrt[n]{a_n} = 1$

**Decide whether the following series are convergent by applying the root test!**

**6.51.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

**6.52.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n}$

**6.53.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}+\frac{1}{n}\right)^n$

**6.54.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{2}-\frac{1}{n}\right)^n$

**6.55.** Let's assume that $b_n \neq 0 \quad (n = 1, 2, 3, \dots)$, and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$. What can we say about the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ for sure, if

**(a)** $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent. **(b)** $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent.

**6.56.** Let's assume that $b_n \neq 0 \quad (n = 1, 2, 3, \dots)$, and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. What can we say about the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ for sure, if

**(a)** $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent. **(b)** $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent.

**6.57.** Let's assume that $b_n \neq 0 \quad (n = 1, 2, 3, \dots)$, and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$. What can we say about the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ for sure, if

**(a)** $\sum_{n=1}^{\infty} b_n$ is convergent. **(b)** $\sum_{n=1}^{\infty} b_n$ is divergent.

**Decide whether the following series are convergent by applying the limit comparison test!**

**6.58.** $\sum_{n=1}^{\infty} \frac{n^2+4}{n^4+3n}$

**6.59.** $\sum_{n=1}^{\infty} \frac{2n^3}{n^2+3}$

**6.60.** $\sum_{n=1}^{\infty} \frac{\sqrt{2n^6}}{n^2+3}$

**6.61.** $\sum_{n=1}^{\infty} \frac{\sqrt[3]{3n^9}}{\sqrt{n^4}+3}$

**6.62.** Let's assume that $\int_1^{\infty} f(x)\,dx = 5$. Does it imply that $\sum_{n=2}^{\infty} f(n) = 5$?

**6.63.** Let's assume that the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. Does it imply that $\sum_{n=2}^{\infty} f(n)$ is convergent?

**6.64.** Let's assume that $f(x) > 0$, and $f(x)$ is monotonically decreasing on $(1,\infty)$, and
$\int_1^{\infty} f(x)\,dx = 5$. Does it imply that $\sum_{n=2}^{\infty} f(n) = 5$?

**6.65.** Let's assume that $f(x) > 0$, $f(x)$ is monotonically decreasing on $(1,\infty)$, and the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. Does it imply that $\sum_{n=2}^{\infty} f(n)$ is convergent?

**Decide whether the following series are convergent by applying the integral test!**

**6.66.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

**6.67.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{4/5}}$

**Decide whether the following series are convergent!**

**6.68.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

**6.69.** $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2+\sqrt{n}}$

**6.70.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

**6.71.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[n]{n}}$

**6.72.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

**6.73.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sqrt[n]{\frac{1}{2}}$

**6.74.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

**6.75.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2} - \frac{1}{n}\right)^n$

**6.76.** $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n$

**6.77.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

**6.78.** $\displaystyle\sum_{n=1}^{\infty} \frac{1000^n}{n!}$

**6.79.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

**6.80.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{3^n - 2^n}$

**6.81.** $\displaystyle\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n$

**6.82.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(-\frac{1}{n}\right)^n$

**6.83.** $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{3}{2}\right)^n$

**6.84.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+200}{2n+7}\right)^n$

**6.85.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^5+3}{n^3-n+2}$

**6.86.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{5^n}$

**6.87.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sqrt[n]{\frac{1}{2}}$

**6.88.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$

**6.89.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$

**6.90.** $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)$

**6.91.** $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n}$

**6.92.** $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$

**6.93.** $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n}$

**6.94.** $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^3}{3^n}$

**6.95.** $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt[n]{n^2+1}}$

**6.96.** $\displaystyle\sum_{n=1}^{\infty}n(n+5)$

**6.97.** $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$

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**6.98.** Let's assume that $\displaystyle\sum_{n=1}^{\infty}a_n = A \in \mathbb{R}$ and $\displaystyle\sum_{n=1}^{\infty}b_n = B \in \mathbb{R}$, and $c$ is a real number. Does it imply the following statements?

**(a)** $\displaystyle\sum_{n=1}^{\infty}c\cdot a_n = c\cdot A$

**(b)** $\displaystyle\sum_{n=1}^{\infty}(a_n+b_n) = A+B$

**(c)** $\displaystyle\sum_{n=1}^{\infty}(a_n-b_n) = A-B$

**(d)** $\displaystyle\sum_{n=1}^{\infty}(a_nb_n) = AB$

**(e)** $\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{b_n} = \frac{A}{B} \quad (b_n \neq 0)$

**(f)** $\displaystyle\sum_{n=1}^{\infty}|a_n| = |A|$

## 6.3 Conditional and Absolute Converge

**6.99.** Is that true that if the terms are alternating in a series, then the series is convergent?

**6.100.** Is that true that if the sequence of the terms of an infinite series is monotonically decreasing, then the series is convergent?

**6.101.** Is that true that if the terms of a series are alternating, and the sequence of the absolute values of the terms is monotonically decreasing, then the series is convergent?

**6.102.** Is that true that if the sequence of the terms of an infinite series monotonically decreasingly converges converges to 0, then the series is convergent?

**6.103.** Is that true that if the terms of a series are alternating, and the sequence of the absolute values of the terms converges to 0, then the series is convergent?

**Which series are Leibniz series? Which series are convergent?**

**6.104.** $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

**6.105.** $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$

**6.106.** $1-\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}-\frac{1}{5\cdot 7}+\cdots$

**6.107.** $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$

**6.108.** $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt[3]{3}}-\frac{1}{\sqrt[4]{4}}+\cdots$

**6.109.** $1-\frac{1}{2\cdot 3}+\frac{1}{4\cdot 9}-\frac{1}{8\cdot 27}+\cdots$

**Are the following series convergent? Are the following series absolute convergent?**

**6.110.** $\sum_{n=1}^{\infty}(-1)^n$

**6.111.** $\sum_{n=1}^{\infty}(-2)^n$

**6.112.** $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n$

**6.113.** $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$

**6.114.** $\sum_{n=1}^{\infty}(-1)^n\frac{1}{2n+1}$

**6.115.** $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n\cdot(n+1)}$

**6.116.** $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^3}$

**6.117.** $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt[5]{n}}$

**6.118.** $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{1,3}}$

**6.119.** $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{0,3}}$

**6.120.** $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$

**6.121.** $\displaystyle\sum_{n=1}^{\infty}\frac{(\sin n)^2}{n^2}$

**6.122.** Can the convergence or the sum of an infinite series with positive terms change, if

**(a)** we insert some new parentheses into the series?

**(b)** we delete some parentheses from the series?

**(c)** we rearrange the sequence of the terms?

## Chapter 7

# Sequences of Functions and Function Series

**7.1 Properties of uniform convergence.**

— **Continuity.** The limit function of a uniformly convergent sequence of continuous functions is continuous.

— **Differentiability.** If a sequence of differentiable functions $f_n$ converges pointwise to $f$, and $f'_n$ converges uniformly to the function $g$ on the interval $(a,b)$, then $f$ is differentiable on $(a,b)$ and $f' = g$, that is

$$\lim_{n\to\infty} f'_n = \left(\lim_{n\to\infty} f_n\right)' = f'.$$

— **Integrability.** If a sequence of integrable functions $f_n$ converges uniformly to the function $f$ on the interval $[a,b]$, then $f$ is integrable, and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,dx = \int_a^b f(x)\,dx.$$

**7.2 Weierstrass criterium for the uniform convergence.** If the function series $\sum_{n=1}^{\infty} f_n(x)$ has a **convergent numerical majorant** on the set $H$, that is, there is a sequence $(M_n)$ such that for all $n \in \mathbb{N}$ and $x \in H$ implies that

$$|f_n(x)| \le M_n \quad \text{and} \quad \sum_{n=1}^{\infty} M_n < \infty,$$

then the function series $\sum_{n=1}^{\infty} f_n(x)$ is absolute and uniformly convergent on $H$.

## 7.3 Range of the convergence of power series.

— The range of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is an interval which (perhaps except the endpoints) is symmetric to the center of the power series $c$.

The theorem above includes the cases in which the range of convergence is the whole number line, or only the point $c$.

— **Radius of convergence.** If $\limsup \sqrt[n]{|a_n|} = L$ or $\limsup \dfrac{|a_{n+1}|}{|a_n|} = L$, then the radius of convergence is

$$R = \begin{cases} \dfrac{1}{L} & \text{if } 0 < L < \infty \\ \infty & \text{if } L = 0 \\ 0, & \text{if } L = \infty \end{cases}$$

— The convergence of a power series is uniform on any closed interval in the interior of the range of convergence, therefore the power series can be integrated and derived by terms.

**7.4** If $f$ is differentiable any times at point $c$, then we call the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

**Taylor series** of the function $f$ around the point $c$, and the coefficients

$$a_n = \frac{f^{(n)}(c)}{n!}$$

are called **Taylor coefficients**.

**7.5 Lagrange-form of the reminder.** Let $f$ be differentiable $n+1$ times on the interval $[c,x]$. Then there is a number $d \in (c,x)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(d)}{(n+1)!}(x-c)^{n+1}.$$

If $f$ is differentiable $n+1$ times on $[x,c]$, then there is a number $d \in (x,c)$ such that the equation above holds.

**7.6** A function with uniformly bounded derivatives is equal to its Taylor series, that is $f$ is differentiable on $(a, b)$ at any times, and there is a number $M$ such that for any $n \in \mathbb{N}$ and $x \in (a, b)$ $\left|f^{(n)}(x)\right| \leq M$, then for all $c \in (a, b)$, the Taylor series around $c$ equals the function on the whole interval $(a, b)$.

**7.7 Taylor series of common functions.**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{if } x \in \mathbb{R};$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
if $x \in \mathbb{R}$;

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$
if $x \in \mathbb{R}$;

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$
if $x \in \mathbb{R}$;

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$
if $x \in \mathbb{R}$;

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$
if $|x| < 1$ ( **Geometric series**);

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$
if $|x| < 1$;

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$
if $|x| < 1$.

**7.8 Fourier series.**

— If $f : \mathbb{R} \to \mathbb{R}$ is periodic with period $2\pi$, and the function is integrable on $[0, 2\pi]$, then the trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called **Fourier series** of the function $f$, where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

are the **Fourier coefficients** of $f$.

— **Cantor's theorem.** If $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ on $[0, 2\pi]$, then the coefficient $a_0$, $a_n$ ĂŠs $b_n$ are defined uniquely, that is, a function has only one Fourier series. If the convergence is uniform, then $f(x)$ is continuous, and it is the Fourier series of the function $f$.

— **Pointwise convergence.** If $f$ is periodic with period $2\pi$, and $f$ is piecewise continuously differentiable, then the Fourier series of $f$ is convergent everywhere, and it is equal to $f$ at the points where $f$ is continuous, and the Fourier series equals to the arithmetic mean of the left-hand side and right-hand side limits at the points where $f$ is not continuous, that is, for all $x \in \mathbb{R}$

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \frac{f(x^+) + f(x^-)}{2}.$$

## 7.1 Pointwise and Uniform Convergence

**Where are the following function sequences convergent, and what are their pointwise limits? On which intervals are the convergences uniform?**

**7.1.** $f_n(x) = x^n$

**7.2.** $f_n(x) = x^n - x^{n+1}$

**7.3.** $f_n(x) = \dfrac{x}{n}$

**7.4.** $f_n(x) = \dfrac{x^n}{n!}$

**7.5.** $f_n(x) = \sqrt[n]{1 + x^{2n}}$

**7.6.** $f_n(x) = \sqrt[n]{|x|}$

**7.7.** $f_n(x) = \sqrt{x + \dfrac{1}{n}}$

**7.8.** $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$

**7.9.** $f_n(x) = \dfrac{\sin x}{n}$

**7.10.** $f_n(x) = \sin \dfrac{x}{n}$

**7.11.** $f_n(x) = \begin{cases} 1, & \text{if } x = \dfrac{1}{n} \\ 0, & \text{otherwise} \end{cases}$

**7.12.** $f_n(x) = \begin{cases} 1 \text{ if } 0 < x < \dfrac{1}{n} \\ 0 \text{ otherwise} \end{cases}$

## Which statement implies the other?

**7.13.** **P:** $\forall x \in [a,b] \quad \lim\limits_{n\to\infty} f_n(x) = 5$ **Q:** $\forall n \in \mathbb{N} \quad \lim\limits_{x\to\infty} f_n(x) = 5$

**7.14.** **P:** $\forall x \in [a,b] \quad \lim\limits_{n\to\infty} f_n(x) = f(x)$ **Q:** $\forall x \in [a,b] \quad \lim\limits_{n\to\infty} (f_n(x) - f(x)) = 0$

**7.15.** **P:** $f_n$ is pointwisely convergent on $I$. **Q:** $f_n$ is uniformly convergent on $I$.

**7.16.** Prove that the sequence $f_n(x) = \cos\, nx$ is convergent at the points $x = 2k\pi \quad (x \in \mathbb{Z})$. Is the sequence convergent at points $x = k\pi \quad (x \in \mathbb{Z})$?

**7.17.** Give 3 different examples of sequences of functions such that all of three sequences converge to the constant function 5 on $[0,1]$.

**7.18.** Are there any sequences of functions such that they converge to the constant function 5 on $[0,1]$, and any functions of the sequence are not continuous on $[0,1]$?

**7.19.** Give an example of a sequence of functions such that every term is discontinuous at every point of $[0,1]$, but the sequence of functions converges uniformly to a continuous function on $[0,1]$.

**7.20.** Give an example of a sequence of functions such that all terms are continuous at every point, and the sequence converges to

$$f(x) = \begin{cases} 1 & \text{if } x = 5 \\ 0 & \text{otherwise} \end{cases}$$

Can the convergence be uniform?

**7.21.** Give 3 different sequences of functions such that all 3 sequences converge uniformly to the Dirichlet-function

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

on $[0,1]$. Can all of the terms of the sequence be continuous functions?

**7.22.** Let $f_n(x) = n^2(x^{n-1} - x^n)$. Prove that

**(a)** $\forall x \in [0,1] \quad \lim\limits_{n\to\infty} f_n(x) = 0$

**(b)** $\lim\limits_{n\to\infty} \int\limits_0^1 f_n(x)\, dx \neq 0$

**(c)** $f_n$ does not converge uniformly to the constant function 0 on $[0,1]$.

**7.23.** Let $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$. Prove that

**(a)** $\forall n \quad f_n(x)$ is differentiable at 0.

**(b)** $\lim\limits_{n\to\infty} f_n(x) = |x|$

**(c)** $f_n$ converges uniformly to $|x|$.

**(d)** $|x|$ is not differentiable at 0.

**7.24.** What is the limit function of $f_n(x) = x^n - x^{n+1}$ on $[0,1]$? Is the convergence uniform?

**Prove that the following function series are uniformly convergent by applying the Weierstrass criterium, that is, find a convergent numerical series of whose terms are larger than the absolute values of the terms of the function series!**

**7.25.** $\sum\limits_{n=1}^{\infty} \frac{1}{n^2 + n^4 x^{2n}}$

**7.26.** $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} \sin^n x$

**Are the following function series convergent or uniformly convergent on $\mathbb{R}$?**

**7.27.** $\sum\limits_{n=1}^{\infty} \frac{1}{x^2 + n^2}$

**7.28.** $\sum\limits_{n=1}^{\infty} (x^n - x^{n-1})$

**7.29.** $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{x^4 + 2^n}$

**7.30.** $\sum\limits_{n=1}^{\infty} (\arctan(n+1)x - \arctan\,(nx))$

**7.31.** $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n!}$

**7.32.** $\sum\limits_{n=1}^{\infty} 2^n x^n$

**7.33.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2[1+(nx)^2]}$

**7.34.** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$

**7.35.** $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n!+2^n}$

**7.36.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^n}$

**7.37.** $\sum_{n=1}^{\infty} \frac{x^4}{x^4+2^n}$

**7.38.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{x^6+3^n}$

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**7.39.** Where is the function series $\sum_{n=1}^{\infty} x^n$ convergent? Is the convergence uniform at the range of convergence?

**7.40.** Is $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$ uniformly convergent?

## 7.2 Power Series, Taylor Series

**7.41.** Is the power series $\displaystyle\sum_{n=0}^{\infty}(2^n+3^n)x^n$ convergent at the points $x = \frac{1}{6}$, $x = 0.3$ $x = \frac{1}{2}$, $x = 1$?

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**For what $x$ are the following series convergent?**

**7.42.** $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n}x^n$

**7.43.** $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n^2}$

**7.44.** $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2} x^n$

**7.45.** $\displaystyle\sum_{n=1}^{\infty} nx^n$

**7.46.** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

**7.47.** $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^7}$

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**Find the radius of convergence of the following power series!**

**7.48.** $\sum_{n=0}^{\infty} n^2 x^n$

**7.49.** $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

**7.50.** $\sum_{n=1}^{\infty} \frac{1000^n}{n!}(x-2)^n$

**7.51.** $\sum_{n=1}^{\infty} \frac{(n-1)!}{n^n}(x-3)^n$

**7.52.** $\sum_{n=1}^{\infty} \frac{n}{2^n} x^n$

**7.53.** $\sum_{n=1}^{\infty} (x+4)^n$

**7.54.** $\sum_{n=1}^{\infty} \frac{(x+5)^n}{n \cdot 2^n}$

**7.55.** $\sum_{n=1}^{\infty} \frac{(x-6)^n}{n!}$

**7.56.** $\sum_{n=1}^{\infty} n!(x+7)^n$

**7.57.** $\sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}}$

**7.58.** $\sum_{n=1}^{\infty} \frac{1000^n}{n^2+1} x^n$

**7.59.** $\sum_{n=1}^{\infty} \frac{2^n+3^n}{n^n} x^n$

**7.60.** $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$

**7.61.** $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

**7.62.** $\sum_{n=1}^{\infty} 2^n x^n$

**7.63.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{\sqrt{n}}$

**7.64.** $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$

**7.65.** $\sum_{n=0}^{\infty} n! x^n$

**Are the following statements true?**

**7.66.** $$\int_{-0,2}^{0,3} \left( \sum_{n=0}^{\infty} x^n \right) dx = \sum_{n=1}^{\infty} \left( \int_{-0,2}^{0,3} x^n \, dx \right)$$

**7.67.** $$\int_{-2}^{3} \left( \sum_{n=0}^{\infty} x^n \right) dx = \sum_{n=1}^{\infty} \left( \int_{-2}^{3} x^n \, dx \right)$$

**7.68.** $$\forall x \in \mathbb{R} \quad \left( \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} \right)' = \frac{1}{1-x}$$

**7.69.** $\forall\,|x|<1 \quad \left(\sum_{n=0}^{\infty}\frac{1}{n+1}x^{n+1}\right)' = \frac{1}{1-x}$

**Find the radius of convergence for the following power series! Find the sum function in the range of convergence!**

**7.70.** $1+x+x^2+x^3+\cdots$

**7.71.** $1-x+x^2-x^3+\cdots$

**7.72.** $x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$

**7.73.** $1-x^2+x^4-x^6+x^8-\cdots$

**7.74.** $x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots$

**7.75.** $x+2x^2+3x^3+\cdots$

**Where are the following power series convergent? Give the sum functions!**

**7.76.** $\sum_{n=0}^{\infty}(\sin x)^n$

**7.77.** $\sum_{n=1}^{\infty}(1+x^2)^n$

**7.78.** $\sum_{n=1}^{\infty}\left(\frac{1}{0,1+0,2\cos^2 x}\right)^n$

**7.79.** $\sum_{n=1}^{\infty}\frac{1}{2^n x^n}$

**Find the radius of convergence and the sum of the following power series!**

**7.80.** $f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$

**7.81.** $\sum_{n=0}^{\infty}\frac{x^n}{n+1}$

**7.82.** $f(x)=\sum_{n=1}^{\infty}nx^n$

**7.83.** $\sum_{n=1}^{\infty}n^2x^n$

**7.84.** $f(x)=\sum_{n=1}^{\infty}n(n+1)x^n$

**7.85.** $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$

**Give Taylor series of the following function around $a=0$. Find the intervals of convergence! Is the sum equal to the function?**

**7.86.** $e^x$

**7.87.** $e^{2x}$

**7.88.** $e^{-x}$

**7.89.** $e^{-x^2}$

**7.90.** $\sinh x$

**7.91.** $\cosh x$

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**7.92.** $\dfrac{1}{1-x}$

**7.93.** $\dfrac{1}{1+x}$

**7.94.** $\dfrac{1}{1+x^2}$

**7.95.** $\dfrac{x^3}{1-x^2}$

**7.96.** $\ln(1+x)$

**7.97.** $\arctan x$

**7.98.** $\dfrac{1}{2+x}$

**7.99.** $f(x) = \dfrac{1}{1+x+x^2}$

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**7.100.** $\sin x$

**7.101.** $\cos x$

**7.102.** $\sin(2x)$

**7.103.** $\cos x^2$

**7.104.** $\sin^2 x$

**7.105.** $\cos^2 x$

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**Let the function $f : \mathbb{R} \to \mathbb{R}$ be differentiable any times at 0. Are the following statements true? Always justify your answers!**

**7.106.** $f$ has always a Taylor series.

**7.107.** The Taylor series of $f$ is convergent at all real $x$.

**7.108.** If the Taylor series of $f$ is convergent at $x_0$, then the sum of the Taylor series at this point is $f(x_0)$.

**7.109.** If $\sum\limits_{n=0}^{\infty} a_n x^n$ is equal to the function $f$, then this power series is the Taylor series of $f$.

**7.110.** If the Taylor series of $f(x)$ is $\sum_{n=0}^{\infty} a_n x^n$, then for all real $x$ $\sum_{n=0}^{\infty} a_n x^n = f(x)$.

**7.111.** Calculate the values of $\sin 1$ and $e$ with precision $10^{-2}$.

**7.112.** Find the sum $1 - \frac{\pi^2}{2} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots + (-1)^n \frac{\pi^{2n}}{(2n)!} + \dots$

**7.113.** Prove that the function $f(x) = \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!}$ satisfies the equation $f'''(x) = f(x)$.

**7.114.** Write down the Taylor series of $f(x) = \int_0^x e^{-t^2}\, dt$. Calculate the value of $\int_0^1 e^{-x^2}\, dx$ with two-digit precision!

**7.115.** Prove that the number $e$ is irrational!

**Find the derivatives at $x = 0$.**

**7.116.** $\left(\frac{1}{1-x}\right)^{(135)}$

**7.117.** $\left(e^{x^2}\right)^{(136)}$

**7.118.** $(\arctan x)^{(356)}$

**7.119.** $(\arctan x)^{(357)}$

**7.120.** Write down the third Taylor polynomial of $f(x) = \tan x$ at 0.

**7.121.** The function $\sin x$ is approximated by $x$ in the law about the movement of pendulum. What can the error at most be if the amplitude is at most $5° < 0.1$ radian?

**7.122.** How many terms of the Taylor series of $\sin x$ should we add such that the error is less than $10^{-6}$ if $|x| < 0.1$?

**7.123.** How many terms of the Taylor series of $\sin x$ should we add such that the error is less than $10^{-3}$, if $x < 1 < \frac{\pi}{3}(= 60°)$?

**7.124.** Write down the power series of the function $f(x) = x^2 + x + 1$

**(a)** around 0; **(b)** around 1.

**7.125.** Write down the power series of the function $f(x) = \dfrac{1}{2+x}$

**(a)** around 0; **(b)** around 1.

## 7.3 Trigonometric Series, Fourier Series

**Write down the Fourier series of the following functions!**

**7.126.** $\sin^2 x$

**7.127.** $\cos^2 x$

**Find the Fourier series of the functions on the interval $(-\pi, \pi)$.**

**7.128.** $f(x) = \operatorname{sgn} x \qquad -\pi < x < \pi$

**7.129.** $f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ 0 & \text{if } -\pi < x < 0 \end{cases}$

**7.130.** $f(x) = |\operatorname{sgn} x| \qquad -\pi < x < \pi$

The three functions above are equal on the interval $(0, \pi)$.

**7.131.** $f(x) = x \qquad -\pi < x < \pi$

**7.132.** $f(x) = |x| \qquad -\pi < x < \pi$

The two functions above are equal on the interval $(0, \pi)$.

**7.133.** $f(x) = \begin{cases} x^2 & \text{if } 0 < x < \pi \\ -x^2 & \text{if } -\pi < x < 0 \end{cases}$

**7.134.** Let $f(x)$ be the periodic function with period $2\pi$ such that $f(x) = \dfrac{\pi - x}{2}$, if $x \in (0, 2\pi)$ and $f(0) = 0$. This function is called *sawtooth function* as well.

**(a)** Find the Fourier series of $f(x)$ on the open interval $(0, 2\pi)$.

**(b)** Find the sum $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$.

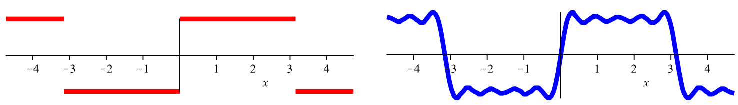
**7.135.** Let $f$ be the periodic function with period $2\pi$ such that $f(x) = x^2$ if $x \in [-\pi, \pi]$.

**(a)** Find the Fourier series of $f(x)$ on $[-\pi, \pi]$.

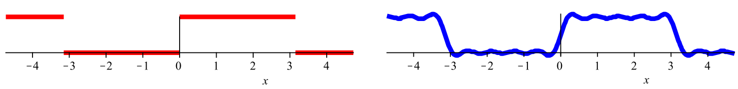
**(b)** Find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

**We plotted the graphs of some functions from the previous problems, and their 10th (in one case 5th) Fourier approximations. What are these functions?**

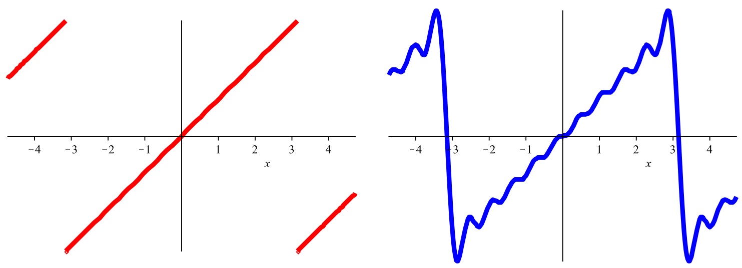
**7.136.**



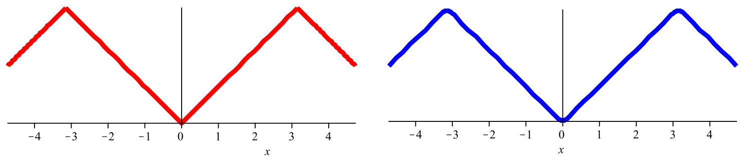

**7.137.**



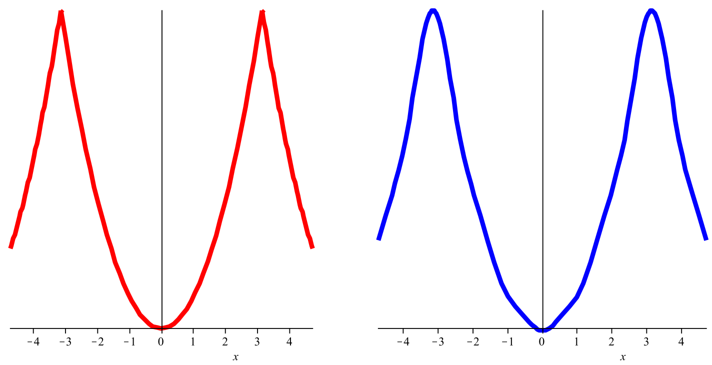
**7.138.**



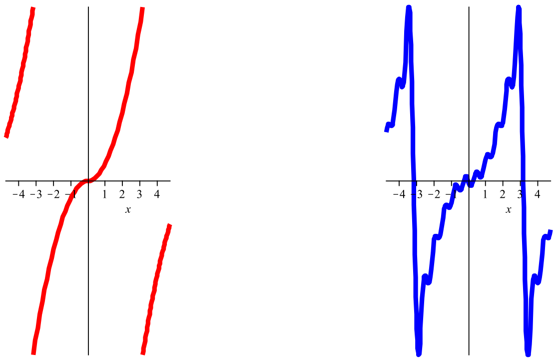

**7.139.**




**7.140.**



**7.141.**




**Find the Fourier series of the following functions on $(-\pi, \pi)$**

**7.142.** $e^x$

**7.143.** $e^{2x}$

**7.144.** $\sinh x$

**7.145.** $\sinh 3x$

**7.146.** $\cosh x$

**7.147.** $\cosh 4x$

Chapter 8

# Differentiation of Multivariable Functions

## 8.1 Topology of the Euclidean Space

— The set $G \subset \mathbb{R}^n$ is **open** if and only if the set does not contain any of its boundary points.

— The following statements are equivalent for the set $F \subset \mathbb{R}^n$:

- The set $F$ is **closed**.
- $F$ contains all of its boundary points.
- A sequence cannot "converge out" from $F$:

$$\{p_n : n \in \mathbb{N}\} \subset F, \text{ if } p_n \longrightarrow p, \text{ then } p \in F.$$

— The following statements are equivalent for the set $K \subset \mathbb{R}^n$:

- The set $K$ is **compact**.
- The set $K$ is bounded and closed.
- For all sequences in $K$ there is a subsequence which converges to a point in $K$.

**8.2 Multivariable Bolzano-Weierstrass theorem.** Every bounded sequence of points has a convergent subsequence.

**8.3 Multivariable Weierstrass theorem.** The image of a compact set is compact. Especially, if a function is continuous on a compact set, then the function has a minimum and a maximum.

**8.4 Multivariable Bolzano's theorem.** If a function is continuous on a connected set, then its image is connected, as well.

**8.5 Multivariable derivative.**

— If $f$ is (totally) differentiable at the point $\mathbf{p}$, then the function is continuous at this point, all of its partial derivatives exist, and the coordinates of the derivative are the partial derivatives.

— If the partial derivatives of $f$ exist on a neighbourhood of point $\mathbf{p}$, and they are continuous at $\mathbf{p}$ (**continuously differentiable at $\mathbf{p}$**), then $f$ is differentiable at $\mathbf{p}$.

— If $f$ is (totally) differentiable at $\mathbf{p}$, then it is differentiable at all $\mathbf{v} \neq \mathbf{0}$ directions, and

$$\frac{\partial}{\partial \mathbf{v}} f(\mathbf{p}) = \frac{1}{|\mathbf{v}|}\mathbf{v} \cdot \operatorname{grad} f(\mathbf{p}).$$

The consequence of this:

$$\max\left\{\left|\frac{\partial}{\partial \mathbf{v}} f(\mathbf{p})\right| : |\mathbf{v}| = 1\right\} = |\operatorname{grad} f(p)|^2.$$

— If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{p}$, then its graph has a **tangent hyperplane** on the point $\mathbf{p}$, and the equation of the tangent plane is

$$y = f(\mathbf{p}) + \operatorname{grad} f \cdot (\mathbf{x} - \mathbf{p}).$$

**8.6 Young's theorem.** If the $n$-variable function $f$ is two times continuously differentiable on a neighbourhood of point $\mathbf{p}$, then for all $1 \leq i, j \leq n$

$$f''_{x_i x_j}(\mathbf{p}) = f''_{x_j x_i}(\mathbf{p}).$$

**8.7 Multivariable extrema.**

— If the $n$-variable function $f : \mathbb{R}^n \to \mathbb{R}$ has a local extremum at the $\mathbf{p}$ inner point of its domain, and the partial derivatives exist at $\mathbf{p}$, then $\operatorname{grad} f(\mathbf{p}) = \mathbf{0}$.

— Let's assume that the function $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable at point $\mathbf{p}$, and $\operatorname{grad} f(\mathbf{p}) = \mathbf{0}$. Let's denote by $H$ the symmetrical Hessian matrix consisting of the second order partial derivatives of $f$ at the point $\mathbf{p}$, and the quadratic formula belonging to $H$ is $Q(\mathbf{x}) = \mathbf{x} \cdot H\mathbf{x}$. In this case

- if $Q$ is positive definite, then there is a local minimum at $\mathbf{p}$;
- if $Q$ is negative definite, then there is a local maximum at $\mathbf{p}$;
- if $Q$ is indefinite, there is no local extremum at (but is a saddle point) $\mathbf{p}$;
- if $Q$ is semidefinite, then this test is inconclusive at $\mathbf{p}$.

Especially for two-variable functions if

$$D = \det H = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{vmatrix}$$

is the determinant of the Hessian matrix at point $\mathbf{p}$, then

- there is a minimum if $D > 0$ and $f''_{xx} > 0$;
- there is a maximum if $D > 0$ and $f''_{xx} < 0$;
- there is no local extremum if $D < 0$;
- this test is inconclusive if $D = 0$.

**8.8 Conditional extrema, Lagrange-multiplier.**
If the function $f : \mathbb{R}^n \to \mathbb{R}$ has a local extremum at point $\mathbf{a}$ subject to the constraints

$$g_k(\mathbf{x}) = 0, \; k = 1, 2, \ldots, p,$$

then there are $\lambda_1, \lambda_2, \ldots, \lambda_p$ real numbers such that

$$\frac{\partial}{\partial x_i} f(\mathbf{a}) + \sum_{k=1}^{p} \lambda_k \frac{\partial}{\partial x_i} g_k(\mathbf{a}) = 0, \quad i = 1, 2, \ldots, n.$$

## 8.1 Basic Topological Concepts

**Find the distance of the given points!**

**8.1.** $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2 \quad \mathbf{p} = (-1, 3) \quad \mathbf{q} = (5, -4)$

**8.2.** $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3 \quad \mathbf{p} = (-1, 3, 5) \quad \mathbf{q} = (5, -4, 0)$

**8.3.** $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n \quad \mathbf{p} = (p_1, p_2, \ldots p_n) \quad \mathbf{q} = (q_1, q_2, \ldots q_n)$

**8.4.** Write down the inequality defining the points of the (open) sphere in $\mathbb{R}^k$ whose center is the origin, and its radius is 1, where $k = 1, 2, 3, n$.

**8.5.** Write down the inequality defining the points of the (open) sphere in $\mathbb{R}^k$ whose center is the origin, and its radius is $r$, where $k = 1, 2, 3, n$.

**Plot the following sets, find the interior, the exterior and the boundary points, and decide which sets are closed, which ones are open, and which ones are bounded!**

**8.6.** $\{h \in \mathbb{R} : -3 < h \leq 5\}$

**8.7.** $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

**8.8.** $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$

**8.9.** $\{(x, y) \in \mathbb{R}^2 : -3 < x < 5\}$

**8.10.** $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 4\}$

**8.11.** $\{(x, y) \in \mathbb{R}^2 : -1 \leq x, y \leq 1\}$

**8.12.** $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 0\}$

**8.13.** $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq -4\}$

**Which of the following sets are open, closed, both open and closed, neither open nor closed? Give the reasons for all answers! Note that $x$ and $y$ are real numbers in these problems.**

**8.14.** $H = \{x : 0 < x < 1\}$

**8.15.** $H = \{(x, 0) : 0 < x < 1\}$

**8.16.** $H = \{x : 0 \leq x \leq 1\}$

**8.17.** $H = \{(x, 0) : 0 \leq x \leq 1\}$

**8.18.** $H = \{x : 0 \leq x < 1\}$

**8.19.** $H = \{(x, 0) : 0 \leq x < 1\}$

**8.20.** $H = \{(x, y) : x^2 + y^2 < 1\}$

**8.21.** $H = \{(x, y, 0) : x^2 + y^2 < 1\}$

**8.22.** $H = \{(x, y) : x \in \mathbb{Q}, y \in \mathbb{Q}\}$

**8.23.** $H = \{(x, y) : 0 < x < 1, 0 < y < 1\}$

**Find the interior, the exterior and the boundary points of the following sets!**

**8.24.** $H = \{(x, y) : 0 \le x \le 1, 0 \le x \le 1\}$

**8.25.** $H = \{(x, y) : 0 \le x < 1, 0 < y \le 1\}$

**8.26.** $H = \{(x, y) : x \in \mathbb{Q}, y \in \mathbb{Q}, 0 \le x \le 1, 0 \le y \le 1\}$

**8.27.** $H = \{(x, y) : 0 < x < 1, y = 0\}$

**Find the interior, the exterior and the boundary points of the following sets in the space!**

**8.28.** $H = \{(x, y, z) : x^2 + y^2 + z^2 < 1\}$

**8.29.** $H = \{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$

**8.30.** $H = \{(x, y, z) : x \in \mathbb{Q}, y \in \mathbb{Q}, z \in \mathbb{Q}, 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

**8.31.** $H = \{(x, y, z) : x \in \mathbb{Q}, 0 < x < 1, y = z = 0\}$

**8.32.** Let $H \subset \mathbb{R}^n$. Are the following statements true?

**(a)** If $x \in H$, then $x$ is an interior point of $H$.
**(b)** If $x \notin H$, then $x$ cannot be an interior point of $H$.
**(c)** If $x \in H$, then $x$ cannot be a boundary point of $H$.
**(d)** If $x \notin H$, then $x$ cannot be a boundary point of $H$.
**(e)** There is a set such that all points of the set are boundary points.
**(f)** There is a set such that all points of the set are interior points.
**(g)** There is a set which has no interior points.

## 8.2 The Graphs of Multivariable Functions

**Find the value of the following functions at the given p points!**

**8.33.** $f(x,y) = x + y^2 \quad \mathbf{p} = (2,3)$

**8.34.** $f(x,y) = \arctan x + \arcsin xy \quad \mathbf{p} = (\pi/4, 0)$

**8.35.** $f(x,y,z) = x^{(y^z)} \quad \mathbf{p} = (4,3,2)$

**8.36.** $f(x,y,z) = {x^y}^z \quad \mathbf{p} = (4,3,2)$

**Find the value of the following functions at the points of the given curves!**

**8.37.** $f(x,y) = x^2 + y^2,$ $\qquad y = x,$ and $x^2 + y^2 = 1$

**8.38.** $f(x,y) = x - y,$ $\qquad y = x,$ and $y = x^2$

**8.39.** $f(x,y) = \sin x,$ $\qquad x = \pi/4,$ and $y = \pi/4$

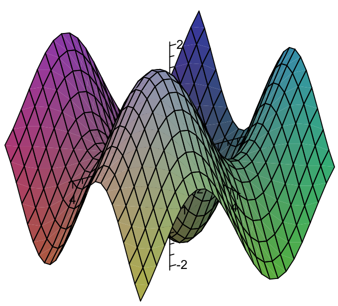
**8.40.** $f(x,y) = \sin xy,$ $\qquad x = \pi/4,$ and $y = \pi/4$

**8.41.** Pair the graphs with the following two-variable functions!

**(a)** $x^2 + y^2$ **(b)** $(x+y)^2$ **(c)** $x^2 - y^2$

**(d)** $xy$ **(e)** $\sin x + \sin y$ **(f)** $\sin x \sin y$

(A)



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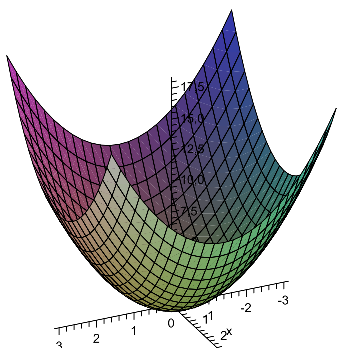

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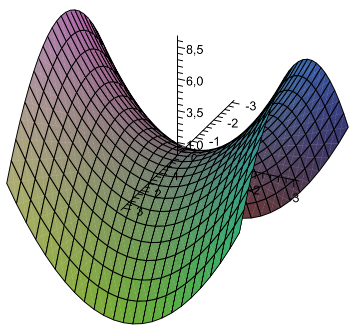

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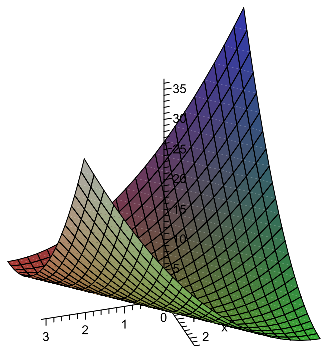

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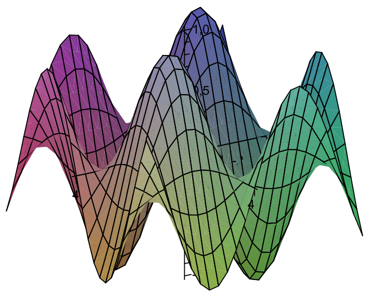

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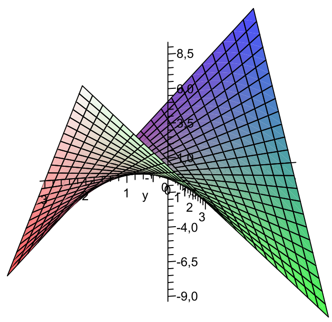
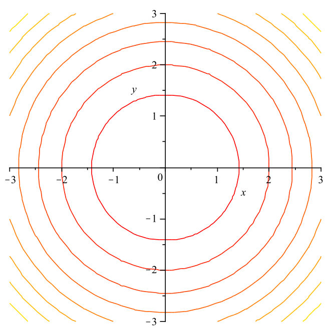


**8.42.** We plotted the contours of the previous functions. Find the formulas belonging to the graphs!

(A)



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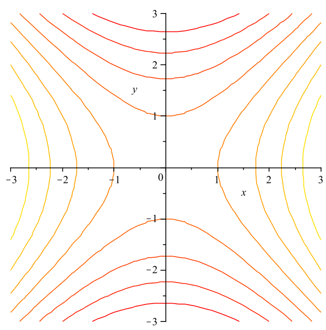

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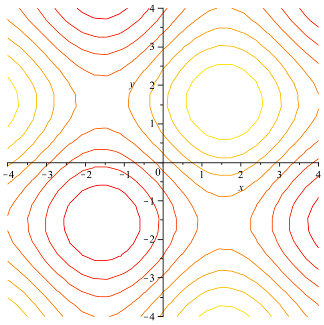

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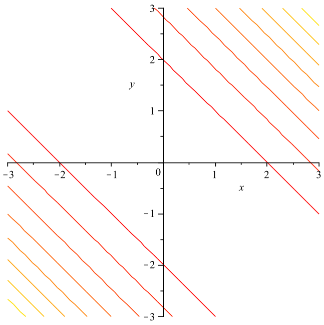

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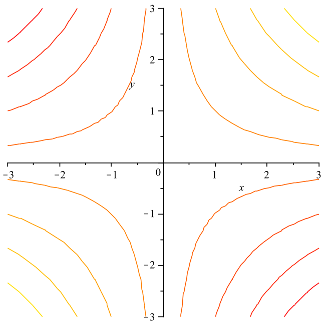

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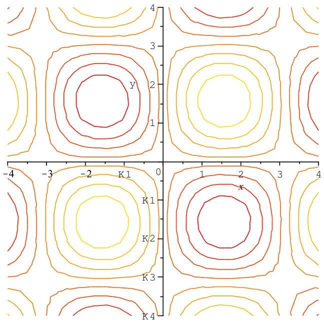

**Plot the contours of the following functions! Make a spatial effect sketch of the graphs!**

**8.43.** $(x+y)^2$

**8.44.** $\sqrt{x^2+y^2}$

**8.45.** $xy$

**8.46.** $(x-y)^2$

**8.47.** $x^2+y^2$,

**8.48.** $|x|$

**8.49.** $x^2-y^2$

**8.50.** $|x+y|$

**Find the contour surfaces of the following functions!**

**8.51.** $f(x,y,z) = x + y + z$

**8.52.** $f(x,y,z) = x^2 + y^2 + z^2$

**8.53.** $f(x,y,z) = x + y$

**8.54.** $f(x,y,z) = y^2$

**Decide whether the following spatial sets can be the graphs of some two-variable functions. If yes, then give examples for the functions!**

**8.55.** plane

**8.56.** surface of a sphere

**8.57.** cone lateral

**8.58.** cylinder lateral

**8.59.** lateral of a half cylinder

**8.60.** surface of a half sphere

**Find the maximal domain of the following functions!**

**8.61.** $f(x,y) = \dfrac{x+y}{x-y}$

**8.62.** $f(x,y) = \sqrt{1-x^2} + \sqrt{y^2+1}$

**8.63.** $f(x,y) = \dfrac{1}{x^2+y^2}$

**8.64.** $f(x,y) = \sqrt{1-x^2-y^2}$

**8.65.** $f(x,y,z) = \dfrac{z}{\sin x} \cos y$

**8.66.** $f(x,y,z) = \dfrac{x}{y-z} + \dfrac{y}{x+z} - \dfrac{z}{x^2-y^2}$

## 8.3 Multivariable Limit, Continuity

**Does the limit of the following functions exist at the given points? If yes, what is the limit? Where are the following functions continuous?**

**8.67.** $f(x,y) = 7$ $\quad \mathbf{p} = (0,0)$

**8.68.** $f(x,y) = x + y$ $\quad \mathbf{p} = (3,5)$

**8.69.** $f(x, y) = \dfrac{x}{y}$ $\qquad$ $\mathbf{p} = (3, 0)$

**8.70.** $f(x, y) = \dfrac{\sin xy}{x}$ $\qquad$ $\mathbf{p} = (0, 2)$

**8.71.** $f(x, y) = \dfrac{\sin xy}{\tan 2xy}$ $\qquad$ $\mathbf{p} = (0, 3)$

**8.72.** $f(x, y) = \dfrac{\sin x - \sin y}{x - y}$ $\qquad$ $\mathbf{p} = (0, 0)$

**8.73.** $f(x, y) = \dfrac{\sin x - \sin y}{e^x - e^y}$ $\qquad$ $\mathbf{p} = (0, 0)$

**8.74.** $\sqrt{x + y - 1}$ $\qquad$ $\mathbf{p} = (0, 0)$

**8.75.** $xy \ln(x^2 + y^2)$ $\qquad$ $\mathbf{p} = (0, 0)$

**8.76.** $f(x, y) = \begin{cases} 1 \text{ if } x \neq 0 \text{ and } y \neq 0 \\ 0 \text{ if } x = 0 \text{ or } y = 0 \end{cases}$ $\qquad$ $\mathbf{p} = (0, 0), \quad \mathbf{q} = (0, 1)$

**8.77.** $f(x, y) = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ $\qquad$ $\mathbf{p} = (0, 0), \quad \mathbf{q} = (0, 1), \quad \mathbf{r} = (1, 0)$

**8.78.** $f(x, y) = \begin{cases} 1 \text{ if } x^2 + y^2 \neq 0 \\ 0 \text{ otherwise} \end{cases}$ $\qquad$ $\mathbf{p} = (0, 0), \quad \mathbf{q} = (0, 1)$

**8.79.** $f(x, y) = \begin{cases} x \text{ if } x = y \\ 0 \text{ otherwise} \end{cases}$ $\qquad$ $\mathbf{p} = (0, 0), \quad \mathbf{q} = (0, 1)$

**8.80.** $f(x, y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ otherwise} \end{cases}$ $\qquad$ $\mathbf{p} = (0, 0), \quad \mathbf{q} = (0, 1)$

---

**8.81.** Let $f(x, y) = \dfrac{x - y}{x + y}$. Do the following limits exist?

**(a)** $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ $\qquad$ **(b)** $\displaystyle\lim_{x\to 0} \left( \lim_{y\to 0} f(x, y) \right)$

**(c)** $\displaystyle\lim_{y\to 0} \left( \lim_{x\to 0} f(x, y) \right)$ $\qquad$ **(d)** $\displaystyle\lim_{x\to 0} f(x, x)$

**8.82.** Let $f(x, y) = (x + y) \sin \dfrac{1}{x} \sin \dfrac{1}{y}$. Do the following limits exist?

**(a)** $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ **(b)** $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right)$

**(c)** $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} f(x,y)\right)$ **(d)** $\lim\limits_{x\to 0} f(x,x)$

**8.83.** Let

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Prove that the functions $g(x) = f(x,0)$ and $h(y) = f(0,y)$ are continuous at $x = 0$ and $y = 0$. Prove that the function $f(x,y)$ is not continuous at $(x,y) = (0,0)$.

**8.84.** Prove that if $f(x,y)$ is continuous on the plane, then

**(a)** $G = \{(x,y) : f(x,y) > 0\}$ is an open set;

**(b)** $F = \{(x,y) : f(x,y) \geq 0\}$ is a closed set!

## 8.4 Partial and Total Derivative

**8.85.** Give an example of a two-variable function such that both partial derivatives exist at the origin, but the function is not totally differentiable at the origin!

**Calculate the partial derivatives of the following functions!**

**8.86.** $f(x,y) = x\sin y$

**8.87.** $f(x,y) = \sin(xy)$

**8.88.** $f(x,y) = (x+2y)\sin(x+2y)$

**8.89.** $f(x,y) = 3x^2y^4 - 4$

**8.90.** $f(x,y) = \dfrac{x+y}{x-y^2}$

**8.91.** $f(x,y) = (5x+y^2)^{e^{x^2}+3y}$

**8.92.** $g(x,y,z) = \sin z \cdot (\cos x)^{\ln y}$

**8.93.** $g(x,y,z) = \left(\dfrac{x}{y}\right)^z$

**8.94.** $f(x,y) = x^2 + xy + y^2$

**8.95.** $f(x,y) = e^{x-y}$

**8.96.** $g(x,y,z) = x^{y^z}$

**8.97.** $g(x,y,z) = \dfrac{z \arctan x^2 y}{1 + \ln(xy^3 + 2x\sqrt{z})}$

**8.98.** $f(x,y,z) = \sin(x^2 + y^3 + z^4)$

**8.99.** $f(x,y) = \arctan \dfrac{1-x}{1-y}$

**8.100.** $f(x,y) = x^y + y^x$

**8.101.** $f(x,y,z) = x^{yz}$

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**8.102.** Is the function $f(x,y) = |x| + |y|$ partially differentiable at the point $(0,0)$?

**8.103.** Which statement implies the other?

**P:** $f(x,y)$ is continuous at $(0,0)$ **Q:** The partial derivatives of $f(x,y)$ exist at $(0,0)$.

**8.104.** We know that $f(1,2) = 3$, and that for all $(x,y)$ points $f'_x(x,y) = 0$ and $f'_y(x,y) = 0$. Find the value of $f(5,10)$.

**8.105.** Where are the partial derivatives of the following function continuous?

$$f(x,y) = \begin{cases} (x+y)\sin\dfrac{1}{x}\sin y & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0 & \text{if } x = 0 \text{ or } y = 0 \end{cases}$$

**Find the second order partial derivatives of the following functions!**

**8.106.** $g(x, y, z) = xy^2 \sin z$

**8.107.** $g(x, y, z) = \dfrac{x}{y + z}$

**8.108.** $g(x, y, z) = 2 + x + y^2 + z^3$

**8.109.** $g(x, y, z) = \ln(x + y^2 + z^3)$

**Find the directional derivatives of the following functions along the given vectors at the given points!**

**8.110.** $f(x, y) = x + y^2$ $\quad \mathbf{p} = (2, 3), \quad \mathbf{v} = (3, 4)$

**8.111.** $f(x, y) = \sin xy$ $\quad \mathbf{p} = (0, 0), \quad \mathbf{v} = (1, -1)$

**8.112.** $f(x, y) = e^{x+y} \cdot \ln y$ $\quad \mathbf{p} = (0, 1), \quad \mathbf{v} = (-4, 3)$

**8.113.** $f(x, y) = \dfrac{x}{y}$ $\quad \mathbf{p} = (1, 1), \quad \mathbf{v} = (-1, -1)$

**Find the directional derivatives of the following functions at the point $(3, 5)$ and the angle of the direction is $\alpha = 30°$.**

**8.114.** $f(x, y) = x^2 - y^2$

**8.115.** $h(x, y) = x\mathrm{e}^y$

**In what direction is the directional derivative maximal or minimal at the point $(-4, 2)$?**

**8.116.** $f(x, y) = (x - y)^2$

**8.117.** $g(x, y) = x^2 + \dfrac{2}{xy}$

**The surface of a model layout is given by the function $f(x, y)$. In what direction does a ball start rolling on the surface at points $A = (1, 2)$, $B = (2, 1)$, $C = (2, 0)$ and $D = (-2, 1)$?**

**8.118.** $e^{-(x^2+y^2)}$

**8.119.** $x^3+y^3-9xy$

**8.120.** The surface of the hill is given by the function $f(x,y)=-x^2+\sin y$ if $-2\le x\le 2, -10\le y\le 10$. In what direction should the skier start from the point $\mathbf{p}=(1,\pi/3)$ if he wants to ski on the maximal angle slope?

**8.121.** Let $f(x,y)=\sqrt{1-x^2-y^2}$. Find the equation of the contour going through the point $(1/2,1/2)$, and write down the tangent line of the contour at the point $(1/2,1/2)$. Calculate the gradient of the function at $(1/2,1/2)$. What is the angle between the tangent line and the contour?

**8.122.** Give an example of a function $f(x,y)$ which has gradient, but the gradient is not perpendicular to the tangent line of the contour!

Hint: Examine the function

$$f(x,y)=\begin{cases}0 & \text{if } (x-1)^2+y^2=1\\ y & \text{if } x=0 \text{ ĂŠs } y\neq 0\\ x & \text{otherwise}\end{cases}$$

at the origin!

**8.123.** Let $f(x,y)=\sqrt{1-x^2-y^2}$. Verify that the gradient and the tangent line of the contour are perpendicular to each other at the point $(1/2,1/2)$.

**8.124.** Write down the gradients of the function $f(x,y)=\operatorname{sgn}x\operatorname{sgn}y$ at the points where the gradient exists!

**8.125.** Let $f(x,y)=\begin{cases}xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } x^2+y^2\neq 0\\ 0 & \text{if } x^2+y^2=0\end{cases}$

**(a)** Find the values of $f'_x(0,0)$ and $f'_y(0,0)$.

**(b)** Find the values of $f'_x(0,y)$, if $y\neq 0$ and the values of $f'_y(x,0)$, if $x\neq 0$.

**(c)** Find the values of $f''_{xy}(0,0)$ and $f''_{yx}(0,0)$.

**(d)** Why doesn't this contradict Young's theorem?

**(e)** Is the function $f$ differentiable two times at $(0,0)$?

**8.126.** Let $f : \mathbb{R}^n \to \mathbb{R}$. Which of the following statements are true, and which ones are false? If a statement is false, give a counterexample!

**(a)** If $f$ is continuous at the point $\mathbf{p}$, then $f$ is differentiable at that point.

**(b)** If $f$ is differentiable at $\mathbf{p}$, then $f$ is continuous at $\mathbf{p}$.

**(c)** If $f$ has partial derivatives at the point $\mathbf{p}$, then $f$ is continuous at $\mathbf{p}$.

**(d)** If $f$ has continuous partial derivatives at $\mathbf{p}$, then $f$ is differentiable at $\mathbf{p}$.

**(e)** If $f$ has continuous partial derivatives at $\mathbf{p}$, then $f$ is continuous at $\mathbf{p}$.

**(f)** If $f$ is differentiable at $\mathbf{p}$, then $f$ has partial derivatives at $\mathbf{p}$.

**(g)** If $f$ has second order partial derivatives at $\mathbf{p}$, then $f$ is differentiable two times at $\mathbf{p}$.

**(h)** If $f$ has continuous second order partial derivatives at $\mathbf{p}$, then $f$ is differentiable two times at $\mathbf{p}$.

**(i)** If $f$ is differentiable two times at $\mathbf{p}$, then $f$ has second order partial derivatives at $\mathbf{p}$.

**(j)** If $f$ is differentiable two times at $\mathbf{p}$, then $f$ has continuous partial derivatives at $\mathbf{p}$.

**Is there a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that**

**8.127.** $f'_x(x,y) = \sin y$, $f'_y(x,y) = x\cos y$

**8.128.** $f'_x(x,y) = e^{xy}$, $f'_y(x,y) = \cos(x-y)$

**Find the equations of the tangent planes or hyperplanes for the following functions at the given points!**

**8.129.** $f(x,y) = x^y$ $\qquad$ $\mathbf{p} = (2,3)$

**8.130.** $f(x,y) = \sin(xy)$ $\qquad$ $\mathbf{p} = (1/2, \pi)$

**8.131.** $f(x,y,z)=xy^2-z^3$ $\quad$ $\mathbf{p}=(3,2,1)$

**8.132.** $f(x_1,\dots,x_n)=x_1\cdot x_2\cdot\ldots\cdot x_n$ $\quad$ $\mathbf{p}=(1,2,\dots,n)$

**Write down the composite functions $G(t)=F(\mathrm{r}(t))$. Find the derivatives $G'(t)$ directly from the formulas, and by using the chain rule!**

**8.133.** $F(x,y)=x^2+y^2$ $\quad$ $\mathbf{r}(t)=\cos t\cdot\mathbf{i}+\sin t\cdot\mathbf{j}$ $\quad$ $t\in[0,2\pi]$

**8.134.** $F(x,y)=x^2-y^2$ $\quad$ $\mathbf{r}(t)=\cos t\cdot\mathbf{i}+\sin t\cdot\mathbf{j}$ $\quad$ $t\in[0,2\pi]$

**8.135.** $F(x,y)=x^2+y^2$ $\quad$ $\mathbf{r}(t)=t\cdot\mathbf{i}+3t\cdot\mathbf{j}$ $\quad$ $t\in[0,10]$

**Write down the Jacobian matrices of the following composite functions!**

**8.136.** $g(t)=(\sin t,\cos t),\quad f(x,y)=x+y,\quad h=f\circ g.$

**8.137.** $f(u,v)=(\sin uv,\cos uv),\quad g(x,y)=x^2+y^2,\quad h=g\circ f$

**8.138.** $f(u,v)=\left(u^2v^2,\dfrac{1}{uv}\right),\quad g(x,y)=\ln x+\ln y,\quad h=g\circ f.$

**Give the Jacobian matrices of the following mappings!**

**8.139.** $\mathbf{v}=\sin xy\cdot\mathbf{i}+x\cos y\cdot\mathbf{j}$

**8.140.** $\mathbf{v}=\dfrac{xy}{1+x^2}\cdot\mathbf{i}+\dfrac{x}{x^2+y^2}\cdot\mathbf{j}$

**8.141.** $\mathbf{v}=(\ln x+\ln y)\cdot\mathbf{i}+x^2\cdot\mathbf{j}$

**8.142.** $\mathbf{v}=\sin(x+3y^2)\cdot\mathbf{i}+e^{xy}\cdot\mathbf{j}$

**8.143.** Let $F(x,y)=xy$, and $f:\mathbb{R}^2\to\mathbb{R}$, $g:\mathbb{R}^2\to\mathbb{R}$ be differentiable functions! Write down the derivative of the composite function $F(f,g)$ by using the chain rule!

**8.144.** Let the function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be differentiable, and $f(x, y) = x + y$. Write down the derivative of $f \circ \mathbf{r}$ by using the chain rule!

**Write down the tangent lines of the contours of the following functions going through the point $P$!**

**8.145.** $f(x, y) = x^y$, $P(3, 5)$

**8.146.** $f(x, y) = x^2 - y^2$ $P(5, 3)$

## 8.5 Multivariable Extrema

**Are there absolute extrema of the following functions on the given sets? Justify your answers!**

**8.147.** $f(x) = \dfrac{1}{x}$ $\qquad H = \{(x) : x \neq 0\}$

**8.148.** $f(x) = \sin^2 \sqrt[7]{x^3}$ $\qquad H = \{(x) : x \in \mathbb{R}\}$

**8.149.** $f(x, y) = \dfrac{y}{x}$ $\qquad H = \{(x, y) : x \neq 0\}$

**8.150.** $f(x, y) = x^2 + e^y \sin\left(x^3 y^2\right)$ $\qquad H = \{(x, y) : x^2 + y^2 \leq 1\}$

**8.151.** $f(x, y) = x^2 + y^2$ $\qquad H = \{(x, y) : x^2 + y^2 < 1\}$

**8.152.** $f(x, y) = x + y$ $\qquad H = \{(x, y) : 0 < x < 1, 0 < y < 1\}$

**8.153.** $f(x, y) = xy$ $\qquad H = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$

**8.154.** $f(x, y, z) = xyz$ $\qquad H = \{(x, y, z) : (x-1)^2 + (y+2)^2 + (z-3)^2 \leq 4\}$

**Find the absolute extrema of the following functions on the given sets!**

**8.155.** $f(x, y) = x^3 y^2 (1 - x - y)$ $\qquad H = \{(x, y) : 0 \leq x, 0 \leq y, x + y \leq 1\}$

**8.156.** $f(x, y) = x^2 + y^2 + (x + y + 1)^2$ $\quad H = \mathbb{R}^2$

**8.157.** $f(x,y) = x - y - 3$ $H = \{(x,y) : x^2 + y^2 \le 1\}$

**8.158.** $f(x,y) = \ln x \cdot \ln y + \dfrac{1}{2}\ln x + \dfrac{1}{2}\ln y$ $H = \{(x,y) : \dfrac{1}{e} \le x \le e, \dfrac{1}{e} \le y \le e\}$

**8.159.** $f(x,y) = \sin x + \sin y + \sin(x+y)$ $H = \{(x,y) : 0 \le x \le \dfrac{\pi}{2}, 0 \le y \le \dfrac{\pi}{2}\}$

**8.160.** $f(x,y) = x^2 - 2xy + 2y^2 - 2x + 4y$ $H = \{(x,y) : |x| \le 3, |y| \le 3\}$

**Find the locations of the local extrema of the following functions, if there are any!**

**8.161.** $f(x,y) = 3x^2 + 5y^2$

**8.162.** $f(x,y) = (2x - 5y)^2$

**8.163.** $f(x,y) = 2x^2 - 3y^2$

**8.164.** $f(x,y) = 2x^2 - y^2 + 4x + 4y - 3$

**8.165.** $f(x,y) = x^2 + y^2 - 6x + 8y + 35$

**8.166.** $f(x,y) = 3 - \sqrt{2 - (x^2 + y^2)}$

**8.167.** $f(x,y) = -y^2 + \sin x$

**8.168.** $f(x,y) = e^{-(x^2+y^2)}$

**8.169.** $f(x,y) = (x - y^2)(2x - y^2)$

**8.170.** $f(x,y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$

**8.171.** Is there any one-variable polynomials whose range is $(0,\infty)$? If there is, then give an example! Is there any two-variable polynomials which range is $(0,\infty)$? If there is, then give an example!

**8.172.** Give an example of a two-variable function, which has infinitely many strict local maximum, but has no local minimum at all!

**8.173.** Find the maximum and the minimum of the function $2x + 3y + 4z$ on the surface of the sphere with origin center, and radius 1.

**8.174.** Find the distance of the lines $\mathbf{p}(t) = 2t \cdot \mathbf{i} + t \cdot \mathbf{j} + (1-t) \cdot \mathbf{k}$ and $\mathbf{q}(t) = 3t \cdot \mathbf{i} + t \cdot \mathbf{j} + (2t-1) \cdot \mathbf{k}$.

**8.175.** Are the following statements true?

**(a)** If $f'_x(x_0, y_0) = 0$, then $f$ has a local extremum at the point $(x_0, y_0)$.

**(b)** If $f'_x(x_0, y_0) = 0$ and $f'_y(x_0, y_0) = 0$, then $f$ has a local extremum at the point $(x_0, y_0)$.

**(c)** If $f''_{xy}(x_0, y_0) = 0$ and $f''_{yx}(x_0, y_0) = 0$, then $f$ has a local extremum at the point $(x_0, y_0)$.

**(d)** If $f''_{xx}(x_0, y_0) f''_{yy}(x_0, y_0) - (f''_{xy}(x_0, y_0))^2 < 0$, then $f$ has no local extremum at the point $(x_0, y_0)$.

**(e)** If $f''_{xx}(x_0, y_0) f''_{yy}(x_0, y_0) - (f''_{xy}(x_0, y_0))^2 \leq 0$, then $f$ has no local extremum at the point $(x_0, y_0)$.

**(f)** If $f''_{xx}(x_0, y_0) < 0$, then $f$ has no local extremum at the point $(x_0, y_0)$.

**At which $(x, y) \in \mathbb{R}^2$ points are both partial derivatives of the function $f(x, y)$ zero? At which $(x, y) \in \mathbb{R}^2$ points has the function $f(x, y)$ local extrema?**

**8.176.** $f(x, y) = x^3$

**8.177.** $f(x, y) = x^2$

**8.178.** $f(x, y) = x^2 - y^2$

**8.179.** $f(x, y) = x^2 + y^2$

**8.180.** $f(x, y) = (x + y)^2$

**8.181.** $f(x, y) = x^3 + y^3$

**8.182.** $f(x, y) = e^{-(x^2+y^2)}$

**8.183.** $f(x, y) = x^2 + \sin y$

**8.184.** $f(x, y) = 3x^2 + 5y^2$

**8.185.** $f(x, y) = \dfrac{2}{3}x^3 + y^4 + xy$

**8.186.** $f(x, y) = xy$

**8.187.** $f(x, y) = e^{y^2 - x^2}$

**8.188.** $f(x, y, z) = xyz + x^2 + y^2 + z^2$

**8.189.** $f(x, y) = x^3 - y^3$

**8.190.** $f(x, y) = x^4 + y^4$

**8.191.** $f(x, y) = -2x^2 - y^4$

**8.192.** $f(x, y) = (2x - 5y)^2$

**8.193.** $f(x, y) = (1 + e^y)\cos x - ye^y$

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**The surface of a hill is given by the function $F(x, y) = 30 - \dfrac{x^2}{100} - \dfrac{y^2}{100}$. Find the maximal height of the path whose coordinates satisfy the following equations:**

**8.194.** $3x + 3y = \pi \sin x + \pi \sin y$

**8.195.** $4x^2 + 9y^2 = 36$

**8.196.** $y = \dfrac{1}{1 + x^2}$

**8.197.** $x^2 + y^2 = 25$

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**Find the maximum of $f$ with the given constraint!**

**8.198.** $f(x, y) = xy,$ $\quad x^2 + y^2 = 1$

**8.199.** $f(x, y, z) = x - y + 3z,$ $\quad x^2 + \dfrac{y^2}{2} + \dfrac{z^2}{3} = 1$

**8.200.** $f(x, y, z) = xyz,$ $\quad x^2 + y^2 + z^2 = 3$

**8.201.** $f(x, y) = xy,$ $\quad x + y + z = 5$

**8.202.** $f(x, y) = xyz,$ $\quad xy + yz + xz = 8$

**8.203.** $f(x, y) = xyz,$ $\quad xy + yz + xz = 8, \quad x, y, z \geq 0$

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**8.204.** A particle can move on the circle path $x^2 + y^2 = 25$ on the plane, where its potential energy at the point $(x, y)$ is $E(x, y) = x^2 + 24xy + 8$. Does the particle have stable equilibrium at any points?

**8.205.** The amount of the products made in a factory depends on the parameters $x$ and $y$:
$M(x,y) = xy$. The product cost is $C(x,y) = 2x + 3y$. What amount can the factory produce at most, if it has $C(x,y) = 10$ money unit for the product cost?

**8.206.** With a given volume, which brick has the minimal surface?

**8.207.** Find the angles of the triangle with maximal area, if its perimeter is $K$.

**8.208.** Find the equation of the tangent plane of the $3x^2+2y^2+z^2 = 9$ ellipsoid, where the point of tangent is $(1,-1,2)$.

**8.209.** Let $P = (3,-7,-1)$, $Q = (5,-3,5)$, and $S$ be a plane going through $Q$, and the plane be perpendicular to the line segment $PQ$.

**(a)** Find the equation of the plane $S$!

**(b)** Write down the distance between a point of the plane and the origin!

**(c)** Which point of the plane $S$ is closest to the origin?

**(d)** Show that the line segment between the previous point and the origin is perpendicular to the plane $S$. Give a geometric reason for this fact!

Chapter 9

# Multivariable Riemann-integral

## 9.1 Properties of Jordan measurable sets.

— If $A \subset \mathbb{R}^n$ is bounded, then $b(A) = b(\operatorname{int} A)$, $k(A) = k\left(\overline{A}\right)$.

— The bounded set $A \subset \mathbb{R}^n$ is Jordan measurable if and only if its boundary is a null set.

— If $A \subset \mathbb{R}^n$ is Jordan measurable and $f : A \to \mathbb{R}$ is bounded, then $f$ is integrable if and only if the graf $f \subset \mathbb{R}^{n+1}$ is a null set.

## 9.2 Properties of the integral.

— If $A$ is a Jordan measurable set, then $t(A) = \int\limits_A \chi_A$, where $\chi_A$ is the **characteristic function** of the set $A$, that is,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

— If the sets $A$ and $B$ are bounded, $\operatorname{int} A \cap \operatorname{int} B = \emptyset$ (**non-overlapping**), and $f$ is integrable both on $A$ and $B$, then $f$ is integrable on the set $C = A \cup B$, and

$$\int\limits_C f = \int\limits_A f + \int\limits_B f.$$

— A continuous function is integrable on a measurable closed set.

— If $f$ and $g$ are equal on the measurable set $A$ except on a null set, and $f$ is integrable on $A$, then $g$ is integrable on $A$, and

$$\int\limits_A f = \int\limits_A g.$$

— If $f$ and $g$ are integrable on the set $A$, and $c$ is an arbitrary real number, then $f+g$ and $c\cdot f$ are also integrable on $A$, and

$$\int_A (f+g) = \int_A f + \int_A g, \qquad \int_A (c\cdot f) = c\int_A f.$$

## 9.3 Integration Methods.

— **Successive integration - Fubini's theorem.**

Let $A \subset \mathbb{R}^{n-1}$ be a closed, Jordan measurable set, $B = [a,b]\times A \subset \mathbb{R}^n$ and $f : B \to \mathbb{R}$ be continuous, then

$$\iint_B f(x,y)\,dx\,dy = \int_a^b \left(\int_A f(x,y)\,dy\right)\,dx$$

— **Integration between continuous functions.**

Let $A \subset \mathbb{R}^{n-1}$ be a closed Jordan measurable set, $\varphi : A \to \mathbb{R}$, $\psi : A \to \mathbb{R}$ two continuous functions, $\varphi \le \psi$ at the points of $A$,

$$N = \{(x,y) : x \in A, \varphi(x) \le y \le \psi(x)\} \subset \mathbb{R}^n, \qquad f : N \to \mathbb{R}$$

be a continuous function. In this case $N$ is Jordan measurable, $f$ is integrable on $N$, and

$$\iint_N f(x,y)\,dx\,dy = \int_A \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right)\,dx.$$

— **Integral Transform.**

Let $A \subset \mathbb{R}^n$ be a closed, Jordan measurable set, $\Phi : A \to \mathbb{R}^n$ be continuous, and on $\operatorname{int} A$ bijection and continuously differentiable, $B = \{\Phi(x) : x \in A\} = \Psi(A)$, and $f : B \to \mathbb{R}$ be a continuous function. In this case $B$ is a (closed) Jordan measurable set and

$$\int_B f(y)\,dy = \int_A |J|\,f(\Psi(x))\,dx,$$

where $J = \det(\Psi')$ is the **Jacobian determinant** of $\Psi$.

## 9.1 Jordan Measure

**Is there area of the boundary of the following sets on the plane? If yes, then calculate the area!**

**9.1.** $H = \{(x,y) : 0 \leq x < 1, 0 < y \leq 1\}$

**9.2.** $H = \{(x,y) : x \in \mathbb{Q}, y \in \mathbb{Q}, 0 \leq x \leq 1, 0 \leq x \leq 1\}$

**Is there volume of the boundary of the following spatial sets? If yes, then calculate the volume!**

**9.3.** $H = \{(x,y,z) : 0 \leq x \leq 1, 0 \leq x \leq 1, 0 \leq z \leq 1\}$

**9.4.** $H = \{(x,y,z) : x \in \mathbb{Q}, y \in \mathbb{Q}, z \in \mathbb{Q}, 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$

**Find the outer and inner Jordan measure of the following sets! Which set is measurable?**

**9.5.** $H = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$

**9.6.** $H = \{(x,y) : 0 \leq x < 1, 0 < y \leq 1\}$

**9.7.** $H = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$

**9.8.** $H = \{(x,y) : x \in \mathbb{Q}, y \in \mathbb{Q}, 0 \leq x \leq 1, 0 \leq x \leq 1\}$

**Find the outer and inner measure of the following sets! Which set is measurable?**

**9.9.** $H = \{(x,y,z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$

**9.10.** $H = \{(x,y,z) : 0 \leq x < 1, 0 < y < 1, 0 < z < 1\}$

**9.11.** $H = \{(x,y,z) : 0 < x < 1, 0 < y < 1, 0 < z < x + y\}$

**9.12.** $H = \{(x,y,z) : x \in \mathbb{Q}, y \in \mathbb{Q}, z \in \mathbb{Q}, 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

**9.13.** Prove that a bounded set is measurable if and only if its boundary is a null set!

**9.14.** Prove that if the set $H_1$ and $H_2$ are measurable, then the sets $H_1 \cup H_2, H_1 \setminus H_2, H_1 \cap H_2$ are measurable, too!

**9.15.** Are there bounded planar sets $A$ and $B$ such that

**(a)** $b(A \cup B) > b(A) + b(B)$? **(b)** $k(A \cup B) < k(A) + k(B)$?

**9.16.** Are there bounded and disjoint planar sets $A$ and $B$ such that

**(a)** $b(A \cup B) > b(A) + b(B)$? **(b)** $k(A \cup B) < k(A) + k(B)$?

**9.17.** Let's assume that the area of the boundary of a bounded set $H$ is 0. Does it imply that the interior of the set $H$ is empty?

**9.18.** Let's assume that the interior of the bounded set $H$ is empty. Does it imply that $H$ is measurable?

**9.19.** Let $R$ be a brick whose edges are parallel to the axes, and let $H \subset R$ an arbitrary set. Prove that $b(H) + k(R \setminus H) = t(R)$!

**9.20.** Let $R$ be a brick whose edges are parallel to the axes, and let $H \subset R$ be an arbitrary set. Prove that $H$ is measurable if and only if $k(H) + k(R \setminus H) = t(R)$!

**9.21.** Is there a bounded set $H$ such that

**(a)** $k(H) > b(H)$ **(b)** $k(H) < b(H)$

**(c)** $t(\partial H) > b(H)$ **(d)** $t(\partial H) > k(H)$?

**9.22.** Is there a measurable $H$ set such that

**(a)** $k(H) > b(H)$ **(b)** $t(\partial H) = 1$?

**9.23.** Let's assume that the set $H$ is bounded. Is it true that if $H$ is measurable, then $H \cup \partial H$ is also measurable?

**9.24.** Let $K_n$ be a circle on the plane with center at the origin and radius $1/n$. Find the area of the set $\bigcup\limits_{n=1}^{\infty} K_n$!

**9.25.** Let $K_n$ be a circle on the plane with center $(1/n, 1/n)$ and radius $1/n$! Find the area of the set $\bigcup_{n=1}^{\infty} K_n$!

**9.26.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and $G_f = \{(x, y) : a \le x \le b, y = f(x)\}$ be the graph of the function. Prove that $G_f$ is Jordan measurable if and only if $f$ is integrable.

**9.27.** Calculate the outer and inner measure of the set of the points with rational coordinates of the unit square!

**9.28.** Find a bounded, open set on the plane, which has no Jordan area.

**9.29.** Find a bounded, closed set on the plane, which has no Jordan area.

**9.30.** Prove that for an arbitrary bounded set $A \subset \mathbb{R}^n$

$$b(A) = 0 \iff \operatorname{int} A = \emptyset.$$

**9.31.** Prove that if $A \subset \mathbb{R}^n$ is Jordan measurable, then $\forall\, \varepsilon > 0\ \exists\, K \subset A$ closed and $\exists\, G \supset A$ open measurable sets such that

$$t(A) - \varepsilon < t(K) \le t(A) \le t(G) < t(A) + \varepsilon.$$

**9.32.** Let $C \subset \mathbb{R}$ be the Cantor set. $H = C \times [0, 1] \subset \mathbb{R}^2$. Is $H$ Jordan measurable? If yes, what is its area?

**9.33.** Let $H = \bigcup_{n=1}^{\infty} K_n$, where $K_n$ is a circle line with the center at the origin, and radius $1/n$.

**(a)** Is $H$ measurable?

**(b)** Is there an $S \subset \mathbb{R}^2$ (measurable) set such that $\partial S = H$?

**(c)** Is there an $S \subset \mathbb{R}^2$ (measurable) set such that $\partial S \supset H$?

## 9.2 Multivariable Riemann integral

**9.34.** Let $H = [-1, 1] \times [0, 1]$, and

$$f(x, y) = \begin{cases} |x| & \text{if } y \in \mathbb{Q} \\ 0 & \text{if } y \notin \mathbb{Q} \end{cases}$$

Show that $\displaystyle\int_0^1 \left( \int_{-1}^1 f(x, y)\, dx \right) dy = 0$, and $f$ is not integrable on $H$!

**9.35.** Let $H = \{x^2 + y^2 \leq 1\}$, and

$$f(x, y) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Calculate the lower and upper integral of $f$ on the set $H$! Is $f$ integrable on the set $H$?

**Are the following functions integrable on the unit square $N = [0, 1] \times [0, 1]$? If the answer is yes, calculate the values of the integrals!**

**9.36.** $f(x, y) = \begin{cases} 0 & \text{if } y > x \\ 1 & \text{if } y \leq x \end{cases}$

**9.37.** $f(x, y) = \begin{cases} 1 & \text{if } y \geq x \\ 0 & \text{if } y < x \end{cases}$

**9.38.** $f(x, y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$

**9.39.** $f(x, y) = \begin{cases} 1 & \text{if } x, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

**9.40.** $f(x,y) = \begin{cases} 1 & \text{if } x \geq 1/2 \\ 2 & \text{if } x < 1/2 \end{cases}$

**9.41.** $f(x,y) = \begin{cases} 1 & \text{if } y \geq 1/2 \\ 2 & \text{if } y < 1/2 \end{cases}$

**9.42.** $f(x) = D(x)D(y)$, where $D(t) = \begin{cases} 1 & \text{if } t \in \mathbb{Q} \\ 0 & \text{if } t \notin \mathbb{Q} \end{cases}$ is the Dirichlet function.

**9.43.** Let $f(x,y) = \begin{cases} n & \text{if } x+y = 1/n,\ n \in \mathbb{N}^+ \\ 0 & \text{otherwise} \end{cases}$.

Show that $f$ is not integrable on the $N$ unit square, but

$$\int_0^1 \left( \int_0^1 f(x,y)\,dx \right) dy = 0 \text{ and } \int_0^1 \left( \int_0^1 f(x,y)\,dy \right) dx = 0.$$

**Calculate the following integrals on the set $N = [0,1] \times [0,1]$! Apply Fubini's theorem!**

**9.44.** $\iint_N \sin x \, dx\, dy$

**9.45.** $\iint_N \sin y \, dx\, dy$

**9.46.** $\iint_N \sin(x+y) \, dx\, dy$

**9.47.** $\iint_N \sin xy \, dx\, dy$

**9.48.** $\iint_N e^{2x+y} \, dx\, dy$

**9.49.** $\iint_N xy \, dx\, dy$

**Let $N = [0,1]^2 \subset \mathbb{R}^2$, the unit square. Integrate the following $f(x,y)$ functions on $N$:**

**9.50.** $f(x,y) = x$

**9.51.** $f(x,y) = x^3 - x^2y + \sqrt{y}$

**9.52.** $f(x,y) = e^{x+2y}$

**9.53.** $f(x,y) = xe^{xy}$

**9.54.** $f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$ **9.55.** $f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x < y \end{cases}$

**9.56.** $f(x,y) = \begin{cases} 1 & \text{if } x + y = 1 \\ 0 & \text{otherwise} \end{cases}$

**9.57.** $f(x,y) = \begin{cases} 1 & \text{if } x = 1/n,\ n \in \mathbb{N}^+ \\ 0, & \text{otherwise} \end{cases}$

**Find the following integrals on the given rectangles:**

**9.58.** $\iint\limits_T (x+y)\,dx\,dy$ $T:\ 0 \le x \le 1,\ 1 \le y \le 3$

**9.59.** $\iint\limits_T xy\,dx\,dy$ $T:\ 0 \le x \le 1,\ 1 \le y \le 3$

**9.60.** $\iint\limits_T e^{x+y}\,dx\,dy$ $T:\ 0 \le x \le 1,\ 0 \le y \le 1$

**9.61.** $\iint\limits_T xe^y\,dx\,dy$ $T:\ 1 \le x \le 2,\ 3 \le y \le 4$

**9.62.** $\iint\limits_T \frac{x}{y}\,dx\,dy$ $T:\ 1 \le x \le 2,\ 3 \le y \le 4$

**9.63.** $\iint\limits_T x\sin y\,dx\,dy$ $T:\ 0 \le x \le 1,\ 2 \le y \le 3$

**Calculate the following integrals on the set $H = \{(x,y) : 0 \le x \le \pi/2, 0 \le y \le \sin x\}$! Apply Fubini's theorem!**

**9.64.** $\iint\limits_H (x-y)\,dx\,dy$ **9.65.** $\iint\limits_H xy\,dx\,dy$

**9.66.** $\iint\limits_H y\sin x\,dx\,dy$ **9.67.** $\iint\limits_H \frac{x}{\sqrt{1+y^2}}\,dx\,dy$

**Find the following integrals on the given sets:**

**9.68.** $\displaystyle\iint_T (x+y)\,dx\,dy$ $\quad T:\ x^2+y^2 \le 1$

**9.69.** $\displaystyle\iint_T xy\,dx\,dy$ $\quad T:\ (x-1)^2+(y+1)^2 \le 4$

**9.70.** $\displaystyle\iint_T xy\,dx\,dy$ $\quad T:\ \begin{matrix}(x-1)^2+y^2=1\\(x-2)^2+y^2=4\end{matrix}$ domain bounded by the circles.

**9.71.** $\displaystyle\iint_T (x-xy)\,dx\,dy$ $\quad T:\ (x-2)^2+(y+3)^2 \le 4$

**9.72.** $\displaystyle\iint_T e^{-x^2}\,dx\,dy$ $\quad T:\ 0 \le x \le 1, 0 \le y \le x$

**9.73.** $\displaystyle\iint_T (x^2+y^2)^{3/2}\,dx\,dy$ $\quad T:\ x^2+y^2 \le 1$

**Let $T = [0,1]\times[0,2]\times[0,3] \subset \mathbb{R}^3$. Calculate the integrals on the brick $T$:**

**9.74.** $\displaystyle\iiint_T (x+y+z)\,dx\,dy\,dz$

**9.75.** $\displaystyle\iiint_T xyz\,dx\,dy\,dz$

**9.76.** $\displaystyle\iiint_T e^{x+y+z}\,dx\,dy\,dz$

**9.77.** $\displaystyle\iiint_T (xz+y^2)\,dx\,dy\,dz$

**Find the area of the domains bounded by the following curves!**

**9.78.** $y=x^2,\qquad x=y^2$

**9.79.** $y=2x-x^2,\qquad y=x^2$

**9.80.** $2y=x^2,\qquad y=x$

**9.81.** $4y=x^2-4x,\qquad x-y-3=0$

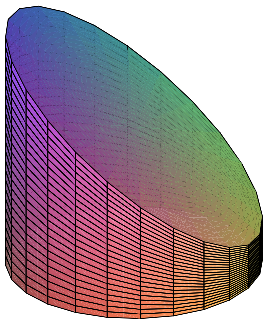
**9.82.** $y=x^2,\ y=2x^2,\ xy=1,\ xy=2$

**9.83.** $x^2 - y^2 = 1,\ x^2 - y^2 = 4,\ xy = 1,\ xy = 2$

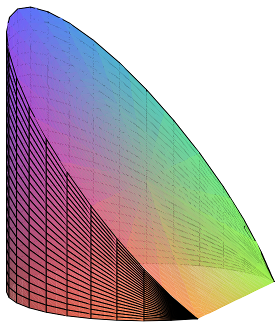
**9.84.** Find the area of the set $H = \{(x, y) : -1 \le x \le 1, 0 \le y \le \sqrt{1 - x^2}\}$.

**9.85.** Calculate the area of the two-dimensional shape bounded by the parabolas $y = x^2$, $y = 2x^2$, and the line $x = 1$.

**9.86.** Calculate the volume of the solid body bounded by the cylinder lateral $x^2 + y^2 = 1$, and the planes $x + y + z = 2$, $z = 0$. The sought solid body:



**9.87.** Calculate the volume of the solid body bounded by the cylinder $x^2 + y^2 = 1$, and the planes $x + y + z = 1$, $z = 0$. The sought solid body:



**9.88.** Calculate the volume of the solid body below the function $f(x, y) = 1 - \frac{x^2}{2} - \frac{y^2}{2}$ and above the set $H$, if $H = [0, 1] \times [0, 1]$!

**9.89.** Calculate the volume of the solid body below the graph of the function $f(x, y) = x + y$ and above the set $H$, if $H = \{0 \le x + y \le 1, 0 \le x, 0 \le y\}$!

**Plot the solid body whose volume can be calculated by the following integrals. Calculate the integrals!**

**9.90.** $$\int_0^1 \left( \int_0^{1-x} \left(x^2+y^2\right)\, dy \right) dx$$

**9.91.** $$\iint_{|x|+|y|\le 1} \left(x^2+y^2\right)\, dy\, dx$$

**Calculate the volume of the solid bodies bounded by the following surfaces!**

**9.92.** $x+y+z=6, \quad x=0, \quad z=0, \quad x+2y=4$

**9.93.** $x-y+z=6, \quad x+y=2, \quad x=y, \quad y=0, \quad z=0$

**9.94.** $z=1-x^2-y^2, \quad x^2+y^2\le 1$

**9.95.** $z=\cos x\cos y, \quad |x+y|\le \dfrac{\pi}{2}, \quad z=0$

**Calculate the volumes of the following solid bodies!**

**9.96.** sphere

**9.97.** ellipsoid

**9.98.** circular cylinder

**9.99.** circular cone

**Let us assume that the two-dimensional range $H$ consist of some material with density $\varrho(x,y)$. The mass of the shape:**

$$M=\iint_H \varrho(x,y)\, dx\, dy,$$

**the coordinates of the center of mass are**

$$S_x=\frac{1}{M}\iint_H x\varrho(x,y)\, dx\, dy, \qquad S_y=\frac{1}{M}\iint_H y\varrho(x,y)\, dx\, dy.$$

**Find the coordinates of the center of mass if $H=[0,1]\times[0,1]$ and**

**9.100.** $\varrho(x, y) = x^2$

**9.101.** $\varrho(x, y) = x + y$

**9.102.** $\varrho(x, y) = xy$

**9.103.** $\varrho(x, y) = x^2 + y^2$

**Find the center of mass of the two-dimensional shape which is bounded by the lines $y = 0$, $x = 2$, $y = 1, y = x$, and its density is**

**9.104.** $\varrho(x, y) = 1$

**9.105.** $\varrho(x, y) = x$

**9.106.** $\varrho(x, y) = y$

**9.107.** $\varrho(x, y) = xy$

**9.108.** $\varrho(x, y) = \dfrac{1}{x + y^3}$

**9.109.** $\varrho(x, y) = e^{x+y}$

**The moment of inertia of a rigid body on the plane $xy$ with respect to $z$ axis is**

$$\Theta = \iint\limits_H r^2(x, y)\varrho(x, y)\, dx\, dy,$$

**where $r(x, y)$ is the distance of the point $(x, y)$ from the $z$ axis. Find the moment of inertia of the unit square on the $xy$ plane with density $\varrho$ with respect to the $z$ axis if one vertex of the square is on the $z$ axis.**

**9.110.** $\varrho(x, y) = 1$

**9.111.** $\varrho(x, y) = xy$

**9.112.** Find the moment of inertia of the squares in the previous problems with respect to the $z$ axis if the midpoint of one side of the square is on the $z$ axis!

**9.113.** A thin membrane is bounded by the line $y = 0, x = 1$ and $y = 2x$. The density of the membrane is $\varrho(x, y) = 6x + 6y + 6$. Find the mass of the membrane and coordinates of the center of mass!

**9.114.** A rigid body is in the first octant, it is bounded by the coordinate planes and the plane $x + y + z = 2$, and its density is $\varrho(x, y, z) = 2x$. Find the mass of the membrane and the coordinates of the center of mass!

**9.115.** We pump the water to the surface from a sump whose depth is 1 meter. What amount of work is needed against the gravitation if the sump is

**(a)** a cube **(b)** a half sphere?

# Chapter 10

# Line Integral and Primitive Function

**10.1 Tangent line.** The equation of the spatial curve $\mathbf{r}(t)$ at the point $\mathbf{r}_0 = \mathbf{r}(t_0)$ is

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v} \cdot t,$$

where $\mathbf{v} = \dot{\mathbf{r}}(t_0)$ is the direction vector of the tangent line.

**10.2 Arc length for planar and spatial curves.**

— If the planar curve $\mathbf{r} : [a, b] \to \mathbb{R}^2$ is continuously differentiable, then it is rectifiable, and the length of its arc is

$$L = \int_a^b |\dot{\mathbf{r}}| \, dt = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2} \, dt.$$

— If $f : [a, b] \to \mathbb{R}$ is continuously differentiable, then its graph is rectifiable, and the length of its arc is

$$L = \int_a^b \sqrt{1 + (f')^2} \, dx.$$

— If the spatial curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is continuously differentiable, then it is rectifiable, and the length of its arc is

$$L = \int_a^b |\dot{\mathbf{r}}| \, dt = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \, dt.$$

**10.3 Tangent plane.** The equation of the tangent plane of the surface $\mathbf{r}(u, v)$ at the point $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$ is

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0,$$

where $\mathbf{n} = \mathbf{r}'_u(u_0, v_0) \times \mathbf{r}'_v(u_0, v_0)$ is the normal vector to the tangent plane. Especially, the tangent plane of the graph of $z = f(x, y)$ at the point $(x_0, y_0)$ is

$$z = f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0).$$

**10.4 Surface area.** If the surface $\mathbf{r} : A \to \mathbb{R}^3$ is continuously differentiable, then the (finite) area of the surface exists, and

$$S = \iint\limits_A |\mathbf{r}'_u \times \mathbf{r}'_v| \, du \, dv.$$

Especially, the surface of the graph of the continuously differentiable function $z = f(x, y)$ over the measurable planar region $A$ is

$$S = \iint\limits_A \sqrt{1 + (z'_x)^2 + (z'_y)^2} \, dx \, dy.$$

**10.5 Calculating the line integral.** If the vector field $\mathbf{v} = v_1(x, y, z) \cdot \mathbf{i} + v_2(x, y, z) \cdot \mathbf{j} + v_3(x, y, z) \cdot \mathbf{k}$ is continuous on the region $G$, and $\mathbf{r} : [a, b] \to G$, $\mathbf{r}(t) = x(t) \cdot \mathbf{i} + y(t) \cdot \mathbf{j} + z(t) \cdot \mathbf{k}$ is continuously differentiable, then the line integral of $\mathbf{v}$ exists along the curve $\Gamma : \mathbf{r}(t)$, and

$$\int\limits_\Gamma v \, d\mathbf{r} = \int\limits_a^b \mathbf{v}(\mathbf{r}(t)) \cdot \dot{\mathbf{r}}(t) \, dt.$$

With coordinates

$$\int\limits_\Gamma v_1(x, y, z) \, dx + v_2(x, y, z) \, dy + v_3(x, y, z) \, dz =$$

$$= \int\limits_a^b [v_1(x, y, z)\dot{x} + v_2(x, y, z)\dot{y} + v_3(x, y, z)\dot{z}] \, dt.$$

In the case of planar vector field and curve

$$\int\limits_\Gamma v_1(x, y) \, dx + v_2(x, y) \, dy = \int\limits_a^b [v_1(x, y)\dot{x} + v_2(x, y)\dot{y}] \, dt.$$

**10.6 Conservative vector field.**

— The vector field $\mathbf{v}$ is conservative on the region $G$ if and only if along all closed and rectifiable $\Gamma$ curves inside $G$

$$\oint_\Gamma \mathbf{v}\, d\mathbf{r} = 0,$$

that is, all closed line integrals are zero.

— **Newton-Leibniz formula** for line integrals.

If the vector field $\mathbf{v}$ is conservative on the region $G$, $U(\mathbf{r})$ is a primitive function of $\mathbf{v}$ on $G$, and $\Gamma$ is a continuously differentiable curve in $G$ with starting point $\mathbf{a}$ and endpoint $\mathbf{b}$, then

$$\int_\Gamma \mathbf{v}\, d\mathbf{r} = U(\mathbf{b}) - U(\mathbf{a}).$$

— If $\mathbf{v}$ is a continuously differentiable and conservative vector field on the region $G$, then

$$\operatorname{curl}\mathbf{v} = \mathbf{0},$$

that is, the vector field is **irrotational**.

— If the vector field $\mathbf{v}$ is continuously differentiable on the **simply connected** domain $G$, and at the point of $G$ $\operatorname{curl}\mathbf{v} = \mathbf{0}$, then $\mathbf{v}$ is conservative.

## 10.1 Planar and Spatial Curves

**Plot the following planar curves!**

**10.1.** $\mathbf{r} = t\cdot\mathbf{i} + t^2\cdot\mathbf{j}$

$t\in[0,4]$

**10.2.** $\mathbf{r} = t^2\cdot\mathbf{i} + t\cdot\mathbf{j}$

$t\in[0,16]$

**10.3.** $\mathbf{r} = \sqrt{t}\cdot\mathbf{i} + t\cdot\mathbf{j}$

$t\in[0,16]$

**10.4.** $\mathbf{r} = t\cdot\mathbf{i} + \sqrt{t}\cdot\mathbf{j}$

$t\in[0,4]$

**10.5.** $\mathbf{r} = 2t \cdot \mathbf{i} + 4t^2 \cdot \mathbf{j}$
$t \in [0, 2]$

**10.6.** $\mathbf{r} = t^2 \cdot \mathbf{i} + t^2 \cdot \mathbf{j}$
$t \in [0, 4]$

**10.7.** $\mathbf{r} = \cos t \cdot \mathbf{i} + \sin t \cdot \mathbf{j}$
$t \in [0, 2\pi]$

**10.8.** $\mathbf{r} = \cos t \cdot \mathbf{i} + \sin t \cdot \mathbf{j}$
$t \in [0, \pi]$

**10.9.** $\mathbf{r} = \cos t \cdot \mathbf{i} + \sin t \cdot \mathbf{j}$
$t \in [-\pi/2, \pi/2]$

**10.10.** $\mathbf{r} = 2\cos t \cdot \mathbf{i} + 4\sin t \cdot \mathbf{j}$
$t \in [0, 2\pi]$

**10.11.** $\mathbf{r} = 4\cos t \cdot \mathbf{i} + 2\sin t \cdot \mathbf{j}$
$t \in [\pi/2, 3\pi/2]$

**10.12.** $\mathbf{r} = \cos t \cdot \mathbf{i} + t\sin t \cdot \mathbf{j}$
$t \in [0, 2\pi]$

**10.13.** $\mathbf{r} = t\cos t \cdot \mathbf{i} + \sin t \cdot \mathbf{j}$
$t \in [0, 6\pi]$

**10.14.** $\mathbf{r} = t\cos t \cdot \mathbf{i} + t\sin t \cdot \mathbf{j}$
$t \in [0, 4\pi]$

**Plot the following spatial curves!**

**10.15.** $\mathbf{r} = t \cdot \mathbf{i} + 2t \cdot \mathbf{j} + 3t \cdot \mathbf{k} \quad t \in [2, 4]$

**10.16.** $\mathbf{r} = -2t \cdot \mathbf{i} + t \cdot \mathbf{j} - (t/3) \cdot \mathbf{k} \quad t \in [2, 4]$

**10.17.** $\mathbf{r} = \cos t \cdot \mathbf{i} + \sin t \cdot \mathbf{j} + t \cdot \mathbf{k} \quad t \in [0, 6\pi]$

**10.18.** $\mathbf{r} = t \cdot \mathbf{i} + \sin t \cdot \mathbf{j} + \cos t \cdot \mathbf{k} \quad t \in [0, 6\pi]$

**10.19.** $\mathbf{r} = 2\sin t \cdot \mathbf{i} - t^2 \cdot \mathbf{j} + \cos t \cdot \mathbf{k} \quad t \in [2, 6\pi]$

**10.20.** $\mathbf{r} = t\cos t \cdot \mathbf{i} + t\sin t \cdot \mathbf{j} + t \cdot \mathbf{k} \quad t \in [2, 6\pi]$

**10.21.** Give an example of a curve which is a

**(a)** cylindrical spiral;
**(b)** conical spiral!

**10.22.** Find the arc length of the previous curves if the height and radius of the cylinder, the circle at the bottom of the cone are given!

**Plot the following curves! Write down the equation of the tangent lines at $t = \pi/4$!**

**10.23.** $\mathbf{r}(t) = 2\cos t \cdot \mathbf{i} + 3\sin t \cdot \mathbf{j} \qquad t \in [0, 8\pi]$

**10.24.** $\mathbf{r}(t) = t\cos t \cdot \mathbf{i} + t\sin t \cdot \mathbf{j} \qquad t \in [0, 8\pi]$

**Write down the equation of the following planar curves at the given points!**

**10.25.** $x^2 - xy^3 + y^5 = 17 \qquad P(5,2)$

**10.26.** $(x^2 + y^2)^2 = 3x^2y - y^3 \qquad P(0,0)$

**Calculate the equations of the tangent lines of the following spatial curves at the given points!**

**10.27.** $\mathbf{r}(t) = (t-3)\cdot\mathbf{i} + (t^2+1)\cdot\mathbf{j} + t^2\cdot\mathbf{k} \quad t = 2$

**10.28.** $\mathbf{r}(t) = \sin t \cdot \mathbf{i} + \cos t \cdot \mathbf{j} + \dfrac{1}{\cos t} \cdot \mathbf{k} \quad \mathbf{p} = \mathbf{j} + \mathbf{k}$

**Find the arc length of the following planar curves!**

**10.29.** (cykloid)

$x = r(t - \sin t) \qquad 0 \le t \le 2\pi$
$y = r(1 - \cos t)$

**10.30.** (Archimedean spiral)

$r = a\varphi \qquad 0 \le \varphi \le 2\pi$

**10.31.** $y = \sqrt{x} \qquad 0 \le x \le a$

## 10.2 Scalar and Vector Fields, Differential Operators

**10.32.** Which geometric transformation corresponds to the mapping

$$f(x, y) = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right)$$

on the plane?

**Give examples of mappings $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that their domain is $H$, and their range is $K$!**

**10.33.** $H = \{(x, y) : 0 < x < 1, 0 < y < 1\}$
$K = \{(x, y) : 0 < x < 2, 0 < y < 4\}$

**10.34.** $H = \{(x, y) : 0 < x < 1, 0 < y < 1, x + y < 1\}$
$K = \{(x, y) : 0 < x < 1, 0 < y < 1\}$

**10.35.** $H = \{(x, y) : 0 < x \leq 1, y = 0\}$ $K = \{(x, y) : x^2 + y^2 = 1\}$

**10.36.** $H = \{(x, y) : 0 < x < 1, 0 < y < 2\}$ $K = \{(x, y) : x^2 + y^2 < 1\}$

**Find grad $f = \nabla f$ if**

**10.37.** $f(x, y) = x^2 - y^2$

**10.38.** $f(x, y) = x^2 + y^2$

**10.39.** $f(x, y) = x^4 - 6x^2y^2 + y^4$

**10.40.** $f(x, y) = \sqrt{x^2 + y^2}$

**Write down the gradients of the following functions at the given points!**

**10.41.** $f(x, y) = x^2 \sin y$ $\mathbf{p} = (\pi/3, -\pi/4)$

**10.42.** $f(x, y) = \sqrt{x} \ln xy$ $\mathbf{p} = (e^2, e^4)$

**10.43.** $f(x, y, z) = x + xy^2 + x^2z^3$ $\quad \mathbf{p} = (2, -1, 1)$

**10.44.** $f(x, y, z) = x \sin y + z^2 \cos y$ $\quad \mathbf{p} = (\pi/2, \pi/6)$

**Calculate the gradients of the following scalar fields, where 'a' is a fixed constant vector, and 'r' is a vector variable:**

**10.45.** $U(\mathbf{r}) = \mathbf{a} \cdot \mathbf{r}$

**10.46.** $U(\mathbf{r}) = |\mathbf{a} \times \mathbf{r}|$

**10.47.** $U(\mathbf{r}) = \mathbf{r}^2 + \dfrac{1}{\mathbf{r}^2}$

**10.48.** $U(\mathbf{r}) = \dfrac{\mathbf{a}^2}{\mathbf{r}^2}$

**Find the tangent planes of the following surfaces at the given points:**

**10.49.** $\mathbf{r} = (u^2 - v) \cdot \mathbf{i} + (u - v^3) \cdot \mathbf{j} - (u + v\cdot)\mathbf{k};$ $\quad u = 1,\ v = 2$

**10.50.** $z = x^2 + y^2;$ $\quad x = 1,\ y = 2$

**Calculate the areas of the following surfaces:**

**10.51.** $\mathbf{r} = u \cos v \cdot \mathbf{i} + u \sin v \cdot \mathbf{j} + u \cdot \mathbf{k}$ $\quad 0 \le u \le 1;\ 0 \le v \le \pi$

**10.52.** $z = \dfrac{x^2}{2y}$ $\quad 0 \le x \le 1; 1 \le y \le 2$

## 10.3 Line Integral

**Let $\mathbf{v} = (x + y) \cdot \mathbf{i} + (x - y) \cdot \mathbf{j}$. Calculate the line integral of v along the following curves!**

**10.53.** $\Gamma : t \cdot \mathbf{i}$ $\quad t \in [0, 1]$

**10.54.** $\Gamma : \begin{cases} t \cdot \mathbf{i} + (t + 1) \cdot \mathbf{j} & \text{if } t \in [-1, 0] \\ t \cdot \mathbf{i} + (-t + 1) \cdot \mathbf{j} & \text{if } t \in (0, 1] \end{cases}$

**10.55.** $\Gamma : \cos t \cdot \mathbf{i} + \sin t \cdot \mathbf{j}$ $\quad t \in [0, \pi]$

**10.56.** $\Gamma : \begin{cases} (1+\cos t)\cdot \mathbf{i} + \sin t \cdot \mathbf{j}\,, & \text{if } t \in [0,\pi] \\ \dfrac{t-\pi}{\pi}\cdot \mathbf{i}\,, & \text{if } t \in (\pi, 2\pi] \end{cases}$

**Let $\mathbf{v} = (x^2 - 2xy)\cdot \mathbf{i} + (y^2 - 2xy)\cdot \mathbf{j}$. Calculate the line integrals of $\mathbf{v}$ along the following curves!**

**10.57.** $\Gamma : t \cdot \mathbf{i} + t^2 \cdot \mathbf{j} \quad t \in [-1,1]$

**10.58.** $\Gamma : t \cdot \mathbf{i} + \mathbf{j} \quad t \in [-1,1]$

**Let the curve $\Gamma_1$ be the line segment between $A(0,0)$ and $B(1,1)$, and $\Gamma_2$ the unit parabola arc between $A(0,0)$ and $B(1,1)$, that is, the graph of the function $y = x^2$ between 0 and 1. Calculate the line integrals of the following mappings along these curves!**

**10.59.** $\mathbf{v} = (x-y)\cdot \mathbf{i} + (x+y)\cdot \mathbf{j}$

**10.60.** $\mathbf{v} = x \cdot \mathbf{i} + y \cdot \mathbf{j}$

**10.61.** $\mathbf{v} = y \cdot \mathbf{i} + x \cdot \mathbf{j}$

**10.62.** $\mathbf{v} = (x^2+y^2)\cdot \mathbf{i} + (x^2-y^2)\cdot \mathbf{j}$

**Let the curve $\Gamma$ be the polygonal chain connecting the points $A(0,0), B(1,0)$ and $C(0,1)$. Calculate line integrals of the following mappings on this curve!**

**10.63.** $\mathbf{v} = -2x \cdot \mathbf{i} + y \cdot \mathbf{j}$

**10.64.** $\mathbf{v} = \mathbf{i} + x^2 \cdot \mathbf{j}$

**10.65.** $\mathbf{v} = y^2 \cdot \mathbf{i} - \mathbf{j}$

**10.66.** $\mathbf{v} = xy \cdot \mathbf{i} + (x+y)\cdot \mathbf{j}$

**10.67.** Let the curve $\Gamma$ be the line segment between the points $A(-2,0)$ and $B(1,0)$. Calculate the line integral of the mapping

$$\mathbf{v} = \frac{2x^3 - 3x}{x^2+y^2}\cdot \mathbf{i} + \frac{1}{x^2+y^2}\cdot \mathbf{j}$$

on this curve!

**Let $\mathbf{v} = (x+y)\cdot\mathbf{i} + (y+z)\cdot\mathbf{j} + (z+x)\cdot\mathbf{k}$. Calculate the line integral of $\mathbf{v}$ on the following curves!**

**10.68.** $\Gamma : t\cdot\mathbf{i} + 2t\cdot\mathbf{j} + 3t\cdot\mathbf{k}$ $\quad t \in [0,1]$

**10.69.** $\Gamma : t\cdot\mathbf{i} + t^2\cdot\mathbf{j}$ $\quad t \in [1,2]$

**10.70.** $\Gamma : \cos t\cdot\mathbf{i} + \sin t\cdot\mathbf{j} + t\cdot\mathbf{k}$ $\quad t \in [0,\pi]$

**10.71.** $\Gamma : \cos t\cdot\mathbf{i} + \sin t\cdot\mathbf{j} + t\cdot\mathbf{k}$ $\quad t \in [\pi,2\pi]$

**Let the curve $\Gamma$ be the line segment between the points $A(0,0,0)$ and $B(1,1,1)$. Calculate the line integrals of the following mappings on that curve!**

**10.72.** $\mathbf{v} = xz\cdot\mathbf{i} + yx\cdot\mathbf{j} + xy\cdot\mathbf{k}$

**10.73.** $\mathbf{v} = (x-y)\cdot\mathbf{i} + (x+y)\cdot\mathbf{j} + z\cdot\mathbf{k}$

**10.74.** $\mathbf{v} = xy\cdot\mathbf{i} + yz\cdot\mathbf{j} + xz\cdot\mathbf{k}$

**10.75.** $\mathbf{v} = y^2\cdot\mathbf{i} + z^2\cdot\mathbf{j} + x^2\cdot\mathbf{k}$

**10.76.** Let the curve $\Gamma$ be the line segment between the points $A(-2,0,1)$ and $B(1,0,3)$. Find the line integral of the mapping

$$\mathbf{v} = \frac{x}{x^2+y^2+z^2}\cdot\mathbf{i} + \frac{1}{x^2+y^2+z^2}\cdot\mathbf{j} + \frac{z}{x^2+y^2+z^2}\cdot\mathbf{k}$$

on this curve!

**Find the line integrals below:**

**10.77.** $\displaystyle\int_C (x^2-2xy)\,dx + (y^2-2xy)\,dy \qquad \Gamma:\ y = x^2 \quad (-1 \le x \le 1)$

**10.78.** $\displaystyle\oint_C (x+y)\,dx + (x-y)\,dy \qquad \Gamma:\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

**10.79.** $\displaystyle\int_C y\,dx + z\,dy + x\,dz$ $\qquad C: \begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases}$ $\qquad 0 \le t \le 2\pi$

**Have the following planar vector fields got primitive functions? If yes, then find them!**

**10.80.** $\mathbf{v} = y \cdot \mathbf{i} + x \cdot \mathbf{j}$

**10.81.** $\mathbf{v} = x \cdot \mathbf{i} + y \cdot \mathbf{j}$

**10.82.** $\mathbf{v} = (x-y) \cdot \mathbf{i} + (y-x) \cdot \mathbf{j}$

**10.83.** $\mathbf{v} = (x^4 + 4xy^3) \cdot \mathbf{i} + (6x^2y^2 - 5y^4) \cdot \mathbf{j}$

**10.84.** $\mathbf{v} = (x+y) \cdot \mathbf{i} + (x-y) \cdot \mathbf{j}$

**10.85.** $\mathbf{v} = e^x \cdot \mathbf{i} + e^y \cdot \mathbf{j}$

**10.86.** $\mathbf{v} = e^y \cdot \mathbf{i} + e^x \cdot \mathbf{j}$

**10.87.** $\mathbf{v} = e^x \cos y \cdot \mathbf{i} - e^x \sin y \cdot \mathbf{j}$

**10.88.** $\mathbf{v} = (x^2 + y) \cdot \mathbf{i} + (x + \cot y) \cdot \mathbf{j}$

**10.89.** $\mathbf{v} = \sin y \cdot \mathbf{i} + \sin x \cdot \mathbf{j}$

**10.90.** $\mathbf{v} = \cos xy \cdot \mathbf{i} + \sin xy \cdot \mathbf{j}$

**10.91.** $\mathbf{v} = y \sin xy \cdot \mathbf{i} + x \sin xy \cdot \mathbf{j}$

**10.92.** $\mathbf{v} = \dfrac{x}{x^2+y^2} \cdot \mathbf{i} + \dfrac{y}{x^2+y^2} \cdot \mathbf{j}$

**10.93.** $\mathbf{v} = \dfrac{x}{x^2+y^2} \cdot \mathbf{i} - \dfrac{y}{x^2+y^2} \cdot \mathbf{j}$

**10.94.** $\mathbf{v} = \dfrac{y}{x^2+y^2} \cdot \mathbf{i} + \dfrac{x}{x^2+y^2} \cdot \mathbf{j}$

**10.95.** $\mathbf{v} = \dfrac{y}{x^2+y^2} \cdot \mathbf{i} - \dfrac{x}{x^2+y^2} \cdot \mathbf{j}$

**10.96.** $\mathbf{v} = \dfrac{y}{(x^2+y^2)^2} \cdot \mathbf{i} + \dfrac{x}{(x^2+y^2)^2} \cdot \mathbf{j}$

**10.97.** $\mathbf{v} = -\dfrac{y}{(x^2+y^2)^2} \cdot \mathbf{i} + \dfrac{x}{(x^2+y^2)^2} \cdot \mathbf{j}$

**Calculate the line integrals of the mappings of the problems from 10.80. to 10.97.**

**10.98.** on the unit circle with center the origin in positive direction;

**10.99.** on the boundary of the square with vertices $A(-1,-1), B(1,-1), C(1,1)$ and $D(-1,1)$ in positive direction.

**Are the following force fields conservative on the whole plane? If yes, give a potential function!**

**10.100.** $\mathbf{E} = 9,81 \cdot \mathbf{j}$

**10.101.** $\mathbf{E} = (y+x) \cdot \mathbf{i} + x \cdot \mathbf{j}$

**10.102.** $\mathbf{E} = (y + \operatorname{sgn} x) \cdot \mathbf{i} + x \cdot \mathbf{j}$

**10.103.** $\mathbf{E} = (x+y) \cdot \mathbf{i} + (x + [y]) \cdot \mathbf{j}$

**10.104.** $\mathbf{E} = x \cdot \mathbf{i} + 2y \cdot \mathbf{j}$

**10.105.** $\mathbf{E} = (x^2 - 2xy) \cdot \mathbf{i} + (y^2 - 2xy) \cdot \mathbf{j}$

**10.106.** $\mathbf{E} = -\dfrac{y}{x^2+y^2} \cdot \mathbf{i} + \dfrac{x}{x^2+y^2} \cdot \mathbf{j}$

**10.107.** $\mathbf{E} = \dfrac{x}{(x^2+y^2)^{3/2}} \cdot \mathbf{i} + \dfrac{y}{(x^2+y^2)^{3/2}} \cdot \mathbf{j}$

**Have the following spatial vector fields got primitive functions? If yes, find them!**

**10.108.** $\mathbf{v} = yz \cdot \mathbf{i} + xz \cdot \mathbf{j} + xy \cdot \mathbf{k}$

**10.109.** $\mathbf{v} = xy \cdot \mathbf{i} + yz \cdot \mathbf{j} + xz \cdot \mathbf{k}$

**10.110.** $\mathbf{v} = (x+y)\cdot\mathbf{i} + (z-y)\cdot\mathbf{j} + xz\cdot\mathbf{k}$

**10.111.** $\mathbf{v} = \dfrac{-x^2+y^2+z^2}{x^2+y^2+z^2}\cdot\mathbf{i} + \dfrac{x^2-y^2+z^2}{x^2+y^2+z^2}\cdot\mathbf{j} + \dfrac{x^2+y^2-z^2}{x^2+y^2+z^2}\cdot\mathbf{k}$

**10.112.** $\mathbf{v} = 2xy^3z^4\cdot\mathbf{i} + 3x^2y^2z^4\cdot\mathbf{j} + 4x^2y^3z^3\cdot\mathbf{k}$

**10.113.** $\mathbf{v} = 3xy^3z^4\cdot\mathbf{i} + 3x^2y^2z^4\cdot\mathbf{j} + x^2y^3z^3\cdot\mathbf{k}$

**10.114.** $\mathbf{v} = \sin y\cdot\mathbf{i} + x\cos y\cdot\mathbf{j} + 2z\cdot\mathbf{k}$

**10.115.** $\mathbf{v} = e^x z\sin y\cdot\mathbf{i} + e^x z\cos y\cdot\mathbf{j} + e^x\sin y\cdot\mathbf{k}$

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**10.116.** Which line integrals of the mappings of the problems between 10.108. and 10.115. are 0 on the boundary of an arbitrary spatial circle with center $(3,4,5)$ and radius 1?

**10.117.** Calculate the line integral below, and show that the cross derivatives are equal:

$$\oint_C \frac{y\,dx - x\,dy}{x^2+y^2}, \qquad \Gamma:\ x^2+y^2=R^2$$

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## Find the primitive function $z(x,y)$:

**10.118.** $dz = (x^2+2xy-y^2)\,dx + (x^2-2xy-y^2)\,dy$

**10.119.** $dz = \dfrac{y\,dx - x\,dy}{3x^2-2xy+3y^2}$

**10.120.** $dz = \dfrac{(x^2+2xy+5y^2)\,dx + (x^2-2xy+y^2)\,dy}{(x+y)^3}$

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## Find the primitive function $u(x,y,z)$:

**10.121.** $du = (x^2-2yz)\,dx + (y^2-2xz)\,dy + (z^2-2xy)\,dz$

**10.122.** $du = \left(1-\dfrac{1}{y}+\dfrac{y}{z}\right)dx + \left(\dfrac{x}{z}+\dfrac{x}{y^2}\right)dy - \dfrac{xy}{z^2}\,dz$

**10.123.** $du = \dfrac{(x+y)\,dx + (x+y)\,dy + z\,dz}{x^2+y^2+z^2+2xy}$

**The gravitation force between the mass point $M$ at the origin and the mass point $m$ at the point $(x, y, z)$ is**

$$c\frac{Mm}{x^2+y^2+z^2},$$

**where $c$ is a constant. The direction of the force is the same as the line segment with initial point $(x, y, z)$ and origin terminal point. Calculate the work of the gravitational force, if the mass point $m$ moves on the following curves:**

**10.124.** $\Gamma : \cos t \cdot \mathbf{i} + \sin t \cdot \mathbf{j} \quad t \in [0, 2\pi]$

**10.125.** $\Gamma : \cos t \cdot \mathbf{i} + \sin t \cdot \mathbf{j} \quad t \in [0, \pi]$

**10.126.** $\Gamma : t \cdot \mathbf{i} + 2t \cdot \mathbf{j} + 3t \cdot \mathbf{k} \quad t \in (0, 1]$

**10.127.** $\Gamma$ : the boundary of the square with vertices

$$A(-1, -1, 0), \quad B(1, -1, 0), \quad C(1, 1, 0), \quad D(-1, 1, 0)$$

in positive direction.

**10.128.** Find the potential function of the gravitational force in the previous problems!

**The force between the point charge $Q$ at the origin and the point charge $q$ at the point $(x, y, z)$ is**

$$c\frac{Mm}{x^2+y^2+z^2},$$

**where $c$ is a constant, and the initial point of the force vector is the origin, and the terminal point is $(x, y, z)$.**

**10.129.** Find the work of the electrostatic force, when the charge $q$ moves from the point $(1, 2, 3)$ to the point $(5, 6, 7)$. Does the work depend on the path?

**10.130.** Find the work of the electrostatic force, when the charge $q$ moves from the point $(1,2,3)$ to infinity! Does the work depend on the path?

**10.131.** Find the potential function of the electrostatic force in the previous problems!

**10.132.** The force of friction between the surface of a table and a slipping body with mass $m$ is $c \cdot m$, where $c$ is a constant. The direction of the force is the opposite of the direction of the displacement. Find the work of the slip force when the body moves from the point $(0,0)$ to the point $(3,4)$ along the line segment! Find the work of the slip force when the body moves along the connecting polygonal chain of the points $(0,0)$, $(3,0)$ and $(3,4)$. Does the work depend on the path?

**10.133.** Has the slip force in the previous problem got a potential function?

# Chapter 11

# Complex Functions

**11.1 Cauchy-Riemann's differential equations.** If $f(z) = f(x+iy) = u(x,y) + i\cdot v(x,y)$ is differentiable at the point $z_0 = x_0 + iy_0$, then

$$u'_x(x_0,y_0) = v'_y(x_0,y_0), \qquad u'_y(x_0,y_0) = -v'_x(x_0,y_0).$$

In reverse, if the equations above are fulfilled at the point $(x_0, y_0)$, and $u$ and $v$ are totally differentiable (as two-variable real functions) at the point $(x_0, y_0)$, then the complex function $f(z)$ is complex differentiable at $z_0$.

**11.2 Cauchy's integral theorem.** If $f$ is analytic on the interior of the simple closed curve $\Gamma$, that is on the set $\Omega$, and $f$ is continuous at the points of $\Gamma$, then

$$\oint_{\Gamma} f(z)\,dz = 0.$$

**11.3 Cauchy's integral formula.** If $f$ is analytic at $a$, $\Gamma$ is a closed circle line going around $a$ in positive direction inside the domain where $f$ is regular, then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{\Gamma}\frac{f(z)}{(z-a)^{n+1}}\,dz.$$

**11.4 Holomorphic functions.**

— **Maximum modulus principle.** If $f$ is a holomorphic function on a simple connected domain, then the modulus $|f|$ cannot have a (local) maximum within the simple connected domain.

— **Liouville's theorem.** Every bounded entire function must be constant.

— **Rouché's theorem.** Let $\Gamma$ be a simple, closed curve on the complex plane, its interior is $\Omega$, $f$ and $g$ are continuous complex functions on

$\overline{\Omega} = \Omega \cup \Gamma$, and $f$ and $g$ are holomorphic on $\Omega$, and assume that for all $z \in \Gamma$

$$|g(z)| > |f(z) - g(z)|.$$

In this case the two functions have the same number of roots counted with multiplicity on $\Omega$.

## 11.5 Meromorphic functions.

— If $f(z)$ can be expressed as a Laurent series around $a$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

then

$$\operatorname{Res}(f,a) = a_{-1} = \frac{1}{2\pi i} \oint_{\Gamma} f(z)\, dz,$$

where $\Gamma$ is a positive directional circle line with radius less than the radius of convergence of the Laurent series.

— **Residue theorem.** If $D \subset \mathbb{C}$ is a simple connected open subset of the complex plane, $f$ is meromorphic on $D$, and $\Gamma$ is a simple, closed curve on $D$, and $\Gamma$ does not meet any of the poles, then

$$\oint_{\Gamma} f(z)\, dz = 2\pi i \sum \{\operatorname{Res}(f,a) : a \in \Omega\},$$

where $\Omega$ is the interior of the curve $\Gamma$.

**11.1.** Prove that the reciprocal of the complex conjugate of a complex number $z$ equals to the complex conjugate of the reciprocal of the complex number $z$!

**11.2.** Let's assume that $|z| < 1$ and $|\alpha| < 1$. Prove that $\left|\dfrac{z-\alpha}{1-z\overline{\alpha}}\right| < 1$.

**Prove that the Cauchy-Riemann's differential equations are fulfilled for the following functions.**

**11.3.** $f(z) = z^2$

**11.4.** $f(z) = z^n, \quad n \in \mathbb{N}^+$

**11.5.** $f(z) = \frac{1}{z}, \quad z \neq 0$

**11.6.** $f(z) = \frac{1}{z^2+1}$

**11.7.** Check whether Cauchy-Riemann's differential equations are fulfilled for the function $f(z) = \sqrt{|xy|}$ at $z = 0$, where $x = \mathrm{Re}(z), y = \mathrm{Im}(z)$. Is the function differentiable at $z = 0$?

**11.8.** Prove that the function $f(z) = 2x^2 + 3y^2 + xy + 2x + i(4xy + 5y)$ is not differentiable on any domains of the plane!

**Find the points, where $f$ is differentiable!**

**11.9.** $f(x+iy) = xy + iy$

**11.10.** $f(x+iy) = \left(2x^2 - y\right) + i\left(x^2 + y^2\right)$

**Find the differentiable function $f(x+iy) = u(x,y) + iv(x,y)$, if**

**11.11.** $u(x,y) = x^2 - y^2 + xy, \quad f(0) = 0$

**11.12.** $v(x,y) = \frac{x^2}{x^2+y^2}, \quad f(2) = 0$

**Find the radius of convergence of the following power series!**

**11.13.** $\sum_{n=1}^{\infty} \frac{1}{n}(z-i)^n$

**11.14.** $\sum_{n=1}^{\infty} 2^n(z+i)^n$

**11.15.** $\sum_{n=1}^{\infty} n^2(z-2-2i)^n$

**11.16.** $\sum_{n=1}^{\infty} \frac{(n-1)!}{n!} z^n$

**11.17.** $\sum_{n=0}^{\infty} \frac{(n^2)!}{3^{n!}} z^n$

**11.18.** $\sum_{n=1}^{\infty} \ln(n!) z^n$

**11.19.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} z^n$

**11.20.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{in}{n+1}\right)^{n^2} z^n$

**Find the radius of convergence and the sum of the following power series.**

**11.21.** $\displaystyle\sum_{n=1}^{\infty} z^n$

**11.22.** $\displaystyle\sum_{n=0}^{\infty} i^n z^n$

**11.23.** $\displaystyle\sum_{n=0}^{\infty} (n+1) z^n$

**11.24.** $\displaystyle\sum_{n=0}^{\infty} (n+2)(n+1) z^n$

**Prove the Euler-formulas using the suitable power series:**

**11.25.** $e^{iz} = \cos z + i \sin z$

**11.26.** $e^{-iz} = \cos z - i \sin z$

**11.27.** $\cos z = \dfrac{1}{2}\left(e^{iz} + e^{-iz}\right)$

**11.28.** $\sin z = \dfrac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

**11.29.** Applying the Euler-formulas prove that $e^z$ has a period $2\pi i$.

**11.30.** Prove that the complex functions $\sin z$ and $\cos z$ have the same roots as the real functions $\sin x$ and $\cos x$.

**Integrate the following functions on the curve $\Gamma$: $|z| = 1$ in positive direction.**

**11.31.** $f(x+iy) = x$

**11.32.** $f(x+iy) = y$

**11.33.** $f(x+iy) = x - iy$

**11.34.** $f(x+iy) = x + iy$

**Integrate the following functions on the curve $\Gamma$: $|z| = R$ in positive direction.**

**11.35.** $f(z) = \dfrac{1}{z}$

**11.36.** $f(z) = \dfrac{1}{z^2}$

**Find the integral of $f(z) = |z|$ on the following curves from the point $z_1 = -1$ to the point $z_2 = i$. Decide whether the value of the integral is independent of the curves.**

**11.37.** $\Gamma = \{e^{-it} : t \in [\pi, 3\pi/2]\}$

**11.38.** $\Gamma = \{t : t \in [-1, 0]\} \bigcup \{it : t \in [0, 1]\}$

**11.39.** Find the line integral $\displaystyle\int (x^2 - y^2)\, dx - 2xy\, dy$ with the help of the real primitive function on the line segment with starting point $1 + i$ and endpoint $3 + 2i$.

**11.40.** Find the line integral $\displaystyle\int (x^2 - y^2)\, dx - 2xy\, dy$ by using the Cauchy's integral theorem on the line segment with starting point $1 + i$ and endpoint $3 + 2i$.

**Let $\Gamma : z(t) = 1 + it, t \in [0, 1]$. Find the integrals of the following functions by using the Cauchy's integral theorem on the curve $\Gamma$.**

**11.41.** $f(z) = 3z^2$

**11.42.** $f(z) = \dfrac{1}{z}$

**11.43.** $f(z) = e^z$

**11.44.** $f(z) = ze^{z^2}$

**Find the value of the complex line integrals of $\displaystyle\int_\Gamma \frac{1}{z^2+1}\, dz$, where $\gamma$ is the following simple closed curve.**

**11.45.** $\Gamma :\ |z| = 1/2$

**11.46.** $\Gamma :\ |z| = 3$

**11.47.** $\Gamma :\ |z - i| = 1$

**11.48.** $\Gamma :\ |z + i| = 1$

**Find the power series of the following function around the given point $a$!**

**11.49.** $\dfrac{1}{(1-z)^2}, \quad a=3$

**11.50.** $\dfrac{1}{(z-2)(z-3)}, \quad a=5$

**11.51.** $\dfrac{1}{1-z+z^2}, \quad a=0$

**11.52.** $\dfrac{3z-6}{(z-4)(z+25)}, \quad a=10$

**Find the residue of the function $f(z)$ at $z_0=0$.**

**11.53.** $f(z)=\dfrac{e^z}{z^2}$

**11.54.** $f(z)=\dfrac{\cos z}{\sin z}$

**Find the residue of the function $f(z)=\dfrac{1}{z^3-z^5}$ at $z_0$.**

**11.55.** $z_0=0$

**11.56.** $z_0=1$

**11.57.** $z_0=-1$

**11.58.** $z_0=i$

**Find the residue of the function $f(z)=\dfrac{z^2}{(z^2+1)^2}$ at $z_0$.**

**11.59.** $z_0=i$

**11.60.** $z_0=-i$

**Find the complex line integrals of the form $\displaystyle\int_{|z|=4} f(z)\,dz$!**

**11.61.** $f(z)=\dfrac{e^z\sin z}{z-1}$

**11.62.** $f(z)=\dfrac{e^{\sin z}}{z^2}$

**11.63.** $f(z)=\dfrac{e^{\sin z}}{z-2}$

**11.64.** $f(z)=\dfrac{e^{\sin z}}{(z-1)(z-2)}$

**11.65.** $f(z)=\dfrac{e^z\cos z}{z-\pi}$

**11.66.** $f(z)=\dfrac{e^{\sin z}}{z^2-1}$

**Let $\Gamma: z(t)=t+it, t\in[0,1]$. Integrate the following functions on the curve $\Gamma$.**

**11.67.** $f(z) = z^2$

**11.68.** $f(z) = e^z$

**Let $\Gamma$ be the circle line $|z - 2i| = 1$, and find the integrals of the following functions in positive direction on the curve $\Gamma$.**

**11.69.** $f(z) = z^2$

**11.70.** $f(z) = \dfrac{1}{z}$

**11.71.** $f(z) = z^2 + \dfrac{1}{z}$

**11.72.** $f(z) = z + \dfrac{1}{z}$

**How many roots do the following equations have on the disk $|z| < 1$? (Hint: apply Rouché's theorem.)**

**11.73.** $z^6 - 6z + 10 = 0$

**11.74.** $z^4 - 5z + 1 = 0$

**11.75.** Find the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx$ by integration over a curve on the complex plane!

**11.76.** Find the image of the circle with center $z = 0$ and radius 1, if the transformation is $f(z) = \dfrac{az+b}{cz+d}$.

**11.77.** Find an $f(z)$ transformation which is a mapping between the upper halfplane and the circle with center $z = 0$ and radius 1.

**Find the curves or domains given by the following conditions!**

**11.78.** $|z - 2| < |z|$

**11.79.** $\left|z^2 - 1\right| < 1$

**11.80.** $\operatorname{Im} \dfrac{1}{z} = 2$

**11.81.** $\operatorname{Re} z = \operatorname{Im} z$

**The function $w = f(z)$ maps the plane $z = x + iy$ to the plane $w = u + iv$. Find the images of the given $T$ domains!**

**11.82.** $w = z^2, \quad T = \{x + iy : x \geq 0, y \geq 0\}$

**11.83.** $w = e^z, \quad T = \{x + iy : 0 < y < \dfrac{\pi}{2}\}$

# Solutions

## Basic Notions, Real Numbers

### 1.1 Elementary Exercises

**1.1.** The solution is the open interval $(2, 8)$.


**1.2.** The solution is the same as in the previous exercise.

**1.3.** Solution: $(4, 6)$.


**1.5.** The original inequality:

$$\frac{1}{5x+6} \geq -1$$

Let's multiply both sides of the inequality by $5x + 6$. We have two cases:

Case I: $5x + 6 > 0$, that is $x > -6/5$. Now the new inequality:

$$1 \geq -(5x+6), \quad 5x \geq -7, \quad x \geq -7/5$$

In this case this is only possible if $x > -6/5$.

Case II: $5x+6 < 0$, that is $x < -6/5$. Now the inequality sign changes:

$$1 \leq -(5x+6), \quad 5x \leq -7, \quad x \leq -7/5$$

In this case this is only possible if $x \leq -7/5$.

Therefore, the solution is the union of a closed and an open half-line:

$$x \in (-\infty, -7/5] \cup (-6/5, \infty).$$

**1.7.** The original inequality:

$$10x^2 + 17x + 3 \leq 0.$$

The main coefficient of the quadratic polynomial on the left side is positive, so the points of the parabola are below the $x$ axis (in the interval) between the roots. Let's find the roots:

$$10x^2 + 17x + 3 = 0$$

$$x_1 = -\frac{3}{2}, \quad x_2 = -\frac{1}{5}$$

Therefore the solution is:

$$x \in [-3/2, -1/5].$$

**1.9.** The original inequality:

$$8x^2 - 30x + 25 \geq 0$$

Let's find the roots of the quadratic equation:

$$8x^2 - 30x + 25 = 0$$

$$x_{1,2} = \frac{30 \pm \sqrt{900 - 800}}{16}, \quad x_1 = \frac{5}{2}, \; x_2 = \frac{5}{4}$$

Since the main coefficient is positive, the quadratic polynomial is positive outside of $[x_2, x_1]$, therefore the solution is:

$$x \in (-\infty, 5/4] \cup [5/2, \infty).$$

**1.11.** The original inequality:

$$9x^2 - 24x + 17 \geq 0.$$

Let's find the roots of the quadratic equation:

$$9x^2 - 24x + 17 = 0.$$

Since the discriminant of the equation is negative $(-36)$, the quadratic polynomial has no roots. Since the main coefficient is positive, for all $x \in \mathbb{R}$

$$9x^2 - 24x + 17 > 0.$$

Therefore the solution is all real numbers: $x \in \mathbb{R}$.

**1.14.** For which $x \in \mathbb{R}$ is it true:

$$|x+1| + |x-2| \leq 12\ ?$$

There are three cases, according to the sign of $x+1$ and $x-2$.

Case I: $x < -1$, that is both terms are negative.

$$-(x+1)-(x-2) \leq 12, \quad -2x \leq 11, \quad x \geq -\frac{11}{2}$$

The solution in this case:

$$x \in [-11/2; -1)$$

Case II: $-1 \leq x < 2$, that is the first term is nonnegative, the second term is negative.

$$(x+1)-(x-2) \leq 12, \quad 3 \leq 12, \quad x \text{ arbitrary}$$

The solution in this case:

$$x \in [-1; 2)$$

Case III: $x \geq 2$, that is neither of the terms are negative.

$$(x+1)+(x-2) \leq 12, \quad 2x \leq 13, \quad x \leq \frac{13}{2}$$

The solution in this case:

$$x \in [2; 13/2]$$

Therefore, the solution of the exercise:

$$x \in [-11/2; 13/2].$$

**1.16.** For which $x \in \mathbb{R}$

$$\left|\frac{x+1}{2x+1}\right| > \frac{1}{2} \; ?$$

Case I: $x < -1$, that is the numerator and the denominator are negative.

$$\frac{x+1}{2x+1} > \frac{1}{2}, \quad 2(x+1) < 2x+1, \quad 2 < 1$$

There is no solution in this case.

Case II: $x > -1/2$, that is the numerator and the denominator are positive.

$$\frac{x+1}{2x+1} > \frac{1}{2}, \quad 2(x+1) > 2x+1, \quad 2 > 1$$

The solution in this case:

$$x \in (-1/2; \infty)$$

Case III: $-1 < x < -1/2$, that is the numerator is positive, the denominator is negative.

$$\left|\frac{x+1}{2x+1}\right| = -\frac{x+1}{2x+1} > \frac{1}{2}, \quad -2(x+1) < 2x+1, \quad 4x > -3, \quad x > -\frac{3}{4}$$

The solution in this case:

$$x \in (-3/4; -1/2)$$

There are no other cases, because the numerator is positive if the denominator is positive. Therefore the solution is:

$$x \in (-3/4; -1/2) \cup (-1/2; \infty).$$

**1.18.**

$$\sqrt{x+3} + |x-2| = 0$$

Both terms are nonnegative, so the sum can be zero only if both terms are zero. Therefore,

$$x + 3 = 0 \text{ and } x - 2 = 0.$$

There is no $x$, for which these two equations are fulfilled at the same time.

## 1.2 Basic Logical Concepts

**1.22.** Yes:
(b)+(c) $\implies$ All animals are either mammals or have a gill.
(b)+(a) $\implies$ If an animal is a mammal, then it has a gill.
Therefore an animal is either a mammal or it is not, in any case it has a gill.

**1.25.** For this sentence we cannot assign a truth value, if there is only one Mohican. If the sentence were true, then the last of the Mohicans would have lied, therefore the sentence would be false. If the sentence were false, then the last of the Mohicans would tell the truth, that is, not all of the Mohicans are liars.

**1.27.** Only the statement **(c)**, moreover the two statements are equivalent:

$$(A \implies B) \iff (\neg B \implies \neg A)$$

**1.29.** For $2^{98}$ subsets it is true, for $2^{100} - 2^{98} = 3 \cdot 2^{98}$ it is not true.

**1.31.** 1 is an element: $2^{99}$ subsets.

2 is not an element: $2^{99}$ subsets.

1 is an element and 2 is not an element: $2^{98}$ subsets.

1 is an element or 2 is not an element: $2^{99} + 2^{99} - 2^{98} = 3 \cdot 2^{98}$

The complement: There are $2^{98}$ subsets, such that 1 is not an element and 2 is an element of the subset.

**1.33.** (b) $\implies$ (a), (c) $\implies$ (d).

The other implications are not true.

## 1.3 Methods of Proof

**1.34.** Let us assume indirectly that there is $p, q \in \mathbb{N}^+$ such that $\sqrt{3} = \dfrac{p}{q}$. We can also assume that $p$ and $q$ are relatively primes.

$$3 = \frac{p^2}{q^2}, \quad 3q^2 = p^2.$$

Therefore, $p^2$ is divisible by 3, but since 3 is a prime number, $p$ is divisible by 3: $p^2 = 9r^2$. Therefore,

$$3q^2 = 9r^2, \quad q^2 = 3r^2.$$

Repeating the previous reasoning for $q$ instead of $p$, we have that $q$ is divisible by 3, which contradicts the assumption that $p$ and $q$ are relatively primes.

**1.36.** Let us assume indirectly that

$$r = \frac{\dfrac{\sqrt{2}+1}{2} + 3}{4} + 5$$

is rational. But in this case

$$(r-5)\cdot 4 = \frac{\sqrt{2}+1}{2} + 3$$

is also rational. Continuing this reasoning, we have that $\sqrt{2}$ rational. This is a contradiction. We can prove that $\sqrt{2}$ is irrational with the same reasoning as we proved that $\sqrt{3}$ is irrational.

**1.38.**

(a) Cannot: if $x+y$ were rational, then $(x+y)-x=y$ would be also rational.

(b) Cannot: if $x-y$ were rational, then $x-(x-y)=y$ would be also rational.

(c) Can, but only if $x=0$.

(d) Can, but only if $x=0$.

**1.40.**

(a) True.

(b) False: e.g. $a=\sqrt{2}, \quad b=-\sqrt{2}$

(c) False: e.g. $a+b$ is irrational.

(d) True.

**1.42.**

Proof by induction:

$n=1: 16|16$

For $n+1$: Since if $k|a-b$ and $k|b$, then $k|a$, it is enough to prove that

$16|(5^{n+2}-4(n+1)-5)-(5^{n+1}-4n-5)$, that is, $16|4\cdot(5^{n+1}-1)$.

It is fulfilled if $4|5^{n+1}-1$. It is easy to prove this by induction.

**1.43.**

Indirect proof: Let us assume that $\tan 1°$ is rational. Now we prove by induction that $\tan n°$ is rational for all $n\in\mathbb{N}^+,\ n<90$. We can prove this by using the trigonometric addition formula:

$$\tan(n+1)° = \frac{\tan n° + \tan 1°}{1+\tan n°\cdot\tan 1°}.$$

In this rational formula all terms are rational, so the result is also rational. Since $\tan 30°$ irrational, we have a contradiction.

**1.44.**

According to the inequality of arithmetic and geometric means

$$\sqrt[n]{n!} = \sqrt[n]{1\cdot 2\cdots n} \le \frac{1+2+\cdots n}{n} = \frac{n+1}{2}.$$

Let's raise both sides to the power of $n$, we have the result.

**1.45.** Yes, it is true. Let us assume indirectly that for all $n$ we have $a_n \geq 10^{-6}$, and therefore $a_n^2 \geq d = 10^{-12} > 0$. Therefore, $a_{n+1} \leq a_n - d$, and in general $a_{n+k} \leq a_n - k \cdot d$. Therefore, let $k = 10^{12} = \dfrac{1}{d}$, so $a_{1+k} \leq 0.9 - 1 < 0 < 10^{-6}$. That is a contradiction.

**1.47.** **(a)** Let us denote the sum by $s_n$, that is

$$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n}.$$

We calculate the value of $s_n$ for $n = 2,\ 3,\ 4$.

$$s_2 = \frac{1}{2},\ s_3 = \frac{2}{3},\ s_4 = \frac{3}{4},\ \cdots$$

The sum is:

$$s_n = \frac{n-1}{n} = 1 - \frac{1}{n}.$$

Proof by induction: It is true for $n = 2$. Let us assume that the formula is valid for $n$.

$$\begin{aligned} s_{n+1} &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} + \frac{1}{n \cdot (n+1)} \\ &= s_n + \frac{1}{n \cdot (n+1)} = \frac{n-1}{n} + \frac{1}{n \cdot (n+1)} \\ &= \frac{(n-1)(n+1)+1}{n \cdot (n+1)} = \frac{n}{n+1} \end{aligned}$$

**(b)** Let $s_n = 1 + 3 + \ldots + (2n-1)$, the sum of the odd numbers.

$$s_1 = 1,\ s_2 = 4,\ s_3 = 9,\ s_4 = 16,\ \cdots$$

The sum: $s_n = n^2$.
By induction: the statement is true for $n = 1$. Let us assume that the statement is true for $n$.

$$\begin{aligned} s_n + 1 = 1 + 3 + \ldots + (2n-1) + (2n+1) &= s_n + (2n+1) \\ = n^2 + 2n + 1 = (n+1)^2 & \end{aligned}$$

**1.58.** By induction: the statement is true if $n = 0$. Let us assume that it is true for $n$, and the positive $k$ is a divisor of both $u_{n+1}$ and $u_{n+2}$. In this case $k$ is also a divisor of $u_{n+2} - u_{n+1} = u_n$. According to the assumption $u_n$ and $u_{n+1}$ are relatively primes, so $k = 1$, therefore $u_{n+1}$ and $u_{n+2}$ are also relatively primes.

**1.59.** By induction:
It is easily proved for $n = 1$ and $n = 2$.
Let us assume that $n > 2$, and the statement is true for $n-1$ and $n-2$.
$u_n = u_{n-1} + u_{n-2} > \frac{1.6^{n-1}}{3} + \frac{1.6^{n-2}}{3} = \frac{1.6^{n-2}}{3}(1.6+1) > \frac{1.6^{n-2}}{3}1.6^2 = \frac{1.6^n}{3}$
On the other hand,
$u_n = u_{n-1} + u_{n-2} < 1.7^{n-1} + 1.7^{n-2} = 1.7^{n-2}(1.7+1) < 1.7^{n-2}1.7^2 = 1.7^n$.

**1.60.** By induction: For $n = 1$ the given statements are true in every exercise. Let us assume that the given statement is true for $n$, and we have to prove the statement for $n+1$.

(a) $(u_1 + u_2 + \cdots u_n) + u_{n+1} = u_{n+2} - 1 + u_{n+1} = u_{n+3} - 1$

(b) $u_{n+1}^2 - u_n u_{n+2} = u_{n+1}^2 - u_n(u_{n+1} + u_n) = u_{n+1}(u_{n+1} - u_n) - u_n^2 = u_{n+1}u_{n-1} - u_n^2 = -(-1)^{n+1} = (-1)^{n+2}$

(c) $(u_1^2 + u_2^2 + \cdots + u_n^2) + u_{n+1}^2 = u_n u_{n+1} + u_{n+1}^2 = u_{n+1}(u_n + u_{n+1}) = u_{n+1}u_{n+2}$

**1.61.** The following expressions are easily proved by induction:

(a) $s_n = \alpha \cdot u_{2n+1} + \beta = u_{2n+1} - 1$.

(b) $s_n = \alpha \cdot u_{2n+2} + \beta = u_{2n+2}$.

(c) $s_n = \alpha \cdot u_{3n+2} + \beta = \frac{u_{3n+2} - 1}{2}$.

(d) $s_n = u_{2n}^2$

**1.63.** The proof only works if $n$ is at least 2. But for $n = 1$ and $n = 2$ the statement is not proved.

**1.67.** $a^2bc$ has four factors. We can write the 3-terms sum as a 4-terms sum, and we can apply the arithmetic and geometric means inequality for the 4 terms:

$$\frac{\frac{a}{2} + \frac{a}{2} + b + c}{4} \geq \sqrt[4]{\frac{a}{2} \cdot \frac{a}{2} \cdot b \cdot c} = \sqrt[4]{\frac{1}{4}a^2bc}.$$

Therefore,

$$\frac{18}{4} = \frac{9}{2} \geq \sqrt[4]{\frac{1}{4}a^2bc}$$

$$a^2bc \leq 4\left(\frac{9}{2}\right)^4.$$

The number on the right-hand side is the maximum, because there is equality, if $\frac{a}{2} = b = c = \frac{18}{4} = \frac{9}{2}$.

**1.69.** Apply the arithmetic and geometric means inequality.

$$\frac{abc}{ab+bc+ac} = \frac{1}{3} \cdot \frac{3}{\frac{1}{c}+\frac{1}{a}+\frac{1}{b}} \leq \frac{1}{3} \cdot \frac{a+b+c}{3} = \frac{18}{9} = 2$$

The equation is true if and only if $a = b = c = 6$.

**1.71.**

$$2a+b+c = 3 \cdot \frac{2a+b+c}{3} \geq 3\sqrt[3]{(2a)bc} = 3\sqrt[3]{2 \cdot 18} = 3\sqrt[3]{36}$$

There is equality if and only if $2a = b = c = \sqrt[3]{36}$.

**1.73.** Apply the arithmetic and geometric means inequality.

$$a^2+b^2+c^2 = 3\left(\sqrt{\frac{a^2+b^2+c^2}{3}}\right)^2 \geq 3\left(\sqrt[3]{abc}\right)^2 = 3\left(\sqrt[3]{18}\right)^2$$

There is equality if and only if $a = b = c = \sqrt[3]{18}$.

**1.75.**

$$a + \frac{1}{a} = 2 \cdot \frac{a+\frac{1}{a}}{2} \geq 2 \cdot \sqrt{a \cdot \frac{1}{a}} = 2$$

**1.79.**

$$f(x) = x(1-x) = \left(\sqrt{x(1-x)}\right)^2 \leq \left(\frac{x+(1-x)}{2}\right)^2 = \frac{1}{4}$$

There is equality if $x = 1 - x = \frac{1}{2}$.

**1.80.**

$$f(x) = x + \frac{4}{x} = 2 \cdot \frac{x + \frac{4}{x}}{2} \geq 2 \cdot \sqrt{x \cdot \frac{4}{x}} = 4$$

The minimum is at the point where the equality holds: $x = \frac{4}{x} = 2$.

**1.82.** Applying the arithmetic and geometric means inequality:

$$x^2(1-x) = 4 \cdot \frac{x}{2} \cdot \frac{x}{2} \cdot (1-x) \leq 4 \cdot \left( \frac{\frac{x}{2} + \frac{x}{2} + (1-x)}{3} \right)^3 = \frac{4}{27}$$

There is equality if $\frac{x}{2} = 1 - x$, that is, $x = \frac{2}{3}$.

Therefore the maximum: $\frac{4}{27}$.

**1.86.** Using the notations of the figure, the area of the rectangle:

$T = 4F$, where $F = x \cdot y$ and $x^2 + y^2 = 1$.

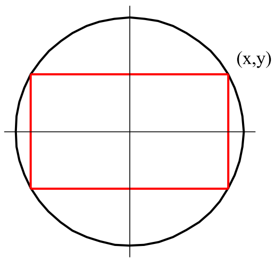
Therefore $F = x\sqrt{1-x^2}$. Calculate the maximum of $F^2$:

$$F^2 = x^2(1-x^2) \leq \left( \frac{x^2 + (1-x^2)}{2} \right)^2 = \frac{1}{4}.$$

There is equality if $x^2 = 1 - x^2$, that is,

$$x = y = \frac{1}{\sqrt{2}}.$$

Therefore the maximal area: $T = 2$, in the case of a square.




## 1.4 Sets

**1.87.** This statement is true.

$$x \in (A \setminus B) \iff (x \in A) \wedge (x \notin B) \iff (x \in A) \wedge (x \in \overline{B}) \iff$$
$$\iff x \in (A \cap \overline{B})$$

**1.90.** This statement is false. Counterexample: $A = B = \{1\} \subset \mathbb{R}$. Therefore, $\overline{A} \setminus B = \overline{A} = \mathbb{R} \setminus \{1\}$, but $A \setminus \overline{B} = A = \{1\}$.

**1.91.** This statement is false. Counterexample: Let $A = B = \{1\}$, then $(A \cup B) \setminus A = \emptyset \neq B$.

**1.94.** This statement is true.

**Proof 1.**

We show that $A \setminus B \subset A \setminus (A \cap B)$, and also that $A \setminus (A \cap B) \subset A \setminus B$.

Let $x \in A \setminus B$ be arbitrary. In this case $x \notin B$, so $x \notin A \cap B$. Since $x \in A$, so $x \in A \setminus (A \cap B)$.

Let $x \in A \setminus (A \cap B)$ be arbitrary. In this case $x \notin A \cap B$. Since $x \in A$, so $x \notin B \setminus A$. Therefore, $x \notin (B \setminus A) \cup (A \cap B) = B$. Therefore $x \in A \setminus B$.

**Proof 2.**

$$
\begin{aligned}
A \setminus (A \cap B) &= A \cap \overline{(A \cap B)} = A \cap (\overline{A} \cup \overline{B}) = (A \cap \overline{A}) \cup (A \cap \overline{B}) \\
&= \emptyset \cup (A \cap \overline{B}) = (A \cap \overline{B}) = A \setminus B
\end{aligned}
$$

**1.98.**

$$A \setminus (B \cup C)$$

**1.100.**

$$((A \cap B) \cup (A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)$$

**1.102.**

$$
\begin{aligned}
x \in \overline{(A \cup B)} &\iff x \notin (A \cup B) \iff (x \notin A) \wedge (x \notin B) \iff \\
&\iff (x \in \overline{A}) \wedge (x \in \overline{B}) \iff x \in (\overline{A} \cap \overline{B})
\end{aligned}
$$

## 1.5 Axioms of the Real Numbers

**1.106.**

**(a)** That is false, counterexample: $x = -1,\ A = 0$.

**(b)** That is true. Since the left-hand side of the first inequality, $|x| < A$, is not negative, we can multiply it by itself, that is, $|x|^2 < A^2$. Since $|x|^2 = |x^2|$, therefore $|x^2| < A^2$.

**1.109.**

**(a)** The set $H$ has no minimum. (The set $H$ has no minimal element.)

**(b)** The set $H$ has no maximum.

**(c)** The set $H$ has a maximum.

**(d)** The set $H$ has a minimum.

**1.111.** Statement (b) is false, the other ones are true.

**1.112.** $\bigcap_{n=1}^{\infty} A_n = \{0\}$

**1.113.** $\bigcap_{n=1}^{\infty} B_n = \emptyset$

**1.117.** Formally:

$$\exists x \in H \; \forall y \in H \; (x > 2 \land y \geq x^2)$$

This statement is not true for any set $H \subset \mathbb{R}$, because if $y = x$, then $x > 2$ $(x > 1)$ implies $x < x^2$.

**1.118.** $M = \bigcap_{n=1}^{\infty} I_n = \{0\}$

For all $n \in \mathbb{N}^+$ implies $-1/n \leq 0 \leq 1/n$ that is $0 \in I_n$, so $0 \in M$. If $x \neq 0$, then there exists $k \in \mathbb{N}^+$ such that $1/k < |x|$. For this $k$ $x \notin I_k = [-1/k, 1/k]$.

**1.119.** $M = \bigcap_{n=1}^{\infty} I_n = \{0\}$

**1.124.** $M = \bigcap_{n=1}^{\infty} I_n = \{0\}$

Since for all $n \in \mathbb{N}^+$ $0 < 1/n$ holds, that is, $0 \in I_n$, so $0 \in M$. If $x \neq 0$, then there exists $k \in \mathbb{N}^+$ such that $1/k < |x|$. For this $k$ $x \notin I_k = [0, 1/k)$.

**1.125.** $M = \bigcap_{n=1}^{\infty} I_n = \emptyset$

Since for all $n \in \mathbb{N}^+$ $0 \notin I_n$ holds, so $0 \notin M$. If $x \neq 0$, then there exists $k \in \mathbb{N}^+$ such that $1/k < |x|$. For this $k$ $x \notin I_k = (0, 1/k]$.

**1.126.** Only the statement 1.126.e is true.

**1.128.** Cannot be, because of the Cantor axiom.

**1.134.** Cannot be. The intersection of any number of closed intervals can be empty, or a single point, or a closed interval. Therefore, the intersection cannot be an open interval.

**1.135.** Can be, but only if the intervals are the same, if the index is large enough.

Let $I_n = (a_n, b_n)$. The word "nested" means that for all $n$ it is true that $a_n \le a_{n+1} < b_{n+1} \le b_n$.

Let $a = \sup a_n, \quad b = \inf b_n$. We know that $a \le b$.

There are four cases:

1) $a = \max a_n$ and $b = \min b_n$. It is true if and only if from a certain index $a_n = a_{n+1}$ and $b_n = b_{n+1}$. Therefore, $\bigcap\limits_{n=1}^{\infty} I_n = I_N = (a, b)$.

2) $a = \max a_n$ and there is no minimal among the $b_n$. Therefore, $\bigcap\limits_{n=1}^{\infty} I_n = (a, b]$, which is empty if $a = b$, and a not empty left-open, right-closed interval if $a < b$.

3) There is no maximal among the $a_n$, but $b = \min b_n$. In this case $\bigcap\limits_{n=1}^{\infty} I_n = [a, b)$.

4) There is no maximal among the $a_n$, and there is no minimal among the $b_n$. In this case $\bigcap\limits_{n=1}^{\infty} I_n = [a, b]$.

**1.136.** All of them except the Cantor's axiom.

**1.140.** All of the finite decimal numbers are rational, but for example 1/3 has no finite decimal form.

Exactly those rational numbers have finite decimal form, which can be written as the quotient of two integers, in which the denominator has no other prime divisor, but 2 and 5.

**1.142.** There are two assumptions for the sequence of intervals $I_n$.

1. The intervals $I_n$ are closed and bounded.

2. The intervals $I_n$ are "nested", that is, the interval of larger index is a subset of the interval of smaller index.

If we omit the first assumption, then, for example, the intersection of the open intervals $I_n = (0, 1/n)$ is empty.

If we omit the second assumption, then, for example, the intersection of the closed intervals $I_n = [n, n+1]$ is empty.

Remark:

The second assumption can be exchanged with the weaker condition that the intersection of any finite number of intervals is not empty.

If the presented intervals are of arbitrary type, and neither the sequences of the left endpoints, nor the sequences of the right endpoints are "stabilized", that is, there are infinitely many left and right endpoints, then the intersection of the intervals is not empty with the second assumption.

## 1.6 The Number Line

**1.145.** $B = (2, 6)$

**1.147.** $D = \{2, 3, 4, 5, 6\}$

**1.148.** $E = [2, 6]$

**1.149.** $F = (2, 6]$

**1.150.** $G = [2, 6)$

**1.151.** $H = [2, 6] \cap \mathbb{Q}$, it is not an interval!

**1.155.** The set $A = \left\{ \frac{1}{n} : n \in \mathbb{N}^+ \right\}$ is bounded from below, the maximal lower bound is 0. The set is bounded from above, because it has a maximum, the maximal element is 1. Since the set is bounded from below and bounded from above, the set is bounded.

**1.160.** The set $Ip = \{n \in \mathbb{N} : n \text{ is prime } \wedge \ n+2 \text{ is prime}\}$, the set of the so-called twin primes, is bounded below, for example, 0 is a lower bound. We still don't know today (March 4, 2014), whether the set is bounded from above, since we don't know if there exist infinitely many twin primes or do not.

**1.162.** For example: $a_n = (-1)^n \cdot \left(1 - \frac{1}{n}\right)$, that is,

$$a_n = \begin{cases} 1 - \dfrac{1}{n} & \text{if } n \text{ is even} \\[2ex] -1 + \dfrac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$

**1.165.** $\forall x \in A\ \exists y \in A\ (y < x)$

**1.168.** $\sup(A \cup B) = \max\{\sup A, \sup B\}$, $\quad \sup(A \cap B) = \min\{\sup A, \sup B\}$.
If $\sup(A \setminus B) \neq \emptyset$, that is, $A \not\subseteq B$, then $\sup(A \setminus B) \leq \sup A$.

**1.173.** $A = \left\{\dfrac{1}{2n-1} : n \in \mathbb{N}^+\right\}$, $\quad \inf A = 0,\ \sup A = \max A = 1$, there is no minimum.

**1.175.** $A = \left\{\dfrac{1}{n} + \dfrac{1}{\sqrt{n}} : n \in \mathbb{N}^+\right\}$ implies $\inf A = 0,\ \sup A = \max A = 2$, there is no minimum.

**1.178.** If $A = \left\{\dfrac{1}{n} + \dfrac{1}{k} : n \in \mathbb{N}^+\right\}$, then $\inf A = 0,\ \sup A = \max A = 2$, and there is no minimum.

**1.181.** Let $A = \left\{\sqrt[n]{2} : n \in \mathbb{N}^+\right\}$. It is obvious that $\sup A = \max A = 2$. We show that $\inf A = 1$. Since $\sqrt[n]{2} > 1$, so 1 is a lower bound. We show that for any $x > 0$ we have that $1 + x$ is not a lower bound:

According to Bernoulli's inequality $(1 + x)^n \geq 1 + nx$ and $1 + nx > 2$ if $n > \dfrac{1}{x}$, therefore there exists an $n$ such that $1 + x > \sqrt[n]{2}$.

**1.182.** Let $A = \left\{\sqrt[n]{2^n - n} : n \in \mathbb{N}^+\right\}$. Since for all $n \in \mathbb{N}^+$ we have $2^n \geq n+1$ according to Bernoulli's inequality, therefore $\sup A = \min A = 1$.

On the other hand, $2^n - n < 2^n$, so 2 is an upper bound of $A$. The number 2 is also the supremum of $A$, that is, $\sup A = 2$, because

$$\sqrt[n]{2^n - n} \geq \sqrt[n]{2^n - 2^{n-1}} = 2 \cdot \frac{1}{\sqrt[n]{2}},$$

and according to the previous exercise

$$\sup\left\{\frac{1}{\sqrt[n]{2}} : n \in \mathbb{N}^+\right\} = \frac{1}{\inf\left\{\sqrt[n]{2} : n \in \mathbb{N}^+\right\}} = 1.$$

Since $2 \notin A$, therefore the set $A$ has no maximum.

**1.186.** Because of the definition of the supremum, it is enough to show that every upper bound of $A$ is an upper bound of $B$, as well.

Let $K$ be an arbitrary upper bound of $B$, and $a \in A$ be arbitrary. According to the assumption, there exists $b \in B$ such that $a \leq b$. Since

$K$ is an upper bound, so $b \leq K$. Therefore, any $a \in A$ implies $a \leq K$, that is, $K$ is an upper bound of $A$.

**1.189.** $\mathbf{Q} \Longrightarrow \mathbf{P}$:

$$|x-y| = |(x-A)+(A-y)| \leq |x-A|+|A-y| = |x-A|+|y-A| < \varepsilon+\varepsilon = 2\varepsilon.$$

$\mathbf{P} \not\Longrightarrow \mathbf{Q}$:

For example, $x = y = 0$, $\varepsilon = 1$ and $A = 2$.

**1.194.** $\mathbf{P} \not\Longrightarrow \mathbf{Q}$: For example, $H = (1, 2]$. In this case $\mathbf{P}$ is true, but with the choice $a = 1$ we can show that $\mathbf{Q}$ is not true.

$\mathbf{Q} \not\Longrightarrow \mathbf{P}$: For example, $H = \{-1\}$.

# Convergence of a Sequence

## 2.1 Limit of a Sequence

**2.1.** Since $a_n \to 1$, we can give a threshold.

**(a)** $\varepsilon = 0.1$

$$|a_n - 1| = \left|1 + \frac{1}{\sqrt{n}} - 1\right| = \frac{1}{\sqrt{n}} < 0.1 \iff \sqrt{n} > 10 \iff n > 10^2$$

Therefore, the choice $n_0 = 10^2$ meets the requirements .

**(b)** $\varepsilon = 0.01$ If we change 0.1 to 0.01 in the previous solution, we got $n_0 = 10^6$.

**2.2.** There is no such $n_0$ threshold. According to the solution of part **(a)** of the previous exercise, if
$n > 10^4$, then $|a_n - 1| < 0.1$. So for these $n$ we have

$$|a_n - 2| = |(a_n - 1) - (2 - 1)| \geq |1 - |a_n - 1|| > 1 - 0.1 = 0.9 > 0.001.$$

**2.3.** These exercises show the importance of the orders and types of the logic symbols in the definition of convergence.

**(a)** True. Generalizing the exercise easily can be seen that $a_n \to 1$, and this is identical with the formula.

**(b)** False. This formula fulfills for a sequence $(a_n)$ if and only if the terms of the sequence equal 1 from some index. The given sequence does not fulfill that.

**(c)** True. This formula fulfills for a sequence $(a_n)$ if and only if the sequence is bounded. The given sequence is bounded.

**(d)** False. This formula fulfills for a sequence $(a_n)$ if and only if there is an open interval around 1 with radius $\varepsilon$, which contains only finitely many terms of the sequence.

**(e)** True. This formula fulfills for a sequence $(a_n)$ if and only if the first term of the sequence $a_1 = 1$ because the choice $n_0 = 1$ corresponds for all $\varepsilon$.

**(f)** False. This formula fulfills for a sequence $(a_n)$ if and only if the first term of the sequence $a_1 \neq 1$.

**2.4.** We show that for all $n$ large enough $b_n > a_n$:

$10n^2 + 25 \le 10n^2 + n^2 = 11n^2$ if $n \ge 5$.

On the other hand $11n^2 < n^3$ if $n > 11$.

So with the choice $N = 11$, we have $b_n > a_n$ if $n > N$.

**2.6.** We show that for all $n$ large enough $b_n > a_n$:

$$3^n - n^2 > 2^n + n \iff 3^n > 2^n + n^2 + n.$$

At first we show that from a certain index $2^n > n^2$. Using the binomial expansion,

$$\begin{aligned} 2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} > \binom{n}{3} = \frac{n(n-1)(n-2)}{6} > \left(\frac{n}{2}\right)^3 \cdot \frac{1}{6} \\ = \frac{n^3}{48} > n^2. \end{aligned}$$

That is fulfilled if $n - 2 > \dfrac{n}{2}$, that is, $n > 4$, and $n > 48$. Therefore, $n > 48 = 2^3 \cdot 6$ implies

$$2^n + n^2 + n < 2^n + 2^n + 2^n = 3 \cdot 2^n < 3^n.$$

The inequality is certainly fulfilled if

$$\left(\frac{3}{2}\right)^n = (1 + 0.5)^n > 3.$$

But according to Bernoulli's inequality that is true if $n > 4$.

Summarizing the requirements for the number $N$, the choice $N = 48$ is a solution.

**2.8.** $(b_n)$ is the greater one from some index:

If $n > 3$, then

$$n! = 6 \cdot (4 \cdot 5 \cdots n) > 6 \cdot 4^{n-3} = \frac{6}{4^3} 2^{2n} > 2^n.$$

The inequality fulfills if $2^n > \dfrac{4^3}{6} = \dfrac{2^7}{3}$, and that fulfills if $n > 8$. Therefore, the choice $N = 8$ works.

**2.16.**

$$\sqrt[n]{2} < 1.01 = 1 + 0.1 \iff 2 < (1 + 0.1)^n$$

According to Bernoulli's inequality

$$(1 + 0.1)^n \geq 1 + 0.1n > 0.1n > 2,$$

if $n > 20$.

**2.17.**

$$\sqrt[n]{n} < 1.0001 \iff n < \left(1 + 10^{-4}\right)^n$$

Using the binomial expansion

$$\left(1 + 10^{-4}\right)^n = \sum_{k=0}^{n} \binom{n}{k} 10^{-4k} > \binom{n}{2} 10^{-8} = \frac{n(n-1)}{2 \cdot 10^8} > \frac{n^2}{8 \cdot 10^8} > n$$

if $n > 8 \cdot 10^8$.

**2.25.**

$$\sqrt{n^2+5} - n = (\sqrt{n^2+5} - n) \cdot \frac{\sqrt{n^2+5} + n}{\sqrt{n^2+5} + n} = \frac{5}{\sqrt{n^2+5} + n} < \frac{5}{n} < 0.01$$

if $n > 500$.

**2.28.** **P** $\implies$ **Q** because the smallest term is a lower bound, and the greatest term is an upper bound.

**Q** $\not\Longrightarrow$ **P**, counterexample: the sequence $a_n = \dfrac{1}{n}$ is bounded, but there is no smallest and there is no greatest term in the sequence.

**2.29.** **(b)** is true, the others are false.

**2.40.**

$$\lim_{n\to\infty} \frac{2n^6 + 3n^5}{7n^6 - 2} = \frac{2}{7}$$

$$\left| \frac{2n^6 + 3n^5}{7n^6 - 2} - \frac{2}{7} \right| = \frac{7(2n^6 + 3n^5) - 2(7n^6 - 2)}{7(7n^6 - 2)} = \frac{21n^5 + 4}{7(7n^6 - 2)} <$$

$$< \frac{21n^5 + 4n^5}{7(7n^6 - 2n^6)} = \frac{25}{35} \cdot \frac{1}{n} < \frac{1}{n} < \varepsilon.$$

The last inequality certainly fulfills if $n > \frac{1}{\varepsilon}$, so a threshold is

$$n_0 = \left[\frac{1}{\varepsilon}\right] + 1.$$

**2.47.** $\lim_{n\to\infty}(\sqrt{n^2+1} - n) = \lim_{n\to\infty}(\sqrt{n^2-1} - n) = 0$, so $\lim_{n\to\infty}(\sqrt{n^2+1} + \sqrt{n^2-1} - 2n) = 0$.

Let's find thresholds for $\frac{\varepsilon}{2}$ separately for $a_n = \sqrt{n^2+1} - n$ and $b_n = \sqrt{n^2-1} - n$.

$$|a_n| = \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{n} < \frac{\varepsilon}{2}$$

fulfills if $n > \frac{2}{\varepsilon}$.

$$|b_n| = n - \sqrt{n^2-1} = \frac{1}{n+\sqrt{n^2-1}} < \frac{1}{n} < \frac{\varepsilon}{2}.$$

This also fulfills if $n > \frac{2}{\varepsilon}$, so $n_0 = \left[\frac{2}{\varepsilon}\right]$ is a good threshold:

$$\begin{aligned}\left|\sqrt{n^2+1} + \sqrt{n^2-1} - 2n\right| &\le \left|\sqrt{n^2+1} - n\right| + \left|\sqrt{n^2-1} - n\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

if $n > n_0$.

**2.49.**
- **(a)** The sequence $(a_n)$ is oscillating divergent.
- **(b)** The sequence $(a_n)$ is convergent, $a_n \to 4$.
- **(c)** The sequence $(a_n)$ is divergent, $a_n \to \infty$.
- **(d)** The sequence $(a_n)$ is oscillating divergent.

**2.55.** For example, $a_n = \frac{1}{n}, \quad b_n = \frac{1}{n^2}$.

**2.58.** Since $a > 0$, the sequence $(a_n)$ has only finitely many negative terms, therefore from some index $\sqrt{a_n}$ is valid.

$$\left|\sqrt{a_n} - \sqrt{a}\right| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \le \frac{|a_n - a|}{\sqrt{a}}$$

Since $a_n \to a$, we can choose such an $n_0$ threshold that $n > n_0$ implies $|a_n - a| < \varepsilon \cdot \sqrt{a}$. This $n_0$ is a good threshold:

$$\left|\sqrt{a_n} - \sqrt{a}\right| \le \frac{|a_n - a|}{\sqrt{a}} < \frac{\varepsilon \cdot \sqrt{a}}{\sqrt{a}} = \varepsilon,$$

if $n > n_0$.

**2.61.** All of the statements are true. Only the statement **(d)** is equivalent to $a_n \to \infty$.

**2.66.** The limit cannot be $\infty$, but it can be $-\infty$ or a real number.

**2.69.** The limit cannot be $-\infty$, but it can be $\infty$ or a real number.

**2.74.**

$$\frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n} > \frac{\sqrt{\left[\frac{n}{2}\right]} + \cdots + \sqrt{n}}{n} > \frac{1}{n} \cdot \frac{n}{2} \cdot \sqrt{\left[\frac{n}{2}\right]} >$$

$$> \frac{1}{n} \cdot \frac{n}{2} \cdot \sqrt{\frac{n}{2} - 1} > K$$

if $n > 8K^2 + 2$. Therefore, a threshold is $n_0 = [8K^2 + 2] + 1$.

**2.81.** According to the condition there exists a number $N$ such that

$$a_{n+1} - a_n > d = \frac{c}{2} > 0.$$

We can prove by induction that $n > N$ implies

$$a_n > a_N + d \cdot (n - N).$$

Since $\lim\limits_{n\to\infty} (a_N + d \cdot (n - N)) = \infty$, so according to the squeezing theorem (or sandwich theorem, or two policemen theorem) $\lim\limits_{n\to\infty} a_n = \infty$.

## 2.2 Properties of the Limit

**2.84.** Since $\frac{1}{n} \to 0$ and $\frac{2}{n} \to 0$, so according to the squeezing theorem $b_n \to 0$.

**2.89.** We cannot say anything about $(b_n)$, the limit can be anything, or the sequence can be oscillating divergent, too.

**2.91.** It is enough to prove that $a_{2n}$ and $a_{2n+1}$ have the same limit because the bigger of the thresholds we got for the two subsequences works for the whole sequence. Since $a_{6n}$ is a common subsequence of $a_{2n}$ and $a_{3n}$, therefore

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{3n}.$$

On the other hand $a_{6n+3}$ is common subsequence of $a_{2n+1}$ and $a_{3n}$, therefore

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} a_{3n}.$$

**2.95.** Since $a > 0$, from some index $\frac{a}{2} < a_n < 2a$, therefore

$$\sqrt[n]{\frac{a}{2}} < \sqrt[n]{a_n} < \sqrt[n]{2a}.$$

Since for arbitrary $c \in \mathbb{R}^+$ implies $\sqrt[n]{c} \to 1$, so according to the squeezing theorem

$$\sqrt[n]{a_n} \to 1.$$

**2.100.**

$$b_n = \frac{a_n - 1}{a_n + 1} = \frac{a_n + 1 - 2}{a_n + 1} = 1 - \frac{2}{a_n + 1} \to 0$$

Let's express $a_n$ with the help of $b_n$:

$$\frac{2}{a_n + 1} = 1 - b_n, \qquad a_n + 1 = \frac{2}{1 - b_n}, \qquad a_n = \frac{2}{1 - b_n} - 1$$

Applying the operational rules of the limit, we get: $a_n \to 1$.

**2.102.** From some index $0 \le \sqrt[n]{a_n} < 0.5$ , so $0 \le a_n < 0.5^n \to 0$.

**2.107.** $\mathbf{P} \not\Longrightarrow \mathbf{Q}$: for example, $a_n = \frac{1}{\sqrt{n}}$

$\mathbf{Q} \Longrightarrow \mathbf{P}$: Let $\lim_{n\to\infty} a_n = a > 0$.

Case 1: $a = \infty$. In this case there is a threshold $N$ such that $n > N$ implies

$$a_n > 1 \ge \frac{1}{n}.$$

Case 2: $0 < a < \infty$: Choose for $\varepsilon = \frac{a}{2}$ a threshold $N$, such that

$$|a_n - a| < \varepsilon \qquad \text{and} \qquad \frac{1}{n} < \varepsilon$$

if $n > N$. But in this case

$$\frac{1}{n} < \varepsilon = \frac{a}{2} = a - \varepsilon < a_n.$$

**2.111.** The statement implies that $a_n \to \infty$, "squeezing theorem for infinity": Let $K \in \mathbb{R}$ be arbitrary. Since $b_n \to \infty$, there exists a threshold $N$ such that $n > N$ implies $K < b_n$. But according to the condition for these $n$ numbers $K < a_n$ is true.

**2.114.** The statement implies nothing:

$(a_n)$ can be convergent, for example if $a_n = 0$,

it can go to $\infty$, for example if $a_n = b_n - 1$

or can go to $-\infty$, for example if $a_n = -n$,

or it can be oscillating divergent, for example if $a_n = (-1)^n$.

**2.118.** The sequence is bounded because it is convergent,

$$\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^2} = 0$$

because

$$0 < \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^2} \leq \frac{\sqrt{n} + \sqrt{n} + \cdots + \sqrt{n}}{n^2} = \frac{n\sqrt{n}}{n^2} = \frac{1}{\sqrt{n}} \to 0.$$

**2.120.** Since from some index $2^n < \dfrac{1}{2}3^n$, therefore

$$\frac{3}{\sqrt[n]{2}} = \sqrt[n]{3^n - \frac{1}{2}3^n} < \sqrt[n]{3^n - 2^n} < \sqrt[n]{3^n} = 3,$$

if $n$ is large enough. The left-hand side of the inequality is $\dfrac{3}{\sqrt[n]{2}} \to 3$, and because of the squeezing theorem

$$\sqrt[n]{3^n - 2^n} \to 3.$$

**2.125.** Simplify the terms of the sequences:

$$a_n = \frac{1 - 2 + 3 - \cdots - 2n}{\sqrt{n^2+1}} = \frac{(1-2) + (3-4) + \cdots + (2n-1-2n)}{\sqrt{n^2+1}}$$
$$= -\frac{n}{\sqrt{n^2+1}}$$

Dividing the numerator and the denominator by the order of magnitude of the denominator, $n$, we get that

$$a_n = -\frac{n}{\sqrt{n^2+1}} = -\frac{1}{\sqrt{1+\dfrac{1}{n^2}}} \to -1.$$

**2.131.** According to exercise 2.181. $\left(1+\dfrac{1}{n}\right)^n$ is a monotonically increasing sequence, therefore $\left(1+\dfrac{1}{n}\right)^n \geq 2$.

$$a_n = \left(1+\frac{1}{n}\right)^{n^2} = \left[\left(1+\frac{1}{n}\right)^n\right]^2 \geq 2^n \to \infty.$$

**2.140.** At this fraction the "order of growth" of the denominator is $7^n$, but the alternating sign causes a problem. We show that $a_n = \dfrac{2^n+3^n}{4^n+(-7)^n} \to 0$. It is enough to show that $|a_n| \to 0$.

$$|a_n| = \left|\frac{2^n+3^n}{4^n+(-7)^n}\right| \leq \frac{2^n+3^n}{7^n-4^n} = \frac{\left(\dfrac{2}{7}\right)^n + \left(\dfrac{3}{7}\right)^n}{1-\left(\dfrac{4}{7}\right)^n} \to 0.$$

**2.146.** $\mathbf{P} \not\Rightarrow \mathbf{Q}$: let

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 1/n & \text{if } n \text{ is odd} \end{cases}, \qquad b_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

$\mathbf{Q} \not\Rightarrow \mathbf{P}$: let $a_n = \dfrac{1}{n}$, $\quad b_n = n$

Remark: If one of the sequences goes to 0, and the other one is bounded, then $a_n \cdot b_n$ goes to 0.

**2.152.** **(a)** $\dfrac{a_n}{b_n}$ is convergent and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} > 0$: $\quad a_n = n, \quad b_n = n+1.$

**(b)** $\dfrac{a_n}{b_n}$ is convergent and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$: $\quad a_n = n, \quad b_n = n^2.$

**(c)** $\dfrac{a_n}{b_n}$ is divergent and $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$: $\quad a_n = n^2, \quad b_n = n.$

**(d)** $\dfrac{a_n}{b_n}$ is oscillating divergent: $a_n = \begin{cases} n & \text{if } n \text{ is even} \\ n^2 & \text{if } n \text{ is odd} \end{cases}$, $b_n = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$

**2.155.** **P** $\not\Longrightarrow$ **Q**: For example $a_n = n+1, \quad b_n = n$.

**Q** $\Longrightarrow$ **P**: Since $b_n \to \infty$, so $\dfrac{1}{b_n} \to 0$, and $\dfrac{a_n - b_n}{b_n} = \dfrac{a_n}{b_n} - 1 \to 0$.

## 2.3 Monotonic Sequences

**2.159.**

– The product of two positive, monotonically increasing/decreasing sequences is monotonically increasing/decreasing.

– The product of two negative, monotonically increasing/decreasing sequences is monotonically decreasing/increasing.

– The product of one positive and one negative, monotonically decreasing sequences is monotonically decreasing.

– The product of one positive, monotonically decreasing and one negative, monotonically increasing sequence is monotonically increasing.

In the other cases we cannot say anything about the monotonity.

**2.164.** We can prove by induction that $a_n > a_1 \cdot (1.1)^{n-1}$. Therefore, it is enough to find an $n$, such that $1.1^{n-1} > \dfrac{10^6}{a_1}$. According to the Bernoulli's inequality

$$1.1^{n-1} = (1+0.1)^{n-1} \geq 1 + (n-1)\cdot 0.1 > (n-1)\cdot 0.1 > \frac{10^6}{a_1}.$$

This certainly holds, if $n - 1 > \dfrac{10^7}{a_1}$, that is $n > \dfrac{10^7}{a_1} + 1$.

**2.168.** At first we prove that all terms of the sequence are positive, which is obvious by induction. Now we can give a better lower estimate for the terms of the sequence: according to the (two terms) inequality between the arithmetic and the geometric means

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) \geq \sqrt{a_n \cdot \frac{a}{a_n}} = \sqrt{a}.$$

We prove that the sequence $(a_n)$ is monotonically decreasing from $n = 2$. Using that $a_n > 0$

$$\frac{1}{2}\left(a_n + \frac{a}{a_n}\right) \le a_n \iff a_n^2 + a \le 2a_n^2 \iff a_n^2 \ge a \iff a_n \ge \sqrt{a}.$$

But we already proved that $a_n \ge \sqrt{a}$ for all $n \ge 2$, therefore the sequence is monotonically decreasing and bounded from below, therefore convergent. Let $\lim\limits_{n\to\infty} a_n = b$, and $b \ge \sqrt{a}$, so $\lim\limits_{n\to\infty} a_{n+1} = b$. On the other hand

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) \to \frac{1}{2}\left(b + \frac{a}{b}\right).$$

Therefore,

$$\frac{1}{2}\left(b + \frac{a}{b}\right) = b \iff b = \sqrt{a}.$$

**2.173.** It is easy to read from the recurrence formula that $a_n \ge 0$ (even more $a_n \ge \sqrt{2}$, if $n > 1$). On the other hand we can prove by induction that $a_n < 2$. It is true for $n = 1$. Let's assume that it is true for $n$. Then

$$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = 2.$$

We'll show that the sequence is monotonically increasing. Let's solve the inequality

$$\sqrt{2 + x} \ge x$$

on the set of nonnegative numbers:

$$\sqrt{2 + x} \ge x \iff 2 + x \ge x^2 \iff x^2 - x - 2 \le 0 \iff 0 \le x \le 2.$$

Since we already proved that $0 \le a_n < 2$, so we can write $a_n$ instead of $x$, so $a_{n+1} \ge a_n$. Therefore, $(a_n)$ is monotonically increasing, and bounded (from above), therefore convergent. Let $a = \lim\limits_{n\to\infty} a_n$. Since the terms of the sequence are nonnegative, so $a \ge 0$. Because of the operational rules of the limit and the recurrence formula

$$a = \sqrt{2 + a} \iff a = 2.$$

**2.180.** $a_1 > 0$, and if $a_n > 0$ then $\quad a_{n+1} = a_n + \dfrac{1}{a_n^3 + 1} > 0$, so for all $n$ $a_n > 1$. According to the recurrence formula

$$a_{n+1} - a_n = \frac{1}{a_n^3 + 1} > 0,$$

therefore the sequence is monotonically increasing. We prove by contradiction that the sequence is not convergent, and therefore it is not bounded. If $\lim\limits_{n\to\infty} a_n = a$, then $a \ge 0$ because $a_n \ge 0$, therefore $a^3 + 1 \ne 0$.

$$a = a + \frac{1}{a^3+1}.$$

But this equation has no solution. Therefore, $(a_n)$ is monotonically increasing and not bounded, so $a_n \to \infty$.

**2.181.** We show that the sequence is strictly monotonically increasing. Applying the inequality between the arithmetic and the geometric means

$$\left(1+\frac{1}{n}\right)^n = 1\cdot\left(1+\frac{1}{n}\right)^n < \left(\frac{1+n\cdot\left(1+\frac{1}{n}\right)}{n+1}\right)^{n+1}$$
$$= \left(1+\frac{1}{n+1}\right)^{n+1}.$$

Now we prove that

$$\left(1+\frac{1}{n}\right)^n < 4 \iff \frac{1}{4}\left(1+\frac{1}{n}\right)^n < 1.$$

Now applying the inequality between the arithmetic and the geometric means for $n+2$ terms

$$\frac{1}{4}\left(1+\frac{1}{n}\right)^n = \frac{1}{2}\cdot\frac{1}{2}\left(1+\frac{1}{n}\right)^n < \left(\frac{\frac{1}{2}+\frac{1}{2}+n\cdot\left(1+\frac{1}{n}\right)}{n+2}\right)^{n+2} = 1.$$

Therefore, the sequence $\left(1+\frac{1}{n}\right)^n$ is convergent. We call the limit of the sequence **Euler's constant**, and denote it by $e$. It can be proved that $2 < e < 3$, $e$ is irrational (moreover transcendent), and $e = 2.71\ldots$.

## 2.4 The Bolzano–Weierstrass theorem and the Cauchy Criterion

**2.186.** **P** $\not\Longrightarrow$ **Q**: let $a_n = (-1)^n$.

**Q** $\Longrightarrow$ **P**: if $(a_n)$ are convergent, then all of its subsequences are convergent (with the same limit).

**2.189.** This condition is not sufficient (but necessary) for the convergence. If, for example, $a_n = \sqrt{n}$, then $\sqrt{n+1} - \sqrt{n} \to 0$, but $\sqrt{n} \to \infty$.

**2.195.** According to the Bolzano-Weiertrass theorem, the sequence $(a_n)$ has no convergent subsequence if and only if the sequence has no bounded subsequence.

And it is true, if for all real number $K > 0$ , there are only finitely many terms of the sequence in the interval $[-K, K]$, that is with the exception of finitely many $n$, $|a_n| > K$.

That means $|a_n| \to \infty$.

**2.198.** We show that the sequence satisfies the Cauchy's criterion. Let $\varepsilon > 0$, and $n < m$.

$$\begin{aligned}|a_n - a_m| &= |(a_{n+1} - a_n) + (a_{n+2} - a_{n+1}) + \cdots + (a_m - a_{m-1})| \leq \\ &\leq |a_{n+1} - a_n| + |a_{n+2} - a_{n+1}| + \cdots + |a_m - a_{m-1}| \leq \\ &\leq 2^{-n} + 2^{-(n+1)} + \cdots + 2^{-(m-1)} = \\ &= 2^{-n} \cdot 2 \cdot \left(1 - 2^{-(m-n)}\right) < 2^{-(n-1)}.\end{aligned}$$

Since $2^{-(n-1)} \to 0$, therefore from some index

$$|a_n - a_m| < 2^{-(n-1)} < \varepsilon.$$

## 2.5 Order of Growth of the Sequences

**2.203.**

$$n^n \sim n! + n^n, \qquad \sqrt{n} \sim \sqrt{n+1}.$$

There is no other asymptotically equal pairs among the sequences. However, $\dfrac{\sqrt[n]{2}}{\sqrt[n]{n}} \to 1$, but these sequences don't go to $\infty$.

**2.210.**

$$\frac{3.01^n}{2^n + 3^n} = \frac{\left(\dfrac{3.1}{3}\right)^n}{\left(\dfrac{2}{3}\right)^n + 1}.$$

Here the numerator goes to $\infty$ because $\dfrac{3.1}{3} > 1$, and the denominator goes to 1, because $\dfrac{2}{3} < 1$. Therefore

$$\lim_{n\to\infty} \frac{3,01^n}{2^n + 3^n} = \infty.$$

**2.216.** The order of growth of the denominator is $2^n$.

$$\frac{n! - 3^n}{n^{10} - 2^n} = \frac{\dfrac{n!}{2^n} - \left(\dfrac{3}{2}\right)^n}{\dfrac{n^{10}}{2^n} - 1}.$$

Since $\dfrac{n^{10}}{2^n} \to 0$, therefore the denominator goes to $-1$. But the numerator is still critical, the difference of two sequences going to infinity. Using that $n! > \left(\dfrac{n}{4}\right)^n$,

$$\begin{aligned}\frac{n!}{2^n} - \left(\frac{3}{2}\right)^n &> \left(\frac{n}{8}\right)^n - \left(\frac{3}{2}\right)^n > \left(\frac{24}{8}\right)^n - \left(\frac{3}{2}\right)^n \\ &> 2 \cdot \left(\frac{3}{2}\right)^n - \left(\frac{3}{2}\right)^n = \left(\frac{3}{2}\right)^n,\end{aligned}$$

if $n \geq 24$. Therefore, the numerator goes to $\infty$, and

$$\frac{n! - 3^n}{n^{10} - 2^n} \to -\infty.$$

## 2.6 Miscellaneous Exercises

**2.222.** The sequence is not the sum of **finitely many** copies of the $(1/n)$ sequences because the number of terms in $a_n$ goes to infinity. Therefore, the first reasoning has the error.

**2.225.** $a_n \to 0$.

Because there is a threshold $N$, such that if $n > N$, then

$$0 < \sqrt[n]{a_n} < \frac{2}{3} \iff 0 < a_n < \left(\frac{2}{3}\right)^n.$$

Since $\left(\dfrac{2}{3}\right)^n \to 0$, so according to the squeezing theorem $a_n \to 0$.

**2.228.** $a_n^n \to 0$.

There exists a threshold $N$ such that if $n > N$, then

$$0 < a_n < \frac{2}{3}, \iff 0 < a_n^n < \left(\frac{2}{3}\right)^n.$$

Since $\left(\frac{2}{3}\right)^n \to 0$, so applying the squeezing theorem $a_n^n \to 0$.

**2.232.** For example, $a_n = \dfrac{1}{n}$.

# Limit and Continuity of Real Functions

## 3.1 Global Properties of Functions

**3.3.** Yes, this is a function. It is the Dirichlet function.

**3.8.** $-\infty < x < 0$.

**3.11.** Let's find the domain of the functions, and let's write the formulas in a more simple way:

**(a)** $f_1(x) = x$,
$D_{f_1} = (-\infty, \infty)$

**(b)** $f_2(x) = \sqrt{x^2} = |x|$,
$D_{f_2} = (-\infty, \infty)$

**(c)** $f_3(x) = \left(\sqrt{x}\right)^2 = x$
$D_{f_3} = [0, \infty)$

**(d)** $f_4(x) = \ln e^x = x$
$D_{f_4} = (-\infty, \infty)$

**(e)** $f_5(x) = e^{\ln x} = x$
$D_{f_5} = (0, \infty)$

**(f)** $f_6(x) = \left(\sqrt{-x}\right)^2 = |x|$
$D_{f_6} = (-\infty, 0]$

Two functions are equivalent if and only if their domain is the same, and the values are the same at every point, therefore only $f_1$ and $f_4$ are equivalent.

**3.14.** Odd.

**3.19.** Even.

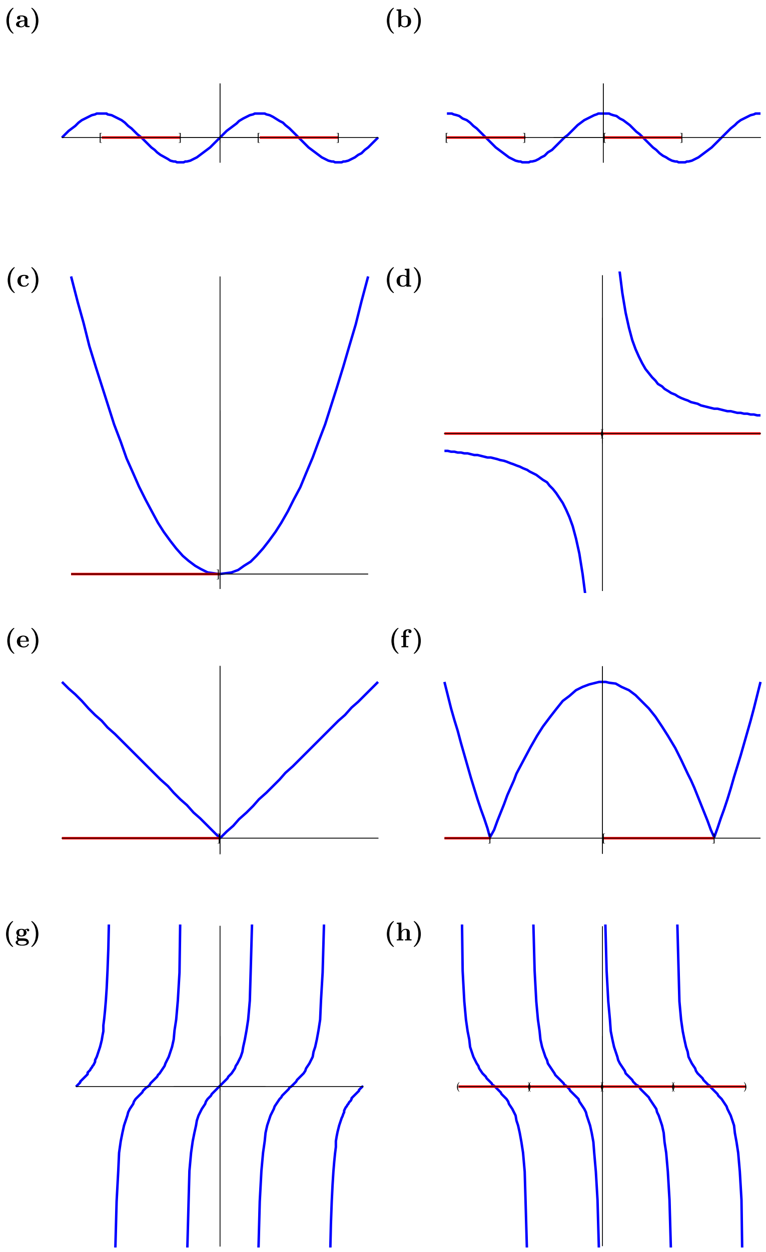
**3.22.** Even and odd.

**3.25.** Neither even, nor odd.

**3.28.** True.

**3.29.** False, e.g. $f(x) = \begin{cases} x & \text{if } x \neq -5 \\ 5 & \text{if } x = -5 \end{cases}$

**3.34.** Functions $\cot x$ and $\dfrac{1}{x}$ are (strictly) monotonically decreasing in the whole domain, the other functions have monotonically increasing intervals, too.




**3.38.** True.

**3.39.** Generally false, e. g. if $f(x) = g(x) = -x$, then $f(x) \cdot g(x) = x^2$ which is not monotonically decreasing.

But if both functions are positive, the statement is true.

**3.43.** Bounded from below, the greatest lower bound is 0. Not bounded from above.

**3.47.** Bounded from below, the greatest lower bound is 0. Bounded from above the least upper bound is 1.

**3.51.** $\forall x \in \mathbb{R}\,(f(x) \leq f(3))$. E.g. $f(x) = -(x-3)^2$.

**3.54.** $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ \ (f(y) < f(x))$. For example $f(x) = x$.

**3.57.** $m = 0$, $M$ does not exist.

**3.60.** $m = -1, \quad M = 1$.

**3.63.** $m = -1, \quad M = 0$.

**3.66.** E. g. $\arctan x$.

**3.68.** For example $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{if } x = -1 \text{ or } x = 1 \end{cases}$

**3.74.** $2\pi$

**3.76.** $4\pi$

**3.78.** $2\pi$

**3.79.** $2\pi$

**3.82.** All nonzero rational numbers are periods of the Dirichlet function, therefore the Dirichlet function has no least positive period.

**3.86.** The $\sqrt{x}$ function is (strictly) concave in the $(0, \infty)$ half-line.

It is enough to prove that for all $0 < a < x < b$

$$\sqrt{x} > \frac{\sqrt{b} - \sqrt{a}}{b - a}(x - a) + \sqrt{a}$$

that is,

$$\frac{\sqrt{x} - \sqrt{a}}{x - a} > \frac{\sqrt{b} - \sqrt{a}}{b - a}.$$

From this

$$\frac{1}{\sqrt{x}+\sqrt{a}} > \frac{1}{\sqrt{b}+\sqrt{a}} \iff \sqrt{b}+\sqrt{a} > \sqrt{x}+\sqrt{a} \iff \sqrt{b} > \sqrt{x}.$$

Since $0 < x < b$, the last inequality is true.

**3.92.** $\mathbf{P} \not\Longrightarrow \mathbf{Q}$: For example $f(x) = \sin \pi x$

$\mathbf{Q} \Longrightarrow \mathbf{P}$: If $f(x)$ is convex in $(-1, 3)$, then for all $-1 < a < b < 3$ and $0 < t < 1$

$$f(ta + (1-t)b) \le tf(a) + (1-t)f(b).$$

With the choice $a = 0,\ b = 2,\ t = \dfrac{1}{2}$ we get the inequality in statement **P**.

**3.96.** The equation of the chord is

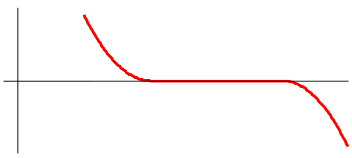
$$h(x) = \frac{\log_7 4 - \log_7 2}{4-2}(x-2) + \log_7 2 = \frac{\log_7 2}{2}x.$$

Substitute 3 for $x$. Since $\log_7 x$ is concave, so

$$\log_7 3 \ge h(3) = \frac{3\log_7 2}{2} = \frac{\log_7 8}{2} = \frac{\log_7 2 + \log_7 4}{2}.$$

**3.100.** Since the function is both convex and concave in the interval $[2, 4]$, therefore the function must be linear in this interval. For example

$$f(x) = \begin{cases} (x-2)^2 & \text{if } 1 \le x \le 2 \\ 0 & \text{if } 2 < x \le 4 \\ -(x-4)^2 & \text{if } 4 < x \le 5 \end{cases}$$



**3.112.** The functions $x,\ x^3,\ \sqrt[3]{x}$ and $f(x) = \begin{cases} 1/x & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ are bijections, the other functions are not.

**3.114.** $f(x) = x^2$ has an inverse in $[0, \infty)$, the inverse is $f^{-1}(x) = \sqrt{x}$, and has an inverse in $(-\infty, 0]$, the inverse is $f^{-1}(x) = \sqrt{-x}$.

**3.116.** $f(x) = \sin x$ has an inverse in intervals $[-\pi/2 + n\pi, \pi/2 + n\pi]$, for any $n \in \mathbb{Z}$. The domain of the inverse function is the closed interval $[-1, 1]$. If we denote by $\arcsin x$ the inverse of $\sin x$ in $[-\pi/2, \pi/2]$, then in the interval $[-\pi/2 + n\pi, \pi/2 + n\pi]$ the inverse is

$$f^{-1}(x) = \begin{cases} n\pi + \arcsin x & \text{if } n \text{ even} \\ n\pi - \arcsin x & \text{if } n \text{ odd} \end{cases}$$

or

$$f^{-1}(x) = n\pi + (-1)^n \arcsin x.$$

**3.122.** **(a)** There exists, but the only one is the constant zero function.

The symmetry line of a function graph is the axis $x$ if and only if $f(x) = -f(x)$ for all $x \in D_f$.

**(b)** For example $f(x) = x^2$.

The symmetry line of a function graph is the axis $y$ if and only if $f(-x) = f(x)$ for all $x \in D_f$, that is, the function is even.

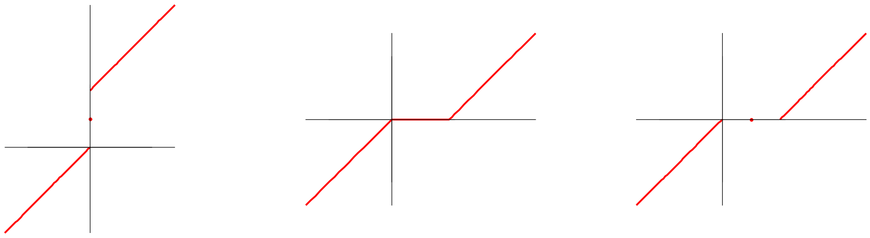
**3.125.** We show that 4 is a period of $f(x)$. Since

$$f(x+2) = \frac{1 + f(x+1)}{1 - f(x+1)} = \frac{1 + \dfrac{1+f(x)}{1-f(x)}}{1 - \dfrac{1+f(x)}{1-f(x)}} = -\frac{1}{f(x)}$$

is true for all $x$, so

$$f(x+4) = -\frac{1}{f(x+1)} = -\frac{1}{-\dfrac{1}{f(x)}} = f(x)$$

**3.128.** The domain of the function $h = g \circ f$ is $\mathbb{R}$, and $h(x) = g(f(x)) = x$. But the function $g(x)$ is not the inverse of $f$ because the domain of $g$ is larger than the range of $f$.



$f(x)$ $\qquad$ $g(x)$ $\qquad$ $f^{-1}(x)$

## 3.2 Limit

**3.130.**

**(a)** $\lim_{x\to-2} f(x) = 0$ **(b)** $\lim_{x\to-1} f(x) = -1$

**(c)** $\lim_{x\to0} f(x)$ does not exist.

**3.133.** True: (b), (d), (e), (f).

False: (a), (c), (g), (h), (i), (j), (k), (l).

**3.136.** $\lim_{x\to3} 5x = 5 \cdot \lim_{x\to3} x = 5 \cdot 3 = 15$

**3.139.** $\lim_{x\to1} \frac{-2}{7x-3} = \frac{-2}{7\cdot1-3} = -\frac{1}{2}$

**3.142.** $\lim_{x\to\pi/2} x\sin x = \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$

**3.148.** $\frac{t^2+t-2}{t^2-1} = \frac{(t-1)(t+2)}{(t-1)(t+1)} = \frac{t+2}{t+1} \xrightarrow[t\to1]{} = \frac{3}{2}$

**3.149.** $\frac{t^2+3t+2}{t^2-t-2} = \frac{(t+1)(t+2)}{(t+1)(t-2)} = \frac{t+2}{t-2} \xrightarrow[t\to-1]{} -\frac{1}{3}$

**3.154.** Rationalizing the numerator:

$$\frac{\sqrt{x}-1}{x-1} = \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} = \frac{x-1}{(x-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1} \xrightarrow[x\to1]{} \frac{1}{2}$$

**3.155.** With the $t = 1 + x^2$ substitution and using the result of the previous problem:

$$\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x^2} = \lim_{t\to 1} \frac{\sqrt{t}-1}{t-1} = \frac{1}{2}.$$

Without substitution, with rationalization:

$$\frac{\sqrt{1+x^2}-1}{x^2} = \frac{\sqrt{1+x^2}-1}{x^2} \cdot \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1} = \frac{x^2}{x^2(\sqrt{1+x^2}+1)}$$
$$= \frac{1}{\sqrt{1+x^2}+1} \xrightarrow[x\to 0]{} \frac{1}{2}.$$

**3.156.** Since $\sin x$ is odd, it is enough to find the right-hand side limit. If $0 < x < \frac{\pi}{2}$, then

$$0 < \sin x < x < \tan x.$$

Let's divide the inequalities by the positive $\sin x$:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

Since all of the expressions are positive, we can take the reciprocals:

$$\cos x < \frac{\sin x}{x} < 1.$$

Since $\cos x$ is continuous at 0, so $\lim_{x\to 0} \cos x = 1$. We can use the squeeze theorem:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

**3.157.**

$$\frac{1-\cos x}{x^2} = \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos x} \to \frac{1}{2}$$

**3.162.** $\dfrac{\tan 2x}{x} = \dfrac{\tan 2x}{2x} \cdot 2 = \dfrac{\sin 2x}{2x} \cdot \dfrac{2}{\cos 2x} = \dfrac{\sin t}{t} \cdot \dfrac{2}{\cos t} \xrightarrow[t\to 0]{} 2.$

We used that $t = 2x \to 0$ if $x \to 0$.

**3.167.** Let's divide the numerator and the denominator by the order of the denominator, $x$. After that the limit is not critical, not the case of $\frac{\infty}{\infty}$. If $x > 0$, then

$$\frac{2x^2 - 7x + 1}{\sqrt{x^2+1}+1} = \frac{2x - 7 + \dfrac{1}{x}}{\sqrt{1 + \dfrac{1}{x^2}} + \dfrac{1}{x}} \to \infty \text{ if } x \to \infty.$$

We get the same limit at $-\infty$, if we choose $|x|$ for the order of denominator when $x < 0$.

**3.172.** $$\frac{2\sqrt{x} + x^{-1}}{3x - 7} = \frac{2\dfrac{\sqrt{x}}{x} + \dfrac{1}{x^2}}{3 - \dfrac{7}{x}} \to \frac{0+0}{3-0} = 0$$

**3.176.** $$\frac{2x^2 - 7x + 1}{\sqrt{x^4+1}+1} = \frac{2 - \dfrac{7}{x} + \dfrac{1}{x^2}}{\sqrt{1 + \dfrac{1}{x^4}} + \dfrac{1}{x^2}} \to 2$$

**3.180.** $$\lim_{x\to 2^-} \frac{3}{x-2} = \frac{3}{0^-} = -\infty, \qquad \lim_{x\to 2^+} \frac{3}{x-2} = \infty.$$

**3.184.** $$\lim_{x\to 7^+} \frac{4}{(x-7)^2} = \lim_{x\to 7^-} \frac{4}{(x-7)^2} = \infty$$

**3.190.** Let $a = \sqrt[k]{e} > 1$.

$$\frac{x^k}{e^x} = \left(\frac{x}{a^x}\right)^k \to 0.$$

**3.191.** With the $t = \ln x$ substitution

$$\sqrt[k]{x} = \sqrt[k]{e^t} = \left(\sqrt[k]{e}\right)^t = a^t,$$

so

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[k]{x}} = \lim_{t\to\infty} \frac{t}{a^t} = 0.$$

**3.196.** For example

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

This function has limits at $x = -1$ and $x = 1$, but has no limits at other points.

**3.198.** Let $p > 0$ a (positive) period, and $a$ and $b$ two real numbers such that $f(a) \neq f(b)$. Therefore

$$x_n = a + n \cdot p \to \infty, \; f(a) = f(x_n) \to f(a),$$

$$y_n = b + n \cdot p \to \infty, \; f(b) = f(y_n) \to f(b).$$

Because of the relationship between the limit of functions and the limit of sequences $\lim\limits_{x\to\infty} f(x)$ does not exist.

**3.203.** **P** $\implies$ **Q**: According to the multiplication rule

$$\lim_{x\to\infty} f^2(x) = \left(\lim_{x\to\infty} f(x)\right) \cdot \left(\lim_{x\to\infty} f(x)\right) = 5 \cdot 5 = 25.$$

**Q** $\not\Longrightarrow$ **P**: For example, $f(x) = -5$.

**3.206.**
- **(a)** $a_n = \sin(n\pi) = 0 \to 0$.
- **(b)** $f(x) = \sin x$ is a periodic and non-constant function, therefore it has no limit at infinity (see problem 3.198.).
- **(c)** $a_n = \left[\dfrac{1}{n}\right] = 0$ if $n > 1$, therefore $a_n \to 0$.
- **(d)** Since $\lim\limits_{x\to 0+} [x] = 0 \neq -1 = \lim\limits_{x\to 0-} [x]$, so $f(x) = [x]$ (integer part of $x$) has no limit at 0.

**3.208.** **P** $\not\Longrightarrow$ **Q**: For example, $f(x) = \begin{cases} 5 & \text{if } x = 1/n \text{ for some } n \\ 0 & \text{otherwise} \end{cases}$.

The function has no limit at 0, but $f\left(\dfrac{1}{n}\right) = 5 \to 5$.

**Q** $\implies$ **P**: Because of the relationship between the limits of functions and sequences, $\dfrac{1}{n} \to 0$ and $\lim\limits_{x\to 0} f(x) = 5$, so $f\left(\dfrac{1}{n}\right) \to 5$.

## 3.3 Continuous Functions

**3.216.**
- **(a)** This function is the Dirichlet-function, which is not continuous at any points, even more the function has no limit at any points. For any $a \in \mathbb{R}$ there are sequences $x_n \in \mathbb{Q}$ and $y_n \notin \mathbb{Q}$ such that $a \neq x_n$, $a \neq y_n$, $x_n \to a$ and $y_n \to a$. So $D(x_n) \to 1$ and $D(y_n) \to 0$. Because of the relationship of the limits of functions and sequences, the $D(x)$ function has no limit at $a$.

**(b)** $f(x)$ is continuous at 0 (but not continuous at other points). For $\varepsilon > 0$, $\delta = \varepsilon$.
If $|x - 0| = |x| < \delta$, then $|f(x) - f(0)| = |f(x)| = |x| < \varepsilon = \delta$.

**3.221.** **(a)** $h = f + g$ is not continuous at 3. Proof by contradiction: if $h = f + g$ would be continuous, then $g = h - f$ would also be continuous.

**(b)** $f \cdot g$ can be continuous at 3, but only if $f(3) = 0$. For example, $f(x) = 0$ and $g(x) = D(x)$, the Dirichlet function.

**3.226.** $\sqrt{x}$ is continuous, if $x > 0$. At 0, it is continuous from right-hand side.

**3.229.** $f(x) = \begin{cases} x^2 + 2 & \text{if } x \geq 0 \\ mx + c & \text{if } x < 0 \end{cases}$ is continuous at 0 if and only if it is continuous both from left-hand side and right-hand side. From right-hand side, that is for $x \geq 0$ $f(x)$ equals to $x^2 + 2$, which is continuous at every point, including 0.

For $x < 0$ $f(x)$ equals to $mx + c$, whose left-hand side limit is $c$ at 0. Therefore $f$ is continuous at 0 if and only if

$$2 = f(0) = c.$$

**3.233.** Let $p(x)$ be a third degree polynomial. It is enough to examine the case in which the main coefficient of $p(x)$ (that is coefficient of the term $x^3$) is positive. In this case

$$\lim_{x \to \infty} p(x) = \infty, \qquad \lim_{x \to -\infty} p(x) = -\infty.$$

Therefore $p(x)$ must have both positive and negative values as well. But because of the intermediate value theorem there is a point at which $p(x) = 0$.

**3.236.** Let $h(x) = f(x) - g(x)$. The function $h$ is continuous in $[a, b]$, and $h(a) \geq 0$, $h(b) \leq 0$. According to the intermediate value theorem there is a $c \in [a, b]$ such that $h(c) = f(c) - g(c) = 0$.

**3.237.** Let $h(x) = g(x) - f(x)$. This $h(x)$ is positive and continuous in $[a, b]$. According to the Weierstrass theorem $h(x)$ has a minimum, that is, there is $c \in [a, b]$ such that for all $x \in [a, b]$

$$0 < m = h(c) \leq h(x) = g(x) - f(x).$$

**3.242.** **P** $\not\Rightarrow$ **Q**: For example, $f(x) = \begin{cases} x^2 & \text{if } x \in (1,2) \\ 2 & \text{if } x = 1 \text{ or } x = 2 \end{cases}$

**Q** $\Longrightarrow$ **P**: The maximum of $f$ is an upper bound, and the minimum is a lower bound.

**3.245.** For example, $f(x) = x$.

**3.249.** $[x]$ is monotonically increasing, therefore it has a maximum in every bounded closed interval. Therefore

$$\max\{[x] : x \in [77, 888]\} = [888] = 888.$$

**3.250.** $f(x) = \{x\}$ has no maximum in such $[a,b]$ intervals whose length is at least 1, since

$$\begin{aligned}\sup\{f(x) : x \in [a,b]\} &= \sup\{f(x) : x \in [a, a+1]\} \\ &= \sup\{f(x) : x \in [0,1]\} = 1,\end{aligned}$$

since number 1 is a period of $\{x\}$. On the other hand, $f(x) = \{x\} \neq 1$.

**3.254.** For example a continuous function:

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1/3 \\ 3x - 1 & \text{if } 1/3 \le x \le 2/3 \\ 1 & \text{if } 2/3 < x < 1 \end{cases}$$

**3.257.** Yes, there is. The solution of problem 3.254. is a continuous example.

**3.258.** Such a function does not exist. Let's assume that $f(x)$ is continuous in the closed interval $[0,1]$, and its range is the open $(0,1)$ interval. According to the Weierstrass theorem $f(x)$ has a maximum. Let's denote this maximum by $M$. Because of the assumption $M < 1$, and since $M$ is maximum, $f(x) \notin (M, 1)$.

This statement can be proved more generally as well, see problem 3.260.

**3.260.** Let $f$ be continuous in $[a,b]$. According to the Weierstrass theorem, $f$ has a minimum $m$, and a maximum $M$. So $R(f) \subset [m, M]$.

On the other hand according to the Bolzano theorem every value between $m$ and $M$ is in the range of $f$, that is, $R(f) \supset [m, M]$.

**3.263.** Let $x_n = \dfrac{\pi}{2} + 2\pi n,\ y_n = \dfrac{3\pi}{2} + 2\pi n$. If $n > 100$, then

$$x_n \sin x_n = x_n > 2\pi n > 100 \quad \text{and} \quad y_n \sin y_n = -y_n < -100.$$

According to the intermediate value theorem for all $n > 100$ there is $z_n \in [x_n, y_n]$ such that

$$z_n \sin z_n = 100.$$

Since the $[x_n, y_n]$ intervals are disjunct, so the roots $z_n$ are different.

**3.271.** $f(x) = \dfrac{1}{x}$

**(a)** is not uniformly continuous in $(0, \infty)$. We'll show that for $\varepsilon = 1$ there is no "good" $\delta > 0$. For any arbitrary $\delta > 0$ we can choose an $n \in \mathbb{N}^+$ positive integer such that

$$\frac{1}{n} - \frac{1}{n+1} < \delta.$$

In this case

$$\left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right) \right| = 1 \not< \varepsilon.$$

**(b)** is uniformly continuous in $[1, 2]$ according to the Heine-Borel theorem.

**(c)** is uniformly continuous in $(1, 2)$ because it is uniformly continuous in the greater $[1, 2]$.

**(d)** is uniformly continuous in $[1, \infty)$. Let $\varepsilon > 0$ be arbitrary, $\delta = \varepsilon$. If $x, y \geq 1$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|x - y|}{xy} \leq |x - y| < \delta = \varepsilon.$$

# Differential Calculus and its Applications

## 4.1 The Concept of Derivative

**4.2.** If $\lim\limits_{x\to 3}\dfrac{f(x)-f(3)}{x-3}=4$, then according to the definition of the derivative $f(x)$ is differentiable at 3 and $f'(3)=4$. According to the theorem 4.2 $f(x)$ is continuous at 3.

**4.3.** It does not imply, for example $f(x)=|x-3|$.

**4.5.** This limit is equal to the derivative of the function $\sqrt{x}$ at any $x>0$ point.

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{\sqrt{x+h}+\sqrt{x}}\xrightarrow[h\to 0]{}\frac{1}{2\sqrt{x}}$$

**4.10.** Let $x_0\neq 0$ be arbitrary.

$$\frac{1/x-1/x_0}{x-x_0}=\frac{x_0-x}{xx_0(x-x_0)}=-\frac{1}{xx_0}\xrightarrow[x\to x_0]{}-\frac{1}{x_0^2}$$

Therefore, the derivative of the function $\dfrac{1}{x}$ is $-\dfrac{1}{x^2}$.

**4.14.** $f(x)=\left|x^2-1\right|$ is continuous everywhere, and differentiable except at 1 and $-1$ because it is composed by "this type" of functions, that is, we can use the differentiation rules. We show that at 1 the left-hand side and the right-hand side limits of the difference quotient are different, so the function is not differentiable at 1. Now we may assume that $x>0$.

$$\begin{aligned}\frac{f(x)-f(1)}{x-1}&=\frac{\left|x^2-1\right|}{x-1}=(x+1)\frac{|x-1|}{x-1}\\&=(x+1)\operatorname{sgn}(x-1)\to\begin{cases}2 & \text{if } x\to 1^+\\ -2 & \text{if } x\to 1^-\end{cases}\end{aligned}$$

Since the function is even, it is not differentiable at $-1$.

**4.18.** At first find the values of $b$ and $c$ such that the function

$$h(x)=\begin{cases}(1-x)(2-x) & \text{if } x\geq -3\\ bx+c & \text{if } x<-3\end{cases}$$

is continuous at $-3$.

$$\lim_{x\to -3^+} h(x) = \lim_{x\to -3}(1-x)(2-x) = 20,$$
$$\lim_{x\to -3^-} h(x) = \lim_{x\to -3}(bx+c) = -3b+c$$

Therefore, the function is continuous at $-3$ if and only if

$$-3b+c = 20.$$

The function $h(x)$ is differentiable at $-3$ if the left-hand side and the right-hand side limits of the difference quotient are equal. Since $h_1(x) = (1-x)(2-x)$ and $h_2(x) = bx+c$ are differentiable at $-3$, the two limits are equal, if $h_1'(-3) = h_2'(-3)$.

$$h_1'(x) = ((1-x)(2-x))' = -(2-x)-(1-x) = 2x-3, \qquad h_1'(-3) = -9,$$
$$h_2'(x) = (bx+c)' = b$$

Therefore, $h(x)$ is differentiable at $-3$ if and only if it is continuous, that is, $-3b+c = 20$ and $b = -9$. From the two equations

$$b = -9, \qquad c = -7.$$

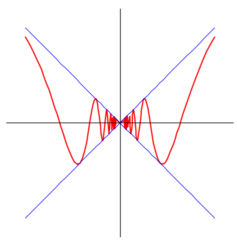
**4.19.** $f(x)$ is differentiable everywhere except $x = 0$, and $f'(x)$ is continuous because of the basic derivative rules. Since $f(x)$ is a product of a function going to zero and a function which is bounded in a neighbourhood of 0, so $\lim_{x\to 0} f(x) = 0 = f(0)$ (see 3.2). Therefore according to 3.3 $f(x)$ is continuous at 0.

$f(x)$ is not differentiable at 0, because the difference quotient

$$g(x) = \frac{f(x)-f(0)}{x} = \sin\frac{1}{x}$$

has no limit at 0.

We can see in the figure that the function $g(x) = \sin\frac{1}{x}$ is between the lines $x$ and $-x$.



**4.20.** $f(x)$ is differentiable at every point. This is trivial for each $x \neq 0$. We show that the limit of the difference quotient at $x = 0$ is 0.

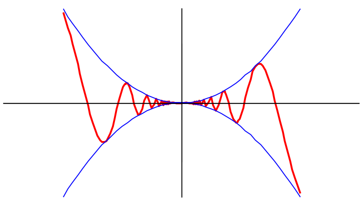
$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} x \sin\frac{1}{x} = 0,$$

since $g(x) = x \sin\frac{1}{x}$ is product of a function going to zero and a bounded function around 0, (see 3.2). The derivative

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous, if $x \neq 0$, but has no limit at 0, therefore it is not continuous at 0.

We can see on the figure that $f(x) = x^2 \sin\frac{1}{x}$ is between the parabolas $x^2$ and $-x^2$.

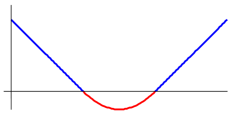


**4.25.** The function $f(x)$ is continuously differentiable except at the points 1 and 2. It is continuous at the two exceptional points. Let's calculate the derivative and the tangent lines of the "middle part", that is, $g(x) = (1-x)(2-x) = x^2 - 3x + 2$ at 1 and 2:

$$g'(x) = 2x - 3, \qquad g'(1) = -1, \qquad g'(2) = 1$$

$$e_1(x) = 1 - x, \qquad e_2(x) = x - 2 = -(2 - x).$$

Since the function $g(x)$ "continues" at both points by the tangent line, therefore the function is continuously differentiable at both points.



**4.31.** If $f(x) = \sin x$, then

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f^{(n)}(x) = \begin{cases} \sin x & \text{if } n = 4k \\ \cos x & \text{if } n = 4k+1 \\ -\sin x & \text{if } n = 4k+2 \\ -\cos x & \text{if } n = 4k+3 \end{cases}$$

**4.32.** If $f(x) = \cos x$, then

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f^{(n)}(x) = \begin{cases} \cos x & \text{if } n = 4k \\ -\sin x & \text{if } n = 4k+1 \\ -\cos x & \text{if } n = 4k+2 \\ \sin x & \text{if } n = 4k+3 \end{cases}$$

## 4.2 The Rules of the Derivative

**4.33.** **P** $\implies$ **Q**: Since $f(x) = f(-x)$, therefore $f'(x) = f'(-x) \cdot (-1) = -f'(-x)$.

**Q** $\implies$ **P**: Let $g(x) = f(-x)$. Since $f'(x) = -f'(-x)$, therefore $g'(x) = -f'(-x) = f'(x)$. So the derivatives of $f$ and $g$ are equal everywhere, so according to the basic theorem of anti-derivatives there is a $c \in \mathbb{R}$ such that

$$g(x) = f(x) + c, \text{ that is, } f(-x) = f(x) + c.$$

Since this statement is true for all $x$, applying it for $x = 0$

$$f(0) = f(0) + c \quad \iff \quad c = 0.$$

**4.34.** **P** $\implies$ **Q**: Since $f(x) = -f(-x)$, so $f'(x) = -f'(-x) \cdot (-1) = f'(-x)$.

**Q** $\not\Longrightarrow$ **P**: For example, $f(x) = x + 1$.

If we also assume that $f(0) = 0$, then it is true that $f$ is odd:

Let $g(x) = -f(-x)$. Since $f'(x) = f'(-x)$, so $g'(x) = f'(-x) = f'(x)$. Therefore, the derivatives of $f$ and $g$ are equal everywhere, so according to the basic theorem of anti-derivatives there is a $c \in \mathbb{R}$, such that

$$g(x) = f(x) + c, \text{ that is } -f(-x) = f(x) + c.$$

Since it is true for all $x$, so applying for $x = 0$

$$0 = f(0) = f(0) + c \quad \Longleftrightarrow \quad c = 0.$$

**4.38.** **P** $\Longrightarrow$ **Q**:

$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h} &= \frac{1}{2}\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}+\frac{f(a)-f(a-h)}{h}\right)\\ &= \frac{1}{2}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to 0}\frac{f(a)-f(a-h)}{h}\\ &= \frac{1}{2}(f'(a)+f'(a)) = f'(a).\end{aligned}$$

**Q** $\not\Longrightarrow$ **P**: For example, $a = 0$, $f(x) = |x|$.

**4.42.** Verify that the given point is on the curve! It means that the value of the function at 1 is 6: $f(1) = 1^3 - 2\cdot 1^2 + 3\cdot 1 + 4 = 6$.

The equation of the tangent line is $y = m(x - x_0) + y_0$, where $x_0 = 1$, $y_0 = 6$ and $m = f'(1)$.

The derivative of the function $f'(x) = 3x^2 - 4x + 3$. Hence, $m = f'(1) = 2$. Therefore, the equation of the tangent line at the given point:

$$y = 2(x-1) + 6, \text{ or otherwise } y = 2x + 4.$$

**4.56.** Let $f(x) = \sqrt{x\sqrt{x\sqrt{x}}}$. In this case $D_f = [0,\infty)$ ĂŠs $D_{f'} = (0,\infty)$. Let's write down $f(x)$ in a "fraction exponent form":

$$\begin{aligned}f(x) = \sqrt{x\sqrt{x\sqrt{x}}} &= \left(x\cdot\left(x\cdot(x)^{1/2}\right)^{1/2}\right)^{1/2}\\ &= \left(x\cdot\left(x^{3/2}\right)^{1/2}\right)^{1/2} = \left(x^{7/4}\right)^{1/2} = x^{7/8}\end{aligned}$$

Therefore,

$$f'(x) = \frac{7}{8}x^{-1/8} = \frac{7}{8\sqrt[8]{x}}$$

**4.57.** Let $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$. So $D_f = [0, \infty)$ and $D_{f'} = (0, \infty)$.

$$f'(x) = \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \left(1 + \frac{1}{2\sqrt{x + \sqrt{x}}} \left(1 + \frac{1}{2\sqrt{x}}\right)\right)$$

**4.62.** Let $f(x) = \dfrac{\sin x - x\cos x}{\cos x + x\sin x}$. The domain is

$$D_f = \{x : \cos x + x\sin x \neq 0\} = \{x : \cot x \neq -x\}.$$

Applying the rules of derivation:

$$f'(x) = \frac{(\sin x - x\cos x)'(\cos x + x\sin x) - (\sin x - x\cos x)(\cos x + x\sin x)'}{(\cos x + x\sin x)^2}.$$

$$(\sin x - x\cos x)' = \cos x - \cos x + x\sin x = x\sin x$$

$$(\cos x + x\sin x)' = -\sin x + \sin x + x\cos x = x\cos x$$

Substituting the results:

$$\begin{aligned} f'(x) &= \frac{x\sin x(\cos x + x\sin x) - (\sin x - x\cos x)x\cos x}{(\cos x + x\sin x)^2} \\ &= \frac{x^2}{(\cos x + x\sin x)^2} \end{aligned}$$

The domain of the derivative: $D_{f'} = D_f$.

**4.63.** Let $f(x) = 4x^3\tan(x^2 + 1)$. The domain

$$D_f = \left\{x : x^2 + 1 \neq (2n+1)\frac{\pi}{2}\right\} = \mathbb{R} \setminus \left\{\pm\sqrt{(2n-1)\frac{\pi}{2} - 1} : n \in \mathbb{N}\right\}.$$

$$f'(x) = 12x^2\tan(x^2 + 1) + 4x^3(\tan(x^2 + 1))'$$

According to the chain rule

$$(\tan(x^2+1))' = \frac{1}{\cos^2(x^2+1)} \cdot 2x = \frac{2x}{\cos^2(x^2+1)}.$$

Therefore

$$f'(x) = 12x^2 \tan(x^2+1) + \frac{2x}{\cos^2(x^2+1)}.$$

The domain of the derivative is $D_{f'} = D_f$.

**4.68.** $f(x) = x^x = e^{x \ln x}, \quad D_f = D_{f'} = (0, \infty).$

$$f'(x) = (x^x)' = \left(e^{x \ln x}\right)' = e^{x \ln x}(\ln x + 1) = x^x(\ln x + 1).$$

**4.69.** $f(x) = \sqrt[x]{x} = x^{1/x} = e^{\frac{\ln x}{x}}, \quad D_f = D_{f'} = (0, \infty).$

$$f'(x) = \left(\sqrt[x]{x}\right)' = \left(e^{\frac{\ln x}{x}}\right)' = \frac{1 - \ln x}{x^2} \sqrt[x]{x}$$

**4.74.** $f(x) = \log_4 x = \dfrac{\ln x}{\ln 4}, \quad D_f = D_{f'} = (0, \infty).$

$$f'(x) = (\log_4 x)' = \left(\frac{\ln x}{\ln 4}\right)' = \frac{1}{x \ln 4}$$

**4.75.** $f(x) = \log_x 4 = \dfrac{\ln 4}{\ln x}, \quad D_f = D_{f'} = (0, 1) \cup (1, \infty).$

$$f'(x) = (\log_x 4)' = \left(\frac{\ln 4}{\ln x}\right)' = -\frac{\ln 4}{x \ln^2 x}$$

**4.84.** Let $f(x) = \dfrac{\sinh x - x \cosh x}{\cosh x + x \sinh x}$. Since the denominator is not 0 at any point, $D_f = \mathbb{R}$.

Applying the rules of derivatives:

$$f'(x) = \frac{(\sinh x - x\cosh x)'(\cosh x + x \sinh x) - (\sinh x - x \cosh x)(\cosh x + x \sinh x)'}{(\cosh x + x \sinh x)^2}.$$

$$(\sinh x - x \cosh x)' = \cosh x - \cosh x - x \sinh x = -x \sinh x$$

$$(\cosh x + x \sinh x)' = \sinh x + \sinh x + x \cosh x = 2 \sinh x + x \cosh x$$

Substituting the results:

$$f'(x) = \frac{-x\sinh x(\cosh x + x\sinh x) - (\sinh x - x\cosh x)(2\sinh x + x\cosh x)}{(\cosh x + x\sinh x)^2} =$$

$$= \frac{x^2 - 2\sinh^2 x}{(\cosh x + x\sinh x)^2}$$

The domain of the derivative is $D_{f'} = D_f = \mathbb{R}$.

**4.90.** $f(x) = \log_3 x \cdot \cos x, \quad D_f = D_{f'} = (0, \infty)$.

$$f'(x) = \frac{\cos x}{x\ln 3} - \log_3 x \cdot \sin x$$

**4.91.** $f(x) = \dfrac{\sin x + 2\ln x}{\sqrt{x}+1}, \quad D_f = D_{f'} = (0, \infty)$.

$$f'(x) = \frac{\left(\cos x + \frac{2}{x}\right)\cdot(\sqrt{x}+1) - \frac{\sin x + 2\ln x}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$$

**4.96.** $f(x) = \ln(\sin x), \quad D_f = D_{f'} = \{x : \sin x > 0\}$.

$$f'(x) = \frac{\cos x}{\sin x} = \cot x$$

**4.97.** $f(x) = x^{\tan x} = e^{\ln x \cdot \tan x}$,
$D_f = D_{f'} = \left\{x : x \neq (2n+1)\frac{\pi}{2},\ \tan x > 0\right\}$.

$$f'(x) = \left(x^{\tan x}\right)' = \left(e^{\ln x \cdot \tan x}\right)' = x^{\tan x}\left(\frac{\tan x}{x} + \frac{\ln x}{\cos^2 x}\right)$$

**4.100.** At first find the point $c$ where the value of the function is $a = 2$, that is, the value of the inverse function at $a = 2$. We can find $c$ by trials: $c = 1$. According to the rule of the derivative of the inverse function:

$$(f^{-1})'(2) = \frac{1}{f'(1)}.$$

The derivative of the function $f'(x) = 5x^4 + 2x$, so $f'(1) = 7$. Therefore

$$(f^{-1})'(2) = \frac{1}{7}.$$

*Remark:* We should examine whether the function has an inverse. Examining the derivative we learn that there is no inverse on the whole number-line, since there is a strict local maximum at point $\sqrt[3]{-\frac{2}{5}}$, and a minimum at 0, therefore the function is not strictly monotonous. Moreover, because the limit of the function at $-\infty$ is $\infty$ and $0 = f(0) < 2$, therefore $f(x) = 2$ somewhere between $-\infty$ and 0. But the function is strictly monotonically increasing on the half-line $(0, \infty)$, containing 1, therefore there is an inverse on this half-line.

**4.104.** Since $\arctan x$ is the inverse of $\tan x$, so $(\arctan x)' = \dfrac{1}{\tan'(\arctan x)} = \cos^2(\arctan x) = \dfrac{1}{1+\tan^2(\arctan x)}$.

Since $\tan(\arctan x) = x$, therefore

$$(\arctan x)' = \frac{1}{1+x^2}$$

**4.108.** $f(x) = \arctan(\sin x), \quad D_{f'} = \mathbb{R}$.

$$f'(x) = \frac{1}{1+\sin^2 x} \cdot \cos x = \frac{\cos x}{1+\sin^2 x}$$

**4.109.** $f(x) = \tan(\arcsin x), \quad D_{f'} = (-1, 1)$.

$$f'(x) = \frac{1}{\cos^2(\arcsin x)} \cdot \frac{1}{\sqrt{1-x^2}} = \frac{1}{1-\sin^2(\arcsin x)} \cdot \frac{1}{\sqrt{1-x^2}}$$
$$= \frac{1}{(1-x^2)\sqrt{1-x^2}}$$

**4.114.**

$$\frac{2\sin x - 1}{6x-\pi} = \frac{2}{6} \cdot \frac{\sin x - \frac{1}{2}}{x - \frac{\pi}{6}} = \frac{2}{6} \cdot \frac{\sin x - \sin\frac{\pi}{6}}{x-\frac{\pi}{6}} \to \frac{2}{6}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{6}$$

**4.119.**

$$\lim_{n\to\infty} n\left(\cos\frac{1}{n} - 1\right) = \lim_{n\to\infty} \frac{\cos\frac{1}{n} - \cos 0}{1/n} = -\sin 0 = 0$$

**4.123.**

$$\left(e^{\sin x}\right)' = \cos x \cdot e^{\sin x}$$

$$\left(e^{\sin x}\right)'' = \left(\cos x \cdot e^{\sin x}\right)' = (\cos^2 x - \sin x)e^{\sin x}$$

## 4.3 Mean Value Theorems, L'Hospital's Rule

**4.128.** If $f(x) = \arctan x, \quad g(x) = \arctan \dfrac{1+x}{1-x}$ and $h(x) = f(x) - g(x)$, then

$$\left(\frac{1+x}{1-x}\right)' = \frac{(1-x)-(1+x)(-1)}{(1-x)^2} = \frac{2}{(1-x)^2}$$

$$g'(x) = \left(\arctan \frac{1+x}{1-x}\right)' = \frac{1}{1+\dfrac{(1+x)^2}{(1-x)^2}} \cdot \frac{2}{(1-x)^2}$$
$$= \frac{2}{(1-x)^2+(1+x)^2} = \frac{1}{1+x^2}.$$

Therefore $h'(x) = 0$ at every point, where $h(x)$ is differentiable. But the domain of $g(x)$, so $h(x)$ as well, does not contain 1, therefore $h(x)$ is not differentiable at 1.

$$h(0) = \arctan 0 - \arctan 1 = -\frac{\pi}{4},$$

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \arctan x - \lim_{x\to\infty} \arctan \frac{1+x}{1-x} = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}.$$

Therefore $h(x)$ is not a constant function because $h(0) \neq h(\infty) = \lim\limits_{x\to\infty} h(x)$. But we can apply the basic theorem of anti-derivatives for the half-lines $(-\infty, 1)$ and $(1, \infty)$.

**4.129.** Let $h(x) = f(x) - g(x)$. Since $h(x)$ is continuous on $[0, \infty)$, $h'(x) > 0$ if $x > 0$, so according to theorem 4.8 it is strictly monotonically increasing on $[0, \infty)$. Therefore

$$0 \leq h(0) = f(0) - g(0) < h(x) = f(x) - g(x) \text{ if } x > 0.$$

**4.131.** Let's find the intervals, where the function $f(x) = x^5 - 5x + 2$ is monotonous.

$$f'(x) = 5(x^4 - 1) = 5(x^2+1)(x^2-1),$$

$$f'(x) > 0 \text{ if } x \in (-\infty, -1) \cup (1, \infty) \text{ and } f'(x) < 0 \text{ if } x \in (-1, 1).$$

Therefore the function is strictly monotonically increasing if $x < -1$ or $x > 1$, and strictly monotonically decreasing on$(-1, 1)$.

$$\lim_{x \to -\infty} f(x) = -\infty, \; f(-1) = 6 > 0, \; f(1) = -2 < 0, \; \lim_{x \to \infty} f(x) = \infty.$$

According to the intermediate value theorem $f(x)$ has three roots on the three disjunct intervals. Because of the strict monotonicity $f(x)$ has exactly one root on each interval, therefore $f(x)$ has three roots.

**4.138.** Since the limits have $\infty/\infty$ form, we can apply L'Hospital's rule.

$$\lim_{x \to 0+} \frac{(\ln x)'}{(\cot x)'} = \lim_{x \to 0+} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = -\lim_{x \to 0+} \frac{\sin x}{x} \cdot \sin x = 0.$$

So $\lim\limits_{x \to 0+} \dfrac{\ln x}{\cot x} = 0$.

**4.143.** The limit $\lim\limits_{x \to 0} \left( \cot x - \dfrac{1}{x} \right)$ is critical (has the form $\infty - \infty$). Let's write the difference in quotient form:

$$\cot x - \frac{1}{x} = \frac{\cos x}{\sin x} - \frac{1}{x} = \frac{x \cos x - \sin x}{x \sin x}.$$

We can apply L'Hospital's rule for this quotient because it has a $0/0$ form.

$$\begin{aligned} \lim_{x \to 0} \frac{(x \cos x - \sin x)'}{(x \sin x)'} &= \lim_{x \to 0} \frac{\cos x - x \sin x - \cos x}{\sin x + x \cos x} \\ &= -\lim_{x \to 0} \frac{\sin x}{\frac{\sin x}{x} + \cos x} = 0. \end{aligned}$$

Therefore $\lim\limits_{x \to 0} \left( \cot x - \dfrac{1}{x} \right) = 0$.

**4.145.** The limit $\lim\limits_{x \to 0} \left( \dfrac{\sin x}{x} \right)^{1/x^2}$ is critical because has a form $1^\infty$. Let's change the expression so that we can apply the limit $\lim\limits_{x \to 0} (1 + x)^{1/x} = e$

(see problem 4.144.):

$$\left(\frac{\sin x}{x}\right)^{\dfrac{1}{x^2}} = \left[\left(1+\frac{\sin x - x}{x}\right)^{\dfrac{x}{\sin x - x}}\right]^{\dfrac{\sin x - x}{x^3}}$$

This formula is not critical. Let's calculate the limit of the "new" exponent with the use of L'Hospital's rule (it has a form 0/0):

$$\lim_{x\to 0}\frac{(\sin x - x)'}{(x^3)'} = \frac{1}{3}\lim_{x\to 0}\frac{\cos x - 1}{x^2} = -\frac{1}{6}.$$

Therefore $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^3} = -\dfrac{1}{6}$, so

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-1/6} = \frac{1}{\sqrt[6]{e}}.$$

**4.148.** The limit has $\infty/\infty$ form, so we can apply L'Hospital's rule.

$$\lim_{x\to\infty}\frac{(\ln x)'}{(\sqrt{x})'} = \lim_{x\to\infty}\frac{\dfrac{1}{x}}{\dfrac{1}{2\sqrt{x}}} = \lim_{x\to\infty}\frac{2\sqrt{x}}{x} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0.$$

So $\lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt{x}} = 0$.

## 4.4 Finding Extrema

**4.155.** Let $f(x) = x^3 - 12x$. The function $f(x)$ can have a global extremum (maximum or minimum) at points where its derivative is zero, or at the endpoints of the closed intervals. Let's find the roots of the derivative:

$$f'(x) = (x^3 - 12x)' = 3x^2 - 12 = 0.$$

The two roots:

$$x_1 = -2 \text{ and } x_2 = 2.$$

For the interval $[-10; 3]$:

$$f(-10) = -880, \quad f(-2) = 16, \quad f(2) = -16, \quad f(3) = -9$$

Therefore, on $[-10; 3]$ the global minimum of $f(x) = x^3 - 12x$ is $-880$ at $x = -10$, and the global maximum is 16 at $x = -2$.

For the interval $[0;3]$:

$$f(0)=0, \quad f(2)=-16, \quad f(3)=-9.$$

(Since $-2$ is not in the interval, so we don't calculate the value at $-2$.)

Therefore the global minimum on $[0;3]$ is $-16$ at $x=2$, and the global maximum is 0 at $x=0$.

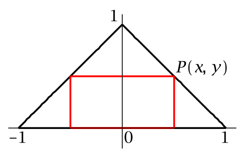
**4.159.** We only want to find the ratio of the sides of the triangle, so we can suppose that the length of the hypotenuse of the triangle is 2. Using the notations of figure, the $T(x)$ area and the $k(x)$ perimeter of the rectangle are:

$$y=1-x,$$
$$T(x)=2xy=2x(1-x), \quad k(x)=2(2x+y)=2(1+x).$$

Because of the geometric meaning of $x$ $0 \le x \le 1$. Since $k(x)$ is a monotonically increasing function, it has the maximum at $x=1$, we got a degenerate rectangle ($y=0$).

The maximum of the continuous function $T(x)$ is not at the endpoints, (see Weierstrass theorem), because $T(0)=T(1)=0$, so the global maximum is also a local maximum, therefore $T'(x)=0$ (see theorem 4.9).

$$T'(x)=2[(1-x)-x]=2-4x=0,$$
$$x=\frac{1}{2}$$




Therefore, the length of one of the sides of the rectangle (the horizontal one by the figure) is half of the other.

**Remark:** According to the inequality between the arithmetic and geometric means $f(x)=\dfrac{T(x)}{2}=x(1-x)$ is maximal on the interval $(0,1)$ if $x=1-x$, that is, $x=\dfrac{1}{2}$.

**4.163.** Let's denote the legs by $x$ and $y$, and the hypotenuse by $z$. The hypotenuse is $z=10-x$, the other leg is

$$y=\sqrt{(10-x)^2-x^2}=\sqrt{100-20x}=2\sqrt{25-5x},$$

and the area of the triangle

$$T(x)=\frac{1}{2}xy=x\sqrt{25-5x}.$$

Since $0 \le x \le 5$, so we have to find the global maximum of the continuous function $T(x)$ on the closed interval $[0,5]$. According to the Weierstrass theorem there is a maximum. Since $T(x) \ge 0$, $T(0) = T(5) = 0$, so the maximum is on the open interval $(0,5)$. In this case the global maximum is also a local maximum, therefore according to theorem 4.9 the derivative is zero at this point. Let's find the roots of $T'(x)$:

$$T'(x) = \sqrt{25-5x} - \frac{5x}{2\sqrt{25-5x}} = 0$$

$$\sqrt{25-5x} = \frac{5x}{2\sqrt{25-5x}} \quad \iff \quad 2(25-5x) = 5x \quad \iff \quad x = \frac{10}{3}$$

Since the derivative has exactly one root, the (local) maximum can be only here. Therefore,

$$\max T = T\left(\frac{10}{3}\right) = \frac{10}{3}\sqrt{25 - \frac{50}{3}} = \frac{50}{3\sqrt{3}}.$$

The area is maximal, if

$$x = \frac{10}{3}, \quad y = \frac{10}{\sqrt{3}} = x\sqrt{3}, \quad z = \frac{20}{3} = 2x.$$

According to the given condition of the problem, the triangle which has maximal area, is the half of an equilateral triangle.

**Remark:** The function $T(x)$ has the maximum at the same point as the function $f(x) = T^2(x) = x^2(25-5x)$. But the maximum of $f(x)$ can be found by "smartly" applying the inequality between the arithmetic and geometric means. There is a 3-factor product on $(0,5)$:

$$\left(\frac{5}{2}x\right)^2 (25-5x)$$

Here all of the factors are positive, and their sum is 25, independently of $x$. Therefore the product is maximal, if the factors are equal, that is,

$$\frac{5}{2}x = 25 - 5x, \quad \iff \quad 15x = 50, \quad \iff \quad x = \frac{10}{3}$$

**4.168.** The surface area $F$ and the volume $V$ of a cylinder are:

$$F = 2\pi R^2 + 2\pi m R, \quad V = mR^2\pi.$$

Let's express $m$ from the formula of the volume, and substitute the results in the formula for surface area:

$$m = \frac{V}{\pi R^2}, \quad F(R) = 2\pi R^2 + \frac{2V}{R}.$$

The domain of $F(R)$ is the open half-line $(0, \infty)$, and here the function is positive. The value of the function is large if $R$ is close to 0 ($R$ is small) or $R$ is close to $\infty$ ($R$ is large), so the global minimum is local minimum as well. Let's find the roots of the derivative function of $F(R)$:

$$F'(R) = 4\pi R - \frac{2V}{R^2} = 0, \text{ that is, } 2\pi R = \frac{V}{R^2}, \ R = \sqrt[3]{\frac{V}{2\pi}}.$$

Therefore, the surface area of a right circular cylinder with given volume is minimal if

$$R = \sqrt[3]{\frac{V}{2\pi}}.$$

The height and the quotient of the height and the diameter:

$$m = \frac{V}{\pi R^2} = \sqrt[3]{\frac{4V}{\pi}}, \quad \frac{m}{2R} = 1.$$

Therefore, the surface area of a right circular cylinder is minimal if the diameter is equal to the height.

**Remark:** We have to show that $f$ really has a global minimum because the reasoning above is not a proof. The correct proof:

Let $F_0 = F(1) = 2\pi + 2V$. Since $\lim\limits_{R\to+0} F(R) = \lim\limits_{R\to\infty} F(R) = \infty$, so an $0 < a < 1$ and a $1 < b$ can be given such that $R \in (0, a]$ or $R \in [b, \infty)$ implies $F(R) > F_0$. According to the Weierstrass theorem the function $F(R)$ has a minimum on the closed interval $[a, b]$. Because of choosing $a$ and $b$ this minimum is the minimum as well on the whole half-line $(0, \infty)$.

Since the point of minimum cannot be any of the endpoints of the interval $[a, b]$, so this minimum is a local minimum as well. According to theorem 4.9, the derivative is 0 at this point. Because there is exactly one such $R$ on the whole half-line, therefore this is the radius which belongs to the minimum.

## 4.5 Examination of Functions

**4.172.** **P** $\iff$ **Q**: see theorem 4.8.

**4.176.** **P** $\not\Rightarrow$ **Q**: For example, $f(x) = x^4,\ a = 0$.

**Q** $\Longrightarrow$ **P**: See the theorems about the relationship of convexity and derivative.

**4.182.** Let's find the roots of $f'(x)$:

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2) = 0,$$

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 2.$$

The sign of the factors of the derivative, so the sign of the derivative can easily be found on the whole number-line:

- If $x < 0$, then $f'(x) < 0$, and therefore $f(x)$ is strictly monotonically decreasing on $(-\infty, 0)$.
- If $0 < x < 1$, then $f'(x) > 0$, and therefore $f(x)$ is strictly monotonically increasing on $(0, 1)$.
- If $1 < x < 2$, then $f'(x) < 0$, and therefore $f(x)$ is strictly monotonically decreasing on $(1, 2)$.
- If $2 < x$, then $f'(x) > 0$, and therefore $f(x)$ is strictly monotonically increasing on $(2, \infty)$.

There are strict local extrema at the joining points of the monotonous intervals, namely

- minimum at 0,
- maximum at 1,
- minimum at 2.

**4.186.** $y' = e^{-x} + xe^{-x}(-1) = (1-x)e^{-x}$.

Let's find the roots of the derivative. From the equation $(1-x)e^{-x} = 0$, we'll get that $x = 1$, therefore the function can have a local extremum only at $c = 1$.
Since $y'(x) > 0$ if $x < 1$ and $y'(x) < 0$ if $x > 1$, so the function has a *strict local extremum*, and the value of the function is $e^{-1}$. Thus the local extremum is the global maximum as well.

**4.194.** The domain of $f(x) = \dfrac{x+1}{1+x^2}$ is $\mathbb{R}$, and the function is any times differentiable.

$$\lim_{x\to-\infty} \frac{x+1}{1+x^2} = \lim_{x\to\infty} \frac{x+1}{1+x^2} = 0.$$

$$f'(x) = \frac{1+x^2-2x(x+1)}{(1+x^2)^2} = \frac{1-2x-x^2}{(1+x^2)^2} = -\frac{(x-x_1)(x-x_2)}{(1+x^2)^2},$$

where $x_1 = -1-\sqrt{2}$ and $x_2 = -1+\sqrt{2}$ are the roots of the numerator.

$$\begin{aligned} f''(x) &= \frac{(-2-2x)(1+x^2)^2 - 4x(1-2x-x^2)(1+x^2)}{(1+x^2)^4} = \\ &= -2\frac{(1+x)(1+x^2)+2x(1-2x-x^2)}{(1+x^2)^3} = \\ &= 2\frac{x^3+3x^2-3x-1}{(1+x^2)^3} = \\ &= 2\frac{(x-X_1)(x-X_2)(x-X_3)}{(1+x^2)^3}, \end{aligned}$$

where $X_1 = -2-\sqrt{3},\ X_2 = -2+\sqrt{3},\ X_3 = 1$ are the roots of the numerator. By examining the sign of the first derivative, we got

- $f(x)$ is strictly monotonically decreasing on $(-\infty, x_1)$,
- there is a strict local minimum at $x_1$,
- $f(x)$ is strictly monotonically increasing on $(x_1, x_2)$,
- there is a strict local maximum at $x_2$,
- $f(x)$ is strictly monotonically decreasing on $(x_2, \infty)$.

By examining the sign of the first derivative, we got

- $f(x)$ is strictly concave on $(-\infty, X_1)$,
- $f(x)$ is strictly convex on $(X_1, X_2)$,
- $f(x)$ is strictly concave on $(X_2, X_3)$,
- $f(x)$ is strictly convex on $(X_3, \infty)$,
- there are inflection points at $X_1,\ X_2$ and $X_3$.

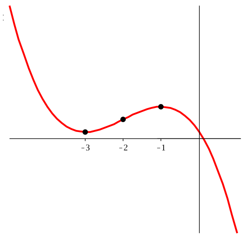
**4.203.** The function $f(x) = 1-9x-6x^2-x^3$ is a polynomial with degree 3, and it is any times differentiable on $\mathbb{R}$. Since the main coefficient is negative,

$$\lim_{x\to-\infty} 1-9x-6x^2-x^3 = \infty, \quad \lim_{x\to\infty} 1-9x-6x^2-x^3 = -\infty.$$

Finding the roots of the first and second derivatives:

$f'(x) = (-9-12x-3x^2 = 0, \quad x_1 = -3, \ x_2$

$$f''(x) = -12 - 6x = 0, \quad X_1 = -2.$$




We can see in the figure, and it can be proved by using the derivatives that there is a minimum at $x_1$ and a maximum at $x_2$, and $X_1$ is an inflection point. The function is decreasing on $(-\infty, -3]$, increasing on $[-3, -1]$, decreasing on $[-1, \infty)$, convex on $(-\infty, -2]$ and concave on $(-2, \infty)$.

## 4.6 Elementary Functions

**4.224.** $2^{\frac{\ln 100}{\ln 2}} = 2^{\log_2 100} = 100.$

**4.226.** $\arcsin \dfrac{\sqrt{3}}{2} = \dfrac{\pi}{3}(= 60^\circ).$

**4.228.** $\arccos(\cos(9\pi)) = \pi$ (and none $9\pi$).

**4.230.** $\tan(\arctan 100) = 100.$

**4.235.** A period of the function $f(x) = \cos \dfrac{x}{2} + \tan \dfrac{x}{3}$ is for example, $p = 12\pi$ because $4\pi$ is a period of $\cos \dfrac{x}{2}$ and $3\pi$ is of $\tan \dfrac{x}{3}$.

# Riemann Integral

## 5.1 Indefinite Integral

**5.1.**

$$
\begin{array}{lll}
f(x) = x^3 - 4x, & f(x) : (2), & f'(x) : (E). \\
f(x) = x^3, & f(x) : (4), & f'(x) : (C). \\
f(x) = x + \sin x, & f(x) : (5), & f'(x) : (A). \\
f(x) = \tan x, & f(x) : (4), & f'(x) : (D). \\
f(x) = e^{-x}, & f(x) : (1), & f'(x) : (B).
\end{array}
$$

**5.5.** At first let's change the integrand:

$$
\begin{aligned}
(\sin x + \cos x)^2 &= \sin^2 x + 2 \sin x \cos x + \cos^2 x \\
&= 1 + 2 \sin x \cos x = 1 + \sin 2x.
\end{aligned}
$$

$$
\begin{aligned}
\int (\sin x + \cos x)^2 \ dx &= \int (1 + \sin 2x)\, dx \\
&= \int dx + \int \sin 2x \, dx = x - \frac{\cos 2x}{2} + C
\end{aligned}
$$

**5.8.** $\displaystyle\int x^{3/2}\, dx = \frac{2}{5} x^{5/2} + C$

**5.10.** $\displaystyle\int \frac{-5}{x-7}\, dx = -5 \ln(x-7) + C$

**5.12.** $\displaystyle\int \sin 2x + 3 \cos x \, dx = -\frac{\cos 2x}{2} + 3 \sin x + C$

**5.14.** $\displaystyle\int e^{2x-3}\, dx = \frac{1}{2} e^{2x-3} + C$

**5.16.** $\displaystyle\int \frac{x^2}{x^3+1}\, dx = \frac{1}{3} \int \frac{3x^2}{x^3+1}\, dx = \frac{1}{3} \int \frac{(x^3+1)'}{x^3+1}\, dx =$
$\frac{1}{3} \ln(x^3+1) + C$

**5.20.** $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\, dx = \frac{1}{2} \int \frac{2x}{\sqrt{x^2+1}}\, dx = \frac{1}{2} \int \frac{(x^2+1)'}{\sqrt{x^2+1}}\, dx =$
$\sqrt{x^2+1} + C$

**5.24.** $\displaystyle\int \frac{1}{2+x^2}\, dx = \frac{1}{2} \int \frac{1}{1 + \left(\frac{x}{\sqrt{2}}\right)^2}\, dx = \frac{1}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) + C$

**5.28.** By partial fraction expansion:

$$\frac{3}{(x+3)(x+2)} = \frac{A}{x+2} + \frac{B}{x+3}$$

Multiply both sides by the denominator of the left-hand side:

$$3 = A(x+3) + B(x+2)$$

By comparison of the coefficients, we got a linear system of equations for $A$ and $B$:

$$\begin{array}{ccccc} 3A & + & 2B & = & 3 \\ A & + & B & = & 0 \end{array}$$

$$A = 3, \qquad B = -3$$

Therefore

$$\int \frac{3}{(x+3)(x+2)}dx = \int \frac{3}{x+2}\,dx - \int \frac{3}{x+3}\,dx = 3\ln\frac{x+2}{x+3} + C$$

**5.32.** The denominator of the integrand has no real roots. In the first step get rid of "$x$" from the numerator by "smuggling in" the derivative of the denominator:

$$\begin{aligned} \int \frac{x-2}{x^2-2x+6}dx &= \frac{1}{2}\int \frac{(2x-2)-2}{x^2-2x+6}\,dx \\ &= \frac{1}{2}\int \frac{2x-2}{x^2-2x+6}\,dx - \int \frac{1}{x^2-2x+6}\,dx \end{aligned}$$

The first integral on the right side has a form $f'/f$, and the denominator is positive, so

$$\int \frac{2x-2}{x^2-2x+6}\,dx = \ln(x^2-2x+6) + C.$$

The second integral can be reduced to the form of $\displaystyle\int \frac{1}{1+t^2}\,dt$:

$$\begin{aligned} \int \frac{1}{x^2-2x+6}\,dx &= \int \frac{1}{(x-1)^2+5}\,dx = \frac{1}{5}\int \frac{1}{\left(\frac{x}{\sqrt{5}} - \frac{1}{\sqrt{5}}\right)^2 + 1}\,dx = \\ &= \frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}} - \frac{1}{\sqrt{5}}\right) + C. \end{aligned}$$

Therefore,

$$\int \frac{x-2}{x^2-2x+6}dx = \frac{1}{2}\ln(x^2-2x+6) - \frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}} - \frac{1}{\sqrt{5}}\right) + C.$$

**5.36.** Write down the integrand as the sum of a polynomial and a real (the grad of the numerator less then the grad of the denominator) fraction:

$$\frac{3x^3+2x-1}{x^2-x-6}=3x+3+\frac{23x+17}{x^2-x-6}=3x+3+\frac{23x+17}{(x-3)(x+2)}$$

$$\int\frac{3x^3+2x-1}{x^2-x-6}dx=\int(3x+3)\,dx+\int\frac{23x+17}{(x-3)(x+2)}\,dx$$

Decompose the rational fraction of the right-hand side integrand to partial fractions:

$$\frac{23x+17}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}$$

$$23x+17=A(x+2)+B(x-3)$$

$$\begin{array}{rcrcl} A & + & B & = & 23 \\ 2A & - & 3B & = & 17 \end{array}$$

Add the three times of the first equation to the second equation:

$$5A=86$$

$$A=\frac{86}{5},\qquad B=\frac{29}{5}$$

$$\int\frac{3x^3+2x-1}{x^2-x-6}dx=\int(3x+3)\,dx+\frac{86}{5}\int\frac{1}{x-3}\,dx+\frac{29}{5}\int\frac{1}{x+2}\,dx$$

$$=\frac{3}{2}x^2+3x+\frac{86}{5}\ln|x-3|+\frac{29}{5}\ln|x+2|+C$$

**5.40.** Decompose the integrand to partial fractions:

$$\frac{1}{x^3+x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

$$1=Ax(x+1)+B(x+1)+Cx^2$$

If we substitute 0 for $x$, we get $B$, and if we substitute 1 for $x$, we get $C$. From these we get $A$:

$$A=-1,\qquad B=1,\qquad C=1$$

$$\int\frac{1}{x^3+x^2}\,dx=-\int\frac{1}{x}\,dx+\int\frac{1}{x^2}\,dx+\int\frac{1}{x+1}\,dx=\ln\left|\frac{x+1}{x}\right|-\frac{1}{x}+C$$

**5.44.**

$$\int \frac{x+2}{x-1}\,dx = \int \frac{(x-1)+3}{x-1}\,dx = \int dx + 3\int \frac{1}{x-1}\,dx$$
$$= x + 3\ln|x-1| + C$$

**5.48.**

$$\int \frac{x+2}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{(2x+2)+2}{x^2+2x+2}\,dx =$$
$$= \frac{1}{2}\ln(x^2+2x+2) + \int \frac{1}{x^2+2x+2}\,dx =$$
$$= \frac{1}{2}\ln(x^2+2x+2) + \int \frac{1}{(x+1)^2+1}\,dx =$$
$$= \frac{1}{2}\ln(x^2+2x+2) + \arctan(x+1) + C$$

**5.50.**

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

**5.54.**

$$\int \frac{5}{\cos^2(1-x)}\,dx = -5\tan(1-x) + C$$

**5.58.**

$$\int \frac{\sin^2 2x + 1}{\cos^2 x}\,dx = \int \frac{4\sin^2 x\cos^2 x + 1}{\cos^2 x}\,dx = 2x - \sin 2x + \tan x + C$$

Here we used the result of problem 5.50.

**5.60.** Let $f(x) = x$, $g'(x) = \cos x$. So

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

**5.64.**

$$\int \arctan x\,dx = \int 1\cdot \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2} =$$
$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

**5.68.**

$$\int x\ln\frac{1+x}{1-x}\,dx = \frac{1}{2}x^2\ln\frac{1+x}{1-x} - \frac{1}{2}\int \frac{x^2}{\frac{1+x}{1-x}}\cdot\frac{2}{(1-x)^2}\,dx$$
$$= \frac{1}{2}x^2\ln\frac{1+x}{1-x} - \int\frac{x^2}{1-x^2}\,dx$$
$$-\int\frac{x^2}{1-x^2}\,dx = \int\frac{1-x^2-1}{1-x^2}\,dx = x - \int\frac{1}{1-x^2}\,dx$$
$$\int\frac{1}{1-x^2}\,dx = \frac{1}{2}\int\left(\frac{1}{1-x}+\frac{1}{1+x}\right)\,dx = \frac{1}{2}\ln\frac{1+x}{1-x} + C$$

We used that the original integrand is defined only for $\frac{1+x}{1-x} > 0$. Therefore

$$\int x\ln\frac{1+x}{1-x}\,dx = \frac{1}{2}x^2\ln\frac{1+x}{1-x} + x - \frac{1}{2}\ln\frac{1+x}{1-x} + C$$

**5.71.**

$$t = x^2, \qquad dt = 2x\,dx$$
$$\int xe^{x^2}\,dx = \frac{1}{2}\int 2xe^{x^2}\,dx = \frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t + C = \frac{1}{2}e^{x^2} + C$$

**5.75.**

$$t = \cos x, \qquad dt = -\sin x\,dx$$
$$\int\frac{1}{\sin x}\,dx = \int\frac{\sin x}{\sin^2 x}\,dx = -\int\frac{-\sin x}{1-\cos^2 x}\,dx = -\int\frac{1}{1-t^2}\,dt$$
$$= \frac{1}{2}\ln\frac{1-t}{1+t} + C = \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x} + C$$

Here we used a part of the result of problem 5.68., and $|t| < 1$ implies $\frac{1-t}{1+t} > 0$.

**5.79.**

$$x = \sin t, \qquad dx = \cos t \, dt, \qquad t = \arcsin x$$

$$\int \frac{1}{(1-x^2)\sqrt{1-x^2}} \, dx = \int \frac{1}{(1-\sin^2 t)\cos t} \cos t \, dt = \int \frac{1}{\cos^2 t} \, dt =$$

$$= \tan t + C = \frac{\sin t}{\sqrt{1-\sin^2 t}} + C = \frac{x}{\sqrt{1-x^2}} + C$$

**5.83.**

$$t = x^2 + x + 1, \qquad dt = 2x + 1$$

$$\int (2x+1) e^{x^2+x+1} \, dx = \int e^t \, dt = e^t + C = e^{x^2+x+1} + C$$

**5.87.**

$$t = e^x, \qquad x = \ln t, \qquad dx = \frac{dt}{t}$$

$$\int \frac{e^x + 2}{e^x + e^{2x}} \, dx = \int \frac{t+2}{t+t^2} \cdot \frac{dt}{t} = \int \frac{t+2}{t^2(t+1)} \, dt$$

Decompose the integrand to partial fractions:

$$\frac{t+2}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$$

$$t + 2 = At(t+1) + B(t+1) + Ct^2$$

$$A = -1, \qquad B = 2, \qquad C = 1$$

$$\int \frac{t+2}{t^2(t+1)} \, dt = \int \left( -\frac{1}{t} + \frac{2}{t^2} + \frac{1}{t+1} \right) dt = -\frac{2}{t} + \ln \frac{t+1}{t} + C$$

We used that $\dfrac{t+1}{t} > 0$. After substituting it back

$$\int \frac{e^x + 2}{e^x + e^{2x}} \, dx = -\frac{2}{e^x} + \ln \frac{e^x + 1}{e^x} + C = -2e^{-x} + \ln(1 + e^{-x}) + C$$

**5.91.**

$$\int \frac{1}{(x+2)(x-1)} \, dx = \frac{1}{3} \int \left( \frac{1}{x-1} - \frac{1}{x+2} \right) dx = \frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C$$

**5.95.**

$$\int \frac{1}{e^x + e^{-x}}\, dx = \frac{1}{2}\int \frac{dx}{\cosh x} = \frac{1}{2}\int \frac{\cosh x}{1+\sinh^2 x}\, dx$$
$$= \frac{1}{2}\arctan \sinh x + C$$

**5.99.**

$$\int \ln(x^2+1)\ dx = \int 1\cdot \ln(x^2+1)\, dx = x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\, dx$$
$$\int \frac{x^2}{x^2+1}\, dx = \int \frac{(x^2+1)-1}{x^2+1}\, dx = x - \arctan x + C$$
$$\int \ln(x^2+1)\ dx = x\ln(x^2+1) - 2x + \arctan x + C$$

**5.103.**

$$\int 2x\sin(x^2+1)\, dx = \int (x^2+1)'\sin(x^2+1)\, dx = -\cos(x^2+1) + C$$

**5.107.**

$$\int \frac{e^{2x}}{1+e^x}\, dx = \int \frac{e^x}{e^x+1} e^x\, dx = \int \frac{t}{t+1}\, dt = t - \ln(t+1) + C$$

ahol $t = e^x$.

$$\int \frac{e^{2x}}{1+e^x}\, dx = e^x - \ln(e^x+1) + C$$

**5.111.**

$$\int (x^2+x)\ln x\, dx = \left(\frac{x^3}{3}+\frac{x^2}{2}\right)\ln x - \int \left(\frac{x^2}{3}+\frac{x}{2}\right)\, dx$$
$$= \left(\frac{x^3}{3}+\frac{x^2}{2}\right)\ln x - \frac{x^3}{9} - \frac{x^2}{4} + C$$

**5.115.**

$$\int 3^{-2x}\, dx = \int e^{(-2\ln 3)x}\, dx = -\frac{1}{2\ln 3} e^{(-2\ln 3)x} + C = -\frac{1}{2\ln 3} 3^{-2x} + C$$

**5.119.**

$$t = \tan x, \qquad dt = \frac{1}{\cos^2 x}\,dx, \qquad \cos^2 x = \frac{1}{1+t^2}, \qquad \sin^2 x = \frac{t^2}{1+t^2}$$

$$\int \frac{1}{\sin^2 x \cos^4 x}\,dx = \int \frac{1}{\sin^2 x \cos^2 x} \cdot \frac{1}{\cos^2 x}\,dx = \int \frac{(t^2+1)^2}{t^2}\,dt =$$

$$= \int \left(t^2 + 2 + \frac{1}{t^2}\right) dt = \frac{t^3}{3} + 2t - \frac{1}{t} + C =$$

$$= \frac{\tan^3 x}{3} + 2\tan x - \cot x + C$$

## 5.2 Definite Integral

**5.124.** **(a)** $\Phi$ is a refinement of $F$.

**(b)** $\Phi$ is not a refinement of $F$ because 1.5 is not among the partition points of $\Phi$.

$F$ is not a refinement of $\Phi$ because $-1$ is not among the partition points of $F$.

**5.125.**

$$S_\Phi = 6, \qquad s_\Phi = 0$$

**5.128.**

$$S_\Phi = -1 \cdot 1 + 0 \cdot 1 + 4 \cdot 4 = 15, \qquad s_\Phi = -2 \cdot 1 - 1 \cdot 1 + 0 \cdot 4 = -3$$

**5.131.** Yes because $f(x)$ is continuous (and monotonic).

**5.135.** No because $f(x)$ is not bounded on $[0, 1]$.

**5.140.** Let $f(x) = \dfrac{1}{x^2 + e^x}$. So $x \in [1, 2]$ implies $0 < f(x) < 1$, so according to problem 5.138.

$$0 = 0 \cdot (2-1) \leq \int_1^2 \frac{1}{x^2 + e^x}\,dx \leq 1 \cdot (2-1) = 1$$

**5.144.** Since every integral functions are continuous, and $\operatorname{sgn} x$ is not continuous at 0, therefore $\operatorname{sgn} x$ cannot be an integral function on $[-1, 1]$.

According to the Darboux theorem $\operatorname{sgn} x$ does not have a primitive function on $[-1, 1]$.

**5.146.** The sum $\sigma_n = \dfrac{\sin\frac{1}{n} + \sin\frac{2}{n} + \cdots + \sin\frac{n}{n}}{n}$ is a Riemann sum of the integrable function $\sin x$ on the interval $[0, 1]$ with an equidistant partition, so

$$\sigma_n = \frac{\sin\frac{1}{n} + \sin\frac{2}{n} + \cdots + \sin\frac{n}{n}}{n} \to \int_0^1 \sin x\, dx = 1 - \cos 1$$

**5.150.**

$$n\sum_{i=1}^{n} \frac{i}{n^2 + i^2} = \sum_{i=1}^{n} \frac{\frac{i}{n}}{1 + \left(\frac{i}{n}\right)^2} \to \int_0^1 \frac{x}{1 + x^2}\, dx = \left[\frac{1}{2}\ln(1 + x^2)\right]_0^1 = \frac{\ln 2}{2}$$

**5.154.**

$$g(x) = G'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Since

$$h(x) = f(x) + g(x) = \begin{cases} 2x \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous everywhere, so it has a primitive function, but in this case the function $f(x) = h(x) - g(x)$ also has a primitive function.

**5.156.**

$$H(x) = \int_2^x \frac{1}{\ln t}\, dt, \qquad H'(x) = \frac{1}{\ln x}$$

**5.158.** Since $\lim\limits_{x\to\infty}\frac{x}{\ln x}=\infty$ and $\lim\limits_{x\to\infty}\int\limits_2^x\frac{1}{\ln t}\,dt=\infty$, so for the quotient

$$\frac{\ln x}{x}\int\limits_2^x\frac{1}{\ln t}\,dt=\frac{\int\limits_2^x\frac{1}{\ln t}\,dt}{\frac{x}{\ln x}}$$

we can apply L'Hospital's rule:

$$\lim_{x\to\infty}\frac{\ln x}{x}\int\limits_2^x\frac{1}{\ln t}\,dt=\lim_{x\to\infty}\frac{\frac{1}{\ln x}}{\frac{\ln x-1}{\ln^2 x}}=\lim_{x\to\infty}\frac{\ln x}{\ln x-1}=1$$

**5.162.**

$$\int\limits_2^3 x^2\,dx=\left[\frac{x^3}{3}\right]_2^3=9-\frac{8}{3}$$

**5.166.**

$$\int\limits_{-2\pi}^0\sin^2 x\,dx=\int\limits_{-2\pi}^0\frac{1-\cos 2x}{2}\,dx=\left[\frac{x}{2}-\frac{\sin 2x}{4}\right]_{-2\pi}^0=\pi$$

**5.168.**

$$x=\sin t,\qquad dx=\cos t\,dt,\qquad \frac{1}{2}=\sin\frac{\pi}{6},\qquad \frac{\sqrt{3}}{2}=\sin\frac{\pi}{3}$$

$$\int\limits_{1/2}^{\sqrt{3}/2}\frac{x^2}{\sqrt{1-x^2}}\,dx=\int\limits_{\pi/6}^{\pi/3}\frac{\sin^2 t}{\sqrt{1-\sin^2 t}}\cos t\,dt=\int\limits_{\pi/6}^{\pi/3}\sin^2 t\,dt$$

$$=\left[\frac{t}{2}-\frac{\sin 2t}{4}\right]_{\pi/6}^{\pi/3}=\frac{\pi}{12}$$

(see the problem 5.50.).

**5.172.**

$$t=\tan x,\qquad x=\arctan t,\qquad dx=\frac{1}{1+t^2}\,dt,\qquad \tan\frac{\pi}{6}=\frac{\sqrt{3}}{3},\qquad \tan\frac{\pi}{2}=\infty$$

$$\int_{\pi/6}^{\pi/2} \frac{1}{1+\tan x}\,dx = \int_{\sqrt{3}/3}^{\infty} \frac{1}{(1+t)(1+t^2)}\,dt = \frac{1}{2}\int_{\sqrt{3}/3}^{\infty} \left(\frac{1}{t+1} + \frac{1-t}{1+t^2}\right) dt$$

$$= \frac{1}{2}\left[\ln \frac{1+t}{\sqrt{1+t^2}} + \arctan t\right]_{\sqrt{3}/3}^{\infty} = -\frac{1}{2}\ln\frac{1+\sqrt{3}}{2} + \frac{\pi}{6}$$

## 5.3 Applications of the Integration

**5.177.** Let's find the intersection points of the parabola and the line:

$$x^2 = -x+2, \qquad x_1 = -2,\ x_2 = 1.$$

Between the two intersection points the line $-x+2$ is "greater".

$$T = \int_{-2}^{1} (-x+2-x^2)\,dx = \int_{-2}^{1} (2-x-x^2)\,dx = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= 6 + \frac{3}{2} + \frac{7}{3}$$

**5.181.**

$$T = \int_{1}^{e} \ln x\,dx = [x\ln x - x]_1^e = 1$$

**5.185.** The two intersection points:

$$\frac{1}{1+x^2} = \frac{x^2}{2}, \qquad x^4+x^2-2=0, \qquad x_1=-1,\ x_2=1$$

$$T = \int_{-1}^{1} \left(\frac{1}{1+x^2} - \frac{x^2}{2}\right) dx = \left[\arctan x - \frac{x^3}{6}\right]_{-1}^{1} = \frac{\pi}{2} - \frac{1}{3}$$

**5.189.** We get this segment of the parabola if we choose the parabola $y =$

$m\left(1-\dfrac{4}{h^2}x^2\right)$ and the $x$-axis.

$$T=\int\limits_{-h/2}^{h/2} m\left(1-\frac{4}{h^2}x^2\right)\,dx = m\left[x-\frac{4}{3h^2}x^3\right]_{-h/2}^{h/2}$$
$$= m\left(h-\frac{h}{3}\right)=\frac{2}{3}mh$$

**5.192.** Here $-\dfrac{\pi}{6}\le\varphi\le\dfrac{\pi}{6}$.

$$T=\frac{1}{2}\int\limits_{-\pi/6}^{\pi/6}\cos^2 3\varphi\,d\varphi=\frac{1}{2}\left[\frac{\varphi}{2}+\frac{\sin 6\varphi}{6}\right]_{-\pi/6}^{\pi/6}=\frac{\pi}{12}$$

**5.196.**

$$V=\pi\int\limits_0^{\pi}\sin^2 x\,dx=\pi\left[\frac{x}{2}-\frac{\sin 2x}{4}\right]_0^{\pi}=\frac{\pi^2}{2}$$

**5.200.**

$$V=\pi\int\limits_0^1\arcsin^2 y\,dy=\pi\int\limits_0^{\pi/2}x^2\cos x\,dx$$

Calculate $\displaystyle\int x^2\cos x\,dx$:

$$\int x^2\cos x\,dx = x^2\sin x-2\int x\sin x\,dx=$$
$$=x^2\sin x+2x\cos x-2\int\cos x\,dx=$$
$$=x^2\sin x+2x\cos x-2\sin x+C.$$

Using the result above

$$V=\pi\int\limits_0^{\pi/2}x^2\cos x\,dx=\pi\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}=\frac{\pi^3}{4}-2\pi.$$

**5.204.**

$$V=\pi\int\limits_{1}^{2}\left(e^{2x}-\frac{1}{x^2}\right)dx=\pi\left[\frac{e^{2x}}{2}+\frac{1}{x}\right]_1^2=\pi\left(\frac{e^2(e^2-1)}{2}-\frac{1}{2}\right)$$

**5.208.**

$$L=\int\limits_{-1}^{1}\sqrt{1+\sinh^2 x}\,dx=\int\limits_{-1}^{1}\cosh x\,dx=[\sinh x]_{-1}^{1}=2\sinh 1$$

## 5.4 Improper integral

**5.220.** If $c\neq 1$, then

$$\int\limits_{1}^{\infty}\frac{1}{x^c}\,dx=\left[\frac{1}{1-c}\cdot\frac{1}{x^{c-1}}\right]_1^{\infty}=\begin{cases}\dfrac{1}{c-1} & \text{if } c>1\\[2ex] \infty & \text{if } c<1\end{cases}$$

If $c=1$, then

$$\int\limits_{1}^{\infty}\frac{1}{x}\,dx=[\ln x]_1^{\infty}=\infty$$

Therefore, the improper integral $\int\limits_{1}^{\infty}\frac{1}{x^c}\,dx$ is convergent if and only if $c>1$.

**5.223.**

$$\int\limits_{3}^{\infty}2^{-x}\,dx=\left[-\frac{2^{-x}}{\ln 2}\right]_3^{\infty}=\frac{1}{8\ln 2}$$

**5.227.** According to the solutions of problem 5.220., $\int\limits_{1}^{\infty}\frac{dx}{\sqrt{x}}$ is divergent.

**5.231.**

$$\int\limits_{\frac{1}{2}}^{1}\frac{dx}{x\ln x}=[\ln|\ln x|]_{1/2}^{1^-}=-\infty$$

**5.235.**

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = [\arcsin x]_0^{1^-} = \frac{\pi}{2}$$

**5.239.**

$$\int_1^\infty \frac{dx}{\sqrt{x}+x^2} < \int_1^\infty \frac{dx}{x^2} < \infty$$

(see problem 5.220.).

**5.243.** The function $\frac{x^2}{x^4-x^2+1}$ is Riemann integrable on $[0,1]$ (because the denominator is not zero), so it is enough to show that the improper integral $\int_1^{+\infty} \frac{x^2}{x^4-x^2+1}\,dx$ is convergent.

Since the order of the functions $f(x) = \frac{x^2}{x^4-x^2+1}$ and $g(x) = \frac{1}{x^2}$ is equal (the ratio of the two functions goes to 1 in $\infty$), and $\int_1^\infty \frac{dx}{x^2}$ is convergent, so according to the limit comparison test $\int_1^{+\infty} \frac{x^2}{x^4-x^2+1}\,dx$ is convergent.

**5.247.** Since $\sin x \sim x$ at 0, and $\int_0^{\pi/2} \frac{dx}{x}$ is divergent, so $\int_0^{\pi/2} \frac{dx}{\sin x}$ is divergent.

**5.251.**

$$\int_0^\infty x e^{-x^2}\,dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^\infty = \frac{1}{2}$$

# Numerical Series

## 6.1 Convergence of Numerical Series

**6.1.** Using the formula about the sum of finite geometric series

$$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}}$$
$$= 2\left(1 - \frac{1}{2^{n+1}}\right) \longrightarrow 2$$

**6.5.**

$$\frac{1}{k(k+3)} = \frac{1}{3}\left(\frac{1}{k} - \frac{1}{k+3}\right)$$

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+3)} = \frac{1}{3}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+3}\right) =$$
$$= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$
$$\longrightarrow \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18}$$

**6.8.**

$$\sum_{n=1}^{\infty} \frac{4^n + 5^n}{9^n} = \sum_{n=1}^{\infty}\left(\frac{4}{9}\right)^n + \sum_{n=1}^{\infty}\left(\frac{5}{9}\right)^n = \frac{4}{9}\frac{1}{1 - \frac{4}{9}} + \frac{5}{9}\frac{1}{1 - \frac{5}{9}} = \frac{4}{5} + \frac{5}{4}$$

**6.13.** Since the terms of the series don't go to 0, therefore the series is divergent. (See the criteria of convergence about the terms converging to 0).

**6.15.** If $A$ is the sum of the series, and $s_n$ is the sum of the first $n$ terms, then

$$s_n \longrightarrow A, \qquad s_{n-1} \longrightarrow A, \qquad a_n = s_n - s_{n-1} \longrightarrow 0.$$

**6.16.** We show that the Cauchy convergence test is not true for the harmonic series because for $\varepsilon = \frac{1}{2}$ there is no "good" threshold:

$$s_{2n} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = n \cdot \frac{1}{2n} = \frac{1}{2}.$$

**6.21.** No, for example the harmonic series is divergent, but the sequence of its terms converges to 0.

**6.27.** The partial sums of $\sum_{n=1}^{\infty} \frac{1}{n+n^2} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ **are so-called telescopic sums**:

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} \longrightarrow 1.$$

**6.31.** Since the improper integral $\int_1^{\infty} \frac{dx}{\sqrt[3]{x}}$ is divergent, so according to the integral test the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ is divergent.

We can also use the direct comparison test because $\frac{1}{\sqrt[3]{n}} \geq \frac{1}{n}$.

**6.35.**

$$\sum_{n=1}^{\infty} \sin(n\pi) = \sum_{n=1}^{\infty} 0 = 0$$

**6.40.** No, for example $b_n = 0,\ a_n = -1$.
But if we assume that $a_n > 0$, then according to the "contrapositive" version of the direct comparison test, $\sum_{n=1}^{\infty} b_n$ is divergent.

## 6.2 Convergence Tests for Series with Positive Terms

**6.43.** According to the integral test (applying for $\frac{1}{x^2}$) the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ dominates the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$, therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ is also convergent.

**6.45.** See the ratio test.

**6.48.**

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n n!} = \frac{3}{(1+1/n)^n} \longrightarrow \frac{3}{e} > 1.$$

Therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$ is divergent.

**6.50.** See the root test.

**6.53.**

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{1}{2}+\frac{1}{n}\right)^n} = \frac{1}{2}+\frac{1}{n} \longrightarrow \frac{1}{2} < 1.$$

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}+\frac{1}{n}\right)^n$ is convergent.

**6.58.** The order of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4}{n^4+3n}$ is $\displaystyle\frac{1}{n^2}$:

$$\frac{n^2+4}{n^4+3n} : \frac{1}{n^2} = \frac{(n^2+4)n^2}{n^4+3n} \longrightarrow 1,$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4}{n^4+3n}$ is convergent.

**6.62.** No. Not even if $f(x)$ is monotonically decreasing. For example, $f(x) = 5e^{-x+1}$. In this case

$$\int_1^{\infty} 5e^{-x+1}\,dx = \left[-5e^{-x+1}\right]_1^{\infty} = 5,$$

$$\sum_{n=1}^{\infty} 5e^{-n+1} = 5\frac{1}{1-1/e} = 5\frac{e}{e-1} \neq 5$$

**6.66.** The derivative of $f(x) = \dfrac{1}{x\ln x}$ is $f'(x) = -\dfrac{\ln x + 1}{x^2 \ln^2 x} < 0$ if $x > 1$, therefore $f(x)$ is positive and monotonically decreasing on $[e, \infty)$. We can apply the integral test for this function:

$$\int_2^\infty \frac{1}{x\ln x}\,dx = [\ln\ln x]_2^\infty = \infty,$$

therefore $\displaystyle\sum_{n=2}^\infty \frac{1}{n\ln n}$ is divergent.

**6.68.**

$$\sum_{n=1}^\infty \frac{1}{n+\sqrt{n}} > \sum_{n=1}^\infty \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n} = \infty,$$

therefore $\displaystyle\sum_{n=1}^\infty \frac{1}{n+\sqrt{n}}$ is divergent (direct comparison test).

**6.74.**

$$\sqrt[n]{\frac{n^2}{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \longrightarrow \frac{1}{2} < 1,$$

therefore $\displaystyle\sum_{n=1}^\infty \frac{n^2}{2^n}$ is convergent (root test).

**6.80.** $\dfrac{n^{10}}{3^n - 2^n} < \dfrac{2n^{10}}{3^n}$ if $2^n < \dfrac{1}{2}3^n$, which is true from some $n$. Applying the root test for the series $\displaystyle\sum_{n=1}^\infty \frac{2n^{10}}{3^n}$,

$$\sqrt[n]{\frac{2n^{10}}{3^n}} = \frac{\sqrt[n]{2}\left(\sqrt[n]{n}\right)^{10}}{3} \longrightarrow \frac{1}{3} < 1,$$

so $\displaystyle\sum_{n=1}^\infty \frac{n^{10}}{3^n - 2^n}$ is convergent.

**6.86.**

$$\sum_{n=1}^\infty \frac{2^n + 3^n}{5^n} = \sum_{n=1}^\infty \left(\frac{2}{5}\right)^n + \sum_{n=1}^\infty \left(\frac{3}{5}\right)^n = \frac{2}{3} + \frac{3}{2} < \infty$$

**6.92.** The sequence of the terms of the series does not converge to 0 (convergence tests):

$$1 + \frac{1}{n} \longrightarrow 1 \neq 0,$$

therefore $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ is divergent.

## 6.3 Conditional and Absolute Converge

**6.99.** No, for example $a_n = (-1)^n$.

**6.104.** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is a Leibniz series because $\frac{1}{n}$ converges to 0 monotonically decreasly, therefore the series is a convergent (but not absolute convergent).

**6.108.** The terms of $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[4]{4}} + \cdots = \displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt[n]{n}}$ are alternating, but the sequence of the terms does not converge to 0 ($|a_n| \to 1$), therefore the series is divergent.

**6.114.** $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{2n+1}$ is convergent Leibniz series, but not absolute convergent.

**6.120.** $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ is absolute convergent, because $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$.

# Sequences of Functions and Function Series

## 7.1 Pointwise and Uniform Convergence

**7.1.** The sequence of functions $f_n(x) = x^n$ is convergent at the points of the interval $(-1, 1]$, and divergent at other points.

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

For all $-1 < a < b < 1$ the convergence is uniform on $[a, b]$ because

$$\max\{|x^n| : x \in [a, b]\} = (\max\{|a|, |b|\})^n \longrightarrow 0.$$

Since the limit function is not continuous (from left-hand side at 1) on $(-1, 1]$, so according to theorem 7.1 the convergence is not uniform on the whole range of convergence.

**7.5.** The sequence $f_n(x) = \sqrt[n]{1 + x^{2n}}$ converges pointwise to the constant 1 on the whole number line. The convergence is uniform on all bounded $[a, b]$ intervals because $1+x^{2n}$ is bounded on $[a, b]$. Because for all $n \in \mathbb{N}$

$$\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty} \sqrt[n]{1 + x^{2n}} = \infty,$$

that is, $f_n$ is not bounded, therefore the convergence of $f_n$ to the bounded function $f(x) \equiv 1$ is not uniform on $\mathbb{R}$ or on $[0, \infty)$. Because the functions are even, the convergence is not uniform on $(-\infty, 0]$ as well.

**7.11.** For all $x \in \mathbb{R}$ implies $\lim\limits_{n\to\infty} f_n(x) = 0$. This convergence is uniform on all intervals or half-lines which contain only finitely many numbers of the form $\dfrac{1}{n}$, but for every $b > 0$ the convergence is not uniform on $(0, b)$ (and any larger) interval because for $\varepsilon = 1/b$ there is no good threshold.

**7.18.** Yes, for example $g_n(x) = f_n(x) + 5$, where $f_n$ is the function from problem 7.11.

**7.23.** **(a)** $f_n'(0) = \lim\limits_{x\to 0} \dfrac{f_n(x) - f_n(0)}{x} = \lim\limits_{x\to 0} \dfrac{\sqrt{x^2 + \frac{1}{n}} - \sqrt{\frac{1}{n}}}{x} =$

$$\lim_{x\to 0} \frac{x}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{\frac{1}{n}}} = 0.$$

**(b)** $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}\sqrt{x^2+\frac{1}{n}} = \sqrt{x^2} = |x|$.

**(c)** $|f_n(x) - |x|| = \sqrt{x^2+\frac{1}{n}} - |x| = \dfrac{\frac{1}{n}}{\sqrt{x^2+\frac{1}{n}}+|x|} \le \sqrt{\frac{1}{n}} \longrightarrow 0$.

**(d)** Since the left-hand side and right-hand side derivatives ($-1$ and $1$) are not equal, $|x|$ is not differentiable at 0.

**7.25.**

$$|f_n(x)| = \frac{1}{n^2+n^4x^{2n}} \le \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^2+n^4x^{2n}} \le \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty.$$

**7.29.** $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{x^4+2^n}$ is uniformly convergent according to the Weierstrass criterium because

$$\left|\frac{(-1)^n}{x^4+2^n}\right| \le \frac{1}{2^n}, \qquad \sum_{n=1}^{\infty}\frac{1}{2^n} < \infty.$$

**7.35.** According to the Weierstrass criterium $\sum\limits_{n=1}^{\infty}\dfrac{\cos nx}{n!+2^n}$ is uniformly convergent on $\mathbb{R}$ because

$$\left|\frac{\cos nx}{n!+2^n}\right| \le \frac{1}{n!+2^n} \le \frac{1}{n!}, \qquad \sum_{n=1}^{\infty}\frac{1}{n!} < \infty.$$

## 7.2 Power Series, Taylor Series

**7.42.** Calculate the $R$ radius of convergence of $\sum\limits_{n=0}^{\infty}\dfrac{n!}{n^n}x^n$. Using theorem 7.3:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e}, \qquad R = e.$$

It is known that (Stirling's formula) for large enough $n$ $n! > \left(\frac{n}{e}\right)^n$, therefore the terms of the series do not converge to 0 at the endpoints of the convergence interval, that is, at $e$ and at $-e$:

$$\frac{n!}{n^n}e^n = n!\cdot\left(\frac{e}{n}\right)^n > 1$$

if $n$ is large enough.

**7.48.** $\sqrt[n]{|a_n|} = \sqrt[n]{n^2} = (\sqrt[n]{n})^2 \to 1$, $\quad R = 1$. The power series is divergent at the endpoint because the terms do not converge to 0.

**7.54.** $\sqrt[n]{\dfrac{1}{n2^n}} = \dfrac{1}{2\sqrt[n]{n}} \longrightarrow \dfrac{1}{2}$, $\quad R = 2$. Therefore, the interior of the range of convergence is the interval $(-7, -3)$. At the right endpoint the series is divergent (harmonic series), and at the left endpoint it is convergent because it is alternating.

**7.60.** $\sqrt[n]{\dfrac{1}{n3^n}} = \dfrac{1}{3\sqrt[n]{n}} \longrightarrow \dfrac{1}{3}$, $\quad R = 3$. Therefore, the interior of the range of convergence is the interval $(-3, 3)$, and at the left endpoint it is convergent because it is alternating.

**7.64.** $\dfrac{|a_{n+1}|}{|a_n|} = \dfrac{((n+1)!)^2}{(2n+2)!} \cdot \dfrac{(2n)!}{(n!)^2} = \dfrac{(n+1)^2}{(2n+1)(2n+2)} \to \dfrac{1}{4}$. Therefore, $R = 4$.

**7.66.** The radius of convergence of $\displaystyle\sum_{n=0}^{\infty} x^n$ is 1, and because of the theorem of integration by terms, the statement is true.

**7.72.** The radius of convergence of $x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ is 1. The sum of the power series is $f(x)$ on $(-1, 1)$. According to the theorem on derivation by terms

$$f'(x) = \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{n} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

therefore $f(x)$ is the primitive function of $\dfrac{1}{1-x}$ and $f(0) = 0$, so

$$f(x) = -\ln(1-x), \qquad -1 < x < 1.$$

**Remark:** Because the series converges at $-1$ (Leibniz series), therefore by the so-called “Abel criterium” one can prove that the expression remains true at $-1$ as well.

**7.76.** Using the sum of the geometric series

$$\sum_{n=0}^{\infty} (\sin x)^n = \frac{1}{1-\sin x} \quad \text{if} \quad |\sin x| \neq 1, \quad \text{that is} \quad x \neq \frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$$

**7.80.** $\sqrt[n]{\dfrac{1}{n(n+1)}} \to 1$, therefore $R = 1$.

Let $g(x) = x \cdot f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$. By derivation by terms $g'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

By another derivation $g''(x) = \displaystyle\sum_{n=1}^{\infty} x^{n-1} = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

So $g'(x) = \displaystyle\int \frac{1}{1-x}\,dx = -\ln(1-x)$, $\quad g(x) = -\displaystyle\int \ln(1-x)\,dx = x + (1-x)\ln(1-x)$.

Therefore, $\quad f(x) = 1 + \dfrac{1-x}{x}\ln(1-x)$ if $|x| < 1$.

**7.82.** $\sqrt[n]{n} \to 1$, therefore $R = 1$.

Let $g(x) = \dfrac{f(x)}{x} = \displaystyle\sum_{n=1}^{\infty} n x^{n-1}$.

Let $h(x)$ be the primitive function of $g(x)$ such that $h(0) = 0$. We can get this function by integration by terms:

$h(x) = \displaystyle\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1 = \frac{x}{1-x}$. Therefore, $g(x) = h'(x) = \dfrac{1}{(1-x)^2}$, so

$$f(x) = x \cdot g(x) = \frac{x}{(1-x)^2}.$$

**7.84.** $\sqrt[n]{n(n+1)} \to 1$, therefore $R = 1$.

Let $g(x)$ be the primitive function of $f(x)$ such that $g(0) = 0$. We can get this function by integration by terms:

$g(x) = \displaystyle\sum_{n=1}^{\infty} n x^{n+1} = x \sum_{n=1}^{\infty} n x^n$. Using the result of the previous problem $g(x) = \dfrac{x^2}{(1-x)^2}$.

Hence $f(x)$:

$$f(x) = g'(x) = \frac{2x(1-x)^2 + 2x^2(1-x)}{(1-x)^4} = 2\frac{x(1-x)+x^2}{(1-x)^3} = 2\frac{x}{(1-x)^3}$$

**7.87.** Substitute $2x$ for $x$ of the Taylor series of $e^x$. (Taylor series of common functions).

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n, \qquad x \in \mathbb{R}.$$

**7.93.**

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1.$$

**7.96.** Let $f(x) = \ln(1+x)$. According to problem 7.93.

$$f'(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad \text{ if } |x| < 1.$$

$$f(x) = \int_0^x \frac{1}{1+t}\, dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n\, dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$

**7.99.** Multiplying both the numerator and denominator by $1-x$:

$$f(x) = \frac{1-x}{(1+x+x^2)(1-x)} = \frac{1-x}{1-x^3}.$$

Using the sum of the geometric series, substituting $x^3$ for $x$

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}, \qquad |x| < 1.$$

So

$$f(x) = \frac{1}{1-x^3} - x\frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n} - \sum_{n=0}^{\infty} x^{3n+1} = \sum_{k=0}^{\infty} a_k x^k,$$

where

$$a_k = \begin{cases} 1 & \text{if } k = 3n \\ -1 & \text{if } k = 3n+1 \\ 0 & \text{if } k = 3n+2 \end{cases}$$

**7.105.**

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!}$$
$$= 1 + \sum_{n=1}^{\infty}(-1)^n 2^{2n-1}\frac{x^{2n}}{(2n)!}, \qquad x \in \mathbb{R}.$$

**7.111.** We estimate the difference between the value of the function and the value of the Taylor polynomial with the help of the Lagrange reminder. For $f(x) = \sin x$ for some $d \in [0,1]$

$$|\sin 1 - T_{2n}(1)| = \left|\sin 1 - \sum_{k=0}^{n-1}(-1)^k \frac{1}{(2k+1)!}\right|$$
$$= \frac{d^{2n+1}}{(2n+1)!} \le \frac{1}{(2n+1)!} < 10^{-2},$$

if $n \ge 2$. Therefore,

$$|\sin 1 - T_4(1)| = \left|\sin 1 - 1 + \frac{1}{3!}\right| = \left|\sin 1 - \frac{5}{6}\right| < 10^{-2}.$$

For $f(x) = e^x$ for some $d \in [0,1]$

$$\left|e^1 - T_n(1)\right| = e - T_n(1) = \frac{e^d d^{n+1}}{(n+1)!} \le \frac{e}{(n+1)!} < 10^{-2},$$

if $n \ge 5$. Therefore

$$e - T_5(1) = e - \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}\right) = e - \left(2 + \frac{43}{60}\right)$$
$$\sim e - 2.716666667 < 10^{-2}.$$

**7.117.** The derivative is the product of 136! and the coefficient of the power $x^{136}$ in the Taylor series around 0 of the function $f(x) = e^{x^2}$.

$$f(x) = e^{x^2} = \sum_{n=0}^{\infty}\frac{1}{n!}x^{2n}, \qquad f^{(136)}(0) = \frac{136!}{68!}$$

**7.120.** $f'(x) = (\tan x)' = \dfrac{1}{\cos^2 x}$

$$f''(x) = (\tan x)'' = \left(\frac{1}{\cos^2 x}\right)' = 2\frac{\sin x}{\cos^3 x}$$

$$f'''(x) = (\tan x)''' = \left(2\frac{\sin x}{\cos^3 x}\right)' = 2\frac{\cos^4 x + 3\sin^2 x\cos^2 x}{\cos^6 x} =$$

$$= 2\frac{\cos^2 x + 3\sin^2 x}{\cos^4 x} = 2\frac{1+2\sin^2 x}{\cos^4 x}$$

Therefore $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 2$. Therefore the 3rd Taylor polynomial is

$$t_3(x) = x + \frac{x^3}{3}.$$

Since $\tan x$ is odd, therefore the 4th derivative at 0 is 0. Therefore $t_3(x) = t_4(x)$.

**7.125.** **(a)** $\dfrac{1}{2+x} = \dfrac{1}{2}\dfrac{1}{1-\dfrac{-x}{2}} = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{n+1}}x^n, \qquad |x| < 2$

**(b)** $\dfrac{1}{2+x} = \dfrac{1}{3+(x-1)} = \dfrac{1}{3}\dfrac{1}{1-\dfrac{-(x-1)}{3}} = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{n+1}}(x-1)^n, \qquad |x-1| < 3$

## 7.3 Trigonometric Series, Fourier Series

**7.126.** $\sin^2 x = \dfrac{1}{2} - \dfrac{1}{2}\cos 2x$

**7.128.** Since $\operatorname{sgn} x$ is odd, so all $a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\operatorname{sgn} x \sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin nx\,dx = -\frac{2}{\pi n}\left[\cos nx\right]_0^{\pi}$$

$$= \begin{cases} \dfrac{4}{\pi n} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Since the function $\operatorname{sgn} x$ is piecewise continuously differentiable and at 0 the value is the arithmetic mean of the two half side limits, therefore the Fourier series equals to the function on the interval $(-\pi, \pi)$.

$$\operatorname{sgn} x = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1} \qquad \text{if } -\pi < x < \pi.$$

**7.134.** **(a)** This function is odd, that is, $f(-x) = -f(x)$. So for all $a_n = 0$.

$$\int_0^{2\pi} \frac{\pi - x}{2} \sin nx \, dx = \left[ -\frac{\pi - x}{2} \cdot \frac{1}{n} \cos nx \right]_0^{2\pi} - \frac{1}{2n} \int_0^{2\pi} \cos nx \, dx$$
$$= \left[ \frac{x - \pi}{2n} \cos nx \right]_0^{2\pi} = \frac{\pi}{n}$$

Therefore,

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{\pi - x}{2} \sin nx \, dx = \frac{1}{n}.$$

Since $f(x)$ is piecewise continuously differentiable, therefore on $(0, 2\pi)$ the Fourier series equals to the function at the point where the function is continuous.

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \qquad \text{if } 0 < x < 2\pi.$$

**(b)** Substitute $\frac{\pi}{2}$ for $x$ in the previous series.

Since $\sin\left(k\frac{\pi}{2}\right) = \begin{cases} (-1)^n & \text{if } k = 2n+1 \\ 0 & \text{if } k = 2n \end{cases}$, therefore

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}.$$

**7.135.** (a) Since $f$ is even,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx \, dx = 0,$$

that is, the coefficients $b_n$ are zero.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

if $n > 0$, integrating by parts twice:

$$\int_{-\pi}^{\pi} x^2\cos nx\,dx = \left[\frac{1}{n}x^2\sin nx\right]_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} x\sin x\,dx =$$

$$= \frac{2}{n^2}\left[x\cos nx\right]_{-\pi}^{\pi} - \frac{2}{n^2}\int_{-\pi}^{\pi}\cos nx\,dx = (-1)^n\frac{4\pi}{n^2} - \frac{2}{n^3}\left[\sin nx\right]_{-\pi}^{\pi} =$$

$(-1)^n\dfrac{4\pi}{n^2}$.

So for $n > 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx = (-1)^n\frac{4}{n^2}$$

Since $\sum \frac{1}{n^2}$ is convergent, therefore the Fourier series of $f$ is uniformly convergent because of the Weierstrass criterium, and $f$ is continuous, so the Fourier series equals to the function.

$$x^2 = \frac{\pi^2}{3} + 4\cdot\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}, \quad \text{if } -\pi \le x \le \pi.$$

(b) If we substitute $\pi$ for $x$ in the previous Fourier series, and use that $\cos n\pi = (-1)^n$, we get that

$$\pi^2 = \frac{\pi^2}{3} + 4\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

After rearrangement

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

**7.142.**

$$e^x = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

Integrating by parts twice, we get

$$\int e^x\cos nx\,dx = \frac{\cos nx + n\sin nx}{n^2+1}e^x + C, \quad \int e^x\sin nx\,dx$$

$$= \frac{\sin nx - n\cos nx}{n^2+1}e^x + C.$$

Using this

$$a_0 = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^x\,dx = \frac{1}{2}\cdot\frac{e^{\pi}-e^{-\pi}}{\pi},$$

$$a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} e^x \cos nx\,dx = (-1)^n\frac{1}{n^2+1}\cdot\frac{e^{\pi}-e^{-\pi}}{\pi}$$

$$b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} e^x \sin nx\,dx = (-1)^{n+1}\frac{n}{n^2+1}\cdot\frac{e^{\pi}-e^{-\pi}}{\pi}.$$

After substitution

$$e^x = \frac{e^{\pi}-e^{-\pi}}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)\right), \qquad -\pi < x < \pi.$$

# Differentiation of Multivariable Functions

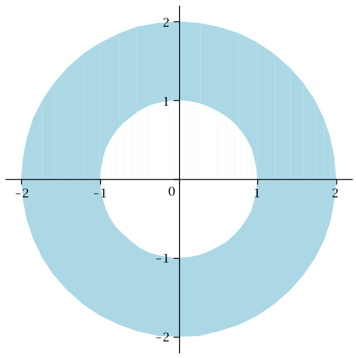
## 8.1 Basic Topological Concepts

**8.2.** $|\mathbf{p}-\mathbf{q}| = \sqrt{(-1-5)^2+(3-(-4))^2+(5-0)^2} = \sqrt{36+49+25} = \sqrt{110}$

**8.5.**

$$k=1: \quad x^2<r^2$$
$$k=2: \quad x^2+y^2<r^2$$
$$k=3: \quad x^2+y^2+z^2<r^2$$
$$k=n: \quad x_1^2+x_2^2+\cdots+x_n^2<r^2$$

**8.8.** This set is the open annulus in the figure. The set is open and bounded,
the boundary points are the points of the circles $x^2+y^2=1$ and $x^2+y^2=4$.




**8.14.** $H=\{x : 0<x<1\}\subset\mathbb{R}$ is open (open interval), because if $x\in H$, $r=\min\{x,1-x\}$, then $\{y : |x-y|<r\}\subset H$.

**8.15.** If $H=\{(x,0) : 0<x<1\}\subset\mathbb{R}^2$, then $\partial H=\{(x,0) : 0\le x\le 1\}$. According to theorem 8.1 $H$ is not open because $H\cap\partial H\neq\emptyset$ and not closed because $\partial H\not\subseteq H$.

**8.22.** If $H=\{(x,y) : x\in\mathbb{Q}, y\in\mathbb{Q}\}\subset\mathbb{R}^2$, then $\partial H=\mathbb{R}^2$ and according to theorem 8.1 $H$ is neither open nor closed.

**8.23.** $H=\{(x,y) : 0<x<1, 0<y<1\}\subset\mathbb{R}^2$ is open (open rectangle) because $p=(x,y)\in H$, $r=\min\{x,1-x,y,1-y\}>0$ implies

$$B(p;r)=\left\{q\in\mathbb{R}^2 : |p-q|<r\right\}\subset H.$$

**8.29.**

$$H = \{(x,y,z) : x^2+y^2+z^2 \le 1\}, \quad \operatorname{int} H = \{(x,y,z) : x^2+y^2+z^2 < 1\},$$

$$\operatorname{ext} H = \{(x,y,z) : x^2+y^2+z^2 > 1\}, \quad \partial H = \{(x,y,z) : x^2+y^2+z^2 = 1\}$$

**8.32.** **(a)** Not true, for example $H = \{\mathbf{0}\}$, $x = \mathbf{0}$.

**(b)** True because $\operatorname{int} H \subset H$.

**(c)** Not true, for example $H = \{\mathbf{0}\}$.

**(d)** Not true, for example $H = \{p \in \mathbb{R}^n : |p| < 1\}$, $x = (1, 0, \dots, 0)$.

**(e)** True, for example $H = \mathbb{Q}^n$.

**(f)** True, for example $H = \{p \in \mathbb{R}^n : |p| < 1\}$.

**(g)** True, for example $H = \{p \in \mathbb{R}^n : |p| = 1\}$.

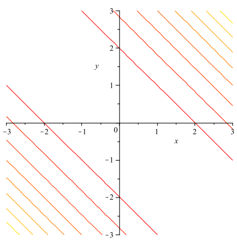
## 8.2 The Graphs of Multivariable Functions

**8.33.** $f(\mathbf{p}) = f(2,3) = 2 + 3^2 = 11$.

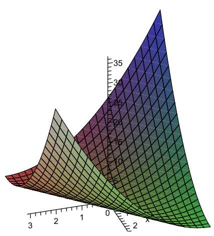
**8.38.** $f(x,y) = x - y, \quad f(x,x) = 0, \quad f(x,x^2) = x - x^2$.

**8.41.** (a) — (B), (b) — (D), (c) — (C), (d) — (F), (e) — (A), (f) — (E)

**8.43.**

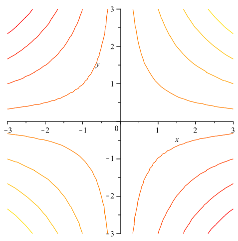


contours

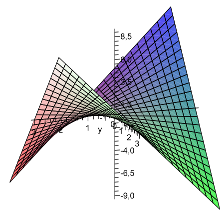


graph

**8.45.**

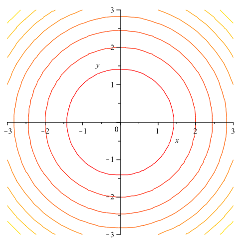


**contours**

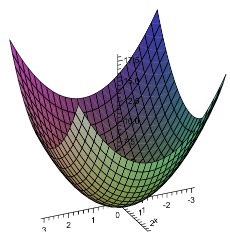


**graph**

**8.47.**

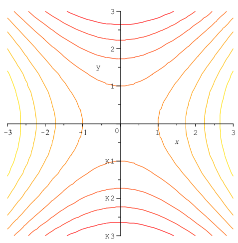


**contours**

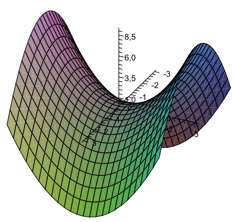


**graph**

**8.49.**



**contours**



**graph**

**8.61.** $f(x,y) = \dfrac{x+y}{x-y}$, $\quad D_f = \big\{(x,y) \in \mathbb{R}^2 : x \neq y\big\}$.

## 8.3 Multivariable Limit, Continuity

**8.67.** $f(x,y) = 7$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} 7 = 7$. The function is continuous everywhere.

**8.73.** The function

$$f(x,y) = \frac{\sin x - \sin y}{e^x - e^y}$$

is not defined if $x = y$, so the domain does not contain a punctured neighbourhood of the origin. Therefore the function has no limit at the origin. The function $f(x,y)$ is continuous at every point of its domain, $\{(x,y) : x \neq y\}$.

**8.79.**

$$f(x,y) = \begin{cases} x \text{ if } x = y \\ 0 \text{ otherwise} \end{cases}$$

The function is continuous everywhere except at the points of the set $H = \{(x,x) : x \neq 0\}$. Therefore it has limit at $(0,0)$ and in $(0,1)$, and the limits are the values of the function.

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0, \qquad \lim_{(x,y)\to(0,1)} f(x,y) = f(0,1) = 0.$$

## 8.4 Partial and Total Derivative

**8.85.** For example, $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } x^2+y^2 > 0 \\ 0 & \text{if } x^2+y^2 > 0 \end{cases}$.

Since $f(x,0) \equiv 0 \equiv f(0,y)$, so both partial derivatives exist at the origin, and they are zero. But since $f(x,x) \equiv 1 \neq 0$, the $f$ function has not even limit at the origin.

**8.90.** If $x \neq y^2$, then $f(x,y) = \dfrac{x+y}{x-y^2}$ is partially differentiable, and

$$f'_x(x,y) = \frac{(x-y^2)-(x+y)}{(x-y^2)^2} = -\frac{y^2+y}{(x-y^2)^2},$$

$$f'_y(x,y) = \frac{(x-y^2)+(x+y)\cdot 2y}{(x-y^2)^2} = \frac{x+y^2+2xy}{(x-y^2)^2}.$$

**8.96.** The domain of $g(x,y,z) = x^{y^z}$ is the set $\big\{(x,y,z) \in \mathbb{R}^3 : x > 0,\ y > 0\big\}$. The function is differentiable at the points of the set, and

$$\frac{\partial}{\partial x}\left(x^{y^z}\right) = x^{(y^z-1)} \cdot y^z, \quad \frac{\partial}{\partial y}\left(x^{y^z}\right) = x^{y^z} \cdot \ln x \cdot z \cdot y^{z-1},$$

$$\frac{\partial}{\partial z}\left(x^{y^z}\right) = x^{y^z} \cdot \ln x \cdot \ln y \cdot y^z$$

**8.102.** Neither of the partial derivatives exist at the origin because the one-variable functions $f(x,0) = |x|$ and $f(0,y) = |y|$ are not differentiable at 0.

**8.108.**

$$g(x,y,z) = 2 + x + y^2 + z^3,$$

$$g'_x(x,y,z) = 1, \quad g'_y(x,y,z) = 2y, \quad g'_z(x,y,z) = 3z^2,$$

$$g''_{xx}(x,y,z) = 0, \quad g''_{xy}(x,y,z) = g''_{yx}(x,y,z) = 0,$$

$$g''_{xz}(x,y,z) = g''_{zx}(x,y,z) = 0. g''_{yy}(x,y,z) = 2,$$

$$g''_{yz}(x,y,z) = g''_{zy}(x,y,z) = 0, \quad g''_{zz}(x,y,z) = 6z$$

**8.112.** The length of the vector $\mathbf{v} = (-4,3)$ is $|\mathbf{v}| = 5$. Since the function $f(x,y) = e^{x+y} \cdot \ln y$ is differentiable at the point $\mathbf{p} = (0,1)$, therefore it is differentiable in any direction (see theorem 8.5) and

$$\frac{\partial}{\partial \mathbf{v}} f(0,1) = \frac{1}{5}\left(-4\frac{\partial}{\partial x} f(0,1) + 3\frac{\partial}{\partial y} f(0,1)\right)$$

$$\frac{\partial}{\partial x}\left(e^{x+y} \cdot \ln y\right) = e^{x+y} \cdot \ln y, \quad \frac{\partial}{\partial y}\left(e^{x+y} \cdot \ln y\right) = e^{x+y} \cdot \left(\ln y + \frac{1}{y}\right).$$

After substitution $\frac{\partial}{\partial \mathbf{v}} f(0,1) = \frac{3e}{5}$.

**8.119.** The absolute value of the directional derivatives at an arbitrary point of a continuously differentiable function is maximal in the direction of the gradient (see theorem 8.5), therefore the ball starts rolling in this direction and downward.

$$f'_x(x,y) = 3x^2 - 9y, \quad f'_y(x,y) = 3y^2 - 9x$$

$$\operatorname{grad} f(1,2) = (-15,3), \quad \operatorname{grad} f(2,1) = (3,-15),$$

$$\operatorname{grad} f(2,0) = (12,-18), \quad \operatorname{grad} f(-2,1) = (3,21)$$

**8.125.**

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } x^2+y^2\neq 0\\[2ex] 0 & \text{if } x^2+y^2=0\end{cases}$$

If $x^2+y^2\neq 0$

$$\begin{aligned} f'_x(x,y)&=y\frac{x^2-y^2}{x^2+y^2}+xy\frac{2x(x^2+y^2)-2x(x^2-y^2)}{(x^2+y^2)^2}\\ &=y\frac{x^2-y^2}{x^2+y^2}+xy\frac{4xy^2}{(x^2+y^2)^2}\\ f'_y(x,y)&=x\frac{x^2-y^2}{x^2+y^2}+xy\frac{-2y(x^2+y^2)-2y(x^2-y^2)}{(x^2+y^2)^2}\\ &=x\frac{x^2-y^2}{x^2+y^2}-xy\frac{4x^2y}{(x^2+y^2)^2}\end{aligned}$$

**(a)** $f(x,0)=f(0,y)=0,\quad f'_x(0,0)=f'_y(0,0)=0$

**(b)** $f'_x(0,y)=-y,\quad f'_y(x,0)=x$

**(c)** $f''_{xy}(0,0)=-1,\quad f''_{yx}(0,0)=1$

**(d)** Because the second order derivatives are not continuous at the origin.

**(e)** We show that $f$ is not differentiable twice, because $f'_x$ is not differentiable at the origin. Using the previous results:

$$f'_x(0,0)=0,\quad f''_{xx}(0,0)=0,\quad f''_{xy}(0,0)=-1$$

$$\frac{f'_x(x,y)-\left[f''_{xx}(0,0)x+f''_{xy}(0,0)y+f'_x(0,0)\right]}{\sqrt{x^2+y^2}}=\frac{f'_x(x,y)+y}{\sqrt{x^2+y^2}}$$

We show that the expression does not converge to 0 at the point $(0,0)$, not even on the line $y=x$.

$$\frac{f'_x(x,x)+x}{\sqrt{x^2+x^2}}=\frac{2x}{\sqrt{2}x}=\sqrt{2}$$

**8.127.** Yes, for example $f(x,y)=x\sin y$.

**8.129.** If $x>0$, then

$$\operatorname{grad} f(x,y)=(yx^{y-1},x^y\ln x),\quad \operatorname{grad} f(2,3)=(12,8\ln 2).$$

The equation of the tangent plane

$$z = 8 + 12(x-2) + 8\ln 2 \cdot (y-3)$$

**8.132.**

$$y = n! + \sum_{k=1}^{n} \frac{n!}{k}(x_k - k)$$

**8.138.** Here $h = g \circ f : \mathbb{R}^2 \to \mathbb{R}$, $\quad h(u,v) = g(f(u,v)) = g(f_1(u,v), f_2(u,v))$ where

$$f_1(u,v) = u^2v^2, \quad f_2(u,v) = \frac{1}{uv}$$

$$\begin{aligned}
h'_u(u,v) &= g'_x(f(u,v))(f_1)'_u(u,v) + g'_y(f(u,v))(f_2)'_u(u,v) \\
&= \frac{1}{u^2v^2} \cdot 2uv^2 + uv \cdot \left(-\frac{1}{u^2v}\right) = \frac{1}{u} \\
h'_v(u,v) &= g'_x(f(u,v))(f_1)'_v(u,v) + g'_y(f(u,v))(f_2)'_v(u,v) \\
&= \frac{1}{u^2v^2} \cdot 2u^2v + uv \cdot \left(-\frac{1}{uv^2}\right) = \frac{1}{v}
\end{aligned}$$

$$J = (h'_u(u,v).h'_v(u,v)) = \left(\frac{1}{u}, \frac{1}{v}\right)$$

Verify the result by calculating the function $h(u,v)$, and the derivative.

$$h(u,v) = \ln(u^2v^2) + \ln\left(\frac{1}{uv}\right) = 2\ln u + 2\ln v - \ln u - \ln v = \ln u + \ln v$$

**8.144.** Let $t$ be the variable of the curve $\mathbf{r}$. $\mathbf{r}(t) = (x(t), y(t))$

$$h = f \circ \mathbf{r}, \quad h(t) = f(\mathbf{r}(t)) = x(t) + y(t),$$

$$h'(t) = h'(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = x'(t) + y'(t)$$

## 8.5 Multivariable Extrema

**8.150.** The function $f(x,y) = x^2 + e^y \sin\left(x^3y^2\right)$ is continuous on the compact (bounded and closed set)

$$H = \{(x,y) : x^2 + y^2 \leq 1\},$$

therefore according to the Weierstrass theorem it has an absolute maximum and minimum on $H$.

**8.155.** The function $f(x,y) = x^3y^2(1-x-y)$ is continuous everywhere, and on the bounded and closed

$$H = \{(x,y) : 0 \le x, 0 \le y, x+y \le 1\}$$

set (closed triangle) $f$ is not negative. On the perimeter of the triangle $f(x,y) = 0$, and on the interior of the triangle $f(x,y) > 0$. Therefore the maximum of $f$ is in the interior of the triangle, and therefore the absolute maximum is also a local maximum. Let's find the stationary points of the function $f$, that is, the roots of the derivative (theorem 8.7) if $x, y > 0,\ x+y < 1$.

$$f'_x(x,y) = 3x^2y^2(1-x-y) - x^3y^2 = 0, \qquad 4x + 3y = 3$$

$$f'_y(x,y) = 2x^3y(1-x-y) - x^3y^2 = 0, \qquad 2x + 3y = 2$$

$$x = \frac{1}{2}, \qquad y = \frac{1}{3}$$

Since we got only one root, that point is the location of the global maximum.

**8.161.** The function $f(x,y) = 3x^2 + 5y^2$ is nowhere negative, and except for the origin, the function is strictly monotonic on the axis. Therefore the only extremum of the function is at the origin, and it is a minimum. Let's check this result by the examination of the derivative (theorem 8.7).

$$f'_x(x,y) = 6x = 0, \qquad x = 0$$

$$f'_y(x,y) = 10y = 0, \qquad y = 0$$

Therefore the only stationary point is the origin.

**8.167.** Find the stationary points of the function $f(x,y) = -y^2 + \sin x$.

$$f'_x(x,y) = \cos x = 0, \qquad x = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$$

$$f'_y(x,y) = -2y = 0, \qquad y = 0$$

Therefore the stationary points of the function are on the $x$ axis:

$$\left\{\mathbf{p}_n = \left((2n+1)\frac{\pi}{2}, 0\right) :\ n \in \mathbb{Z}\right\}.$$

Let's examine the corresponding quadratic forms: (theorem 8.7).

$$f''_{xx}(x,y) = -\sin x, \quad f''_{xy}(x,y) = f''_{yx}(x,y) = 0. \quad f''_{yy}(x,y) = -2$$

The Hessian matrix at these points is

$$H_n = \begin{pmatrix} (-1)^{n+1} & 0 \\ 0 & -2 \end{pmatrix}$$

The determinant of the $H_n$ is negative if $n$ is odd, therefore there is no extremum at these points.

If $n$ is even, then $f$ has maximum at the points $\mathbf{p}_n$, because $f''_{xx} < 0$ at these points.

**8.174.** The distance of two arbitrary points of the two lines is

$$d(\mathbf{p}(t), \mathbf{q}(s)) = \sqrt{(2t-3s)^2 + (t-s)^2 + (2-t-2s)^2}, \qquad t, s \in \mathbb{R}$$

The distance and the square of the distance have minimum at the same point. Therefore we want to find the minimum of the two-variable function

$$f(t,s) = (2t-3s)^2 + (t-s)^2 + (2-t-2s)^2 = 6t^2 - 10ts + 14s^2 - 4t - 8s + 4$$

on the plane.

$$f'_t(t,s) = 12t - 10s - 4 = 0, \qquad 6t - 5s = 2$$

$$f'_s(t,s) = -10t + 28s - 8 = 0, \qquad -5t + 14s = 4$$

$$t = \frac{48}{59}, \qquad s = \frac{34}{59}, \qquad \min\{f(t,s) : t, s \in \mathbb{R}\} = \frac{4}{59}$$

Therefore the distance of the two lines is $\dfrac{2}{\sqrt{59}}$.

**8.180.**

$$f(x,y) = (x+y)^2, \quad f'_x(x,y) = f'_y(x,y) = 2(x+y)$$

Therefore all of the points of the line $x + y = 0$ are critical points. The value of the function is zero at these points, therefore $f$ has minimum at the points of $x + y = 0$. We note that at these points the examination of the second order partial derivatives is inconclusive because the quadratic form is semidefinite.

$$f''_{xx} = f''_{xy} = f''_{yy} = 2, \qquad D = \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0.$$

**8.188.**

$$f(x,y,z) = xyz + x^2 + y^2 + z^2$$
$$f'_x(x,y,z) = yz + 2x = 0, \quad f'_y(x,y,z) = xz + 2y = 0,$$
$$f'_z(x,y,z) = xy + 2z = 0$$

After rearranging the equation

$$xyz + 2x^2 = 0, \quad xyz + 2y^2 = 0, \quad xyz + 2z^2 = 0$$

Hence $x^2 = y^2 = z^2$. It is easy to see that if one of the variables is zero, the the two other variables are zero, too. Therefore one of the critical points is

$$\mathbf{a} = (0,0,0) \text{ (origin)} .$$

In the other cases $xyz$ should be negative, therefore either all the three factors are negative, so

$$\mathbf{b} = (-2,-2,-2)$$

is solution, or one factor is negative, and the two other factors are positive, so

$$\mathbf{c}_1 = (-2,2,2), \ \mathbf{c}_2 = (2,-2,2), \ \mathbf{c}_3 = (2,2,-2)$$

are critical points. Since the variables of the functions are interchangeable, so it is enough to examine one of the last three roots.

$$f''_{xx} = 2, \quad f''_{xy} = f''_{yx} = z, \quad f''_{xz} = f''_{zx} = y, \quad f''_{yz} = f''_{zy} = x,$$
$$f''_{yy} = 2, \quad f''_{zz} = 2$$

The Hessian matrix at $\mathbf{a}$ is

$$H_a = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

Since the leading principal minors are positive, therefore there is a minimum at the origin.

The Hessian matrix at $\mathbf{b}$ is

$$H_b = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} .$$

The quadratic form is indefinite because it has a negative $(-2)$ and a positive $(4)$ eigenvalue as well. Therefore there is no extremum at the point $\mathbf{q} = (-2, -2, -2)$. We can prove this in the way that we show that $f$ has greater and smaller values than $f(-2, -2, -2) = 4$ arbitrary close to the point $\mathbf{q}$. Because the function $g(x) = f(x, x, x) = x^3 + 3x^2$ has strict local maximum at $-2$, and the function $h(x) = f(x, -2, -2) = x^2 + 4x + 8$ has strict local minimum at $-2$.

At last there is no extremum, for example, at the point $\mathbf{c}_1$ because the function

$$g(x) = f(x, -x, -x) = x^3 + 3x^2$$

has strict local maximum at $-2$, and the function

$$h(x) = f(x, 2, 2) = x^2 + 4x + 8$$

has a local minimum at $-2$.

**8.195.** According to the Lagrange multiplier method, we find the critical points of

$$L(x, y) = 30 - \frac{x^2}{100} - \frac{y^2}{100} + \lambda(4x^2 + 9y^2 - 36)$$

with the constraint $4x^2 + 9y^2 = 36$.

$$L'_x(x.y) = -\frac{x}{50} + 8\lambda x = \left(8\lambda - \frac{1}{50}\right) x = 0$$

$$L'_y(x.y) = -\frac{y}{50} + 18\lambda y = \left(18\lambda - \frac{1}{50}\right) y = 0$$

$$4x^2 + 9y^2 = 36$$

The solutions of the 3-variable system of equations:

$$x_1 = 0, \quad y_1 = 2, \quad \lambda_1 = \frac{1}{18 \cdot 50}$$

$$x_2 = 3, \quad y_2 = 0, \quad \lambda_2 = \frac{1}{8 \cdot 50}$$

Since the points satisfying the constraint are the points of a compact set (closed circle line), therefore here $F$ is a maximum and a minimum as well. Since the closed circle line is a closed curve, therefore the extrema are at the two points above, and they are local extrema.

$$F(0, 2) = 30 - \frac{4}{100}, \quad F(3, 0) = 30 - \frac{9}{100}, \quad F(0, 2) > f(3, 0)$$

Therefore the maximal height point of the path is above the point $(0, 2)$, and the minimal height point of the path is above the point $(3, 0)$.

**8.200.** Since $f(x, y, z)$ is odd in its every variable, so the values of the minimums are the negatives of the maximum values. Therefore it is enough to find the maximum of $xyz$ with the constraint

$$x > 0, \quad y > 0, \quad z > 0, \quad x^2 + y^2 + z^2 = 3.$$

Using the inequality between the geometric and quadratic means

$$xyz \le \left(\sqrt{\frac{x^2 + y^2 + z^2}{3}}\right)^3 = 1$$

and there is equality if and only if $x = y = z$. But in this case $x = y = z = 1$.

**8.206.** If the volume of the brick is $V$, then find the minimum of the function

$$F(x, y, z) = 2(xy + xz + yz)$$

with the constraint

$$xyz = V, \quad x > 0, \quad y > 0, \quad z > 0.$$

It is at the same point, where the function $f(x, y, z) = xy + xz + yz$ has a minimum. We find with the Lagrange multiplier method the critical points of the function

$$L(x, y, z) = xy + xz + yz + \lambda(xyz - V)$$

$$L'_x(x, y, z) = y + z + \lambda yz = 0$$
$$L'_y(x, y, z) = x + z + \lambda xz = 0$$
$$L'_z(x, y, z) = x + y + \lambda xy = 0$$

Dividing the first equation by $yz$, the second one by $xz$, and the third one by $xy$, we get that

$$\frac{1}{z} + \frac{1}{y} = \frac{1}{z} + \frac{1}{x} = \frac{1}{y} + \frac{1}{x} = -\lambda,$$

hence $x = y = z = \sqrt[3]{V}$.

# Multivariable Riemann-integral

## 9.1 Jordan Measure

**9.1.** $H=\{(x,y): 0\le x<1, 0<y\le 1\}$, $\quad t(h)=1$.

**9.2.**

$$H=\{(x,y): x\in\mathbb{Q}, y\in\mathbb{Q}, 0\le x\le 1, 0\le x\le 1\},$$
$$\operatorname{int} H=\emptyset, \quad \overline{H}=N=[0,1]\times[0,1].$$

According to theorem 9.1 $b(H)=0\ne 1=k(H)$, so $H$ is not measurable.

**9.8.** See problem 9.2.

**9.12.** $H=\{(x,y,z): x\in\mathbb{Q}, y\in\mathbb{Q}, z\in\mathbb{Q}, 0\le x\le 1, 0\le x\le 1, 0\le z\le 1\}$
Similarly to problem 9.2. $b(H)=0, \quad k(H)=1$.

**9.18.** No, see the problem 9.2.

**9.24.** For all $n\in\mathbb{N}^+$ the contour $K_n=\left\{\mathbf{p}\in\mathbb{R}^2 : |\mathbf{p}|=\dfrac{1}{n}\right\}$ is a null set. We show that the set $H=\bigcup\limits_{n=1}^{\infty} K_n$ is measurable, and it is a null set, too. It is enough to show that the outer measure of $H$ is zero. Let $\varepsilon>0$ be arbitrary, and for $r>0$ let $G_r=\{\mathbf{p}\in\mathbb{R}^2 : |\mathbf{p}|<r\}$ an open circle with origin the center and radius $r$,

$$H_r=G_r\cup\bigcup\left\{K_n : r\le\frac{1}{n}\right\}.$$

In this case $H_r$ is the union of finitely many measurable disjoint sets, therefore it is measurable. Since for all $r>0$ $H\subset H_r$, therefore

$$k(H)\le k(H_r)=t(H_r)=t(G_r)=\pi r^2<\varepsilon$$

if $r$ is small enough.

**9.30.** We can use that every (non-degenerate) brick contains a sphere, and every sphere contains a (non-degenerate) brick. Let $A\subset\mathbb{R}^n$ be an arbitrary bounded set.

$b(A) = 0$ if and only if $A$ does not contain a brick,
if and only if $A$ does not contain a sphere,
if and only if $\mathrm{int}(A) = \emptyset$.

## 9.2 Multivariable Riemann integral

**9.34.**

$$f(x,y) = \begin{cases} |x| & \text{if } y \in \mathbb{Q} \\ 0 & \text{if } y \notin \mathbb{Q} \end{cases}$$

$$g(y) = \int_{-1}^{1} f(x,y)\,dx = 0, \qquad \int_0^1 \left( \int_{-1}^{1} f(x,y)\,dx \right) dy = \int_0^1 g(y)\,dy = 0.$$

Let's decompose the set $H$ into two parts:

$$H_1 = \{(x,y) \in H : x \le 0\} \text{ and } H_2 = \{(x,y) \in H : x \ge 0\}.$$

In this case every upper sum is zero on $H_1$, and every upper sum is at least $1/2$ on $H_2$, so

$$\overline{\int_H} f \ge \overline{\int_{H_1}} f + \overline{\int_{H_2}} f \ge \frac{1}{2}.$$

Since every lower sum is zero on $H$, therefore the Darboux integrals are not equal,

$$\underline{\int_H} f = 0 < \frac{1}{2} \le \overline{\int_H} f,$$

therefore $f$ is not integrable on $H$.

**9.40.** Let $A = \left\{ (x,y) \in N : x \ge \frac{1}{2} \right\}, \quad B = N \setminus A = \left\{ (x,y) \in N : x < \frac{1}{2} \right\}$.
Both sets are measurable (rectangles) and on them the function

$$f(x,y) = \begin{cases} 1 & \text{if } x \ge 1/2 \\ 2 & \text{if } x < 1/2 \end{cases}$$

is constant. Therefore $f$ is integrable on $N$ and

$$\iint_N f(x,y)\,dx\,dy = \iint_A 1\,dx\,dy + \iint_B 2\,dx\,dy = \frac{1}{2} + 1 = \frac{3}{2}$$

**9.46.**

$$\iint_N \sin(x+y)\,dx\,dy = \int_0^1 \left( \int_0^1 \sin(x+y)\,dx \right) dy =$$
$$= \int_0^1 \left[-\cos(x+y)\right]_{x=0}^1 \,dy =$$
$$= \int_0^1 (\cos y - \cos(1+y))\,dy =$$
$$= \left[\sin y - \sin(1+y)\right]_0^1 = 2\sin 1 - \sin 2$$

**9.52.**

$$\iint_N e^{x+2y}\,dx\,dy = \left( \int_0^1 e^x\,dx \right) \cdot \left( \int_0^1 e^{2y}\,dy \right) = (e-1)\frac{e^2-1}{2}$$

**9.58.**

$$\iint_T (x+y)\,dxdy = \int_1^3 \left( \int_0^1 (x+y)\,dx \right) dy = \int_1^3 \left[\frac{x^2}{2} + xy\right]_{x=0}^1 dy =$$
$$= \int_1^3 \left(\frac{1}{2} + y\right) dy = \left[\frac{y}{2} + \frac{y^2}{2}\right]_{y=1}^3 = 5$$

**9.60.**

$$\iint_T e^{x+y}\,dxdy = \left( \int_0^1 e^x\,dx \right) \cdot \left( \int_0^1 e^y\,dy \right) = \left( \int_0^1 e^x\,dx \right)^2$$
$$= \left( \left[e^x\right]_0^1 \right)^2 = (e-1)^2$$

**9.69.** Apply the integral transform formula for the circles:

$$\iint_T xy\,dxdy = \int_0^2 \left( \int_0^{2\pi} r(r\cos\varphi + 1)(r\sin\varphi - 1)d\varphi \right) dr =$$
$$= \int_0^2 \left( \int_0^{2\pi} \left( \frac{r^3 \sin(2\varphi)}{2} + r^2(\sin\varphi - \cos\varphi) - r \right) d\varphi \right) dr =$$
$$= \left( \int_0^2 \frac{r^3}{2}\,dr \right) \left( \int_0^{2\pi} \sin(2\varphi)\,d\varphi \right) + \left( \int_0^2 r^2\,dr \right) \left( \int_0^{2\pi} (\sin\varphi - \cos\varphi)\,d\varphi \right)$$
$$- \left( \int_0^2 r\,dr \right) \left( \int_0^{2\pi} 1\,d\varphi \right) =$$
$$= - \left( \int_0^2 r\,dr \right) \left( \int_0^{2\pi} 1\,d\varphi \right) = -4\pi$$

We used that the primitive functions of $\sin(2\varphi)$ and $(\sin\varphi - \cos\varphi)$ are periodic with $2\pi$, so their altering on the interval $[0, 2\pi]$ is 0.

**9.74.**

$$\iiint_T (x+y+z)\,dx\,dy\,dz = \int_0^3 \left( \int_0^2 \left( \int_0^1 (x+y+z)\,dx \right) dy \right) dz =$$
$$= \int_0^3 \left( \int_0^2 \left[ \frac{x^2}{2} + xy + xz \right]_{x=0}^1 dy \right) dz = \int_0^3 \left( \int_0^2 \left( \frac{1}{2} + y + z \right) dy \right) dz =$$
$$= \int_0^3 \left[ \frac{y}{2} + \frac{y^2}{2} + yz \right]_{y=0}^2 dz = \int_0^3 (3+2z)\,dz = \left[3z + z^2\right]_0^3 = 18$$

**9.82.** Let's denote by $H$ the two-dimensional shapes bounded by the curves $y = x^2$, $y = 2x^2$, $xy = 1$, $xy = 2$. Look at the transform

$$\Psi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \Psi(u,v) = (x(u,v), y(u,v))$$

which is defined by the system of equations

$$y = ux^2, \qquad xy = v,$$

that is,

$$x = u^{-1/3}v^{1/3}, \qquad y = u^{1/3}v^{2/3}.$$

**The $H$ two-dimensional shape**

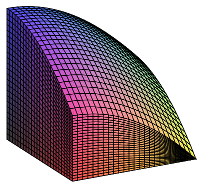
This $\Psi$ transform satisfies the conditions of the integral transform theorem. The set $H$ is the transform of the square $T = [1,2] \times [1,2]$, $H = \Psi(T)$. Calculate the Jacobian determinant of $\Psi$.

$$\Psi'(u,v) = \begin{pmatrix} -\frac{1}{3}u^{-4/3}v^{1/3} & \frac{1}{3}u^{-1/3}v^{-2/3} \\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3} \end{pmatrix}, \qquad J = -\frac{1}{3u}, \qquad |J| = \frac{1}{3u}$$

$$t(H) = \iint_H dx\,dy = \iint_T \frac{1}{3u}\,du\,dv = \frac{1}{3}\left(\int_1^2 \frac{1}{u}\,du\right)\left(\int_1^2 dv\right) = \frac{\ln 2}{3}$$

**9.88.** $f(x,y) = 1 - \dfrac{x^2}{2} - \dfrac{y^2}{2}$, $H = [0,1] \times [0,1]$,
$N = \{(x,y,z) : (x,y) \in H, 0 \le z \le f(x,y)\}$

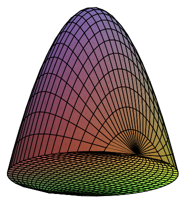
$$t(N) = \iiint\limits_{N} dx\,dy\,dz =$$
$$= \iint\limits_{H} \left( \int\limits_{0}^{f(x,y)} dz \right) dx\,dy =$$
$$= \iint\limits_{H} \left( 1 - \frac{x^2}{2} - \frac{y^2}{2} \right) dx\,dy =$$
$$= \int\limits_{0}^{1} \left( \int\limits_{0}^{1} \left( 1 - \frac{x^2}{2} - \frac{y^2}{2} \right) dx \right) dy =$$
$$= \int\limits_{0}^{1} \left( \frac{5}{6} - \frac{y^2}{2} \right) dy = \frac{1}{3}$$



**solid body below** $\boldsymbol{1 - \frac{x^2}{2} - \frac{y^2}{2}}$

**9.94.** $H = \left\{ (x,y,z) : x^2 + y^2 \leq 1, 0 \leq z \leq 1 - x^2 - y^2 \right\}.$

$$t(H) = \iiint\limits_{H} dx\,dy\,dz =$$
$$= \iint\limits_{x^1+y^2\leq 1} \left( \int\limits_{0}^{1-x^2-y^2} dz \right) dx\,dy =$$
$$= \iint\limits_{x^1+y^2\leq 1} \left( 1 - x^2 - y^2 \right) dx\,dy =$$
$$= \int\limits_{0}^{1} \left( \int\limits_{0}^{2\pi} r(1-r^2)\,d\varphi \right) dr =$$
$$= 2\pi \left[ \frac{r^2}{2} - \frac{r^3}{3} \right]_0^1 = \frac{\pi}{3}$$



**solid body below** $\boldsymbol{1 - x^2 - y^2}$

**9.100.** $\varrho(x,y)=x^2, \quad H=[0,1]\times[0,1]$

$$M=\iint\limits_H \varrho(x,y)\,dx\,dy=\int\limits_0^1\left(\int\limits_0^1 x^2\,dx\right)\,dy=\frac{1}{3}$$

$$S_x=3\int\limits_0^1\left(\int\limits_0^1 x^3\,dx\right)\,dy=\frac{3}{4}, \qquad S_y=3\int\limits_0^1\left(\int\limits_0^1 x^2y\,dx\right)\,dy=\frac{1}{2}$$

**9.106.** The $H$ shape bounded by the lines $y=0$, $x=2$, $y=1, y=x$, is a trapezoid, which is bounded by two continuous functions with variable $y$.

$$H=\{(x,y):0\le y\le 1, y\le x\le 2\}$$

If $\varrho(x,y)=y$, then

$$M=\iint\limits_H y\,dx\,dy=\int\limits_0^1\left(\int\limits_y^2 y\,dx\right)\,dy=\int\limits_0^1 y(2-y)\,dy=\frac{2}{3}$$

$$S_x=\frac{3}{2}\iint\limits_H xy\,dx\,dy=\frac{3}{2}\int\limits_0^1\left(\int\limits_y^2 xy\,dx\right)\,dy=$$

$$=\int\limits_0^1 y\left((2-\frac{y^2}{2}\right)\,dy=\frac{3}{2}\cdot\frac{7}{8}=\frac{21}{16}$$

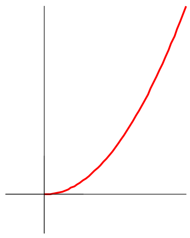
$$S_y=\frac{3}{2}\iint\limits_H y^2\,dx\,dy=\frac{3}{2}\int\limits_0^1\left(\int\limits_y^2 y^2\,dx\right)\,dy=$$

$$=\int\limits_0^1 y^2\,(2-y)\,\,dy=\frac{3}{2}\left(\frac{2}{3}-\frac{1}{4}\right)=\frac{5}{8}$$

# Line Integral and Primitive Function

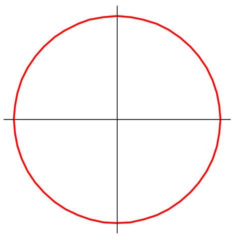
## 10.1 Planar and Spatial Curves

**10.1.**



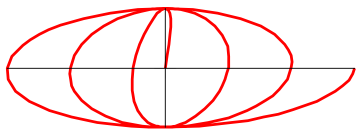
$$\mathbf{r} = t \cdot \mathbf{i} + t^2 \cdot \mathbf{j}$$
$$t \in [0, 4]$$

**10.7.**



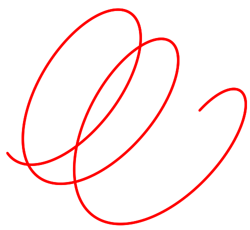
$$\mathbf{r} = \cos t \cdot \mathbf{i} + \sin t \cdot \mathbf{j}$$
$$t \in [0, 2\pi]$$

**10.13.**



$$\mathbf{r} = t\cos t\cdot \mathbf{i} + \sin t\cdot \mathbf{j}$$
$$t \in [0, 6\pi]$$

**10.19.**



$$\mathbf{r} = 2\sin t\cdot \mathbf{i} - t^2\cdot \mathbf{j} + \cos t\cdot \mathbf{k}$$
$$t \in [2, 6\pi]$$

**10.25.** The planar curve $x^2 - xy^3 + y^5 = 17$ around the point $P(5, 2)$ defines an implicit $y(x)$ function. We should find the equation of the tangent line of that function at the point $P$. By deriving the implicit equation we get the slope of the tangent line:

$$2x - y^3 - 3xy^2y' + 5y^4y' = 0, \qquad y' = \frac{y^3 - 2x}{5y^4 - 3xy^2} = \frac{8 - 10}{80 - 60} = -\frac{1}{10}.$$

Therefore, the equation of the tangent line at the point $P(5, 2)$ is

$$y = -\frac{1}{10}(x - 5) + 2, \text{ or in normal form } x + 10y = 25.$$

**10.27.** Calculate the direction vector $\mathbf{v}$ of the tangent line, that is, the derivative of the curve if the value of the parameter is $t = 2$:

$$\mathbf{r}\,(t) = (t-3)\mathbf{i} + (t^2+1)\mathbf{j} + t^2\mathbf{k}$$
$$\dot{\mathbf{r}}\,(t) = \mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k}$$

The direction vector: $\mathbf{v} = \dot{\mathbf{r}}\,(2) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

The point of tangency: $\mathbf{r}_0 = \mathbf{r}\,(2) = -\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$

The tangent line with the direction vector:

$$\mathbf{r}_0 + \mathbf{v}\,t = (t-1)\mathbf{i} + (4t+5)\mathbf{j} + (4t+4)\mathbf{k}$$

**10.29.** According to formulas 10.2 the arc length of a cykloid

$$L = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = \int_0^{2\pi} \sqrt{r^2(1-\cos t)^2 + r^2\sin^2 t}\,dt =$$
$$= r\int_0^{2\pi} \sqrt{2-2\cos t}\,dt = r\int_0^{2\pi} \sqrt{4\sin^2\frac{t}{2}}\,dt =$$
$$= 2r\int_0^{2\pi} \sin\frac{t}{2}\,dt = 4r\left[-\cos\frac{t}{2}\right]_0^{2\pi} = 8r.$$

## 10.2 Scalar and Vector Fields, Differential Operators

**10.35.** For example,

$$f(x,y) = \cos(2\pi x)\cdot\mathbf{i} + \sin(2\pi x)\cdot\mathbf{j}\,,$$
$$(x,y) \in H = \{(x,y) : 0 < x \le 1, y = 0\}.$$

In this case $R_f = K = \{(x,y) : x^2+y^2 = 1\}$.

**10.39.** $f(x,y) = x^4 - 6x^2y^2 + y^4$, $\quad \operatorname{grad} f = \dfrac{\partial}{\partial x}f\cdot\mathbf{i} + \dfrac{\partial}{\partial y}f\cdot\mathbf{j} = (4x^3-12xy^2)\cdot \mathbf{i} + (4y^3-12x^2y)\cdot\mathbf{j}$

**10.43.** $f(x,y,z) = x + xy^2 + x^2z^3$, $\quad \mathbf{p} = (2,-1,1)$

$$\operatorname{grad} f(x,y,z) = (1+y^2+2xz^3)\cdot\mathbf{i} + 2xy\cdot\mathbf{j} + 3x^2z^2\cdot\mathbf{k}$$
$$\operatorname{grad} f(2,-1,1) = 6\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$$

**10.47.**

$$\operatorname{grad} U(\mathbf{r}) = \nabla U(\mathbf{r}) = \nabla\left(\mathbf{r}^2 + \frac{1}{\mathbf{r}^2}\right) = 2\left(1 - \frac{1}{\mathbf{r}^4}\right)\mathbf{r}$$

**10.50.** The tangent plane of the graph of the function $z = f(x,y) = x^2 + y^2$ over the point
$x = 1,\ y = 2$ is

$$z = f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0) = 5 + 2(x-1) + 4(y-2).$$

**10.51.**

$$\mathbf{r} = u\cos v\cdot\mathbf{i} + u\sin v\cdot\mathbf{j} + u\cdot\mathbf{k}, \qquad A = [0,1]\times[0,\pi]$$

$$\mathbf{r}'_u = \cos v\cdot\mathbf{i} + \sin v\cdot\mathbf{j} + \mathbf{k}, \qquad \mathbf{r}'_v = -u\sin v\cdot\mathbf{i} + u\cos v\cdot\mathbf{j}$$

$$\mathbf{r}'_u \times \mathbf{r}'_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = -u\cos v\cdot\mathbf{i} - u\sin v\cdot\mathbf{j} + u\cdot\mathbf{k}$$

$$|\mathbf{r}'_u \times \mathbf{r}'_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = u\sqrt{2}$$

According to the theorem of calculating the area of surface

$$S = \iint_A |\mathbf{r}'_u \times \mathbf{r}'_v|\, du\, dv = \sqrt{2}\int_0^{\pi}\left(\int_0^1 u\, du\right) dv = \frac{\sqrt{2}}{2}\pi.$$

## 10.3 Line Integral

**10.55.** According to the theorem about calculating the line integral

$$\int_{\Gamma} (x+y)\, dx + (x-y)\, dy =$$

$$= \int_0^{\pi} \left[(\cos t + \sin t)(-\sin t) + (\cos t - \sin t)\cos t\right] dt =$$

$$= \int_0^{\pi} (\cos 2t - \sin 2t)\, dt = \left[\frac{\sin 2t}{2} + \frac{\cos 2t}{2}\right]_0^{\pi} = 0.$$

**Remark:**

Since $U(x,y)=\dfrac{x^2}{2}+xy-\dfrac{y^2}{2}$ is a primitive function of the vector field $\mathbf{v}=(x+y)\cdot\mathbf{i}+(x-y)\cdot\mathbf{j}$, therefore the integral of the conservative force field is an alteration of the primitive function:

$$\int_\Gamma \mathbf{v}\,d\mathbf{r}=U(-1,0)-U(1,0)=0.$$

**10.61.** Parameterizing the curves $\Gamma_1$ $x=t,\ y=t$ and $\Gamma_2$ $x=t,\ y=t^2$, where $0\le t\le 1$. The integrals of the vector field $\mathbf{v}=y\cdot\mathbf{i}+x\cdot\mathbf{j}$ is

$$\int_{\Gamma_1} y\,dx+x\,dy=\int_0^1 2t\,dt=1,\qquad \int_{\Gamma_2} y\,dx+x\,dy=\int_0^1 3t^2\,dt=1.$$

**Remark:**

The vector field $\mathbf{v}=y\cdot\mathbf{i}+x\cdot\mathbf{j}$ is conservative, and its primitive function is $U(x,y)=xy$.

**10.67.** The line integral does not exist because $\mathbf{v}$ is not defined at the origin, but the curve $\Gamma$ goes through the origin.

**10.70.**

$$\begin{aligned}
&\int_\Gamma (x+y)\,dx+(y+z)\,dy+(z+x)\,dz=\\
&=\int_0^\pi [(\cos t+\sin t)(-\sin t)+(\sin t+t)\cos t+(t+\cos t)]\,dt=\\
&=\int_0^\pi (-\sin^2 t+t\cos t+t+\cos t)\,dt=\\
&=\left[-\frac{t}{2}+\frac{\sin 2t}{4}+t\sin t+\cos t+\frac{t^2}{2}+\sin t\right]_0^\pi=\\
&=-\frac{\pi}{2}+0+0-2+\frac{\pi^2}{2}+0=\frac{\pi^2}{2}-\frac{\pi}{2}-2.
\end{aligned}$$

**10.77.** The parametric form of the curve $\Gamma$ is:

$$\mathbf{r}\,(t) = t \cdot \mathbf{i} + t^2 \cdot \mathbf{j}\,, \quad -1 \le t \le 1$$

that is,

$$x = t,\ y = t^2, \qquad dx = \dot{x}(t) = 1,\ dy = \dot{y}(t) = 2t, \quad -1 \le t \le 1.$$

According to the formula of the calculation of the line integral

$$\int_\Gamma (x^2 - 2xy)\,dx + (y^2 - 2xy)\,dy =$$

$$= \int_{-1}^{1} \left[(t^2 - 2t^3) \cdot 1 + (t^4 - 2t^3) \cdot 2t\right]\,dt =$$

$$= \int_{-1}^{1} (t^2 - 2t^3 - 4t^4 + 2t^5)\,dt =$$

$$= \left[\frac{t^3}{3} - \frac{2t^4}{4} - \frac{4t^5}{5} + \frac{2t^6}{6}\right]_{t=-1}^{1} = \frac{2}{3} - \frac{8}{5} = -\frac{14}{15}$$

**10.78.** Parameterizing the ellipse:

$$x = a\cos t,\ y = b\sin t, \quad 0 \le t \le 2\pi.$$

In this case we replace $dx$ and $dy$ by

$$dx = a\sin t,\ dy = b\cos t.$$

Therefore, we got a one-variable Riemann integral:

$$\oint_\Gamma (x + y)\,dx + (x - y)\,dy =$$

$$= \int_0^{2\pi} [-a(a\cos t + b\sin t)\sin t + b(a\cos t - b\sin t)\cos t]\,dt =$$

$$= \int_0^{2\pi} (-ab\sin^2 t + ab\cos^2 t - a^2\sin t\cos t - b^2\sin t\cos t)\,dt =$$

$$= \int_0^{2\pi} (ab\cos 2t - \frac{a^2 + b^2}{2}\sin 2t)\,dt = 0.$$

**Remark:**

It is easy to prove that $U(x,y) = \dfrac{x^2}{2} + xy - \dfrac{y^2}{2}$ is a primitive function of the integrand, therefore the integral on the closed curve is 0.

**10.84.** Yes, the primitive functions of the vector field $\mathbf{v} = (x+y)\cdot\mathbf{i} + (x-y)\cdot\mathbf{j}$ are

$$U(x,y) = \frac{x^2}{2} + xy - \frac{y^2}{2} + C.$$

**10.90.** There is no primitive function, since

$$\frac{\partial}{\partial y}(\cos xy) = -x\sin xy \neq \frac{\partial}{\partial x}(\sin xy) = y\cos xy.$$

**10.95.** For the vector field $\mathbf{v} = \dfrac{y}{x^2+y^2}\cdot\mathbf{i} - \dfrac{x}{x^2+y^2}\cdot\mathbf{j}$ it is true that

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}, \qquad \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

But there is no primitive function because if $\Gamma : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$ is the unit circle, then

$$\oint_\Gamma \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy = \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\,dt = -2\pi \neq 0.$$

The reason is that $\mathbf{v}$ is not defined at the origin, but the curve $\Gamma$ goes around the origin.

**10.101.** The force field $\mathbf{E} = (y+x)\cdot\mathbf{i} + x\cdot\mathbf{j}$ is conservative, its primitive function:

$$U(x,y) = \frac{x^2}{2} + xy, \text{ the potential } \Phi(x,y) = -U(x,y) = -\frac{x^2}{2} - xy.$$

**10.107.** Since it is not defined at the origin, therefore it has no potential function on the whole plane. But it has potential function on the punctured plane $\mathbb{R}^2 \setminus \{\mathbf{0}\}$:

$$\Phi(x,y) = \frac{1}{(x^2+y^2)^{1/2}}.$$

**10.108.** The curl of the vector field $\mathbf{v} = yz \cdot \mathbf{i} + xz \cdot \mathbf{j} + xy \cdot \mathbf{k}$ is zero at all points of the space,

$$\frac{\partial}{\partial y}(yz) = \frac{\partial}{\partial x}(xz) = z, \quad \frac{\partial}{\partial z}(yz) = \frac{\partial}{\partial x}(xy) = y,$$

$$\frac{\partial}{\partial z}(xz) = \frac{\partial}{\partial y}(xy) = x,$$

therefore it has a primitive function, $U(x, y, z)$. Because $U'_x = yz$, so

$$U(x, y, z) = \int yz\, dx = xyz + f(y, z).$$

$$U'_y = \frac{\partial}{\partial y}(xyz + f(y, z)) = xz + f'_y(y, z) = xz.$$

Hence $f'_y(y, z) = 0$, that is, $f(y, z)$ is independent of $y$, $f(y, z) = g(z)$.

$$U'_z = \frac{\partial}{\partial z}(xyz + g(z)) = xy + g'(z) = xy,$$

so $g'(z) = 0$, that is, the function $g$ is constant. Therefore, $U(x, y, z) = xyz + C$, where $C$ can be an arbitrary constant.

Although it is easy to find the primitive functions of $xyz$ without any calculations.

**10.113.** The vector field $\mathbf{v} = 3xy^3z^4 \cdot \mathbf{i} + 3x^2y^2z^4 \cdot \mathbf{j} + x^2y^3z^3 \cdot \mathbf{k}$ has no primitive function because for example,

$$\frac{\partial}{\partial y}(3xy^3z^4) = 9xy^2z^4 \neq \frac{\partial}{\partial x}(3x^2y^2z^4) = 6xy^2z^4.$$

**10.118.** Let's find a two-variable function $z(x, y)$ such that

$$z'_x(x, y) = p(x, y) = x^2 + 2xy - y^2, \quad z'_y(x, y) = q(x, y) = x^2 - 2xy - y^2,$$

$$z(x, y) = \int p(x, y)\, dx = \int (x^2 + 2xy - y^2)\, dx = \frac{x^3}{3} + x^2y - xy^2 + g(y).$$

Here $g(y)$ is a not known (differentiable) function of $y$. The function $z(x, y)$ satisfies the equation

$$z'_y(x, y) = x^2 - 2xy + g'(y) = q(x, y) = x^2 - 2xy - y^2.$$

From this

$$g'(y) = -y^2, \text{ therefore } g(y) = -\frac{y^3}{3} + C.$$

Therefore, all primitive functions:

$$z(x,y) = \frac{x^3}{3} + x^2y - xy^2 - \frac{y^3}{3} + C.$$

**10.123.**

$$u'_x(x,y,z) = \frac{x+y}{x^2+y^2+z^2+2xy},$$

$$u(x,y,z) = \int \frac{x+y}{x^2+y^2+z^2+2xy}\,dx = \frac{1}{2}\ln\left|x^2+y^2+z^2+2xy\right| + f(y,z)$$

$$u'_y(x,y,z) = \frac{x+y}{x^2+y^2+z^2+2xy} + f'_y(y,z) = \frac{x+y}{x^2+y^2+z^2+2xy},$$

therefore $f(y,z) = g(z)$.

$$u'_z(x,y,z) = \frac{z}{x^2+y^2+z^2+2xy} + g'(z) = \frac{z}{x^2+y^2+z^2+2xy},$$

therefore $g'(z) = 0$, and $g$ is constant.

$$u(x,y,z) = \frac{1}{2}\ln\left|x^2+y^2+z^2+2xy\right| + C$$

are all of the primitive functions.

## Complex Functions

**11.3.** $z^2 = (x^2 - y^2) + 2xyi$, therefore $u_x = v_y = 2x$ and $v_x = -u_y = 2y$.

**11.6.**

$$\frac{1}{z^2+1} = \frac{1}{x^2 - y^2 + 1 + 2xyi} = \frac{x^2 - y^2 + 1 - 2xyi}{(x^2 - y^2 + 1)^2 + 4x^2y^2} =$$
$$= \frac{x^2 - y^2 + 1 - 2xyi}{(x^2 - y^2 + 1)^2 + 4x^2y^2}.$$

$$u(x,y) = \frac{x^2 - y^2 + 1}{(x^2 - y^2 + 1)^2 + 4x^2y^2}, \quad v(x,y) = -\frac{2xy}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$$

After calculating the partial derivatives, we can see that Cauchy-Riemann's differential equations are fulfilled, if $z^2 + 1 \neq 0$.

**11.7.**

$$\operatorname{Re} f(z) = u(x,y) = \sqrt{|xy|}, \qquad \operatorname{Im} f(z) = v(x,y) = 0$$

Therefore $u(x.0) = 0 = v(0,y)$ and $u(0,y) = 0 = v(x,0)$, we have $u'_x(0,0) = 0 = v'_y(0,0)$ and $u'_y(0,0) = 0 = v'_x(0,0)$, so $u_x(0,0) = u_y(0,0) = 0$, therefore Cauchy-Riemann's differential equations are fulfilled at $z = 0$. But at the line $x = y \quad f(z) = (|x|)^2$, therefore $f(z)$ is not differentiable at $z = 0$.

**11.8.**

$$u(x,y) = 2x^2 + 3y^2 + xy + 2x, \qquad v(x,y) = 4xy + 5y.$$

According to Cauchy-Riemann' differential equations

$$4x + y + 2 = 4x + 5, \text{ and } 6y + x = -4y$$

at all points, where $f$ is differentiable. The system of equations is fulfilled only at $x = -30,\ y = 3$, therefore the function is not differentiable at any other points.

**11.9.**

$$\frac{\partial}{\partial x}(xy) = y, \qquad \frac{\partial}{\partial y}(y) = 1, \qquad \frac{\partial}{\partial y}(xy) = x, \qquad \frac{\partial}{\partial x}(y) = 0$$

Therefore Cauchy-Riemann' differential equations are fulfilled only at the point $z = i$.

**11.10.** Cauchy-Riemann's differential equations:

$$\frac{\partial u}{\partial x} = 4x = 2y = \frac{\partial v}{\partial y}$$

if and only if $y = 2x$, and

$$\frac{\partial u}{\partial y} = -1 = -2x = -\frac{\partial v}{\partial x}$$

if and only if $x = \frac{1}{2}$. Therefore, the function is differentiable only at the point $\frac{1}{2} + i$.

**11.11.** From Cauchy-Riemann's differential equations

$$v'_y(x,y) = u'_x(x,y) = 2x+y, \qquad v(x,y) = \int (2x+y)\,dy = 2xy+\frac{y^2}{2}+g(x)$$

$$v'_x(x,y) = 2y + g'(x) = -u'_y(x,y) = 2y - x,$$

$$g'(x) = -x, \qquad g(x) = -\frac{x^2}{2} + C$$

$$f(z) = (x^2 - y^2 + xy) + i\left(2xy - \frac{x^2}{2} + \frac{y^2}{2} + C\right).$$

Since $f(0) = 0$, therefore $C = 0$

$$f(z) = (x^2 - y^2 + xy) + i\left(2xy - \frac{x^2}{2} + \frac{y^2}{2}\right).$$

**11.13.** $R = 1$

**11.14.** $R = \frac{1}{2}$

**11.15.** $\lim\limits_{n\to\infty} \sqrt[n]{n^2} = 1$, so $R = 1$

**11.20.** Applying the root test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{in}{n+1}\right)^{n^2}\right|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e}, \text{ so } R = e.$$

**11.21.** $\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}, \quad R=1$

**11.22.** Applying the root test: $\lim_{n\to\infty} \sqrt[n]{|i^n|} = 1$, therefore $R=1$.

$$\sum_{n=0}^{\infty} i^n z^n = \sum_{n=0}^{\infty} (iz)^n = \frac{1}{1-iz}$$

**11.23.** $\sum_{n=0}^{\infty} (n+1)z^n = \left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2}, \quad R=1$

**11.24.** $\sum_{n=0}^{\infty} (n+2)(n+1)z^n = \left(\frac{1}{1-z}\right)'' = \frac{1}{(1-z)^2}, \quad R=1$

**11.29.** Apply the formula: $e^{x+iy} = e^x \cdot e^{iy} = e^x \cdot (\cos y + i \sin y)$.

**11.31.** $\int_\Gamma f(x+iy)\,dz = i\pi$

**11.32.** $\int_\Gamma f(x+iy)\,dz = -\pi$

**11.33.** $\int_\Gamma f(x+iy)\,dz = 2i\pi$

**11.34.** $\int_\Gamma f(x+iy)\,dz = 0$

**11.35.** Parameterizing the curve: $z(t) = e^{it}, t \in [0, 2\pi]$. In this case $\frac{dz}{dt} = ie^{it}$. So

$$\int_\Gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

**11.36.**

$$\int_\Gamma \frac{1}{z^2}\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{2it}}\,dt = \int_0^{2\pi} ie^{-it}\,dt = \left[-e^{-it}\right]_0^{2\pi} = -1-(-1) = 0$$

**11.37.** $\int_\pi^{3\pi/2} \left|e^{-it}\right|(-i)e^{-it}\,dt = -i+1$

**11.38.** $\int_{-1}^{0} |t|\,dt + \int_0^1 |it|\,dt = 0$

**11.39.** $$\int_{(1,1)}^{(3.2)} (x^2-y^2)\,dx - 2xy\,dy = \left[\frac{x^3}{3} - xy^2\right]_{(1,1)}^{(3,2)} = -\frac{7}{3}$$

**11.40.** $$\int_{1+i}^{3+2i} (x^2-y^2)\,dx-2xy\,dy = \operatorname{Re}\int_{1+i}^{3+2i} z^2\,dz = \operatorname{Re}\left(\frac{(3+2i)^3}{3} - \frac{(1+i)^3}{3}\right) = -\frac{7}{3}$$

**11.41.** $$\int_\Gamma 3z^2\,dz = \left[z^3\right]_1^{1+i} = -3+2i$$

**11.42.** $$\int_1^{1+i} \frac{1}{z}\,dz = [\operatorname{Log} z]_1^{1+i} = \frac{1}{2}\ln 2 + i\frac{\pi}{4}$$

**11.43.** $$\int_1^{1+i} e^z\,dz = [e^z]_1^{1+i}$$

**11.44.** $$\int_1^{1+i} ze^{z^2}\,dz = \left[\frac{e^{z^2}}{2}\right]_1^{1+i}$$

**11.65.** Apply the residue theorem: $\displaystyle\oint_{|z|=4} f(z)\,dz = 2\pi\cdot i\cdot \operatorname{Res}\left(\frac{e^z\cos z}{z-\pi},\pi\right) =$ $2\pi\cdot i\cdot e^\pi\cos\pi$.

**11.67.** $$\int_\Gamma z^2\,dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3}{3}$$

**11.68.** $$\int_0^{1+i} e^z\,dz = [e^z]_0^{1+i} = e^{1+i}-1$$

**11.73.** Let $g(z) = 10-6z$, $f(z) = z^6-6z+10$. If $z$ is an arbitrary point of the unit circle line, that is $|z|=1$, then

$$|f(z)-g(z)| = \left|z^6\right| = 1, \qquad |g(z)| = |10-6z| \geq 10-6 = 4,$$

the conditions of Rouché's theorem are fulfilled. Since the function $g(z) = 10-6z$ has no root on the unit circle, therefore the function $(z) = z^6-6z+10$ has no root either.

**11.75.** The integral $\int\limits_{-\infty}^{\infty} \frac{1}{\left(x^2+1\right)^2}\, dx$ is convergent, therefore

$$\int\limits_{-\infty}^{\infty} \frac{1}{\left(x^2+1\right)^2}\, dx = \lim_{R\to\infty} \int\limits_{-R}^{R} \frac{1}{\left(x^2+1\right)^2}\, dx.$$

Let $\Gamma_R = \Phi_R + \Psi_R$ be a closed curve such that $\Phi_R$ is the segment $[-R, R]$ of the real axis, and $\Psi_R$ is such the part of the circle with radius $R$ and center origin which lies in the upper halfplane. If $f(z) = \frac{1}{(z^2+1)^2}$, then according to the residue theorem

$$\oint\limits_{\Gamma_R} f(z)\, dz = \oint\limits_{\Gamma_R} \frac{dz}{(z^2+1)^2} = 2\pi i \operatorname{Res}(f, i) = \oint\limits_{|z-i|=1} \frac{dz}{(z^2+1)^2} =$$

$$= \oint\limits_{|z-i|=1} \frac{1}{(z+i)^2} \cdot \frac{dz}{(z-i)^2} = 2\pi i \cdot g'(i) = -4\pi i \frac{1}{(2i)^3} = \frac{\pi}{2}.$$

Here we applied Cauchy's integral formula for the function $g(z) = \frac{1}{(z+i)^2}$, which is regular at $i$.

On the upper half circle $|f(z)| < \frac{1}{R^2}$ if $R$ is large enough, therefore

$$\lim_{R\to\infty} \int\limits_{\Psi_R} \frac{1}{(z^2+1)^2}\, dz = 0$$

$$\frac{\pi}{2} = \lim_{R\to\infty} \int\limits_{\Gamma_R} \frac{dz}{(z^2+1)^2} = \lim_{R\to\infty} \int\limits_{\Phi_R} \frac{dz}{(z^2+1)^2} + \lim_{R\to\infty} \int\limits_{\Psi_R} \frac{dz}{(z^2+1)^2} =$$

$$= \int\limits_{-\infty}^{\infty} \frac{1}{\left(x^2+1\right)^2}\, dx.$$

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