The application of the theory of dynamical systems in conceptual models of environmental physics

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Chapter 1

Introduction

1.1 Overview

1.1.1 Motivation

Environmental physics is an important area of modern physics which applies the tools of classical physics. Environmental fluid dynamics [5, 6] grew out from general hydrodynamics in the first part of the 20th century. Compared to the latter it is special in that its phenomena are considerably influenced, what is more, typically determined by the following factors: the Earth’s rotation and spherical geometry, and the investigated medium’s shallowness and stratification. Taking into account these effects environmental fluid dynamics has proven to be considerably successful in understanding the atmospheric and oceanic processes: we could discover only this way why flow is approximately parallel to isobars, what is the flow like in cyclones and anticyclones, that these formations are in fact generated from the so-called planetary waves, how western boundary currents are formed in the oceans, that oceanic horizontal circulation is mainly driven by wind, and that considerable transport processes take place in the boundary layers the direction of which differs from the direction of the flow in the bulk. The list is, moreover, not at all complete. This knowledge is, on the one hand, necessary for understanding how real weather and, consequently, climate works, and, on the other hand, for interpreting appropriately the solutions obtained numerically from the basic equations of the atmosphere and the ocean. The full exploration of this range of phenomena is, however, a duty for the future.

Within environmental fluid dynamics, phenomena on a rotating sphere still need a more
precise understanding. My research, aimed at investigating the motion of vortex pairs and the chaotic advection in their velocity fields, and providing a testbed for a generally used approximation, the so-called \( \beta \)-plane approximation, contributes to this issue. The description of climate dynamics, in turn, is not at all matured, this poses in fact one of the most important tasks of our age. Its probabilistic approach is based on the so-called snapshot attractor picture. My aim was to work out in detail the method of applying snapshot attractors, and to quantify their deviation from the attractors of time-independent systems.

The new scientific results achieved in these two, more or less separate research topics (which are, however, linked by the theory of dynamical systems and by their environmental origin) are presented in Chapters 2 and 3. A correspondence to the particular thesis points of the PhD work is given in the section titlings, numbered from [T1] to [T8]. The text of the thesis points [T1–T8] is found in their dedicated booklet, along with how the own publications, [1, 2, 3, 4], are associated to these thesis points. In [1, 2, 3, 4] the scientific results are presented in the same form as in the present dissertation, i.e., here.

In the present Chapter, a general introduction is given first. Then, a theoretical background, a review of the relevant literature, and a discussion of the particular models and methods for the two research topics appear organized into Sections 1.2 and 1.3.

### 1.1.2 Fluid dynamics on the surface of a rotating sphere

The most important characteristics of environmental fluid dynamics are the presence of rotation, the shallowness of the fluid and the spherical geometry [5, 6]. It is obviously practical to describe the dynamics in a reference frame that is co-rotating with the planet so that rotation must be taken into account via inertial forces. It turns out that the only relevant component from an environmental point of view is the local vertical component of the Coriolis force. A huge class of environmental phenomena, mainly those involving a large-scale horizontal motion of the fluid, can be described by incorporating the enumerated three effects, and assuming an ideal, constant-density fluid otherwise. In particular, the main phenomenon that is responsible for the variability of weather on midlatitudes: the presence of planetary or Rossby waves, can easily be understood by taking into account the mentioned characteristics.

Nowadays’ quantitative analyses of environmental circulation, which are mainly case studies, usually incorporate all relevant effects into the equations of motion which are solved numerically. These equations are, however, partial differential equations which are expensive to solve,
and the precision of the solution might be limited by the available computational resources. For interpreting the results of these simulations and for obtaining a more general view on environmental fluid flows, it is thus necessary to investigate simplified models which incorporate only those effects that are essential from the point of view of the given phenomenon. (Compare this conclusion with [7] on the role of ‘minimal models’. ) Although the most important environmental phenomena have become understood via such investigations, the details have often remained unexplored.

One reason for this is the technical difficulty of treating vorticity. In an ideal fluid without rotation, this is constant along the trajectories of the particular fluid elements, but it is not so in the presence of rotation accompanied by a spherical geometry or by some topography. In such cases it is the so-called potential vorticity, denoted by $q$, that is conserved [5, 6]:

$$q = \zeta + \frac{2|\Omega| \sin \varphi}{h} H = \text{constant}, \quad (1.1)$$

where $\zeta = (\text{curl} \mathbf{v})_z$ is the vertical component of the fluid’s vorticity, $\Omega$ is the Earth’s angular velocity, $\varphi$ is the geographical latitude, $h$ is the depth of the fluid and $H$ is a typical vertical extension of the fluid. This conservation expresses in the co-rotating frame the conservation of angular momentum in a non-moving reference frame. This conservation implies the variation of the “intrinsic” vorticity, $\zeta$, along a trajectory that moves through a topography, or, what is more important, that displaces latitudinally [note the dependence on $\varphi$ in (1.1)]. Since this conservation holds for all fluid elements, it makes hard any simplification. This is the reason why vortical solutions in environmental fluid dynamics may still exhibit a large range of unexplored behavior.

### 1.1.3 The climate and its change

According to the Glossary (Annex III of Working Group I) of the Fifth Assessment Report of the Intergovernmental Panel on Climate Change [8], ‘climate’ is defined as follows: “Climate in a narrow sense is usually defined as the average weather, or more rigorously, as the statistical description in terms of the mean and variability of relevant quantities over a period of time ranging from months to thousands or millions of years. The classical period for averaging these variables is 30 years, as defined by the World Meteorological Organization. The relevant quantities are most often surface variables such as temperature, precipitation and wind. Climate
in a wider sense is the state, including a statistical description, of the climate system.” The narrow-sense definition inherently involves averaging along a single trajectory of the dynamics underlying climatic phenomena (often called the ‘climate system’ or ‘climate dynamics’), and the averaging is done over time windows of finite length. This view of climate has become widely accepted and applied in the climate science community, both for the interpretation of registered data and for making climate projections based on numerical simulations (see Annex I of Working Group I in [8]).

This approach, however, hardly gives an answer for what kind of weather can be or is expected to be observed under given circumstances. More precisely, the probability distribution in the phase space of the climate system that is subject to a particular forcing can only be defined via an ensemble picture in which different realizations represent the appropriate probability distribution. By its definition, the single-trajectory approach does not catch all the variability implied by the probability distribution. In case if this probability distribution changes in time (which is obviously a ‘climate change’) it is also not guaranteed that this change is reflected appropriately by analyzing a single trajectory. Since knowing the possible behavior of weather is the goal of any practical application, it is a more and more widespread opinion that research should concentrate on an ‘a priori’ probability distribution of the climate system, rather than investigating particular realizations which may not suffice practical purposes [9, 10, 11, 12].

In particular, there is an increasing number of researchers who think that the relevant quantities in the climate dynamics are the statistics taken over an ensemble of possible realizations evolved from various initial conditions in the distant past [13, 14, 15, 16, 17, 18]. This naive view is supported and clarified by the theory of nonautonomous dynamical systems [19, 20, 21] in which snapshot [22] or pullback [23] attractors with their natural measures provide the desired probabilistic description of the dynamics that is subject to a particular forcing.

A motivation for applying temporal averages along single trajectories for characterizing the climate may stem from the fact that asymptotically long temporal averages coincide with ensemble averages on the attractors of autonomous dynamical systems (this property is called ‘ergodicity’ [24]). This is, however, not the case in nonautonomous systems with a generic time-dependence. In climate science, deterministic, smooth and monotonic parameter shifts are particularly relevant, and ergodicity is expected to not hold in this situation, in contrast to a stationary climate with constant parameters. An important question is the degree of the
deviation from ergodicity which may characterize the strength of the climate change itself.

1.1.4 The theory of dynamical systems, chaos

A dynamical system is, loosely speaking, anything that evolves in time, i.e., that has an equation of motion. More precisely, a dynamical system is defined either by differential equations involving temporal derivatives or by a map that can be iterated [25]. In most cases, one considers dynamical systems in a more strict sense: they are systems of ordinary differential equations or maps for a finite number of variables [26], i.e., finite-dimensional systems. Partial differential equations are obtained from a system of ordinary ones in the limit of infinitely many variables, and this is why results obtained in finite-dimensional cases are considered to be relevant for systems described by partial differential equations (e.g. hydrodynamics or the climate system, see [27] for the particular case of the advection-diffusion problem). In fact, numerically discretized solving schemes for partial differential equations are maps for a finite but large number of variables. Most results in dynamical systems theory concern the strict, finite-dimensional definition, and we shall consider this one in what follows.

An autonomous dynamical system can be written as

$$x_{n+1} = M(x_n)$$ \hspace{2cm} (1.2a)

or

$$\frac{dx(t)}{dt} = f(x(t)).$$ \hspace{2cm} (1.2b)

In both cases, $x$ is a vector of all independent variables of the dynamical system. Independence means that there is no functional dependence between this variables, via e.g. a conserved quantity (a constant of motion). These independent variables span the phase space of the dynamical system in which $x$ lives (hence the dimension of the phase space is equal to the number of the independent variables). $M$ and $f$ are functions mapping the phase space to itself. In (1.2a) $x_n$ is the $n^{th}$ iterate, according to the map $M$, of some initial condition $x_0$. Applying $M$ gives the $(n + 1)^{th}$ iterate $x_{n+1}$. Knowing the vector $x_n$ for any particular $n$ determines uniquely all $x_m$, $m > n$. In other words, $x_n$ is always a full set of initial conditions. As for (1.2b), $x(t)$ is the vector of the variables at time $t$, the derivative $\frac{dx(t)}{dt}$ of which is given by the function $f$. The knowledge of $x(t)$ at any particular time instant $t$ is again a full set of initial conditions, note that (1.2b) is a system of first-order differential equations.
Since (1.2a) and (1.2b) are deterministic, the solution to them — which is called a trajectory or an orbit — is unique for given initial conditions. Nevertheless, the trajectory can be nonperiodic and irregular (i.e., having a short autocorrelation time) under rather general circumstances. Such a trajectory is called chaotic [26, 28]. The necessary conditions for chaotic trajectories to appear are (a) at least 3 independent variables and a nonlinear $f$ in (1.2b), (b) at least 2 independent variables and a nonlinear $M$ in (1.2a) if $M$ is invertible or (c) a nonlinear $M$ in (1.2a) if $M$ is not invertible (possibly for 1 variable). These conditions are, however, not sufficient. Whether chaos occurs in a particular dynamical system for particular initial conditions can usually be decided only by numerical investigation. Chaos is induced by an infinite number of periodic orbits that are unstable, i.e., hyperbolic, and there are two criterions for calling a dynamical system chaotic. The first is exponential sensitivity on initial conditions: the distance $\Delta$ between two closely initiated trajectories grows exponentially with time:

$$
\Delta \sim \exp(\lambda t)
$$

(1.3)

where $\lambda$ is the so-called largest Lyapunov exponent [26]. The second is the exponential growth of the number $m$ of existing unstable periodic orbits with the period length $T$:

$$
m \sim \exp(hT)
$$

(1.4)

where $h$ is the topological entropy [26]. Chaotic trajectories are basically wandering among these unstable periodic orbits.

We distinguish between permanent and transient chaos [28]. The former [29] is observable for arbitrarily long times, while in the latter case trajectories escape the domain of chaotic behavior. The escape process in transient chaos is characterized by an exponential decrease with time of the number of trajectories $N$ remaining in the domain of chaotic behavior:

$$
N \sim \exp(-\kappa t)
$$

(1.5)

where $\kappa$ is the escape rate. In such cases the chaotic behavior, including the escape process itself, is governed by a chaotic saddle, a powder-like fractal that consists of unstable periodic orbits. Slowly escaping trajectories, which are chaotic, first come close to the saddle, spend some time near it, but eventually they are repelled from it.
The character of permanent chaos \cite{26, 28} depends on the volume preserving or dissipative nature of the dynamics. This is determined by the phase space contraction factor: for a continuous-time dynamics of the form (1.2b) it is calculated locally as

\[\sigma(x) = \text{div} f(x),\]

(1.6)

and describes the local contraction of the phase phase:

\[\frac{d\Gamma(x)}{dt} = -\sigma(x)\Gamma(x)\]

(1.7)

where \(\Gamma\) is an infinitesimally small phase space volume at the position \(x\). \(\sigma(x) > 0\) usually describes an unphysical situation. If \(\sigma(x) = 0\) everywhere in the phase space, the dynamics is volume preserving, otherwise it is dissipative. The phase space volume is conserved and shrinks in these two cases, respectively. In a volume preserving dynamics chaotic trajectories are wandering in a chaotic sea which is a dense set of hyperbolic trajectories in the phase space. In a dissipative dynamics the hyperbolic trajectories are organized on a filamentary fractal structure which is a chaotic attractor. Trajectories that are not initiated on the attractor converge to the attractor and stay there forever, exhibiting chaotic behavior. They become distributed on the attractor according to a special probability measure, the so-called natural measure of the attractor which is a fractal measure \cite{26}.

Finally, we must note that any dynamical system of the form (1.2b), i.e., one defined in continuous time and given by a system of ordinary differential equations, can be represented by a map of the form (1.2a). For this, one can take the intersection of the \(d\)-dimensional phase space of the continuous dynamics with an arbitrarily defined \((d - 1)\)-dimensional surface (in most cases it is obtained by prescribing a particular value for one of the dynamical variables: \(x_i = x_i^{(0)}\) where \(i\) is one of \(1, 2, \ldots, d\)), and register the passages (usually restricted to one direction) of the trajectories through this surface (which is often called a Poincaré surface of section). On this \((d - 1)\)-dimensional surface the positions of the subsequent passages of a particular trajectory correspond to the subsequent iterates of an initial condition where the iteration rule, i.e., the map \(M\) is defined by the continuous dynamics. The phase space of this map, called a Poincaré map \cite{26, 28}, has \((d - 1)\) dimensions.

So far we have considered autonomous dynamical systems, i.e., ones without explicit dependence on time. Nonautonomous dynamical systems, on the other hand, depend on time (or
on the number of the iteration) explicitly:

\[ x_{n+1} = M(x_n, n) \] (1.8a)

for maps, or

\[ \frac{dx(t)}{dt} = f(x(t), t) \] (1.8b)

for continuous-time dynamics. These are forced (or driven) dynamics with a time-dependent forcing (or driving). A special case is that of periodic time-dependence. For simplicity let us consider a \( d \)-dimensional dynamics in continuous time. If the time-dependence is periodic, one can replace the time variable \( t \) by the phase \( \phi = t \mod T \) within the period of length \( T \), and consider the phase \( \phi \) as an additional dynamical variable (yielding a new phase space of \((d + 1)\) dimensions). By prescribing a particular value \( \phi = \phi(0) \) one obtains the corresponding Poincaré map with \( d \) dimensions (which is called in this case a stroboscopic map [28]). This way one can convert the \( d \)-dimensional, periodically forced nonautonomous dynamical system to an autonomous dynamical system.

The description of nonautonomous dynamical systems with a nonperiodic, i.e., generic time-dependence requires the introduction of a completely new framework. This relies heavily on regarding the initial time instant as a further time variable (in mathematical terms, this means the introduction of the two-time evolution operator of the dynamics [30, 31, 32, 33, 34, 35, 23, 36]). More details on attractors appearing in such systems are given in Section 1.3.1.

1.2 Background for the results related to the modulated point vortex pairs

1.2.1 Chaotic advection

Particles suspended in fluids are advected with the fluid due to an inevitable drag. The advection of particles in fluid flows has an enormous range of applications from fluid dynamics in the environment on small [37, 38, 39, 40, 41] and large [42, 43, 44, 45, 46, 47] scale to industrial hydrodynamics (see e.g. [48, 49]). Advected particles can be passive tracers (i.e., fluid elements) and inertial particles (when the relaxation to the fluid’s velocity is due to a finite drag, described in most cases by the Maxey–Riley equations [50]). A common feature of all advec-
tive problems is that the equation of motion for the advected particles is a system of ordinary
differential equations in which the velocity field of the fluid, which is supposed to be given,
appears via an explicit dependence on time, i.e., as a forcing. Neglecting interaction between
particles (in particular, collisions [51, 52, 53] and physical [54], chemical [55] or biological [56]
reactions) the motion of an individual particle is described by a low-order system of differential
equations. (Denoting the dimension of the configuration space by \( d \), the phase space has \( d + 1 \)
dimensions for passive tracers and \( 2d + 1 \) dimensions for inertial particles if the particles do
not have internal degrees of freedom e.g. rotation [57, 58].) This low-order description implies
that the dynamics of advected particles can be treated by the traditional theory of dynamical
systems.

As follows from what is discussed above, a two-dimensional configuration space and some
nonlinearity in the flow field is enough for chaos to appear [59, 60, 61, 62, 63, 64, 65] even in the
motion of passive tracers. For passive tracers, for which the phase space without the time axis
coincides with the configuration space, chaotic and chaos-related sets appear as the pattern
that is traced out by the position of the particles. This leads to an easy observability of chaotic
advection either in laboratory experiments [66] or in nature [67]. In fact, chaotic advection is
one of the very few examples when low-order chaos is exhibited by natural phenomena.

It is useful to distinguish between different types of advection dynamics, because the chaotic
set is of different nature for the different types. In particular, the advection may takes place
in closed containers or finite domains of the phase space (closed advection) [59, 60, 61, 62, 64]
which leads to the appearance of permanent chaos. The advection can also be followed up
to some escape process (open advection) [68, 69, 70] in which case the chaos is transient.Transient chaos is always governed by a chaotic saddle [29]. For closed advection, when the
chaos is permanent, the dynamics of passive tracers is volume-preserving in incompressible
flows, and the emerging chaotic set is a space-filling chaotic sea [26, 28]. The closed advection
of inertial particles is, however, dissipative, and chaos can thus appear via either a chaotic
saddle or a chaotic attractor [26, 28]. In these cases the patterns that are traced out by the
particles are typically filamentary. This is also true for volume-preserving closed cases at the
beginning of the advection process if the initial conditions are sufficiently localized.

In our work we shall relate a globally closed advection problem to a locally open one for
passive tracers.
1.2.2 Equations of motion for a fluid on a rotating spherical surface

The equations of motion for a fluid are, of course, given by the conservation of momentum, the continuity, and the equation of state. Since fluid velocities in our context are always smaller than the speed of the sound by at least one order of magnitude, incompressibility holds so that

\[ \text{div} \mathbf{v}(r, t) = 0 \]  \hspace{1cm} (1.9)

where \( \mathbf{v}(r, t) \) is the velocity of the fluid at position \( r \) and time \( t \). Furthermore, a huge class of environmental phenomena, mainly those involving a large-scale horizontal motion of the fluid, can be described by assuming a constant density \( \rho_0 \) which is independent of the pressure or the position\(^1\). In the bulk of the fluid, viscosity is negligible\(^2\) so that the conservation of momentum is expressed by Euler’s equation [71]:

\[ \frac{d\mathbf{v}}{dt} = -\frac{1}{\rho_0} \text{grad} p + \mathbf{g} - 2 \Omega \times \mathbf{v}, \]  \hspace{1cm} (1.10)

where \( p = p(r, t) \) is the pressure of the fluid at position \( r \) and time \( t \), \( \mathbf{g} \) is the gravitational acceleration and \( \Omega \) is the angular velocity of the Earth. The last term is the Coriolis force which has basic importance in environmental fluid dynamics. Equation (1.10) is often simplified by eliminating the hydrostatic component of the pressure via introducing the dynamical pressure \( p' \):

\[ p' := p - \rho_0 g z \]  \hspace{1cm} (1.11)

where \( z \) is the negatively signed distance from the surface of the fluid. Equation (1.10) then becomes

\[ \frac{d\mathbf{v}}{dt} = -\frac{1}{\rho_0} \text{grad} p' - 2 \Omega \times \mathbf{v}. \]  \hspace{1cm} (1.12)

Equation (1.12) can also be formulated for the vorticity \( \omega = \text{curl} \mathbf{v} \). Taking the curl of

---

\(^1\)Stratified fluids are often treated by choosing the density depend directly on position via temperature and salinity.

\(^2\)This is a consequence of the relative importance of the Rossby number \( \text{Ro} = \frac{U}{\Omega L} \), describing the ratio of the hydrodynamic acceleration to the Coriolis acceleration, compared to that of the Reynolds number \( \text{Re} = \frac{UL}{\nu} \), corresponding to the ratio of the hydrodynamic acceleration to the viscous acceleration, where \( U \) and \( L \) are the characteristic velocity and spatial extension of the fluid, \( \Omega \) is the angular velocity of the Earth, and \( \nu \) is the kinematic viscosity of the fluid. In the atmosphere, \( \text{Ro} \approx 7 \times 10^{-2} \) and \( \text{Re} \approx 7 \times 10^{11} \). In the ocean, \( \text{Ro} \approx 9 \times 10^{-2} \) and \( \text{Re} \approx 1 \times 10^{11} \). See [5] for details.
\[
\frac{d(\omega + 2\Omega)}{dt} = ((\omega + 2\Omega) \text{grad}) \mathbf{v}. 
\] (1.13)

It is then useful to introduce a new quantity called \textit{total vorticity} as

\[
\omega_T = \omega + 2\Omega. 
\] (1.14)

Now let us suppose that the fluid is two-dimensional. In particular, we assume it to be a spherical surface, but the results hold also for planar geometry. Two-dimensionality is anyway obtained by requiring the velocity \( \mathbf{v} \) to lie in the horizontal plane, which has the consequence that \( \omega \) is vertical so that it is fully characterized by its vertical component, denoted simply by \( \zeta \). Two-dimensionality is reasonable from an environmental point of view since both the atmosphere (or, to be more precise, the troposphere) and the ocean has a much larger horizontal extension than their vertical one. In environmental fluid dynamics this is usually taken into account by the shallow fluid approximation, but for studying horizontal motion only a strict two-dimensionality is also expected to give meaningful results. In the two-dimensional case, Euler’s equation (1.13) for the total vorticity simplifies\(^4\) to a \textit{conservation law}:

\[
\frac{d\zeta_T}{dt} = \frac{d(\zeta + 2\Omega_z)}{dt} = 0, 
\] (1.15)

where \( \zeta_T \) is the vertical component of the total vorticity which is composed of \( \zeta \) and twice the vertical component \( \Omega_z \) of the angular velocity. Equation (1.15) expresses that the total vorticity is conserved along the trajectory of a fluid element. Note that this conservation is a consequence of imposing Euler’s equation for the conservation of momentum. In fact, the conservation of the total vorticity in a rotating reference frame is equivalent to the conservation of angular

\[^3\text{Using div} \mathbf{v} = 0 \text{ and well-known vector calculus identities:}\]

\[
\text{curl} \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \text{grad}) \mathbf{v} \right] = -2\text{curl} (\mathbf{\Omega} \times \mathbf{v}),
\]

\[
\frac{\partial \omega}{\partial t} + \text{curl} (\text{curl} \mathbf{v} \times \mathbf{v}) + \frac{1}{2} \text{curl} \text{grad} \frac{\mathbf{v}^2}{2} = 2 (\mathbf{\Omega} \text{grad}) \mathbf{v},
\]

\[
\frac{\partial \omega}{\partial t} + (\text{grad}) \omega - (\text{grad}) \mathbf{v} = 2 (\mathbf{\Omega} \text{grad}) \mathbf{v},
\]

\[
\frac{d\omega}{dt} = ((\omega + 2\Omega) \text{grad}) \mathbf{v}.
\]

\[^4\text{The vector on the right-hand side of (1.13) lies in the horizontal plane, and any horizontal gain in } \omega \text{ would contradict its being vertical.}\]
momentum in a reference frame at rest. We note that this conservation can be generalized to
the shallow water approximation where it also involves the full depth of the fluid and is called
the conservation of potential vorticity (see Section 1.1.2).

Finding solutions for the two-dimensional dynamics of an incompressible rotating fluid
basically means satisfying the conservation of total vorticity (1.15) and the continuity (1.9).
The former is trivial in the case of a planar geometry where the vertical component \( \Omega_z \)
of the angular velocity is constant so that the conservation law coincides with the conservation
of “intrinsic” vorticity \( \zeta \). On a spherical surface, however, \( \Omega_z \) depends on the position.
In particular,

\[
f := 2\Omega_z = 2|\Omega|\sin \varphi = 2\Omega \sin \varphi
\]

is the so-called Coriolis parameter which is proportional to the modulus of the full angular
velocity and to the sine of the geographical latitude \( \varphi \). The conservation of total vorticity then
results in the generation of “intrinsic” vorticity \( \zeta \) along any trajectory of a fluid element that dis-
places latitudinally. Since generic trajectories are such, this implies that typical environmental
flows are vortical.

As for the continuity (1.9), in a two-dimensional fluid it is equivalent to the existence of a
scalar streamfunction \( \Psi \) for which

\[
\mathbf{v} = \text{curl} (\Psi \mathbf{n})
\]

where \( \mathbf{n} \) is the unit vector normal to the surface. Consequently,

\[
\zeta = \Delta \Psi,
\]

i.e., Poisson’s equation holds for the streamfunction \( \Psi \) in which the source is the “intrinsic”
vorticity \( \zeta \). This way continuity can also be formulated in terms of the vorticity \( \zeta \). This is
independent of the particular geometry of the surface. Nevertheless, there is a restriction for
the vorticity field \( \zeta(\mathbf{r}) \) on the sphere, the so-called Gauss condition [72, 73]:

\[
\int_S \zeta(\mathbf{r})dA = 0
\]

where \( S \) stands for the full spherical surface, and the integral is taken over the area of this
surface.
1.2.3 The $\beta$-plane approximation

In environmental fluid dynamics, the $\beta$-plane approximation is a common tool [5, 6] for facilitating the construction of analytical solutions to the equations of motion when the latitudinal extension of some phenomenon is relatively small. In particular, it consists of two simplifications. The first is the linearization of the latitudinal dependence of the Coriolis parameter (1.16) around some latitude $\varphi_r$ that is in the middle of the domain of interest:

$$f = 2\Omega \sin \varphi \simeq 2\Omega \sin \varphi_r + 2\Omega \cos \varphi_r (\varphi - \varphi_r). \quad (1.20)$$

The linearization is appropriate if $\cos \varphi_r (\varphi - \varphi_r) \ll \sin \varphi_r$. On midlatitudes, this yields a range of validity $|\varphi - \varphi_r| < 0.15$ at most. The second simplification in the $\beta$-plane approximation is the assumption of a planar geometry. For this, the following substitutions are made to the geographical coordinates $(\lambda, \varphi)$:

$$x = R \cos \varphi_r (\lambda - \lambda_r), \quad (1.21a)$$
$$y = R (\varphi - \varphi_r), \quad (1.21b)$$

where $R$ is the radius of the sphere that represents the planet, and $\lambda_r$ is an arbitrarily chosen longitude. Besides these formal substitutions, $x$ and $y$ are considered to be Cartesian coordinates on a plane. Note that the factor $\cos \varphi_r$ in (1.21a) does not depend on $\varphi$, which implies that the $(x, y)$ Cartesian plane does not approximate the surface of the sphere consistently with the linearized Coriolis parameter. This is what we shall investigate in more detail in our work.

With (1.21b), the linearized Coriolis parameter can be written as

$$f = 2\Omega \sin \varphi_r + \frac{2\Omega \cos \varphi_r}{R} y =: f_0 + \beta y \quad (1.22)$$

where the $\beta$ parameter has been introduced as

$$\beta := \frac{2\Omega \cos \varphi_r}{R}. \quad (1.23)$$

The $\beta$-plane approximation has proven to be useful in qualitatively understanding and explaining many atmospheric and oceanic phenomena, ranging from the Rossby waves to the
1.2.4 Point vortex solutions

The simplest vortical solutions in two-dimensional fluids with planar geometry in the absence of the Coriolis force are point vortices [74]: for one point vortex, the vorticity $\zeta$ is zero throughout the fluid and is singular at the (fixed) position $r_0$ of the vortex such that the circulation $\Gamma$ along any curve around this position is finite. The streamfunction of one point vortex reads as

$$\Psi(r) = \frac{\Gamma}{2\pi} \log |r - r_0|.$$  \hfill (1.24)

In fact, dividing (1.24) by $\Gamma$ gives the Green function of (1.18), i.e., the solution to $\Delta \Psi(r) = \delta(r_0)$ where $\delta$ is Dirac’s delta distribution centered at $r_0$. (Convoluting the Green function using $r_0$ with any vorticity distribution in $r_0$ gives the corresponding streamfunction in $r$.) Equation (1.24), outside the position $r_0$, is an exact solution to Euler’s equation in planar geometry since the vorticity $\zeta$ is constantly zero along the trajectory of any fluid element (see the previous Subsection), outside $r_0$.

Kimura and Okamoto [75] showed that all these findings can be transferred to spherical geometry with the same streamfunction (1.24).\(^5\) In this case, $|r - r_0|$ corresponds to the chord distance from the position of the point vortex.

The velocity field (either with planar or with spherical [76] geometry) of one point vortex is obtained from (1.24) by (1.17) as

$$v(r) = \frac{\Gamma}{2\pi} \frac{n \times (r - r_0)}{|r - r_0|^2}.$$  \hfill (1.25)

where $n$ is the vertical unit vector at the position $r$. Due to the linearity of (1.18) point vortex solutions can be superposed. In this case, the position of any point vortex is advected in the velocity field of all other point vortices. In other words, point vortices move in this case, and their equation of motion is given by the system of ordinary differential equations

$$\frac{dr_i}{dt} = \frac{1}{2\pi} \sum_{j \neq i} \frac{\Gamma_j n_j \times (r_i - r_j)}{|r_i - r_j|^2}, \quad i = 1, \ldots, N,$$  \hfill (1.26)

\(^5\)The Green function is, in this case, the solution to $\Delta \Psi(r) = \delta(r_0) - \frac{1}{4\pi}$. This way the Gauss condition (1.19) is satisfied.
where \( N \) is the number of point vortices, \( \Gamma_i \) and \( r_i \) are the circulation and the position of the \( i^{th} \) point vortex, and \( n_i \) is the vertical unit vector at the position \( r_i \).

### 1.2.5 The principle of modulation

As pointed out in Section 1.1.2, the precise description of the vortex dynamics on a rotating sphere requires the proper handling of the variation in the vorticity along the fluid elements’ trajectories. In order to keep the low-dimensional character of the usual point vortex dynamics, a widely used approximation [77, 78, 79, 80] considers a solid body rotation with the angular velocity of the sphere as the background flow, and neglects the feedback of the point vortices of constant circulation on the background flow. In this setting the conservation (1.15) of the total vorticity cannot be clearly incorporated.

Another, phenomenological approach in which this drawback can be avoided is based on a modulation of the vortex circulations. Motivated by vortices moving over sloping bottoms and subjected to the Coriolis force, there is an extended literature [81, 82, 83, 84, 85] on how the conservation of potential vorticity [5] can be introduced into the point vortex picture by applying a modulation of the vortex circulation with the coordinate along the slope. The circulation \( \Gamma_j \) of any vortex is made linearly location dependent. This is done [84] in the spirit of the \( \beta \)-plane approximation. With this approach, one neglects the vorticity production due to the transport of fluid elements outside the vortices, as discussed by [84]. The modulated point vortex model is therefore valid as long as this vorticity gain is negligible. The price of this approach is the introduction of a phenomenological quantity, called the vortex radius. Although such point vortices are not exact solutions of the hydrodynamical equations, they have been shown to be useful in understanding several features, like e.g. the existence of modon-like excitations [81]. In a number of experiments, laboratory generated vortices on a topographic \( \beta \)-plane (sloping bottom) could be approximated quite well by the modulated point vortex model over a considerable time span [86, 84, 87, 63, 88].

As described in [1], the principle of modulation has been generalized to vortices on a rotating sphere by making the vortex circulation nonlinearly depend on the latitudinal angle \( \varphi_j \) of vortex \( j \). The idea, similarly to what is done in the context of sloping bottoms in [84], is to take into account the conservation of total vorticity (1.15) in an integrated form:

\[
\zeta(\varphi, \lambda) + 2\Omega \sin \varphi = C
\]  

(1.27)
along a fluid element trajectory, where $\zeta$ denotes the vertical component of the vorticity vector expressed in geographical coordinates $\lambda$ (longitude) and $\varphi$ (latitude), and $C$ is a constant. Under the assumption that a point vortex represents a small patch of vorticity, of an area $a^2 \pi$, the circulation of vortex $j$ with coordinates $\varphi_j, \lambda_j$ is given as $\Gamma_j = a^2 \pi \zeta(\varphi_j, \lambda_j)$. From (1.27) we then obtain

$$\Gamma_j(\varphi_j) = \Gamma_{jr} - 2\Omega a^2 \pi (\sin \varphi_j - \sin \varphi_{jr}), \quad (1.28)$$

where

$$\Gamma_{jr} = a^2 \pi C - 2\Omega a^2 \pi \sin \varphi_{jr} \quad (1.29)$$

is the circulation at a reference latitude $\varphi_{jr}$. We shall call $\Gamma_{jr}$ the vortex strength (at the reference latitude), and $a$ the vortex radius, the latter being assumed to be the same for all point vortices. Equation (1.28) sets the modulation of circulation on a rotating sphere.

### 1.2.6 The modulated point vortex pair model

We consider point vortices whose location is specified by angles $\lambda, \varphi$ in geographical coordinates on the surface of a sphere of radius $R$ ($\lambda$ being the longitude, $\varphi$ the latitude). The equations of motion for $N$ modulated point vortices are the same as for constant-circulation vortices (1.26) just the circulation of vortex $j$ is given by $\Gamma_j(\varphi_j)$ as expressed in (1.28). We thus have

\[
\frac{d\varphi_i}{dt} = \frac{1}{4\pi R^2} \sum_{j \neq i} \frac{\Gamma_j(\varphi_j) \cos \varphi_j \sin(\lambda_i - \lambda_j)}{1 - \cos \gamma_{ij}}, \quad i, j = 1, \ldots, N, \quad (1.30a)
\]

\[
\frac{d\lambda_i}{dt} = \frac{1}{\cos \varphi_i} \frac{1}{4\pi R^2} \sum_{j \neq i} \frac{\Gamma_j(\varphi_j) \left[ \cos \varphi_i \sin \varphi_j - \sin \varphi_i \cos \varphi_j \cos(\lambda_i - \lambda_j) \right]}{1 - \cos \gamma_{ij}}, \quad (1.30b)
\]

where

$$\cos \gamma_{ij} = \sin \varphi_i \sin \varphi_j + \cos \varphi_i \cos \varphi_j \cos(\lambda_i - \lambda_j), \quad (1.31)$$

and $2R^2(1 - \cos \gamma_{ij}) = r_{ij}^2$ is the chord distance between vortex $i$ and $j$.

A natural choice of the length and time scales, $L$ and $T$, is obtained if the radius is taken as the length unit and $1/\Omega$ is chosen to be proportional to the time unit, i.e. $L = R$ and $T = 1/(2\Omega)$. As a consequence, the dimensionless circulations are

$$\Gamma_j'(\varphi_j) = \Gamma_{jr}' - a'^2 \pi (\sin \varphi_j - \sin \varphi_{jr}), \quad a' = \frac{a}{R}, \quad \Gamma_{jr}' = \frac{\Gamma_{jr}}{2\Omega R^2}. \quad (1.32)$$
We always consider two modulated vortices. Their chord distance $D'$ is a constant of motion, as shown in [1]. We write down the dimensionless equations for two vortices by explicitly indicating that the dimensionless distance is a constant $D'$:

\[
\frac{d\varphi_i}{dt} = \frac{1}{2\pi D'^2} \left[ \Gamma'_{j\!r} - a'^2 \pi (\sin \varphi_j - \sin \varphi_{j\!r}) \right] \cos \varphi_j \sin (\lambda_i - \lambda_j),
\]

\[
\frac{d\lambda_i}{dt} = \frac{1}{\cos \varphi_i} \frac{1}{2\pi D'^2} \left[ \Gamma'_{j\!r} - a'^2 \pi (\sin \varphi_j - \sin \varphi_{j\!r}) \right] \times \left[ \cos \varphi_i \sin \varphi_j - \sin \varphi_i \cos \varphi_j \cos (\lambda_i - \lambda_j) \right].
\]

Here $i, j = 1, 2$, $i \neq j$, and the initial conditions are given by the initial values $\lambda_i0, \varphi_i0, i = 1, 2$. Alternatively, we can use the initial center of mass coordinates $\lambda_0, \varphi_0$ and the initial angle, denoted by $\alpha_0$, between the line connecting the elements of the pair and the local meridian ($\lambda = \text{const}$). $\alpha_0$ is chosen to be 0 when vortex 1 is closer to the North Pole than vortex 2 while both vortices are along the same $\lambda = \text{const}$ line.

We define a vortex pair as a two-vortex system whose elements have oppositely equal vortex strengths at a common reference latitude:

\[
\varphi_{1r} = \varphi_{2r} \equiv \varphi_r
\]

and

\[
\Gamma'_{1r} = -\Gamma'_{2r} \equiv \Gamma' > 0.
\]

This choice gives $\varphi_r$ a well-defined physical meaning, and $\varphi_r$ becomes an important parameter of the vortex pair model. (We note that this definition of a vortex pair permits the elements of the pair to have the same sign of circulation at latitudes far enough from the reference latitude for certain parameter values. However, in most of the cases considered in this work, and in particular, in the dipole limit [see Section 2.1.2], the circulations have different signs for all times.)

We express the velocity $\mathbf{u}$ of the center of mass of the vortex pair in terms of the positions and the circulations of the elements of the pair as follows. Assuming that the signs of the circulations of the elements are opposite, the dimensionless velocity modulus of the center of mass at any instant of time is given by the average of the velocity moduli of the elements of
the pair, after taking into account the spherical geometry:

\[ |u(t)| = \frac{|\Gamma_1'(\varphi_1(t))| + |\Gamma_2'(\varphi_1(t))|}{4\pi D'\sqrt{1 - (D'/2)^2}} = \frac{2\Gamma' + a^2\pi (\sin \varphi_2(t) - \sin \varphi_1(t))}{4\pi D'\sqrt{1 - (D'/2)^2}}. \quad (1.36) \]

The direction of the velocity is perpendicular to the line connecting the vortices.

1.2.7 The $\beta$-plane approximation in the modulated vortex pair model

In any traditional $\beta$-plane approximation (in brief: ‘$\beta$-plane approximation’), as mentioned in Section 1.2.3, one has to choose a latitude where the origin of a local Cartesian reference frame is put. This choice should be made such that the fluid element trajectories under investigation stay close to this latitude when compared to the radius of the sphere. The latter being unity, ±0.15 is traditionally taken as the limit for the deviation from the chosen latitude in the variable $\varphi$, or, equivalently, for the absolute value of the coordinate $y$ in the Cartesian reference frame [5]. In our particular case, we naturally choose $\varphi_r$ to be the origin of the $\beta$-plane $y$ axis since we are interested in the dynamics taking place in the vicinity of the reference latitude $\varphi_r$ of the circulations. As for the origin of the $x$ axis, any longitude $\lambda$ is equivalent, so we choose $\lambda = 0$.

As discussed in Section 1.2.3, the $\beta$-plane equations of motion for a vortex pair are obtained by assuming planar geometry with the coordinates $x$ and $y$ defined as

\[ x = \lambda \cos \varphi_r, \quad y = \varphi - \varphi_r. \quad (1.37) \]

In the $(x, y)$ plane simply the planar equations of motion for a vortex pair are considered. The only modification is the modulation of the vortex circulation as a function of the coordinate $y$, as formulated in [84, 85, 86, 87, 63, 88]. When referring to the form (1.28) of the modulation, only the leading order term is kept in the Taylor expansion of $\sin \varphi_i - \sin \varphi_r$ around $\varphi_r$:

\[ \sin \varphi_i - \sin \varphi_r \simeq \cos \varphi_r y_i. \quad (1.38) \]

The dimensionless Coriolis parameter (the local vertical component of the angular velocity of the sphere) is, in our case, $\sin \varphi_i$, thus the dimensionless $\beta$ parameter ($\beta'$), the derivative of the Coriolis parameter at $\varphi_r$, is $\beta' = \cos \varphi_r$. The $\beta$-plane equations of motion for a vortex pair
are thus (see e.g. [84])

\[
\frac{dx_i}{dt} = -\frac{1}{2\pi D^r} \left( \Gamma'_j - \beta' a^2 y_j \right) (y_i - y_j), \quad (1.39a)
\]

\[
\frac{dy_i}{dt} = \frac{1}{2\pi D^r} \left( \Gamma'_j - \beta' a^2 y_j \right) (x_i - x_j), \quad (1.39b)
\]

for \( i, j = 1, 2, i \neq j \), and \( \Gamma'_1 = -\Gamma'_2 = \Gamma' \).

1.3 Background for the results related to snapshot attractors and climate dynamics

1.3.1 Snapshot attractors and their nonergodicity

The snapshot attractor is the generalization of the concept of a usual attractor to nonautonomous dynamical systems with a forcing that depends arbitrarily on time. The most important feature of snapshot attractors that makes them different from usual ones is their possible aperiodic dependence on time which is completely determined by the forcing. In an autonomous system the attractor is obtained by evolving an ensemble of trajectories to the \( t \to \infty \) limit [26]. In contrast, the snapshot attractor of a particular time instant \( t \) corresponds to the endpoints in \( t \) of an ensemble of trajectories that was initiated in the infinite past, i.e., in \( s < t \) with \( s \to -\infty \) [22]. The infinite limits are, of course, neither meaningful nor accessible in practice. The convergence to a usual attractor is, however, exponentially fast [26, 28] and is characterized by a relaxation time \( \tau \), so that the attractors are traced out by any ensemble that has evolved a few times \( \tau \). One expects a similar convergence to a snapshot attractor, but its practical implications have not yet been addressed specifically in the literature.

A pullback attractor as defined by Ghil and coworkers [23, 89] is a similar object to a snapshot attractor, but it is associated to the entire real time axis \( (-\infty < t < \infty) \). A rigorous mathematical definition only exists for the pullback attractor, and, according to this definition, given by [23, 89], an initialization of the ensemble in the infinite past is strictly required. This definition requires also the consideration of the dynamics’ two-time evolution

\[ A_n \text{ natural question is how (1.39) can be obtained from (1.33). The metric factors (next to the circulations) in (1.33) turn out to be convertible to the differences of the Cartesian coordinates appearing in (1.39), but only with assuming the longitude difference } \lambda_i - \lambda_j \text{ to be of the same order as } \varphi_i - \varphi_j \text{. Furthermore, taking the product of the linearly approximated metric factors with the linearly approximated circulations proves to handle quadratic terms inconsistently. Without any quadratic terms, however, the variation of the Coriolis-parameter with latitude can not be incorporated into the model at all.} \]
operator [30, 31, 32, 33, 34, 35, 23, 36]. As follows from all this, a snapshot attractor can be viewed as an instantaneous slice (corresponding to a given time instant) of a pullback attractor.

In the dynamical systems community, the concept of snapshot attractors has been known for many years [22, 90, 91, 92, 93, 94, 95]. It was used to explain the distribution of floaters on the surface of a turbulent flow in a paradigmatic fluid dynamical experiment [96, 66]. From certain aspects, random and deterministic forms of the forcing can be considered on equal footing, and the random view dominated the first papers on the subject of snapshot attractors. A precursor of this was the discovery of synchronization by common noise by Pikovsky [97, 98], a case when — in the current terminology — the snapshot attractor turns out to be regular, which can also be called a random fixed point.

In our work we shall only consider irregular snapshot attractors underlying a complex dynamics. An appealing feature of such a snapshot attractor is that it carries a unique probability measure (the natural measure, the analogue of the SRB measure of usual attractors). The ensemble representing the natural measure is provided by trajectories evolving from a set of different initial conditions; these trajectories shall thus also be called different realizations. The natural measure is independent of the particular choice of the set of initial conditions used for its representation.

The snapshot attractor and also its natural measure can have a general time-dependence, determined solely by the forcing. Even a time-dependent snapshot attractor traces out a clear, generally fractal, structure in the phase space (corresponding to a given time instant $t$), whereas a traditional single trajectory plot (arising from the consideration of a time interval over which the snapshot attractor changes) would lead in the same problem to an unstructured pattern. This is a clear indication for the breaking down of ergodicity in systems possessing time-dependent snapshot attractors. A similar qualitative conclusion on the necessary breakdown of ergodicity was drawn in [99, 18].

To investigate the degree of the breakdown of ergodicity and to introduce a framework for its quantification, and to relate this breakdown to climate change, we turn to a very simple conceptual climate model. It is Lorenz’s 1984 model of midlatitude atmospheric circulation [100, 101] described by three ordinary differential equations. Additionally to the originally incorporated seasonal cycle [101] we consider a monotonic (linear) shift (called the ramp) in the temperature contrast parameter, in order to mimic the observed increase of greenhouse gases.
1.3.2 The internal variability and the forced response of the climate

Let us consider a climate system with a stationary forcing. Due to the complexity (or the “chaotic” nature) of the solutions of this climate system, a generic realization will depend on time aperiodically. In particular, in a time instant $t$ this realization will be found in the phase space of the system according to the natural probability measure of the attractor. Obviously, such time-dependence in the trajectories will occur also with a generic forcing of the climate system. The short term wandering of the trajectory on the attractor is responsible for the time-dependent weather. A similar variability is also observed in longer term temporal averages along the trajectory, and, according to the definition of [8] for the climate (see Section 1.1.3), this is regarded as a variability of the climate. To emphasize its independence of the system’s systematic response on the forcing, it is called the natural or internal variability of the climate [8]. At the same time, temporal changes that are induced by a non-stationary forcing (e.g. the increase in the atmospheric CO$_2$ concentration) are called the forced response of the system.

One of the most important aims of climate science is the separation of the forced response from the internal variability [9].

From a dynamical systems point of view, it is rather clear that the use of an ensemble of trajectories the members of which differ in their initial conditions will be able to do the separation. This naive expectation is reflected in the works of [9] and [11, 12]. Nevertheless, an appropriate interpretation can only be obtained in the snapshot attractor framework. This is so because the natural measure is the unique probability measure that describes appropriately the probabilities according to which the dynamical variables take on particular values. In the snapshot attractor framework, the internal variability is represented by the natural measure of the snapshot attractor, and the forced response is its temporal change within a given forcing scenario.

This correspondence, however, has not yet been recognized clearly in the literature. The snapshot (or pullback) attractor framework has been explicitly used so far in the climatic context only with random forcings [23, 89]. At the same time, [8] claims internal variability to be hard to estimate in climate projections, and treats it together with what they call model uncertainty, i.e., the deviation of the current climate models and their parameters from the real world and from each other. In particular, they claim that using single realizations from a

\footnote{This also implies that the temporal averages, deviating from the ensemble averages, lack any probabilistic interpretation when evaluated in nonergodic systems like a climate system with a non-stationary forcing. This is why temporal averages are not suited for characterizing the climate and its changes.}
range of different models represents appropriately also the internal variability, but they do not give a strict argumentation for this.

In our work, we wish to clarify the conceptual basis of separating the forced response from the internal variability in a dynamical system that is subject to a smooth deterministic forcing. For this purpose, we turn, as mentioned in the context of nonergodicity, to Lorenz’s 1984 low-order model for midlatitude circulation [100]. In fact, we shall also see that the strength of the forced response, i.e., of the time-dependence of the relevant probability measure, is naturally quantified by our indicators of nonergodicity.

1.3.3 The Lorenz 84 model

The physical content of Lorenz’s atmospheric circulation model for the midlatitudes on one hemisphere [100], generally referred to as the ‘Lorenz 84 model’, is the following. Solar forcing creates a temperature difference between the Equator and the pole, which is proportional to the model variable $F$, and it influences most directly the wind speed of the Westerlies represented by $x$. As an effect of baroclinic instability, cyclonic activity facilitates poleward heat transport, two modes of which are represented by $y$ and $z$. This appealing low-order model was studied in different contexts [102, 103, 104, 105, 106, 107, 108, 109, 28, 110, 111, 112]. The model reads as follows:

$$
\begin{align*}
\dot{x} &= -y^2 - z^2 - ax + aF(t), \\
\dot{y} &= xy - bxz - y + G, \\
\dot{z} &= xz + bxy - z.
\end{align*}
$$

For the parameter setting we take the common choice: $a = 1/4$, $b = 4$, $G = 1$ [100]. The equations appear in a dimensionless form with the time unit corresponding to 5 days, according to [100]. The constant value of $F(t) = F_0 = 6$ (8) was regarded by Lorenz to be an appropriate value for permanent summer (winter), and the system with this value exhibits only periodic (chaotic) attractors.

As for the relevance of this model in environmental fluid dynamics, we note that the model was tailored to mimic the thermal wind relation $x \sim F$ for $F < 1$ at vanishing asymmetry parameter $G$. For positive $G$ and larger values of $F$ not even the fixed point solutions of (1.40) reflect the thermal wind relation. The overall approximate validity of geostrophic balance,
characterizing high-dimensional hydrodynamic models, cannot thus be expected to be present in this low-order model described by three ordinary differential equations. It follows that the quantitative results of the model should not be interpreted from a geophysical point of view. Instead, it serves as a testbed for new concepts in the context of environmental fluid dynamics, due to its structure exhibiting terms up to the second degree on the right-hand side of the equations which is characteristic to atmospheric problems [113, 114].

To model seasonality, Lorenz uses a forcing [101]:

$$F(t) = F_0(t) + A \sin(\omega t)$$

with $A = 2$ and a constant $F_0(t)$. The periodic form (1.41) taken by him is centered on the fixed mean value $F_0(t) = 7$ [101] to describe the variation of the forcing over a year. We define 1 year to be $T = 73$ time units, and thus $\omega = 2\pi/73$ is set. Our calendar starts with year 0, and this year begins at the time instant $t = 0$. Note that this time instant corresponds to an autumnal equinox, according to expression (1.41). Similarly, any time $t \mod T = 0$ coincides with autumnal equinoxes. Midwinters and midsummers then correspond to $t \mod T = T/4 = 18.25 = 0.25$ years and $3T/4 = 54.75 = 0.75$ years, respectively. As for the physical origin of the forcing we note that $F(t)$ may, among others, also contain the contribution of the varying CO$_2$ content in association with the greenhouse effect. From this point on, $F(t)$ will be called the temperature contrast parameter. The model with (1.41) and $F_0(t)$ being constant is considered to represent a stationary climate.
Chapter 2

Modulated point vortex pairs on a rotating sphere

2.1 Basic investigations

2.1.1 Trajectory forms

Numerical solutions of equations (1.33) show that only a few trajectory forms are possible. The most important ones can be found by keeping the parameters constant and varying only one initial condition. A common feature of almost all trajectories is that they repeat their shape in $\lambda$ and also in time. The corresponding temporal period of a trajectory will be denoted by $T_0$.

First we investigate vortex pairs characterized by some particular value of $\varphi_r$. We vary the initial latitudinal angle $\varphi_0$, and choose $\alpha_0 = 0$ which implies an eastward initial velocity. The numerically obtained trajectory forms are shown in Fig. 2.1. At $\varphi_0 = \varphi_r$ (Fig. 2.1(a)), a small amplitude meandering motion, called wobbling, is initiated. Increasing $\varphi_0$ makes the amplitude also increase (Fig. 2.1(b)), a manifestly wobbling motion occurs. The deviation from the eastward direction can grow so large that the trajectory turns back, to the west, intersecting itself (Fig. 2.1(c)). This new type of motion is called tumbling. Increasing $\varphi_0$ further, we find a critical value $\varphi_c$ corresponding to a separatrix (Fig. 2.1(d)) when the trajectory asymptotically approaches a special latitude, denoted by $\varphi_-$ ($\approx \varphi_r$), corresponding to uniform westward propagation. For $\varphi_0 < \varphi_c$, the motion extends to both sides of $\varphi_-$, while for $\varphi_0 > \varphi_c$, the vortices stay on the northern side of $\varphi_-$, tracing out tumbling loops (Fig. 2.1(e), red curve).
Figure 2.1: A summary of the different trajectory forms. The trajectories of the vortices (solid lines) and of their center of mass (dotted, dashed or dot-dashed line) are plotted on the $(\lambda, \varphi)$ plane for vortex pairs with different initial latitudes $\varphi_0$ in the different panels. The arrows indicate the direction of propagation. Parameters: $\varphi_r = 0.65$, $D' = 0.01$, $a' = 0.01$, $\Gamma' = 5 \cdot 10^{-6}\pi$. Initial conditions: $\alpha_0 = 0$, $\lambda_0 = 0$, and $\varphi_0$ is indicated in the panels. The special latitude to which the trajectory of panel (d) converges is $\varphi_- = 0.65027$. The other special latitude (not shown) is $\varphi_+ = 0.64979$.

All the forms described here are repeated in reverse order, in a mirrored way, when initiated well below $\varphi_r$ (Fig. 2.1(e), dot-dashed curve, and Fig. 2.1(f) which represents in a topological sense a mirrored pair to Fig. 2.1(c)). There exist another special latitude $\varphi_+$ ($< \varphi_r$ on the northern hemisphere) which as an initial condition belongs to a uniform eastward propagation.

Vortex pairs initiated near $\varphi_+$ with an eastward velocity always exhibit a wobbling motion. Whether the trajectory initially bends to the south or to the north depends on whether $\varphi_0$ is on the northern or the southern side of $\varphi_+$.

One might note an interesting analogy between the dynamics of a vortex pair with some particular value of $\varphi_r$ and that of a single inertial orbit of a point mass on a rotating Earth worked out by Paldor and coworkers [115, 116, 117, 118, 119]. The center of mass orbits of Fig. 2.1 are remarkably similar to the ones shown in figure 1 of [115]. This is surprising since the equations of motion are rather different. The analogy was shown to be a full correspondence with $\varphi_r = 0$ and $a' = D'$ in [1].

2.1.2 Equations of motion for a dipole

In order to unfold the essence of the vortex dynamics, in this section we introduce dipoles, very close and very weak vortex pairs whose velocity is finite. In view of (1.36), we are interested
in the limit $\Gamma', D' \to 0$. As $a'$ refers to the radius of a patch of vorticity, for nearby vortices it should also go to zero. Since both $D'$ and $a'$ correspond to distances, we consider them to be of the same order. Let us write the time-dependent vortex coordinates as $\varphi_{1,2} = \varphi \pm d\varphi$, and $\lambda_{1,2} = \lambda \pm d\lambda$, where $\varphi$ and $\lambda$ are the center of mass coordinates in the dipole limit. For a dipole, $d\varphi$ and $d\lambda$ are infinitesimally small, and we assume that $d\varphi, d\lambda, D', a', \Gamma'$ are all of the same order.

From the equations of motion (1.33) of a vortex pair one can obtain the equations of motion for $\varphi$, $d\varphi$, $\lambda$ and $d\lambda$ via a systematic Taylor expansion up to leading order. An important element is that the circulations of the vortices have to be calculated up to second order:

$$\Gamma'_{1,2}(\varphi_{1,2}) = \pm \Gamma' + a'^2 \pi \sin \varphi_r - a'^2 \pi \sin(\varphi \pm d\varphi) \approx \pm \Gamma' + a'^2 \pi \sin \varphi_r - a'^2 \pi \sin \varphi. \quad (2.1)$$

From the equations of motion for $\varphi$, $d\varphi$, $\lambda$ and $d\lambda$, a closed dynamics follows for the center of mass coordinates $\varphi$ and $\lambda$:

\begin{align*}
\frac{d}{dt} (\cos \varphi \lambda) &= \varphi \sin \varphi \left( \gamma \delta(\varphi) + \dot{\lambda} \right), \quad (2.2a) \\
\frac{d}{dt} \dot{\varphi} &= -\dot{\lambda} \cos \varphi \sin \varphi \left( \gamma \delta(\varphi) + \dot{\lambda} \right), \quad (2.2b)
\end{align*}

where

$$\gamma = \frac{a'^2}{D'^2} \quad (2.3)$$

and

$$\delta(\varphi) = 1 - \frac{\sin \varphi_r}{\sin \varphi}. \quad (2.4)$$

The details of the calculation are found in the Appendix. Using the zonal and meridional center of mass velocity components

$$u \equiv \dot{\lambda} \cos \varphi, \quad v \equiv \dot{\varphi}, \quad (2.5)$$

the dynamics of the center of mass is written as

\begin{align*}
\dot{u} &= \gamma \delta(\varphi) v \sin \varphi + \tan \varphi uv, \quad \dot{v} = -\gamma \delta(\varphi) u \sin \varphi - \tan \varphi u^2. \quad (2.6)
\end{align*}

Equations (2.5), (2.6) for a dipole depend only on the parameters $\varphi_r$ (through the function
δ(ϕ)) and γ = a′2/D′2. Γ′ does not appear in these equations. Γ′ and D′ together can be used to set the initial velocity by means of (1.36). In the dipole limit, the precise formula is (see the Appendix)

$$|\mathbf{u}| = \frac{\Gamma'}{2\pi D'}.$$  

(2.7)

This shows that the velocity modulus can be calculated from the reference vortex strengths.

### 2.1.3 Equations of motion for a β-plane dipole

For a β-plane dipole, we write $x_{1,2} = x \pm dx$, $y_{1,2} = y \pm dy$, where $(x, y)$ is the position of the center of mass in the β-plane, and $dx$, $dy$, $a'$, $D'$ and Γ' are considered to be infinitesimally small and to be of the same order. A consistent Taylor expansion of the equations of motion (1.39) leads to

$$\ddot{x} = \frac{a'^2}{D'^2} \beta' \beta y \dot{y} = \gamma \cos \varphi_r \gamma \dot{y},$$  

(2.8a)

$$\ddot{y} = -\frac{a'^2}{D'^2} \beta' \beta x \dot{x} = -\gamma \cos \varphi_r \gamma \dot{x}. $$  

(2.8b)

[The derivation of these equations is very similar to that of (2.2) which is discussed in detail in the Appendix.] These are the β-plane equations of motion for a dipole moving near the reference latitude $\varphi_r$.

The velocity of the dipole is the derivative of the position vector $(x, y)$ as defined by the planar geometry:

$$\mathbf{u} = (\dot{x}, \dot{y}).$$  

(2.9)

To see this, one can also refer to definition (2.5) and substitute $\cos \varphi$ by $\cos \varphi_r$ in the spirit of the “planarization” of the dynamics:

$$\mathbf{u} = (\dot{\lambda} \cos \varphi_r, \dot{\varphi}) = (\dot{x}, \dot{y}).$$  

(2.10)

It is clear then that this is the only reasonable velocity vector associated to a β-plane dipole.
2.2 Time scale separation and a crossover to global mixing during the chaotic advection in the velocity field of a modulated vortex pair [T1]

2.2.1 Short term advection

In a frame co-moving with a modulated vortex pair, the flow field is time-periodic (with period $T_0$) and, as a consequence of the time-dependence, the advection dynamics is typically chaotic. (An exception is the case of the separatrix motion of Fig. 2.1(d).) A feature of interest remains to be the type of advective chaos.

The dynamics of a passive tracer can be considered as that of a third vortex of zero circulation $\Gamma'_3(\varphi_3) \equiv 0$. We deal then with a restricted 3-vortex problem where the inactive vortex has no influence on the others. For illustrative purposes we consider vortex pairs with the reference latitude at the equator $\varphi_r = 0$, and with a relatively large distance $D'$. The dimensionless equations of motion for the coordinates $(\varphi, \lambda)$ of the tracer particle are thus

\[
\frac{d\varphi}{dt} = \frac{1}{4\pi} \sum_{j=1}^{2} \frac{(\Gamma' - a^2 \pi \sin \varphi_j) \cos \varphi_j \sin(\lambda - \lambda_j)}{1 - \cos \gamma_j}, \tag{2.11a}
\]

\[
\frac{d\lambda}{dt} = \frac{1}{\cos \varphi} \frac{1}{4\pi} \sum_{j=1}^{2} \frac{(\Gamma' - a^2 \pi \sin \varphi_j) [\cos \varphi \sin \varphi_j - \sin \varphi \cos \varphi_j \cos(\lambda - \lambda_j)]}{1 - \cos \gamma_j}, \tag{2.11b}
\]

where

\[
\cos \gamma_j = \sin \varphi \sin \varphi_j + \cos \varphi \cos \varphi_j \cos(\lambda - \lambda_j) \tag{2.12}
\]

is related to the chord distance between the tracer and vortex $j$. Note that the dynamics (2.11) of a tracer can be considered as a dynamical system of two variables driven by the vortex pair dynamics.

On the one hand, we may naively follow the positions of the tracers on the sphere in the original reference frame. On the other hand, however, the closed-open character of the flow can best be recognized in a reference frame co-moving with the vortex centers. The origin in the $\lambda$ coordinate is chosen to be the center of mass of the vortex pair (i.e. vortices 1 and 2). In this co-moving reference frame the forcing entering into the two-variable tracer dynamics turns out to be periodic. This allows us to define a two-variable stroboscopic map fully describing the tracer dynamics, taken at integer multiples of $T_0$. Consequently, this corresponds to a
configuration in which the maximum of the vortex center of mass trajectory is reached in the vortex pair dynamics.

From the point of view of the tracer advection, there is always a region close to any of the vortices which is isolated from the surroundings. Here the circulational flow of a single vortex dominates, and the influence of the other can practically be neglected [120, 121, 122]. These regions are called the vortex cores. Whether advection is chaotic turns out by investigating regions outside the cores, but being not far away from the vortices. In-between the vortices one always finds such a region because the vortex cores obviously cannot overlap.

If the tracers of a small droplet placed in-between the vortices at \( t = 0 \) remain distributed in a finite range around the vortices after arbitrarily long times, the flow is closed; otherwise it is open. From a dynamical-systems point of view, the basic difference between closed and open advection dynamics lies in the structure of the chaotic set. For closed chaotic advection, the chaotic set extends over a two-dimensional area of the fluid surface. The region filled in asymptotically by the droplet points is part of the chaotic set, and other such areas might also exist, reachable from other initial droplet positions. In contrast, the chaotic invariant set of the open advection dynamics contains fractal parts of zero area. This chaotic saddle [68, 69] is formed by an infinity of unstable particle orbits which are trapped by the vortices forever, both forward and backward in time. In such cases points of a droplet come to a close neighborhood of the chaotic saddle, but leave it sooner or later. Their asymptotic form is determined by the unstable manifold [68, 69], itself a fractal, of the chaotic saddle.

In order to study the short term advection dynamics, we initiate a small droplet of tracer particles between the two vortices, and follow its evolution. One option is to plot the positions of the tracers after some time to visualize the spatial pattern characterizing the advection process (i.e. the unstable manifold of the saddle in the open case). As an other option, we define an escape circle of radius \( \rho \) centered in the center of mass of the two vortices, and measure the escape time (the time needed to leave the circle) of each tracer. We choose this radius \( \rho \) to be considerably smaller than the diameter of the sphere in order to assure that the tracers detrained along the wake leave the escape circle before they could reenter again. This way we investigate the short term detrainment of the tracers from the neighborhood of the pair. High escape times from the circle correspond to closed advection and to initial conditions situated either in the vortex cores or close to the stable manifold of the chaotic saddle in the case of open advection. Plotting the escape time as a function of the initial position draws
Figure 2.2: Basic advection patterns. The positions of $N \approx 70000$ advected tracers and of the two vortices at time $t$ are plotted as dots (the vortices are denoted by larger dots). The center of mass trajectory of the vortices is marked by a thin solid line. Thick lines indicate the equator and the $\lambda = 0$ meridian. The tracers were initiated at the colored and the light gray grid points in Fig. 2.3. Vortex pair parameters: $\varphi_r = 0$, $D' = 0.1$, $a' = 0.1$, $\Gamma' = 5 \cdot 10^{-3}$.$\pi$. Vortex initial conditions: $\alpha_0 = 0$, $\lambda_0 = 0$, and $\varphi_0$ is indicated in the panels.

Thus out the stable manifold of the saddle as ridges in the plot.

When numerically investigating the advection generated by vortex pairs initiated with an eastward velocity, we find that the open or closed character of the flow can be altered by a mere change in the initial latitude (as also found on a topographic $\beta$-plane in [85]). Vortex initial conditions closer to the equator than this critical latitude $\varphi_{0c}$ lead to open advection, while the others correspond to closed advection.

Numerical results are presented in Figs. 2.2-2.3. For wobbling motion (Fig. 2.2(a)) the tracers are detrained along a simple tail, an intricate lobe structure is not recognizable (without magnification). When the vortex trajectories cross themselves (Fig. 2.2(b)), large lobes are formed on both sides of the equator in a symmetrical manner. Finally, when the trajectory is tumbling and is confined to one of the hemispheres (Fig. 2.2(c)), southward traveling lobes are created. In each case the fractal pattern of the stable manifold of the saddle can be
Figure 2.3: The escape times (indicated on the color scale, measured in units of the time period $T_0$ for an escape radius $\rho = 0.8$) of the tracers of Fig. 2.2, as a function of their initial position. Homogeneous light gray color indicates that the corresponding tracers did not leave the neighborhood of the vortices during the simulation time of $250T_0$. $\varphi_0$ and $T_0$ are indicated in the panels. Panel (d) corresponds to the closed case. It presents the shape of the initial tracer droplet in homogeneous light gray color. The trajectories of the vortices (solid lines) and of their center of mass (dotted line) are also shown to help visualize the relative size and location of the droplet compared to the vortex motion.

seen as filamentary structures in the escape time distributions (Figs. 2.3(a)-2.3(c)). Above a critical initial latitude $\varphi_{0c} \approx 0.40$, the tracers are confined to the neighborhood of the vortices (Fig. 2.2(d)), the advection dynamics becomes closed. It is worth noting that the critical latitude $\varphi_{0c}$ differs from the separatrix latitude $\varphi_c$ defined for the vortex pair dynamics.

2.2.2 Long term advection with a crossover to global mixing

It is an interesting consequence of spherical topology that the open character seen on short time scales crosses over into a closed advection for asymptotically long times. The reason for this is that the vortex pair meets its wake after one global period $T_1$ corresponding to the time needed for the vortex pair to go around the sphere. The characteristic periods $T_0$ and $T_1$ are rather different (in the case of Fig. 2.2(a) they are $T_0 = 50.6$ and $T_1 \approx 988$, respectively), therefore qualitatively different advection patterns are expected on the time scales $t \leq T_1$ and $t \gg T_1$. As the vortex pair propagates through its own wake, it mixes the tracers which already went through a similar process when they were located in the vicinity of the vortex pair one
global period earlier. This is numerically illustrated in Figs. 2.4 and 2.5, in a spherical view and in a planar representation, respectively. After several such mixing events we find the tracers to continuously fill a zonal band around the sphere (see Fig. 2.5(b)). This band is somewhat narrower near the current location of the vortex pair. By time $t = 11050$ the band is populated by a nearly space-filling distribution, indicating the tendency towards a complete mixing. The slight increase of the density of tracers in some localized region in $\lambda$ is a consequence of the initially localized tracer distribution, and will slowly disappear with a further increase of time.

Figure 2.4: Positions of $N \approx 70000$ advected tracers at $t = 1000 \approx T_1$. The fact that the center of mass of the vortices has just passed the $\lambda = 0$ meridional indicates that the time taken is slightly more than one global period $T_1$. This is the time when the vortex pair enters its own wake. The tracers were initiated at the colored and the light gray grid points in Fig. 2.3(a). Vortex pair parameters: $\varphi_r = 0$, $D' = 0.1$, $a' = 0.1$, $\Gamma' = 5 \cdot 10^{-3}\pi$. Vortex initial conditions: $\alpha_0 = 0$, $\lambda_0 = 0$ and $\varphi_0 = 0.25$.

Figure 2.5: Same as Fig. 2.4 for (a) $t = 2050 \approx 2T_1$ and (b) $t = 11050 \approx 11T_1$ in a planar view.

The easiest way to characterize these phenomena is the investigation of a chaotic scattering process [68, 26, 29] in the stroboscopic map defined in Section 2.2.1 in a reference frame co-moving with the vortices. On the map’s surface in this reference frame the fluid moves uniformly (in zonal direction) except for the vicinity of the $\lambda = 0$ coordinate where the vortex pair is located in a meridional direction. Chaotic features in the tracer advection can only occur in
this localized region. One can thus define this region as a scattering region. Due to the zonal
direction of the uniform movement of the fluid far from the origin, one may decide to rely only
on the coordinate $\lambda$ when declaring boundaries for the scattering region: we define this region
as the interval $\lambda \in (-\lambda_b, \lambda_b)$, indicated in Fig. 2.6.

The investigation of the short term behavior corresponds to taking into account only one
scattering event for some particular tracer. We shall call this scattering process an elementary
scattering. This elementary scattering event is characterized by a chaotic saddle localized near
the vortices, which we shall call the elementary saddle. It has stable and unstable manifolds
which extend outside the scattering region.

The elementary saddle and its invariant manifolds can be numerically constructed by the
application of the sprinkler method [29]: We initiate a droplet of tracers in the incoming
asymptotic region, and start a time counter when they first step inside the scattering region.
We stop the counter when they leave the scattering region to obtain a (discrete) delay time
for each tracer. Particles with a large delay time approach the elementary saddle during the
scattering event. Therefore, their positions in the incoming asymptotic region mark the stable
manifold of this saddle.\(^1\) Their positions corresponding to about half time of the scattering
process trace out the elementary saddle, and their positions in the outgoing asymptotic region
represent the unstable manifold of this saddle.\(^2\) As a first approach, we only follow the evolution
of the tracers up to their first encounter with $\lambda = -\pi$. We show the corresponding results in
Fig. 2.6(a). The tracers are detrained from the vicinity of the vortex pair along the unstable
manifold (marked by red in Fig. 2.6(a)) of the elementary saddle. The elementary saddle itself
can be considered as the intersection of its own stable and unstable manifolds (although this
is not visible in Fig. 2.6(a) since we do not plot the manifolds for $|\lambda| < \lambda_b$ for better visibility.)

Due to the periodic nature of the sphere in $\lambda$, the invariant manifolds do not have an end
at $\pi$ or $-\pi$ but they reenter the domain $\lambda \in [-\pi, \pi)$ on the other side. If we just follow them
up to their next encounter with the interval $(-\lambda_b, \lambda_b)$ (without taking into account a second
scattering event), we find that they intersect each other. The creation of new homoclinic and
heteroclinic points imply the appearance of new components of the chaotic saddle underlying
the dynamics on long time scales. These new components consist of orbits which are permitted

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\(^1\)One may note that the set of points with large escape times in Fig. 2.3 have a similar meaning. In
Section 2.2.1, however, the region under investigation is only a close vicinity of the vortices and of the elementary
saddle.

\(^2\)Technically, the stable manifold is obtained as the mirror image of the unstable manifold to the $\lambda = 0$ line, due
to the chosen special configuration of the vortex pair.
Figure 2.6: (a) Blue (dark gray): the stable, red (light gray): the unstable manifold of the elementary saddle, followed up to $\lambda = \pi$ and $-\pi$, respectively. The elementary saddle is shown in black in the scattering region $\lambda \in (-\lambda_b, \lambda_b)$. These sets are numerically obtained by means of the sprinkler method with $N = 1.9 \times 10^6$ tracers. (b) Magenta (gray): the new components of the chaotic saddle as described in the text, obtained as intersection points between the stable and the unstable manifolds of the elementary saddle. For comparison the elementary saddle is also plotted. The inset in (b) is a magnification of a well-populated region of the new saddle components. Vortex pair parameters: $\varphi_r = 0$, $D' = 0.1$, $a' = 0.1$, $\Gamma' = 5 \cdot 10^{-3}\pi$. Vortex initial conditions: $\alpha_0 = 0$, $\lambda_0 = 0$ and $\varphi_0 = 0.25$. $\lambda_b \approx 0.42$.

to leave the close vicinity of the vortices but return after the number of iterations that corresponds to one winding around the sphere. Note that the investigated intersections of the stable and the unstable manifold of the elementary saddle appear in the whole domain $\lambda \in [-\pi, \pi]$ (i.e., also outside $(-\lambda_b, \lambda_b)$), implying that the chaotic saddle has a zonally global extension. This is illustrated in Fig. 2.6(b). In the inset of Fig. 2.6(b) one can see the fractal structure of the new components of the chaotic saddle.

The stable and unstable manifolds of the elementary saddle do not have an end after one full winding (entire going around the sphere). After each full winding new intersection points are created. For simplicity, let us suppose that the invariant manifolds of the elementary saddle do not change their spatial pattern shown in Fig. 2.6(a) when continuing the manifolds arbitrarily long. Since the ratio $T_0/T_1$ of the local and global periods is typically irrational, the created intersection points will densely fill the whole domain $\lambda \in [-\pi, \pi]$ in the limit of infinitely many windings.

The real scenario is, however, more complicated. For convenience, let us define $\lambda$ as a rotational angle to see clearly how the invariant manifolds of the elementary saddle evolve after subsequent windings. In Fig. 2.7, one can observe that the unstable manifold of the elementary saddle is distorted (or mixed) when meeting the vortices at $\lambda = -2\pi$. Every time the invariant manifolds of the elementary saddle meet the vortices results in a mixing due to
chaotic effects originating from the elementary scattering. After such an event the invariant manifolds cannot become less dense than before mixing. This process repeats itself at integer multiples of \((-2\pi)\). As a consequence, the argument corresponding to the simplified scenario remains valid and the full domain \(\lambda \in [-\pi, \pi)\) becomes densely filled by intersection points in the limit of infinitely many windings in the real scenario as well. This implies that the exact chaotic “saddle” of the full global problem of tracer advection is space-filling. This corresponds to a chaotic sea characterizing the asymptotically long time advection of the tracers in the velocity field of a vortex pair. The space-filling behavior starts to emerge on time scales much longer than the global period \(T_1\).

Since a drop of tracers of any shape is smeared into a space-filling distribution after a long time, this is true also for the simple pattern of the unstable manifold of the elementary scattering shown in Fig. 2.6(a). We can see three steps of this process in Fig. 2.7: the unstable manifold becomes denser when crossing integer multiples of \(2\pi\) from the right to the left. This shows that the transition to a space-filling distribution is a step-by-step process, controlled by the elementary scattering. From Fig. 2.7 one also learns that the chaotic sea only extends to a finite interval in the latitudinal variable \(\varphi\): though the unstable manifold expands in \(\varphi\) at \(\lambda = -2\pi\), the expansions occurring later are weaker, and the widths converges to a finite size. Then we can say that the long term global advection is characterized by a chaotic sea that extends to a whole zonal band on the sphere.

It is important to note that this conclusion is true only in cases when the short term tracer advection is open. If the short term advection is closed, the invariant manifolds of the chaotic set do not leave the vicinity of the vortices and the chaotic set remains localized around the vortices in both \(\lambda\) and \(\varphi\). This implies that tracers initiated near the vortices always stay in a
localized region around the vortices and never fill a whole zonal band (see Fig. 2.2(d)).

2.3 The breakdown of the $\beta$-plane approximation in the modulated point vortex pair model

2.3.1 The inconsistency of the $\beta$-plane dipole equations [T2]

Now we turn to the investigation of the validity of the $\beta$-plane approximation. In order to obtain a reference, we first derive the consistently linearized dipole equations around the reference latitude $\varphi_r$. Any expressions depending explicitly on the variable $\varphi$, including the metric factors, are taken into account up to the same order. Although it turns out to be useful to define new variables, denoted again by $x$ and $y$, for a more convenient representation of the dynamics, these new variables are not derived from any planar geometry, in contrast to the $\beta$-plane approximation.

The Taylor expansion in $\Delta \varphi \equiv \varphi - \varphi_r$ of the full spherical dipole equations (2.2) results in leading order in

\[
\begin{align*}
\cos \varphi_r \ddot{\lambda} &= 2 \sin \varphi_r \Delta \varphi \dot{\lambda} + (2 / \cos \varphi_r) \Delta \varphi \dot{\lambda} \Delta \varphi + \gamma \cos \varphi_r \Delta \varphi \Delta \varphi, \\
\dot{\Delta} \varphi &= -\sin \varphi_r \cos \varphi_r \lambda^2 - \cos(2 \varphi_r) \lambda^2 \Delta \varphi - \gamma \cos^2 \varphi_r \lambda \Delta \varphi.
\end{align*}
\]

(2.13a)

(2.13b)

Defining

\[
x = \cos \varphi_r \lambda, \quad y = \Delta \varphi,
\]

(2.14)

we obtain

\[
\begin{align*}
\dot{x} &= \gamma \cos \varphi_r \dot{y} y + 2 \tan \varphi_r \dot{y} \dot{x} + (2 / \cos^2 \varphi_r) \dot{y} \dot{x} y, \\
\dot{y} &= -\gamma \cos \varphi_r \dot{x} y - \tan \varphi_r \dot{x}^2 + (\tan^2 \varphi_r - 1) \dot{x}^2 y.
\end{align*}
\]

(2.15a)

(2.15b)

These are the correct equations for dipoles moving close to $\varphi_r$. We emphasize that $x$ and $y$ are not Cartesian coordinates but are arbitrarily defined variables. The conversion, however, between the $(\lambda, \varphi)$ and the $(x, y)$ variable pairs is exactly the same as in the $\beta$-plane approximation. As only the $(\lambda, \varphi)$ coordinates are of physical relevance, the coincidence of the conversion ensures a direct comparability of the current equations with those of the $\beta$-plane approximation.
approximation.

Observe that the second and the third term on the right hand side of both (2.15a) and (2.15b) are missing in the $\beta$-plane equations of motion (2.8). The origin of this is in the metric factors of the equations of motion of a vortex pair (1.33), which are inconsistently omitted in the $\beta$-plane approximation.

Let us now express the velocity of the dipole in terms of $x$, $y$ and their derivatives, applying consistent linearizations. Definition (2.5) yields

$$u = (\dot{\lambda}\cos(\varphi_r + \Delta\varphi), \dot{\Delta}\varphi),$$

(2.16)

from which, by taking the linear Taylor expansion of the cosine function around $\varphi_r$ and substituting the definition (2.14) of $x$ and $y$, we obtain

$$u = \left((1 - \tan \varphi_r y)\dot{x}, \dot{y}\right).$$

(2.17)

It is important that the zonal (i.e. the $\lambda$-) component $u$ of the velocity is not equal to $\dot{x}$.

In the spirit of the $\beta$-plane approximation it is worth taking into account that the components of the velocity vector $u$ should be small in comparison with the equatorial velocity of the surface of the sphere. In a geophysical context, the characteristic dimensional velocity modulus $U$ of the fluid should be much smaller than the equatorial velocity $R\Omega$ of the sphere. The global Rossby number $Ro_g = U/(2\Omega R)$ is then small. This means that the dimensionless $|u|$ should also be restricted to small values. As a consequence of (2.17), $\dot{x}$ (regardless of the particular value of the small variable $y$) and $\dot{y}$ are considered to be of the order of the small global Rossby number $Ro_g$.

The global Rossby number $Ro_g$ is chosen to be around 0.025 or 0.0025 in our numerical simulations. $y$, as a small quantity, with an absolute value of 0.15 at the very most, is considered to be of the same order as $Ro_g$ (see [5]). Then the first and the second terms in the system (2.15) are of order $Ro_g^2$, and the third terms are of order $Ro_g^3$, the latter ones then being not relevant. Thus, as long as $\tan \varphi_r$ is of order unity, omitting the second terms, as done in the $\beta$-plane approximation, leads to a considerable deviation of the trajectories from the correct ones. (A close agreement is expected around the equator, i.e. for $\tan \varphi_r \ll 1$ only, in agreement with Pedlosky’s argumentation for the momentum equation in [5].)

The numerical examples shown in Section 2.3.3 exhibit striking differences for certain cases.
In order to understand these plots, however, a further observation is needed first, leading to the definition of the special latitudes.

2.3.2 Special latitudes in the spherical dipole dynamics [T3]

Now we consider a fixed $\varphi_r \neq 0$ and ask if there exists any latitude that is a dynamical analog of the origin of the $\beta$-plane approximation, a latitude corresponding to an eastward or westward uniform propagation. Intuitively, one might think that this special latitude is $\varphi_r$. Here we show that the special latitudes $\varphi_{\pm}$, describing eastward and westward uniform propagation of a general dipole, respectively, differ from $\varphi_r$ and also from each other.

We are looking for solutions of the dipole equations of motion (2.2) of the form

$$\varphi(t) = \varphi_{\pm}, \quad \dot{\varphi}(t) = 0.$$ (2.18)

Substituting this into equation (2.2a), we obtain

$$\ddot{\lambda} = 0, \quad \lambda(t) = \lambda_0 + \omega t,$$ (2.19)

that is, this propagation is indeed uniform. The constant $\omega$ is the angular velocity and is thus geometrically related to the zonal velocity $u$ as

$$\omega = \frac{u}{\cos \varphi_{\pm}}.$$ (2.20)

Since both $\omega$ and $\varphi_{\pm}$ are constants, $u$ is constant in time and is equal to its initial value $u_0$. Equation (2.2b) then yields

$$0 = -\omega \cos \varphi_{\pm} (\gamma \sin \varphi_{\pm} - \gamma \sin \varphi_r + \omega \sin \varphi_{\pm})$$

$$= -u_0 \gamma \sin \varphi_{\pm} + u_0 \gamma \sin \varphi_r - u_0^2 \tan \varphi_{\pm}$$ (2.21)

from which, for $\varphi_{\pm} \neq \pm \pi/2$,

$$\sin \varphi_{\pm} = \frac{\sin \varphi_r}{1 + u_0/(\gamma \cos \varphi_{\pm})}$$ (2.22)

holds for the special latitudes $\varphi_{\pm}$. According to equation (2.7), $|u_0|$ is obtained from the vortex
strength $\Gamma'$ and the vortex distance $D'$ (remember that $v = \dot{\varphi} = 0$):

$$u_0 = \pm \frac{\Gamma'}{2\pi D'},$$

where the upper (lower) sign corresponds to the eastward (westward) propagation. The above equations implicitly determine $\varphi_\pm$ as a function of $\gamma$, $\varphi_r$, $\Gamma'$ and $D'$.\(^3\)

From equation (2.22) one sees that the latitude of the uniform propagation does not coincide with $\varphi_r$, and is different for the eastward and the westward direction of the propagation. Relative to $\varphi_r$, this latitude is shifted toward and away from the equator in these two cases, respectively.

Figure 2.8: Eastward (a) and westward (b) initiated (wobbling and tumbling) dipole trajectories in a ($\lambda, \varphi$) representation, compared to the relevant special latitude [red dot-dashed line: $\varphi_+ = 0.627$ (a) and $\varphi_- = 0.675$ (b), respectively]. Initial latitudes for the dipole trajectories: dotted: $\varphi_0 = \varphi_r = 0.65$ in both panels, intermittently dashed: $\varphi_0 = 0.617 < \varphi_+$ (a) and $\varphi_0 = 0.683 > \varphi_-$ (b). Further initial conditions: $\lambda_0 = 0$, $v_0 = 0$, and $u_0$ is indicated in the panels. The length $t_{\text{max}}$ of the trajectories is also indicated in the panels, in dimensionless time units. Parameters: $\gamma = 1$, and $\varphi_r = 0.65$ is marked by a gray dot-dot-dashed line. Note the different scales in panels (a) and (b). These two plots illustrate that $\varphi_+$ serves as a “center”, and that $\varphi_-$ is a separatrix.

Numerical results in Fig. 2.8 highlight the role of the special latitudes in the full spherical dipole dynamics. In Fig. 2.8(a), we can see that a dipole initiated close to $\varphi_+$ with an eastward velocity moves to the east on average, meandering in the vicinity of $\varphi_+$. This trajectory type is called an eastward wobbling [79]. As illustrated in the plot, small amplitude eastward wobbling trajectories have inflexion points close to $\varphi_+$. $\varphi_+$ thus serves as a “center” for such trajectories, latitudinally it “attracts” them. Eastward wobbling trajectories initiated on the northern (southern) side of $\varphi_+$ bend initially to the south (north). As Fig. 2.8(b) illustrates, $\varphi_-$ plays an opposite role for dipoles initiated with a westward velocity: they are “repelled” from

\[^3\]For small (geostrophic) velocities and latitudes not too close to the pole, an appropriate estimation for the positions of the special latitudes is $\sin \varphi_\pm \simeq \sin \varphi_r/[1 + u_0/(\gamma \cos \varphi_r)]$. The difference $\sin \varphi_- - \sin \varphi_+ \simeq 2 \tan \varphi_+/u_0/\gamma$ is then always small, of the order of $Ro_\theta$, for a given velocity modulus $|u_0|$.\]
\(\varphi_-,\) resulting in circle-like trajectories slowly drifting to the west. These are called westward tumbling trajectories [79]. It is clear that the dipole moves along in a negative (positive) rotational direction on the northern (southern) side of \(\varphi_-\). We can thus say that \(\varphi_-\) is a separatrix for northern and southern type westward tumblings.

Based on the previous paragraph, the difference between \(\varphi_+\) and \(\varphi_-\) has the following implication: for any initial latitude \(\varphi_0 \in (\varphi_+,\varphi_-)\) there exist both southern type westward tumbling trajectories (for westerly initial velocities), and eastward wobbling trajectories (for easterly initial velocities) that initially exhibit a northern type behavior (i.e. they initially bend to the south). For example, \(\varphi_0 = \varphi_r = 0.65 \in (\varphi_+,\varphi_-)\) behaves differently for \(u_0 > 0\) and \(u_0 < 0\), as Fig. 2.8(a) and Fig. 2.8(b) illustrate, respectively.

The difference of the respective special latitudes from \(\varphi_r\) and from each other is also present in the correctly linearized form of the dipole equations. From (2.13), substituting \(\dot{\lambda} = u/\cos \varphi \simeq u/\cos \varphi_r (1 + \tan \varphi_r \gamma)\) for the angular velocity, one obtains

\[
y_{\pm} = -\frac{\sin \varphi_r}{(1/u_0)\gamma \cos^2 \varphi_r + 1/\cos \varphi_r} \tag{2.24}\]

for the positions of the uniform eastward and the uniform westward propagation with constant latitudinal velocity \(u_0\) (having positive and negative sign in the two cases, respectively).

The existence of the special latitudes \(\varphi_\pm \neq \varphi_r\) under the full spherical treatment (and that of \(y_{\pm} \neq 0\) under the correctly linearized treatment) implies in itself a qualitative breakdown of the \(\beta\)-plane approximation via the clear spatial separation of the eastward and the westward uniform propagation which both belong to \(\varphi_0 = \varphi_r\) under the \(\beta\)-plane treatment. Although this separation is small, it leads to the strongest consequences exactly for trajectories moving near \(\varphi_r\) (for example, initiated in between \(\varphi_+\) and \(\varphi_-\)) where we would naively expect the \(\beta\)-plane approximation to work the best.

### 2.3.3 Differences in the trajectories [T3]

In this section we compare dipole trajectories (i.e., trajectories corresponding to the \(\Gamma', a', D' \rightarrow 0\) limit) obtained numerically from the exact spherical (2.2), the correctly linearized (2.15) and the \(\beta\)-plane (2.8) equations. Note that these equations represent three different \textit{dynamics}, one of which [(2.2)] is considered to be “true” with the other two approximating it. The question is how the latter ones perform in different situations. For obtaining the answer, we fix the initial
meridional velocity component to be \( v_0 = 0 \), choose an approximately geostrophic velocity modulus \( |u_0| \ll 1 \) (see the discussion at the end of Section 2.3.1), and vary systematically the initial latitude \( \varphi_0 \). We discuss eastward and westward initialized velocities separately. We choose \( \gamma = 1 \) throughout our investigations without loss of generality, see [1].

\[
\begin{align*}
\text{(a) } & \varphi_r = 0.65, \ u_0 = 0.025, \ \varphi_0 = 0.64 \\
& \varphi_r = 0.65, \ u_0 = 0.025, \ \varphi_0 = 0.655 \\
& \varphi_r = 0.65, \ u_0 = 0.025, \ \varphi_0 = 0.7 \\
& \varphi_r = 1.1, \ u_0 = 0.0025, \ \varphi_0 = 1.07 
\end{align*}
\]

\( t_{\text{max}} = 160 \) \( t_{\text{max}} = 160 \) \( t_{\text{max}} = 900 \) \( t_{\text{max}} = 900 \)

Figure 2.9: Eastward initiated (wobbling) dipole trajectories in a \((\lambda, \varphi)\) representation, under full spherical (black continuous line), correctly linearized (green dashed line) and \( \beta \)-plane (magenta short dashed line) treatment. The special latitude \( \varphi_+ \) is marked by a red dot-dashed line. Initial conditions: \( \lambda_0 = 0 \), \( v_0 = 0 \), and \( u_0 \) and \( \varphi_0 \) are indicated in panels (a)-(d). The length \( t_{\text{max}} \) of the trajectories in dimensionless time units, and the value of \( \varphi_r \) (marked by a gray dot-dot-dashed line) are also indicated. \( \gamma = 1 \). We conclude that the \( \beta \)-plane approximation is qualitatively wrong near \( \varphi_r \).

In Fig. 2.9 we explore the behavior of trajectories initiated with an eastward velocity, i.e. with \( u_0 > 0 \). In Fig. 2.9(a) with an initial latitude \( \varphi_0 \in (\varphi_+, \varphi_r) \) the exact and the correctly approximated trajectories bend initially to the south, towards \( \varphi_+ \), in accordance with the previous Section. Meanwhile, the \( \beta \)-plane trajectory bends initially to the north, towards \( \varphi_r \). This indicates that the \( \beta \)-plane approximation does not reflect the relevance of \( \varphi_+ \), this special latitude being more relevant for the correct dynamical description than \( \varphi_r \) alone. Although the trajectory is very close to \( \varphi_r \) during the entire motion under the exact and either of the approximated treatments, the \( \beta \)-plane approximation leads to a qualitatively incorrect behavior in this situation. The message of Fig. 2.9(b), with \( \varphi_0 > \varphi_r \) but \( |\varphi_0 - \varphi_r| \ll 1 \), is similar. Although any trajectory bends initially to the south here, the amplitude of the eastward wobbling is an order of magnitude smaller in the \( \beta \)-plane approximation than under the exact and the correctly approximated treatments. Fig. 2.9(c) corresponds to an initial latitude being
relatively far away from $\varphi_\tau$ but inside the validity range of any linear approximation in the variable $\varphi$. In this case all three trajectories are qualitatively similar. This leads to the counterintuitive conclusion that the investigated motion should have a considerable part far away from $\varphi_\tau$ for a qualitative applicability of the $\beta$-plane approximation. In Fig. 2.9(d) we demonstrate that the breakdown of the $\beta$-plane approximation is present also for a different value of $\varphi_\tau$ and $u_0$, and for trajectories initiated on the southern side of $\varphi_+$ but not far away from it.

![Diagram](image)

Figure 2.10: Westward initiated (tumbling) dipole trajectories in a $(\lambda, \varphi)$ representation, under full spherical (black continuous line), correctly linearized (green dashed line) and $\beta$-plane (magenta short dashed line) treatment. The special latitude $\varphi_-$ is marked by a red dot-dashed line. Initial conditions: $\lambda_0 = 0$, $v_0 = 0$, and $u_0$ and $\varphi_0$ are indicated in panels (a), (b). The length $t_{\text{max}}$ of the trajectories in dimensionless time units, and the value of $\varphi_\tau$ (marked by a gray dot-dot-dashed line) are also indicated. $\gamma = 1$. Similarly as in Fig. 2.9, the $\beta$-plane approximation is found to perform poorly near $\varphi_\tau$.

Fig. 2.10 is similar to the previous one but shows trajectories initiated to the west. Fig. 2.10(a) provides an example for an initial latitude $\varphi_0 \in (\varphi_\tau, \varphi_-)$, leading to a completely different direction of evolution in the $\beta$-plane approximation compared to the other two trajectories. This is due only to the fact that the uniform westward propagation (taking place along $\varphi_\tau$ and $\varphi_-$ in the $\beta$-plane approximation and in the full spherical description, respectively) is a separatrix of southern type and northern type westward tumblings, see Fig. 2.8(b) and the related discussion. We note again, however, that trajectories initiated close to $\varphi_\tau$, as in the current setting, should be described correctly at least in the beginning when treated under any consistent first order approximation around $\varphi_\tau$, and this is not the case for the $\beta$-plane approximation. Fig. 2.10(b) exhibits an initial condition with $\varphi_0 < \varphi_\tau$ but $|\varphi_0 - \varphi_\tau| \ll 1$. In this setting we could expect a large error for the $\beta$-plane approximation, based on our experience for trajectories moving near $\varphi_\tau$ (see Fig. 2.9). Now, however, the trajectories spend a long time far away from $\varphi_\tau$ (but still in the traditional validity range of linearized approximations). As a result, the $\beta$-plane treatment does not perform much worse than the correctly linearized one.
Figure 2.11: Phase space portraits of the dipole dynamics under (a) full spherical and (b) β-plane treatment. \( \alpha = \arctan(v/u) \) if \( u > 0 \), \( \alpha = \arctan(v/u) + \pi \) if \( u < 0 \) and \( v > 0 \), and \( \alpha = \arctan(v/u) - \pi \) if \( u < 0 \) and \( v \leq 0 \). Trajectories are shown in the \((\alpha, \varphi)\) plane corresponding to several initial latitudes: \( \varphi_0 \in \{0.25, 0.3, \ldots, 0.6\} \) in red color (continuous lines) and \( \varphi_0 \in \{0.7, 0.75, \ldots, 1.05\} \) in blue color (dashed lines). The further initial conditions are \( u_0 = 0.025, v_0 = 0 \) and \( \lambda_0 = 0 \). \( \varphi_r = 0.65 \) is indicated by a gray dot-dot-dashed line. Uniform eastward [westward] propagation is marked by a filled [open] black circle at the position \((0, \varphi_+)\) \([-\pi, \varphi_-]\) in panel (a) and \((0, \varphi_r)\) \([-\pi, \varphi_r]\) in panel (b). Note the symmetry breaking of panel (a) compared to panel (b). \( \gamma = 1 \).

In order to gain a global view, we also show phase space portraits in Fig. 2.11 for the full spherical and for the β-plane treatment in one particular setting. We focus on the \( \varphi \) dynamics and introduce a new variable \( \alpha \), defined in the figure caption, for characterizing the orientation of the dipole. The special latitude \( \varphi_+ \) \([\varphi_-]\) corresponds to a stable [unstable] fixed point of this dynamics (denoted by a filled [open] circle). Westward tumbling trajectories are seen as lines stretching through all values of \( \alpha \) (representing “rotation”), and wobbling trajectories are seen as closed lines around the stable fixed point (representing “oscillation”). When monitored in time, all trajectories are traced out in a counterclockwise direction. Initiation is eastward (i.e., with \( \alpha = 0 \)), and continuous red [dashed blue] trajectories correspond to an initial value of \( \varphi \) below [above] \( \varphi_+ \) \([\varphi_-]\). We observe that the phase space of the β-plane treatment [in panel (b) of Fig. 2.11] is symmetric to the lines \( \alpha = 0 \) and \( \varphi = \varphi_r \) which formally coincides in this case with \( \varphi_+ = \varphi_- \). As a consequence of the latter symmetry, fixed points are found on the latitude \( \varphi = \varphi_r \), and trajectories initiated symmetrically in \( \varphi \) on the two sides of this latitude coincide. The symmetry to the line \( \varphi = \varphi_r \) breaks down under the full spherical treatment [in panel (a) of Fig. 2.11]: even the fixed points happen to be on different sides of the latitude \( \varphi = \varphi_r \), and the coincidence of trajectories initiated symmetrically to the this latitude does not hold any more. This figure illustrates in a pictorial way how different the two dynamics are.

So far we have only included examples for dipoles, having had mathematical discussions
Figure 2.12: Center-of-mass trajectories of finite-sized vortex pairs in a $(\lambda, \varphi)$ representation, under full spherical (black continuous line) and $\beta$-plane (magenta short dashed line) treatment. Initial conditions and parameters: $\lambda_0 = 0$, $v_0 = 0$; $D' = a' = 0.1$; $\varphi_0$, the sign of $u_0$ and $\Gamma'$ are indicated in the panels. The length $t_{\text{max}}$ of the trajectories in dimensionless time units, and the value of $\varphi_1$ (marked by a gray dot-dot-dashed line) are also indicated. $|u_0|$ is approximately 0.025 (a) and 0.0025 (b). We conclude that the $\beta$-plane approximation exhibits the same qualitative errors for finite-sized vortex pairs as for dipoles.

only for this limiting case of vortex pairs. Nevertheless, the basic findings hold for finite-sized vortex pairs, as illustrated in Figs. 2.12(a) and 2.12(b), exhibiting $\beta$-plane and full spherical trajectories with initial conditions corresponding to those in Figs. 2.9(a) and 2.10(a), respectively.\(^4\) In both plots for finite-sized vortex pairs (of distance $D' = 0.1$) one can observe exactly the same qualitative behavior for the center-of-mass trajectories as that in the corresponding plots for dipoles.

Based on the results of this Section (i.e., Section 2.3.3), we can say that there is a strong difference, on the order of the whole latitudinal extension of the trajectories, between $\beta$-plane and full spherical treatments for any vortex pair staying in a range $|\varphi - \varphi_1| < 0.05$ during the motion (which is not the case under a consistently linearized treatment). The difference is much weaker further away where the $\beta$-plane approximation appears to be reasonable, in a qualitative sense at least, regardless of the fact that linearizations have larger errors farther from the reference latitude. As the $\beta$-plane approximation can be considered to be valid within a range of $\varphi$ about 0.15 at most, we can say that the $\beta$-plane approximation gives qualitatively incorrect results in the middle one third of its validity range around the investigated reference latitude $\varphi_1$.

\(^4\)The derivation of any consistently approximated equations of motion around $\varphi_1$ for a finite-sized vortex pair with linear modulation is ill-defined and is therefore beyond the scope of the present paper. The reason is the unnecessary but traditional restriction to small longitudinal coordinate differences, see Footnote 6.
2.3.4 Deviations in the advection of passive tracers in the field of finite-sized vortex pairs [T4]

We now turn to the investigation of the $\beta$-plane approximation in the dynamics of passive tracers that are advected in the velocity field determined by the moving vortices. To this end, it is necessary to deal with finite-sized pairs instead of dipoles. We now only compare the full spherical and the $\beta$-plane dynamics for the sake of simplicity. The finite-sized vortex pair dynamics is described by the equations of motion (1.33) under the full spherical and (1.39) under the $\beta$-plane treatment. The equations of motion for the position of a passive tracer is given by (2.11). Under the $\beta$-plane treatment of the modulation of the vortex circulations \[85\] the tracers’ equations of motion read as

\[
\frac{dx}{dt} = -\frac{1}{2\pi} \sum_{j=1}^{2} \left( \Gamma' - \beta' a'^2 \pi y_j \right) \left( y - y_j \right) \sqrt{\left( x - x_j \right)^2 + \left( y - y_j \right)^2}, \tag{2.25a}
\]

\[
\frac{dy}{dt} = \frac{1}{2\pi} \sum_{j=1}^{2} \left( \Gamma' - \beta' a'^2 \pi y_j \right) \left( x - x_j \right) \sqrt{\left( x - x_j \right)^2 + \left( y - y_j \right)^2}. \tag{2.25b}
\]

As in Section 2.2, the observation region is a neighborhood of the center of mass of the vortex pair, taken with a reasonable radius $\rho$, called the escape radius. We initiate particles in this region, and concentrate on those particles that exhibit a chaotic behavior. We are interested in the associated (space-filling or fractal) patterns, and also ask if the particles leave the observation region. As pointed out in Section 2.2, in the close vicinity of the position of any point vortex the velocity field of the particular vortex dominates any other velocity contribution, and the particles exhibit a regular circulatory motion around that vortex: these subregions are the vortex cores \[120, 121, 122\]. These are, however, not interesting from our current point of view. Farther away from the vortices, but still inside the escape radius $\rho$, there is a subregion that is weakly affected by the vortices’ velocity field. Particles in this subregion escape the observation region soon and without exhibiting any fractal pattern, they are thus also not relevant. Regular and confined motion occurring outside the vortex cores could be relevant, but this phenomenon is rather rare in our experience. What remains of interest are only the particles subject to chaoticitity, found to be typical in Section 2.2.

For finding such passive tracers, the best choice is, following the considerations of Section 2.2, to put them initially in-between the vortices. A tracer is then either confined to an easily recognizable vortex core, or behaves in a chaotic way, contributing to the interesting
patterns. We remind the reader that we consider the chaotic advection open if the tracers are observed to be left in the wake of the vortex pair, otherwise we consider it closed. The patterns and the openness can easily be investigated by simply plotting the positions of the tracers in various time instants. Using this algorithm, a “phase transition” was found in 2.2 in the advection pattern between open and closed “phases” as a function of the initial latitude $\varphi_0$ of the center of mass of the vortex pair. A similar “phase transition” is known to exist under the $\beta$-plane treatment [85].

A more delicate subject is the characterization of the escape in the open advection. For this purpose, we follow again what is outlined in 2.2: we fill a larger subregion of the observation region by tracers, and we calculate the escape time $\tau$, the time needed to leave the circle of radius $\rho$ (i.e., the escape radius) for each tracer. Plotting the escape time at the initial position of each particular tracer draws out the stable manifold of a chaotic saddle [68, 69, 29] as ridges with higher values, organized into fractal filaments, in a sea of low values of the escape time. The higher are the values found near the ridges, the longer is the escape process of the tracers. The dominant part of the escape time probability density function is always exponential with an exponent $\kappa$ called the escape rate [68, 69, 29], as already introduced in Section 1.1.4. The escape rate characterizes the long term rate of the depletion of the observation region by tracers.

Our aim is to point out important alterations of the $\beta$-plane approximation from the full spherical dynamics in the above aspects of the advection. It is not a surprise to obtain qualitatively different advective properties when the vortex trajectories are also qualitatively different under the two treatments. It adds, however, a new insight into the poorness of the $\beta$-plane approximation if we find strong differences in the advective properties when the vortex trajectories are rather similar.

Such a case is shown in Fig. 2.13 where the vortex initial conditions correspond to those in Fig. 2.10(b). Here the advection pattern is open and consists of lobes, formed after every time period of the $\varphi$ dynamics, and migrating to the north in the farther wake. In panel (a) the positions of the tracers are shown shortly after the initialization, whereas in panel (b) they are shown later on, when they even approach the North Pole in the full spherical dynamics. Although the $\beta$-plane tracer dynamics is not expected to work far away from $\varphi_r$, the way in which the tracers leave the vicinity of the vortices and the beginning of their migration is restricted to a region $|\varphi - \varphi_r| \ll 1$, and thus allows for a fair comparison of the two treatments. Our first observation is that the particular positions and geometry of the
lobes are rather different. Apart from this, the main difference is a slower migration of the β-plane wake to the north. This is related to the more southern position (which is $\varphi_r$) of the vortex motion separatrix in the β-plane approximation compared to that in the full spherical dynamics. Under the β-plane treatment, the vortex center-of-mass trajectory thus stays closer to its separatrix than under the full spherical treatment, which results in a faster movement to the west. This implies that the tracers, left in the wake, get farther away from the vortex pair in a particular time interval, where they are influenced less by the velocity field of the pair. This mechanism, providing a background for our experience, is considered to be general when comparing advection under the β-plane and the full spherical treatment. We emphasize, however, that the similarity of the β-plane and the full spherical vortex trajectories is by far stronger than the similarity of the advection patterns.

Figure 2.13: The positions of $N \approx 17500$ tracers (dots) and the two vortices (symbols) at the time instants $t$ indicated in panels (a), (b). The tracers are initiated at $t = 0$ in between the elements of the vortex pair. Blue (red) tracers correspond to full spherical (β-plane) treatment. The center-of-mass trajectory of the vortex pair is also shown up to the time instant $t$ as a black (magenta) continuous line, corresponding to full spherical (β-plane) treatment. Vortex pair initial conditions: $\lambda_0 = 0$, $\varphi_0 = 1.07$, $v_0 = 0$ and $u_0 < 0$. $\varphi_r = 1.1$ is marked by a gray dot-dot-dashed line. Vortex pair parameters: $D' = a' = 0.1$, and $\Gamma' = \pi/2 \times 10^{-3}$. Note that the particular patterns are quite different under the two treatments.

In fact, one can find initial conditions for the vortex pair when the similarity in the vortex trajectories is of the same degree as before, but the advection patterns are also similar under the two treatments. This situation is illustrated in Fig. 2.14, exhibiting closed advection patterns. It is thus hard or maybe even impossible to predict the agreement of the advection pattern between the two treatments via only the visual inspection of the vortex trajectories.

The background mechanism described above leads also to enhanced escape under the β-plane treatment. The vortex pair’s faster movement to the west under the β-plane treatment involves a smaller importance of self-intersections of its trajectory. Under the full spherical
The positions of $N \approx 17500$ tracers (dots) and the two vortices (symbols) at the time instant $t = 1200$. The tracers are initiated at $t = 0$ in between the elements of the vortex pair. Blue (red) tracers correspond to full spherical ($\beta$-plane) treatment. The center-of-mass trajectory of the vortex pair is also shown up to the time instant $t$ as a black (magenta) continuous line, corresponding to full spherical ($\beta$-plane) treatment. Vortex pair initial conditions: $\lambda_0 = 0$, $\varphi_0 = 0.99$, $v_0 = 0$ and $u_0 < 0$. $\varphi_r = 1.1$ is marked by a gray dot-dot-dashed line. Vortex pair parameters: $D' = \alpha' = 0.1$, and $\Gamma' = \pi/2 \times 10^{-3}$. Unlike in Fig. 2.13, the patterns are rather similar under the two treatments.

treatment, the close revisiting of the same spatial positions by the vortex pair make the escape of tracers more difficult. (This is also the mechanism responsible for the closure of the advection when initiating the vortex pair trajectory farther away from the separatrices.)

The background mechanism of the enhanced escape is valid only for the southern side of the separatrices in its presented form. A similar mechanism works on the northern side but with opposite result.

A direct numerical comparison of the escape times of the tracers confirms that the $\beta$-plane treatment is characterized by lower escape times than the full spherical one, as seen in Fig. 2.15,
corresponding to the same vortex pair setting as that of Fig. 2.13, via the darker colors of panel (b) than those of panel (a). The escape rates are found to be $\kappa = 0.0011$ and $\kappa = 0.0036$ for the $\beta$-plane approximation and for the full spherical treatment, respectively. We emphasize that the escape properties originate mainly in the dynamics taking place in the vicinity of the vortices (where linearization is expected to be applicable), and are therefore not affected by the behavior of the tracers far away from $\varphi_r$ (where linearization would be incorrect in itself). The reason for this is that the migration of the particles from the vicinity of the vortices out to the farther wake is a regular, fast and mostly uniform process, whereas they enter this process only after their chaotic wandering in the vicinity of the vortex pair, in a rate dictated by the chaotic saddle located in this region. A factor more than 3 appearing in an exponent (the escape rate) characterizing material transport and the rather different patterns of the escape time distributions of Fig. 2.15 are perhaps the most striking effects to which the use of a $\beta$-plane approximation instead of the full spherical treatment can lead to. This is even more surprising when taking into account that the time period $T_p$ of the $\varphi$ dynamics is $T_p = 165.6$ and $T_p = 141.2$ under the $\beta$-plane approximation and the full spherical treatment, respectively. These values also support the observation that the Eulerian properties of the $\beta$-plane and the full spherical flows are rather similar.

An example in Fig. 2.16 shows that one can even find cases in which the full spherical...
treatment already exhibits closed advection when the $\beta$-plane advection is still open. This observation is particularly surprising given that the trajectories are now far away from the separatrices during the entire motion. Although we are a little bit too far from $\varphi_r$ in this particular example for the applicability of any linear approximation in $\varphi$, our experience shows the possibility of the existence of such a behavior. Our finding is interpreted as a consequence of the following considerable difference: the value of the critical point (in the initial latitude $\varphi_0$ of the vortex pair) describing the “phase transition” between the open and the closed “phases” is estimated to be approximately $\varphi_{0c} \approx 0.38$ under the $\beta$-plane treatment and $\varphi_{0c} \approx 0.41$ under the full spherical treatment. This shift may be regarded as a further indicator for the inappropriateness of the $\beta$-plane approximation.

2.4 Discussion

In Section 2.2 we studied the advection of passive tracers in the velocity field of modulated vortex pairs. The advection dynamics has been found to be typically chaotic, which would not be the case on a non-rotating sphere. We have pointed out a transition from closed to open chaotic advection in the field of vortex pairs when the initial latitude of the pair becomes closer to the reference latitude than a threshold distance. On asymptotically long times, however, even the open advection becomes converted into a closed one since the vortex pair periodically reenters its own wake. The chaotic saddle governing the open dynamics gradually becomes space filling. Such a time scale separation is expected to occur in any chaotic advection problem where the flow is locally open but globally closed. Relevant situations are provided by the advection by planar vortex pairs moving inside a disk [123], and when understanding mixing in an infinite array of cylinders, as defined in [124], based on the elementary saddle formed in the wake of a single cylinder [68].

In another vein, our approach indicates that previous studies of point vortices on a sphere, which were mainly devoted to the stability of different vortex configurations (see e.g. [77, 79, 80, 125, 126]), might be worth extending to dynamical cases. Both these and the vortex pair dynamics understood on general curved surfaces [72, 73] would be highly interesting to investigate in the presence of rotation.

In Section 2.3 our aim was the investigation of the validity of the well-known and widely used approximation in geophysical fluid dynamics describing cases with a small scale latitudi-
nal motion of fluid elements, the so-called $\beta$-plane approximation. In the traditional $\beta$-plane approximation one replaces the sinusoidal dependence of the Coriolis parameter (the locally vertical component of the angular velocity of the planet) by its linearly approximated form, while the geometry is assumed to be fully planar. We have pointed out that the latter choice neglects even linear metric terms originating from the spherical geometry. The discrepancy in the order of the approximation for the Coriolis parameter and for the geometry is a mathematical inconsistency of any traditional $\beta$-plane approximation.

For a point vortex dipole, we have shown that the traditionally derived $\beta$-plane equations of motion do not contain certain terms that inevitably appear in a consistent first-order approximation of the full spherical equations around the origin of the linearization. Interestingly, the effect of these terms proved to be the most important near this latitude. In particular, under full spherical treatment the special latitudes $\varphi_{\pm}$ which correspond to eastward and westward uniform propagation have been found to generally differ from the origin of the linearization, in contrast to the case of the $\beta$-plane treatment.

Chaotic advection in the velocity field of modulated vortex pairs has also been compared under the $\beta$-plane and the full spherical treatment. We have found a latitudinal shifting effect in the advection under the full spherical treatment when compared to the $\beta$-plane treatment. The importance of the shift cannot be estimated from the pure observation of the vortex trajectories. In particular, this shift can be reflected in an enhanced escape of fluid elements from the neighborhood of the vortex pair, including cases when the open character of the advection, obtained under the $\beta$-plane treatment, contradicts the closed character of the full spherical treatment.

It is not excluded that one can find an appropriately chosen planar projection of the spherical coordinates and a corresponding nonlinear rescaling of the velocity component (as Gill and others did [127, 128, 129] in the basic hydrodynamical context), in which the $\beta$-plane approximation of our vortex problem would appear without any further curvature terms. As we have, however, illustrated, in the widely used flat geometry approach the $\beta$-plane approximation is inconsistent, and leads to results qualitatively different from those of a consistent local linear approximation in which curvature terms appear unavoidably.

Although the $\beta$-plane approximation is, of course, not used today in the analysis and the simulation of real flows, it is a widespread tool in studying basic phenomena in environmental flows, both in theoretical and experimental works. In theoretical works ([130, 131]) the use of
the traditional $\beta$-plane approximation should be avoided and be replaced by mathematically consistent approximations. Similarly, the results of experiments with a topographic $\beta$-plane in a planar geometry ([84]) would have to be analyzed with a certain care when drawing conclusions for situations with a spherical geometry. In experiments applying vessels with a cylindrical geometry [obtaining a $\beta$-effect either by a sloping topography [132, 133, 134] or by an appropriate combination of the shape and the angular velocity of the vessel [135]] the results can be interpreted in a precise way only if a theoretical background is available that treats the cylindrical geometry consistently. Though the observations from such experiments are expected to be closer to the spherical phenomena, the different geometry of the sphere might still lead to significant deviations from these observations. We conjecture that the dynamics on a rotating sphere can be consistently approximated even in linear order by an experimental setting with a cylindrical geometry only if the parameters are adjusted appropriately. The exposition of this idea may be the subject of a future work.
Chapter 3

Snapshot attractors in a climatic context

3.1 The modification of the Lorenz 84 model

In climate science climate changes are of basic relevance. A climate change is most simply induced by a deterministic, smooth shift of parameters in time, without including any stochasticity [99, 18]. In this spirit, we modify the Lorenz 84 model by setting the mean value $F_0$ of the temperature contrast parameter to be time-dependent. In particular, $F_0(t)$ decreases linearly (according to an increase of the greenhouse gas content) by 2 units over $100T = 100$ years:

$$F_0(t) = \begin{cases} 
9.5 & \text{for } t \leq 0, \\
9.5 - \frac{2t}{100T} & \text{for } t > 0. 
\end{cases}$$

(3.1)

Climate change sets in at time $t = 0$. Before this time instant a stationary climate is present, governed by $F_0 = 9.5 = \text{constant}$. Then, in a period of 150 years, the mean value $F_0$ goes down to 6.5. The amplitude of annual oscillations is the same $A = 2$ at all times as in Lorenz’s model which has annual forcing only [101].

We note that a similar ramp, but without a seasonal cycle, was applied by Daron and Stainforth [99] to a simple coupled atmosphere-ocean model in which the atmosphere was also represented by Lorenz’s 1984 model [100]. In our terminology, they also investigated snapshot attractors and their natural measures, and investigated certain aspects of ergodicity (which they call ‘kairodicity’). Their qualitative findings are similar to ours, but our investigations also provide detailed statistical characterizations, which are explained from the point of view of statistical mechanics via quantitative analyses, leading to also offering measures for noner-
godicity and for climate change. Our setup also consists of a rampless regime before the onset of the climate change (i.e., the starting instant of the ramp), which also allows us to compare stationary and time-dependent climates.

The equations (1.40)-(1.41) with (3.1) are numerically solved by the classical 4th-order Runge-Kutta method with a fixed time-step $\Delta t = 0.005 \approx 6.85 \times 10^{-5}$ years. In order to generate the natural measure of the snapshot attractor at time $t$, we start a large number, $N = 10^6$, of trajectories distributed uniformly in a box $[-1.5, 3.5] \times [-2.5, 2.5] \times [-2.5, 2.5]$ at a negative time, $t_0 = -250$ years, monitor the full ensemble up to time $t$ (which can be either positive or negative), and generate a probability density function (pdf) over the phase space $(x, y, z)$. In what follows, this ensemble of trajectories will be used. Before the onset of the climate change, $t = 0$, the snapshot attractor coincides with a usual attractor (which corresponds to a hypothetic forcing with $F_0 = 9.5 = \text{constant eternally}$) since the forcing is periodic. The attractor and its natural measure is of course time-dependent in the climate change period, i.e., for $t > 0$. Numerically we shall investigate the snapshot attractor in the time interval $(-150, 150)$ in years. After 150 years the dynamics loses internal variability along the ramp (3.1), and we do not consider time instants before $-150$ years in order to provide a symmetric investigation.

### 3.2 The usual attractor of the stationary climate

Before investigating the snapshot attractor that occurs during the ramp of the modified Lorenz 84 model, we first understand the usual attractor of the periodically forced model [governed by the dynamics (1.40)-(1.41) with $F_0(t) = \text{constant}$] which corresponds to a stationary climate. In such a three-variable system forced periodically in time with period $T = 1$ year, there exists, at any fixed phase of the period, a unique (stroboscopic) attractor in the $(x, y, z)$ phase space. This can be generated, assuming that $F_0(t)$ is constant eternally, by a single long trajectory, after cutting out the initial transients. Fig. 3.1(a) shows the 3D chaotic attractor at midwinter ($t \mod T = T/4$). The choice of taking a single “day” (more precisely, a particular phase) of the year is only a technical detail and is only aimed to eliminate the effects of seasonality.

In order to obtain a planar view, a Poincaré surface of section can also be taken on the plane $z = 0$ with $\dot{z} > 0$ [100], and we restrict ourselves to this traditional choice. Points of the trajectory of Fig. 3.1(a) trace out in this section the pattern seen in Fig. 3.1(b). Note that due
Figure 3.1: Stroboscopic view at midwinter \((t \mod T = T/4)\) of the chaotic attractor of a stationary climate in the full phase space (the \(z\)-component is color-coded monotonically in the spectrum, reddish colors marking \(z > 2\)) (a) and on the \((x,y)\) plane (b). [The \(z = 0\) plane used to define the section of panel (b) is marked in panel (a) by a black line.] The mean temperature contrast parameter is \(F_0(t) \cong 9.5\) eternally. A single trajectory of length of \(10^6\) years is monitored and a transient segment of 5 years is cut out before plotting. For better visibility, we only included the first \(10^5\) years in panel (a).

To the periodicity of the forcing, the chaotic attractor is the same when viewed after integer multiples of \(T = 1\) year, and the stroboscopic picture both in 3D and on the planar section does not change [see [26], for an illustration within our particular system see also [15]]. Had we taken a different “day” of the year for generating this set, the same would be true, just with a different pattern.

In Fig. 3.1(b), we see clearly the characteristic filamentary, fractal pattern of chaotic attractors. The extension in both directions is larger than 2 units, which is an indication of the considerable internal variability of the dynamics. One particular feature is an “island” separated from the main body. It is located at about \(x = 0, y = 1.4\), and can be considered as a sign of extremal behavior associated with very weak westerlies and a below than average cyclonic activity. Most recently [136] has given a definition of climate sensitivity by comparing such internal variabilities of stationary climates.

Note that in this case exactly the same attractor can also be generated by an ensemble of trajectories started in the past and stopped at any time instant \(t\) with \(t \mod T = T/4\), after an initial transient time interval.
3.3 The snapshot attractor of the changing climate [T5]

3.3.1 Basic properties

As detailed in Section 3.1, the endpoints of an ensemble, initiated in the past, of \( N = 10^6 \) trajectories in the time instant \( t \) trace out the snapshot attractor belonging to that time instant. Its form depends, of course, on the particular choice of \( t \). After initiation, there is some time \( t_c \) needed to come sufficiently close to the attractor. This convergence is expected, as in any dissipative system, to be exponentially fast. As we shall see in Section 3.3.3, we estimate this time in the model to be \( t_c = 5 \) years.

![Figure 3.2: The 3D snapshot attractor in the phase space. Numerically, it is traced out by an ensemble of \( 10^6 \) realizations at the time instants \( t = 150.25 \) (a) and \( t = 150.75 \) (b) years, corresponding to midwinter and midsummer, respectively. The \( z \)-component is color-coded monotonically in the spectrum, reddish colors marking \( z > 2 \). (The \( z = 0 \) plane used to define a slice of the snapshot attractor in Fig. 3.3 is marked in both panels.)](image)

As one of our typical examples, the snapshot attractor obtained this way for the last year of the investigated period, year 150, is shown in the phase space in Fig. 3.2. The midwinter and midsummer instants belong to the left and the right panels, respectively. Both patterns are coil-like and are of the size of a few units in all three directions. The winter snapshot attractor appears, however, to be more extended. Besides illustrating how strong changes a snapshot attractor can undergo within a seasonal cycle, the figure shows that winter is much more active than summer, as also observed in the periodically forced Lorenz model [101, 15], i.e., under a stationary climate.

It is to be emphasized that the scattered dots appearing in both panels (but more typical for the summer case) are not signs of slowly converging points, having not yet reached the
attractor. As we will show in Section 3.3.3, taken at a different time instant (in year 50) for a better overview, scattered dots appear exactly in the same region even if different sets of initial conditions are chosen for the ensemble, and even the convergence time remains to be about 5 years. The scattered dots are thus signs of weakly occupied regions on both the midwinter and the midsummer snapshot attractors (and are consequences of the numerical use of a finite ensemble for the representation of the attractors).

In order to gain a clearer view into the structure of the snapshot attractors, it is worth taking intersections of the 3D snapshot attractors with a surface, thus generalizing Poincaré surfaces of section. Planar slices taken with the conditions $z = 0$, $\dot{z} > 0$ are well defined, and the object obtained this way we call the $(z = 0)$ slice of the 3D snapshot attractor.

The tableau of Fig. 3.3 shows midwinter slices of the snapshot attractors in the first quadrant. Similarly as before, a particular “day” of the year is chosen in order to give an impression of the long-term temporal evolution of the snapshot attractor, not influenced by seasonality. This choice (i.e., that the system is viewed stroboscopically with respect to the periodic component of the forcing) is merely technical. This sequence of plots reflects fractal structures as clear as on the chaotic attractor of Fig. 3.1(b), illustrating that they are the proper generalizations of this latter object. It also shows that on the same “day” of the years there are considerable changes in the variability over decades. A comparison of Figs. 3.1(b) and 3.3(a) leads to the conclusion that the “island” of special winter weather is still there after the onset of the climate change but its size becomes somewhat smaller, and the shrinking goes on such that a gap opens up eventually in the possible wind speeds. In Fig. 3.3(b) $x$ takes on values in $(0, 0.4)$ and $(0.6, 2.6)$ only. Parallel to this, the details of the filamentary structure of the bulk also change. The island is hardly visible in the next panel, and fully disappears by the 100th year after the onset of the climate change. No $x$ values are then realized below 0.4. In the remaining 50 years (Figs. 3.3(e)-3.3(f)) the leftmost branch of the attractor is moving towards the $y$ axis, making the forbidden region in the wind speed smaller. Besides a continuous structural deformation, the maximal extension of the attractor changes in $x$ and $y$ from 2.8 to 2.5 and from 2.5 to less than 2, respectively, during the whole climate change period. The evolution of the shape of the snapshot attractor in Fig. 3.3 is an indication of the change of the internal variability in a changing climate.
Figure 3.3: Midwinter slices of snapshot attractors in the plane of $z = 0$ (conditioned by $\dot{z} > 0$), generated with the ensemble of realizations of Fig. 3.2. The first item of the tableau belongs to the time instant $t = 25.25$ years (midwinter of year 25), the others, analogously, to the midwinter instants of years 50, 75, 100, 125, and 150. Panel (f) happens to be the slice of the 3D snapshot attractor shown in Fig. 3.2(a). Observe the significant dependence on time.

3.3.2 The natural measure of the snapshot attractor

The scattered dots in Fig. 3.2 clearly indicate that certain regions of the 3D snapshot attractor are visited much less probably than others. More systematically, defining a fine grid over the snapshot attractor, one can determine the number of points of the ensemble falling into the different boxes. This occupation probability can be considered, in the limit of a very fine grid
and large ensemble, to be the natural probability of the snapshot attractor. This measure itself over the whole coil of the 3D snapshot attractor changes with time.

Since it is hardly possible to visualize a probability distribution defined on a three-dimensional support, it is worth evaluating analogous occupation probabilities over a two-dimensional grid covering the planar slices of the snapshot attractors.

Figure 3.4: Natural measure on $z = 0$, $\dot{z} > 0$ slices of the snapshot attractor (generated with the ensemble of realizations of Fig. 3.2) in the midwinter point of years 25 (a), 50 (b), 88 (c) and 89 (d). For better visibility, the occupation numbers are truncated at 1500 in the bins, whose linear size is chosen to be 0.01. The number of points from which the distributions are generated in the slices of panels (a)-(c) and (d) is found to be 22167, 25611, 30142, and 53741, respectively. The distributions projected on the $x$ and $y$ axes appear in gray in the back planes. Panels (c) and (d) illustrate well that the time-dependent natural measure can rearrange completely within 1 year while its support remains approximately the same.

Fig. 3.4 exhibits a few numerical examples of such distributions. Note that they are all rather irregular, orders of magnitude differences can be present in neighboring boxes, which is
a strong indication for their being of fractal measures [26]. Due to the parameter shift, the natural measure turns out to change year by year even when sampled on the same “day” of the years. The first two panels belong to the slices of panels (a) and (b) of Fig. 3.3. The other two panels exhibit distributions corresponding to time instants being in-between panels (c) and (d) of Fig. 3.3. These instants are intentionally chosen to be separated by a single year in order to illustrate how dramatic differences can show up in the natural measure of the snapshot attractor in such a short time. It is remarkable that the support (the geometry of the snapshot attractor) hardly changes within the same time. In addition, the strong temporal dependence of the 3D snapshot attractor and its measure is also reflected by the fact that the number of points falling into a neighborhood of the \( z = 0 \) surface conditioned with \( \dot{z} > 0 \) (i.e., the slice) changes considerably over the period investigated, as the data in the caption of Fig. 3.4 indicate.

We emphasize that the existence of the natural measure of snapshot attractors is of central importance. As the example of Fig. 3.4 illustrates, this measure is changing with time, i.e., it gives a response to the forcing. Nevertheless, it is unique at any time instant. Numerically, the only condition for this is that the ensemble used for the generation of the probability measure is initiated in the past further back than a characteristic convergence time \( t_c \), but otherwise both the time instant \( t_0 \) of initiation and the choice of the ensemble (the shape of the volume in the phase space in which the points are distributed, and the density of the points within this volume) is arbitrary, as we will illustrate in Section 3.3.3. There we will also see that the convergence time \( t_c \) is on the order of 5 years, even if the ensemble is initiated after the onset of the climate change. The time \( t_c \) can be interpreted as the time needed for a distribution to reach and become spread on the snapshot attractor\(^1\). It is only this snapshot concept that makes possible a sound evaluation of statistical quantities, like averages and variances, and this can be done at any instant of time in a changing climate. We are not aware of any other tool that could be used for this purpose, i.e., for characterizing internal variability. (Ensemble runs of short duration, i.e., ones lacking the convergence to the natural measure of the snapshot attractor, can carry information that is very specific to the particular initial conditions, and also, of course, to the interval \([t_0, t]\) (where \( t - t_0 < t_c \)).)

The average of a quantity \( \varphi \) taken with respect to the natural measure of the snapshot attractor, belonging to a time instant \( t \), shall be denoted as \( A_\mu(\varphi(t)) \). In formal notation, if

\(^1\)The whole convergence is expected to be exponentially fast and \( t_c \) is related in mathematical terms to the second discrete eigenvalue of the time evolution operator [30, 31].
\( \mu(t) \) is the time-dependent natural measure at time \( t \), the ensemble average of \( \varphi \) is obtained as

\[
A_\mu(\varphi(t)) = \int \varphi \, d\mu(t).
\]  

(3.2)

In order to obtain an overview on how the snapshot attractor and its natural measure \( \mu(t) \) evolve in time, in Fig. 3.5 we show its projection onto the variable \( y \) as a function of time. In particular, we numerically approximate the natural measure by a histogram, coded by the brightness, in every considered time instant. We also mark the time evolution of the numerically obtained ensemble average \( A_\mu(y(t)) \) of the variable \( y \) by a “continuous” line. (As earlier, we wish to eliminate seasonal effects, we thus take only one time instant from every time period \( T \), by which we obtain a stroboscopic map in the stationary climate, and a similar construction in the changing climate.)

![Figure 3.5: Histogram over \( y \) (coded by the brightness), and the ensemble average \( A_\mu(y(t)) \) (red “continuous” line) as a function of time, but restricted to \( t \mod T = T/4 \) time instants (i.e., to midwinters), calculated over a numerical ensemble of \( 10^6 \) trajectories. Note that the amplitudes exhibit a rough time-dependence, but the support exhibits a smooth time-dependence. The marked time instants (by \( \times \) marks) along the \( t \) axis correspond to the time instants considered in Fig. 3.4.](image)

The stationary climate appears very clearly as a time-independent pattern in Fig. 3.5 before \( t = 0 \). For \( t > 0 \) a very complicated time evolution can be seen. Besides of a smooth change of the support of the natural measure, the density on this support changes dramatically all the time (as discussed in the context of Fig. 3.4), and this underlies the irregular time-dependence of \( A_\mu(y(t)) \).
Figure 3.6: Initial condition-independence I. Panels (a)-(b): The set of trajectory endpoints in the full phase space of two ensembles (of $10^6$ points each) at the midsummer time instant $t$ of year 150, initiated in the box $[-1.5, 3.5] \times [-2.5, 0] \times [-2.5, 2.5]$ at time $t_0 = 10.75$ years (a) and $[-1.5, 3.5] \times [0, 2.5] \times [-2.5, 2.5]$ at time $t_0 = 30.75$ years (b). The $z$-component is color-coded monotonically in the spectrum. Panels (c) and (d): The numerically determined distribution in bins of linear size 0.01 on the $z = 0, \dot{z} > 0$ slices of the ensembles of panels (a) and (b), respectively. For better visibility, the occupation numbers are truncated at 1500. The distributions projected on the $x$ and $y$ axes appear in gray in the back planes. [The $z = 0$ plane used to define a slice of the snapshot attractor is marked in panels (a) and (b).] Note that not even the times of initiation are identical. As, however, both time evolutions are longer than $t_c$, the distributions are found to be identical.

3.3.3 The characteristic time for convergence

A basic property of any dissipative system is that the long-term dynamics is independent of initial conditions. This is why the underlying object can be called an attractor: it attracts all trajectories within a large region of the phase space, the basin of attraction. In spite of the nonautonomous character, this holds also for snapshot attractors [23, 89]. Here we illustrate the attracting property for our model. To this end, we take two disjoint large boxes at different initialization times $t_0$ in which many initial conditions are distributed uniformly, and monitor both ensembles up to a given time instant $t$ in the set of equations (1.40)-(1.41) with (3.1).
numerics does not indicate any simultaneous coexistence of different snapshot attractors.) We show results for the midsummer of year 50 in Fig. 3.6: panels (a) and (b) present the position of the members of the two ensembles in 3D. Hardly any difference can be observed, indeed.

A perhaps even more relevant (but of course related) property is that the natural measure should also be independent of the initial conditions. For a clear visualization, as in the previous Subsections, we take the $z = 0$, $\dot{z} > 0$ slices of the attractors. The occupation numbers in grid cells over the $(x, y)$ plane are plotted for the two ensembles in panels (c) and (d) of Fig. 3.6. The agreement is striking again. It is because of this agreement that we are allowed to speak about the natural measure of the snapshot attractor.

![Figure 3.7](image_url)

Figure 3.7: Initial condition-independence II. (a) The difference in the variable $x$ and (b) the distance in the phase space $(x, y, z)$ of the ensemble averages taken with respect to the newly initiated ensembles (those of Fig. 3.6) from the ensemble average taken with respect to the original ensemble (that of Fig. 3.2) as a “continuous” function of time (restricted to midsummer time instants), plotted in the first fifteen years after the new initializations. Red and blue correspond to the ensembles of panels (a) and (b) of Fig. 3.6, respectively. The convergence time $t_c$ is found to be about 5 years. Panel (c) is the same as panel (b) on a semi-logarithmic scale. The dashed lines mark exponential fits to years $10, 11, 12, 13$ and to years $30, 31, 32, 33$, yielding relaxation times (i.e., the reciprocals of the exponents) $\tau = 0.5$ years and $\tau = 0.8$ years, respectively.

This observation is further reinforced by Fig. 3.7 which illustrates the temporal convergence of the newly initiated ensembles (those of Fig. 3.6) to the original ensemble (that of Fig. 3.2). The plots show clearly that the ensemble averages taken with respect to both newly initiated
ensembles converge within $t_c = 5$ years to the ensemble average taken with respect to the original ensemble. The fluctuating, nonzero values of the differences that are observable after the convergence time $t_c$ indicate only the uncertainty of the numerical representation of the natural measure, this is why the newly initiated ensembles can be regarded to be independent of their initial conditions after $t_c$ has passed from their initializations. We find in Fig. 3.7(c) that the temporal convergence is, as expected, approximately exponential. The fitted relaxation times (i.e., the reciprocals of the exponents), $\tau = 0.5$ years and $\tau = 0.8$ years, are in reasonable agreement with $t_c = 5$ years for our numerical representation of the natural measure, and also with each other. The slight difference between the two fitted values may originate in the ensembles’ rather far initialization from the natural measure (as a consequence of which the relaxation is not exactly exponential at the beginning). It may also indicate, however, that the relaxation time $\tau$ can have a slight dependence on time.

The findings illustrated by Figs. 3.6 and 3.7 illuminate the distinguishing relevance of the snapshot concept. The geometry and the natural measure of snapshot attractors are free of any subjective choice, like e.g. the region over which and the distribution according to which the initial conditions are taken. There exists (for $t - t_0 > t_c$) an objective probability measure belonging to any given time instant over which any statistics of interest can be evaluated.

3.4 The difference between single-realization temporal statistics and ensemble statistics

3.4.1 A visual illustration

It is important to illustrate first the contrast between the amount of information one can extract from the single-realization and the snapshot pictures in cases with shifting parameters. Fig. 3.8 shows the stroboscopically generated points of a single trajectory [i.e., a single realization of the dynamics (1.40)-(1.41) with (3.1)] initiated on the chaotic attractor of the $F_0 = 9.5$ stationary climate at midsummer in year -1. From $t = 0$ on, the ramp in (3.1) is active, so the trajectory is subject to a continuous shift of the temperature contrast parameter. Storing the values of the single trajectory at the midwinter and the midsummer time instants of each year, the sets of points shown in panels (a) and (b), respectively, of Fig. 3.8 are obtained. Note that such data sets are the ones corresponding to the traditional treatment of the climate and of its change,
see e.g. [8].

Figure 3.8: Midwinter (a) and midsummer (b) points of a 151 years long trajectory started on the chaotic attractor of the $F_0 = 9.5$ stationary climate with $(x_0 \approx 0.55, y_0 \approx 0.47, z_0 \approx 1.29)$ at $t = -0.25$ years. The $z$-component is color-coded monotonically in the spectrum.

In spite of their conceptual difference, it is worth comparing these single-realization data sets with the results of the snapshot approach, the attractors of Fig. 3.2. The difference is striking: not only that no structure is traced out in the single-realization sets, in addition, when overlaying the corresponding plots, one observes that some points of Fig. 3.8(a) lie outside the corresponding snapshot attractor of Fig. 3.2(a). There are so few points (151 in number) that no reliable statistics can be based on them. Moreover, there is practically no point of the single-realization set with its $z$-coordinate being close to zero, and hence no analog of the section on the $(x, y)$ plane of Fig. 3.1 can be generated, in strong contrast with the case of a stationary climate. Note that the selected points (one from every year) of a single trajectory can be considered to be the union over all different years of the sets consisting of a single point each, with this point being on the snapshot attractor of the particular year. It is thus not a surprise that the patterns of the snapshot attractors become washed out by forming a union over all snapshot attractors via a single trajectory.

3.4.2 A simple comparison of single-realization temporal statistics and ensemble statistics

We now turn to the investigation of snapshot-attractor-based statistics (i.e., ensemble statistics) and temporal statistics (calculated along single realizations). As pointed out in Section 1.3.2, the former carry a relevant probabilistic interpretation, unlike the latter which is, however, a
central concept in the traditional definition of climate (see also Section 1.1.3).

For characterizing temporal averages, we have to choose the length of the time interval (in what follows: the time window) over which the average is taken. Let us denote this by \( \tau \). We consider it to be an essential part of the problem that this window cannot be taken arbitrarily large in practice. Firstly, if one is interested, e.g., in the properties of a climate change period, \( \tau \) is limited by the time elapsed since the onset of the climate change, otherwise properties of stationary and changing climates would be mixed up. Secondly, we shall see (Section 3.4.5) that the convergence of the time average in \( \tau \) to a limiting value is rather slow (without any characteristic time), thus no practically accessible time window exists which would represent an infinite window length faithfully, not even in an ergodic dynamics. In this Section, we ask the question if one can find a criterion for distinguishing stationary and changing climates (corresponding to ergodic and nonergodic cases, respectively, as anticipated in Sections 1.3.1 and 1.3.2 and shall be discussed in detail here) even without carrying out the \( \tau \to \infty \) limit. Therefore, in what follows we use finite window lengths.

In case of a forced dynamics, it should also be decided which time instant the temporal average is ordered to. In this and the next Section this will be chosen to be the midpoint of the time window, but in Section 3.5 we shall see an approach in which the endpoint is more appropriate.

Let \( \varphi(t) \) denote a time series emerging from a single initial position \( x_0 = (x_0, y_0, z_0) \) of the phase space corresponding to some initial time instant \( t_0 \). That is, \( \varphi(t) \) is a single member of the ensemble, or a single realization. The time average \( A_\tau \) of \( \varphi(t) \) taken over the time window of length \( \tau \), ordered to time \( t \) in the midpoint convention (\( t \) shall be called the ‘time of observation’), reads as:

\[
A_\tau(\varphi(t)) = \frac{1}{\tau} \int_{t-\tau/2}^{t+\tau/2} \varphi(t')dt'.
\]

(3.3)

Any \( A_\tau \) average is, by definition, a property of the particular realization emerging from \( x_0 \) in \( t_0 \), and we shall call it a single-realization temporal (SRT) average. For our stroboscopic-type investigation, eliminating seasonal effects, we replace the integral in (3.3) by a yearly sum\(^2\).

\(^2\)In particular, we decided to use the convention that \( \tau = 0 \) corresponds to considering the value of \( \varphi \) at the time instant \( t \). Accordingly, \( A_\tau(\varphi(t)) \) is computed as

\[
A_\tau(\varphi(t)) = \frac{1}{\tau+1} \sum_{t'=t-\tau/2}^{t+\tau/2} \varphi(t'),
\]

(3.4)

where \( t' \) is an index of the year and is hence an integer, and \( \tau \) is an even number. This implies that the number
For a first impression, we present in Fig. 3.9 numerically obtained averages. The black line corresponds to the ensemble average of the variable $x$ at midwinters, similarly to what is shown in Fig. 3.5 for the variable $y$ (in red). Observe that the black graph here has two regions of different character, just as the red graph in Fig. 3.5: there is a plateau spanning the period of stationary climate illustrating the time-independence of the ensemble average, while the time evolution is irregular in the climate change period.

![Figure 3.9](image_url)

Figure 3.9: The time evolution of the ensemble average (black line) and of three different 30-year SRT (single-realization temporal) averages (blue, red and green lines), calculated for the midwinter values of the variable $x$. The numerical ensemble consists of $10^6$ individual realizations, three of which are used for generating the SRT 30-year averages.

SRT averages are similarly irregular. In Fig. 3.9, the three colored lines represent the 30-year SRT averages of three single realizations (plotted yearly at the midpoints of the 30-year intervals). These three realizations are, along with their 99997 companions, individual members of the ensemble. Note that any of the colored lines can be considered as a “historically registered” climate [8]. We can observe that these SRT averages do not follow any fine structures in the ensemble average, and even in the coarse structure substantial alterations occur. This is partially due to the large amount of information contained in 30-year SRT averages that is obsolete from the point of view of the characterization of the present probabilities. What is more, they behave very differently in comparison with each other as well, indicating the lack of the representativity of one single realization for the ensemble behavior.

We note at this point that SRT averages sometimes produce strong signals suggesting changes which are false from the point of view of the snapshot picture. This is the most obvious in the stationary period of the climate, before $t = 0$. For example, the red SRT average line in Fig. 3.9 indicates an increase in the strength of the Westerlies from 0.6 to 0.9 of years included in the temporal average is $\tau + 1$. 

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70
between years -60 and -20. This might be interpreted as a climate change if one does not intend to think in terms of averages taken with respect to the natural probability measure of the snapshot attractor. Unlike any SRT averages, the ensemble average in Fig. 3.9 indicates properly the stationary nature of the dynamics and of the corresponding probability measure.

### 3.4.3 Probability distribution of a signed ergodicity deficit [T6]

A possible quantity characterizing a deviation of the SRT average from the ensemble average is the difference between these averages. Note that this difference corresponds to nothing else but the deviation from an ergodic behavior within the context of the finite-length time window applied for evaluating the SRT average around a particular time $t$ of observation. It is then clear that nonergodicity plays an important role in our problem. Therefore, this difference between the averages we shall call the *signed ergodicity deficit* in the time window of length $\tau$ and associated to time $t$:

$$\delta_\tau(t) = A_\tau(\varphi(t)) - A_\mu(\varphi(t)).$$

(3.5)

This deficit clearly depends, too, on which particular realization is chosen (i.e., on $x_0$ in $t_0$).³

In other words, each realization (initial condition) provides a different value for $\delta_\tau(t)$, even in an ergodic dynamics, because each trajectory has opportunity to visit only a subset of the accessible phase space positions during the time window $\tau$.

When making choices for $\tau$, as anticipated, we shall constrain temporal averages to belong exclusively either to the stationary or to the changing climate (an autonomous and a nonautonomous dynamics), which, in particular, restricts the possible simultaneous choices of the window length $\tau$ and the time instant $t$ of observation (given that $|t| < 150$ years in the simulations). For $t < 0$ the attractor is a usual chaotic attractor which exhibits ergodicity for temporal averages that obey the constraint. Therefore, we shall refer to a time instant $t$ in the stationary climate as an *ergodic case*. The time instants of the changing climate ($t > 0$) will be called *nonergodic cases*, because, as we will show later, ergodicity cannot be satisfied by temporal averages taken around such time instants.

Table 3.1 illustrates, via three different realizations considered in two different time instants $t$ of observation (one in the stationary climate, $t < 0$, and one in the changing climate, $t > 0$)⁴

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³For $\tau \to 0$ the SRT average $A_\tau(\varphi(t))$ becomes the instantaneous value of $\varphi(t)$ so that, irrespectively of whether the dynamics is ergodic, $A_\mu(\delta_{t=0}(t)) = 0$.

⁴A maximal window length satisfying the constraint of the previous paragraph for both year 75 and year 76 (which we shall compare later) is $\tau = 148$ years.
<table>
<thead>
<tr>
<th>$\tau$</th>
<th>Year $-75$</th>
<th>Year 76</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.301</td>
<td>0.180</td>
</tr>
<tr>
<td>72</td>
<td>0.018</td>
<td>0.137</td>
</tr>
<tr>
<td>148</td>
<td>-0.076</td>
<td>0.117</td>
</tr>
</tbody>
</table>

Table 3.1: Three examples (corresponding to three different realizations) for $\delta_\tau$ in two different time instants of observation, for three different window lengths $\tau$. The signed ergodicity deficit $\delta_\tau$ is based on the variable $y$ [i.e., $\varphi = y$ in (3.5)]. The first (second) time instant, year $-75$ (year 76), corresponds to an ergodic (nonergodic) case.

that the typical values of $\delta_\tau$ depend strongly on the window length $\tau$. As examples, a short, an intermediate and a long window length are chosen. For a small length (e.g. $\tau = 4$ years) the variability among different realizations is very large so that investigating only one realization is not representative. For increasing $\tau$ the convergence of $\delta_\tau$ towards some common value for the different realizations is rather slow, even in the stationary climate. Even using a length $\tau$ that can encompass the whole climate change period of our model leads to considerable differences.

Given that the individual signed ergodicity deficit values are typically nonzero for finite time windows $\tau$ in any case, and depend heavily on the trajectory chosen, i.e., on the realization, it is natural to consider the probability density function of $\delta_\tau(t)$, at different time instants $t$. They will be called the \textit{signed ergodicity deficit distributions}, and shall be denoted by $P(\delta_\tau(t))$. It is interesting to see if these distributions of the stationary climate and the changing climate differ in time instants $t$ that correspond to ergodic and nonergodic cases, respectively. We determine these distributions numerically for ensembles of size $N = 10^6$. In particular, we compare here the pdfs corresponding to the two time instants $t$ of observation that we already chose in Table 3.1: year $-75$ and year 76.

For illustrative purposes we choose the variable $y$ as the quantity $\varphi$, and approximate numerically the pdf $P(\delta_\tau)$ of the signed ergodicity deficit $\delta_\tau$ by determining a histogram. In particular, we divide the $\delta_\tau$ axis into small bins and count the number of single realizations out of an ensemble of trajectories, initiated with different initial positions $x_0$ at $t_0 = -250$ years, that produce a $\delta_\tau$ value falling into a particular bin. Histograms obtained this way are shown in Fig. 3.10 for the window lengths $\tau$ and time instants $t$ of Table 3.1.

The results show a clear difference between stationary and changing climates, representing ergodic and nonergodic cases, respectively. Some of the characteristics, like bias and spread, are detailed in the next subsections.
Figure 3.10: Signed ergodicity deficit distributions $P(\delta_\tau(t))$ based on the variable $\varphi = y$, calculated over an ensemble of $N = 10^6$ trajectories. In each panel we compare three different values of the window length $\tau$. The time instants $t$ of observation are indicated above the panels. Panels (a) and (b) show an ergodic and a nonergodic case, respectively. The time instants $t$ of observation is 0.025. The ensemble size appears to realize the asymptotic limit in the sense that we do not find any considerable change in the graphs when plotting the results for $N = 10000$, $20000$ or $10^6$.

3.4.4 Driving-induced bias \[T7\]

The most striking difference between the two panels of Fig. 3.10 is that the distributions are centered around zero in the ergodic case, while there is a shift towards larger values with increasing $\tau$ in the nonergodic case. In other words, in the ergodic case the average signed ergodicity deficit, $A_\mu(\delta_\tau)$, is zero for any of the time windows investigated. We have thus found a tool to decide if a system exhibits ergodicity around a time instant $t$ based on using time averages of finite length only. The price for this is the use of an ensemble instead of individual realizations, exactly in the spirit of snapshot attractors. To be more precise, a time instant $t$ quite surely represents a nonergodic case if the average signed ergodicity deficit associated to this time instant differs from zero,

$$| A_\mu(\delta_\tau(t)) | > 0,$$

for any particular time window $\tau$ (with the exception of a few isolated values of $\tau$, perhaps).

If a distribution is symmetric with respect to zero (so that the average signed ergodicity deficit $A_\mu(\delta_\tau)$ is zero), then we see the following property to hold: although $A_\tau(\varphi(t))$ typically differs from the ensemble average $A_\mu(\varphi(t))$ of $\varphi$, the ensemble average $A_\mu(A_\tau(\varphi(t)))$ of the SRT average coincides with the ensemble average in view of (3.5). As observed in Fig. 3.10(a), this property holds for any finite $\tau$. This can be considered as an extension of the ergodic
theorem. It can be understood by considering the values along a single time series to be samples drawn from the same distribution (which is the natural measure of a usual attractor in our case): the sample average is known to estimate the ensemble average without any bias. In the nonergodic case, however, one can see the symmetry property with respect to zero to be missing in Fig. 3.10(b). In this case an SRT average is expected to give a biased value compared to the ensemble average.

This bias is a consequence of the SRT average incorporating more and more information corresponding to time instants that are different from and are farther and farther away from the time instant \( t \) of interest. Such information is obsolete or not up-to-date (originating partially in the future) in the case when the snapshot attractor and its natural measure change in time. Consequently, \( A_{\mu}(\delta_t) \) is expected, in generic time-dependent cases, to differ from zero for any (generic) finite window length \( \tau \), and to remain nonzero even for \( \tau \to \infty \) if this limit can meaningfully be carried out. The latter finding means the breakdown of ergodicity in a formal sense. Given that \( A_{\mu}(\delta_t) = 0 \) in an ergodic case for any \( \tau \), it also follows that the modulus of the average signed ergodicity deficit \( |A_{\mu}(\delta_t)| \) provides an appropriate measure of nonergodicity (i.e., for how different the system is from an ergodic system) when observed on a time interval of \( \tau \) around a particular time instant \( t \).

![Figure 3.11: The bias \( A_{\mu}(\delta_t) \) as a “continuous” function of the window length \( \tau \), calculated for a numerical ensemble of \( 10^6 \) trajectories, where the signed ergodicity deficit \( \delta_t \) is based on the variable \( y \), as in Fig. 3.10. The time instants \( t \) of observation are year \(-75\) (black line, ergodic case) and year \(76\) (magenta line, nonergodic case).](image)

These considerations are illustrated in Fig. 3.11 where the average \( A_{\mu}(\delta_t) \) of the signed ergodicity deficit is plotted as a continuous function of the window length \( \tau \). This quantity can be seen to be identically zero in the ergodic case, while it increases considerably with \( \tau \) in the nonergodic case. In the latter case, it starts from a small value, because short time windows
contain little amount of inappropriate information. After a rapid increase in value from about 0 to 0.2, it is seen to switch at $\tau = 25$ years to a moderate increase from about 0.2 to 0.25, in harmony with the fact that the ensemble average in Fig. 3.5 typically fluctuates in time $t$ within a band of width 0.5.

### 3.4.5 Spread due to the time window [T7]

The next important feature of the signed ergodicity deficit distributions $P(\delta_\tau)$ is their considerable width, observable in all histograms of Fig. 3.10. It means that there is a considerable spread among single realizations, and, therefore, one single realization is not representative for the ensemble behavior sought for any finite value of $\tau$. In other words, it is not sufficient to investigate the SRT average of one particular realization in order to draw any meaningful conclusion related to the appropriate statistics of a quantity $\varphi$. A common feature of the ergodic and the nonergodic cases is the decrease of this width with increasing $\tau$. In the ergodic case the sampling argumentation mentioned in the previous Subsection explains again this observation. In fact, $P(\delta_\tau)$ approximates a Gaussian for larger values of $\tau$ (i.e., for larger samplings) according to the central limit theorem [137]. It is also known that the standard deviation $\sigma_\mu$ of this Gaussian decreases with increasing window length: in particular, it decreases according to a one-over-square-root law. This behavior is illustrated numerically by the black line of Fig. 3.12.\(^5\) Although this argumentation cannot be transferred to the nonergodic case easily, larger samplings provide sharper pdfs according to one’s intuition even in this case. Since the shape of the pdf is also Gaussian-like, the standard deviation $\sigma_\mu$ can again be used as a measure of the width. The numerical calculation for the dependence of $\sigma_\mu(\delta_\tau)$ on $\tau$, shown in light blue in Fig. 3.12, indicates a surprisingly good agreement with a one-over-square-root law. This appears to be in harmony with a recent mathematical result on generalized central limit theorems in nonautonomous systems by [138].

From a more general point of view, the convergence of the signed ergodicity deficit pdfs to sharp distributions turns out to be in both ergodic and nonergodic cases a scale free problem, i.e., no characteristic times can be defined for the convergence with the window length $\tau$. One can see from Fig. 3.12 that the standard deviation of year 5 falls to 10 percent of its original

\(^5\) $\sigma_\mu$ has been calculated as: $\sigma_\mu = \frac{(N-3)!!}{(N-2)!!} \sqrt{\sum_{i=1}^{N} \left( \delta_{\tau,i} - \frac{1}{N} \sum_{i=1}^{N} \delta_{\tau,i} \right)^2}$ (which is the unbiased estimator for Gaussian distributions from sample data) where $\delta_{\tau,i}$ is the signed ergodicity deficit $\delta_\tau$ corresponding to the $i^{th}$ member of the ensemble. We use the same estimator in the nonergodic case because of the similar shape of the signed ergodicity deficit pdfs of the two cases.
Figure 3.12: The spread $\sigma_\mu(\delta_\tau)$ of the signed ergodicity deficits as a function of the number of years included in the temporal average ($\tau + 1$ years), plotted on a doubly logarithmic scale, calculated for a numerical ensemble of $10^6$ trajectories, where the signed ergodicity deficit $\delta_\tau$ is based on the variable $y$, as in Fig. 3.10. The time instants $t$ of observation are year $-75$ (black line, ergodic case) and year 76 (light blue line, nonergodic case). The dashed line is of slope $-1/2$ to guide the eye.

value in about 200 years, and would fall to one percent of it in about 20 000 years. This property is in strong contrast with the convergence in time $t$ of an ensemble towards a snapshot attractor (examples can be seen in Fig. 3.2) which was shown to be exponential in Section 3.3.3 and in [139]. The latter process can be characterized by a characteristic time $t_c$ ($t_c = 5$ years in our example). This indicates a convergence to the snapshot attractor with an accuracy of about $10^{-3}$ in $t_c$. Reaching a snapshot attractor with an ensemble of trajectories in time is thus much faster than reaching the ergodic property for individual trajectories in window length.\(^6\) The fact that it is hopeless to choose time windows sufficiently long to observe $\delta_\tau = 0$ supports “a posteriori” our choice to focus on cases with finite window lengths $\tau$. Note that our findings provide a quantitative explanation of the rather qualitative picture of [99, 18] regarding the speed of the convergence of the ensemble-based and single-trajectory-based distributions.

3.4.6 The absolute ergodicity deficit [T7]

We emphasize that both a large $A_\mu(\delta_\tau)$ and a large $\sigma_\mu(\delta_\tau)$ lead to the inapplicability of SRT averages for estimating ensemble averages in nonautonomous dissipative systems with a parameter shift. One can reduce the first and the second quantity by decreasing and increasing $\tau$,

\(^6\)When reaching the ergodic property is principally possible, i.e., in an ergodic dynamics, the computational demand for reaching a particular precision in approximating some statistics is, in fact, of the same order for following an ensemble for a short time and for following an individual trajectory for a long time. The practical importance of the snapshot attractor picture is the ability of describing nonautonomous dynamics in which ergodicity does not hold at all.
respectively: this is a trade-off situation. Let us now investigate, as a function of the window length $\tau$, how good an SRT average is expected to perform “overall”. To this end, we plot in Fig. 3.13 the dependence on $\tau$ of the average $A_\mu(\delta_\tau)$ and of the standard deviation $\sigma_\mu(\delta_\tau)$ together for the same two time instants $t$ as in Fig. 3.10, and we include into the plots a new characteristic, $A_\mu(|\delta_\tau|)$, which measures how large deviation is expected to occur in an absolute value. $|\delta_\tau|$ shall be called the absolute ergodicity deficit. Note that this quantity is nonzero in both ergodic and nonergodic cases.

The average absolute ergodicity deficit $A_\mu(|\delta_\tau|)$ characterizes the expected modulus of the deviation of an SRT average from the ensemble average. This is so even in unbiased ergodic cases when this expected absolute ergodicity deficit is related solely to the nonzero width of the distribution, i.e., to the spread of the individual realizations. In such cases $A_\mu(|\delta_\tau|)$ is expected to carry the same information as $\sigma_\mu(\delta_\tau)$. Indeed, $A_\mu(|\delta_\tau|)$ is not zero in Fig. 3.13(a), but it is practically a rescaled version\footnote{By assuming a Gaussian shape for $P(\delta_\tau)$, the rescaling factor is $\sqrt{2/\pi} \approx 0.798$. Numerically the ratio is found to be approximately 0.809 for small $\tau$, and to converge rapidly to $\sqrt{2/\pi}$ for increasing $\tau$: it is already 0.799 for $\tau = 25$ years.} of $\sigma_\mu(\delta_\tau)$. Both $A_\mu(|\delta_\tau|)$ and $\sigma_\mu(\delta_\tau)$ are seen to converge toward zero with increasing $\tau$, the stationary climate thus represents an ergodic case indeed.

We point out that a nonzero expected absolute ergodicity deficit, $A_\mu(|\delta_\tau|) > 0$, even in the case of ergodic autonomous dynamics due to $\sigma_\mu(\delta_\tau) > 0$, prompts that the nonergodicity quantified by $A_\mu(|\delta_\tau|)$ is a useful property of the observation over a particular time window.

In the changing climate case of Fig. 3.13(b) $A_\mu(|\delta_\tau|)$ is seen to describe the spread of different realizations as long as $A_\mu(\delta_\tau)$ is small (up to $\tau = 20$ years). For increasing $\tau$, however,
the bias, quantified by $A_\mu(\delta_\tau)$, plays a more and more important role in determining $A_\mu(|\delta_\tau|)$. Finally, $A_\mu(|\delta_\tau|)$ and $A_\mu(\delta_\tau)$ become approximately the same, as the spread $\sigma_\mu(\delta_\tau)$ decreases. In our particular case this is already observed when reaching $\tau = 100$ years, as a consequence of the negligible probability for $\delta_\tau < 0$, as Fig. 3.10(b) illustrates. In total, one can observe that the time evolution of $A_\mu(|\delta_\tau(\tau)|)$ follows very close the larger out of the values of $\sigma_\mu(\delta_\tau(t))$ and the modulus $|A_\mu(\delta_\tau(t))|$. The quantity $A_\mu(|\delta_\tau|)$ thus incorporates in general both effects (the spread and the bias) that lead to the deviation of an SRT average from the corresponding ensemble average, and provides a natural quantification for this deviation. In the particular case of Fig. 3.13(b) this deviation is never small, any statistics extracted from the time evolution of a single realization is thus always meaningless from a probabilistic point of view.

### 3.4.7 Statistics beyond the average

The signed ergodicity deficit $\delta_\tau$ can be evaluated based on any variable $\varphi$, one may also choose e.g. $\varphi = x$ (the speed of the Westerlies). Furthermore, the signed ergodicity deficit can be defined not only for averages in (3.5), but also for other statistical quantities of interest. For example, the signed ergodicity deficit $\delta^{(n)}_\tau$ for the $n$th cumulant $C^{(n)}_\mu(\varphi)$ of the variable $\varphi$ is defined as

$$\delta^{(n)}_\tau(t) = C^{(n)}_\tau(\varphi(t)) - C^{(n)}_\mu(\varphi(t))$$

(3.7)

where $C^{(n)}_\tau$ is the estimator of the $n$th cumulant evaluated on the time window of length $\tau$.\(^8\) [Note that $\delta^{(1)}_\tau(t) = \delta^{(1)}_\tau$.] The advantage of using $\delta^{(n)}_\tau$, $n > 1$, is that $A_\mu(\delta^{(n)}_\tau)$ is never zero in a nonergodic case, not even in the very unlikely situation when the time evolution of the ensemble average $A_\mu(\varphi(t))$ is exactly a linear function, and therefore $A_\mu(\delta^{(1)}_\tau) = 0$, at least with the midpoint convention.

For illustrative purposes we plot in Fig. 3.14 the dependence of the signed ergodicity deficit $\delta^{(2)}_\tau$ based on the variable $\varphi = x$ on the window length $\tau$. Two neighboring time instants $t$ are chosen, both corresponding to nonergodic cases. The graphs of the $\tau$-dependence in Fig. 3.14(a) are similar to that of Fig. 3.13(b), this picture can thus be considered generic. In Fig. 3.14(b), however, $A_\mu(\delta^{(2)}_\tau)$ remains significantly smaller than $\sigma_\mu(\delta^{(2)}_\tau)$ for any considered value of $\tau$. This leads to completely different $A_\mu(|\delta^{(2)}_\tau|)$ functions in the two plots. As these two plots are separated by a single year, this experience implies that certain aspects of the

\(^8\)In order to see that $A_\mu(\delta^{(n)}_\tau(t)) = 0$ in ergodic cases we require the use of the unbiased estimators of the population statistics from sample data.
Figure 3.14: $A_\mu(\delta^{(2)}_\tau)$, $\sigma_\mu(\delta^{(2)}_\tau)$ and $A_\mu(|\delta^{(2)}_\tau|)$ as functions of the window length $\tau$, calculated for a numerical ensemble of $10^6$ trajectories, where $\delta^{(2)}_\tau$ is based on the variable $x$. The time instants $t$ of observation are indicated in the panels. Note that $A_\mu(\delta^{(2)}_\tau)$ is not identically zero even in year 76. This indicates nonergodicity for this year, too.

3.4.8 Time-dependence of averages and variances

So far we have investigated only few different time instants $t$ of observation. From Figs. 3.5 and 3.14, however, one may expect the time evolution of the pdf of any signed ergodicity deficit $\delta^{(n)}_\tau$ to be rich. In Fig. 3.15 we plot histograms of $\delta^{(n)}_\tau$, $n = 1, 2$, such that the time window $\tau$ is fixed but different time instants $t$ of observation are chosen from both the stationary and the changing climate. In the two panels we take $\delta_\tau$ for the variable $y$ and $\delta^{(2)}_\tau$ for the variable $x$. The first two histograms in each panel (red and green), belonging to the stationary case, practically coincide and are symmetric with respect to zero. The next three are, however, completely different. The maxima (mean values) do not change here monotonically in time: in Fig. 3.15(a), for example, the histogram is centered on a negative value right after the climate change, then it becomes shifted to a large positive one, and ends at a moderate positive value. Smaller but significant
changes can be seen in Fig. 3.15(b). The pdf $P(\delta_\tau^{(n)})$ of a signed ergodicity deficit $\delta_\tau^{(n)}$ is thus dramatically changing in time.

![Histograms of $\delta_\tau$ and $\delta_\tau^{(2)}$](image)

Figure 3.15: The time-dependence of the histograms (a) $P(\delta_\tau(t))$ of the variable $y$ and (b) $P(\delta_\tau^{(2)}(t))$ of the variable $x$, with a fixed window length of $\tau = 72$ years, calculated over an ensemble of $N = 10^6$ trajectories. In any single panel we compare different time instants $t$ of observation. The bin size is 0.025. Note that the lines for the two first time instants almost overlap.

Fig. 3.16 explores the time evolution of $P(\delta_\tau(t))$ for the variable $y$ and of $P(\delta_\tau^{(2)}(t))$ for the variable $x$ via their ensemble average $\mathcal{A}_\mu$ and standard deviation $\sigma_\mu$. For a better understanding, the two terms composing $\mathcal{A}_\mu(\delta_\tau(t))$ [or $\mathcal{A}_\mu(\delta_\tau^{(2)}(t))$ in panel (b)], i.e., the ensemble averages $\mathcal{A}_\mu(y(t))$ and $\mathcal{A}_\mu(\mathcal{A}_\tau(y(t)))$ [or $\mathcal{C}_\mu^{(2)}(x(t))$ and $\mathcal{A}_\mu(\mathcal{C}_\tau^{(2)}(x(t)))$ in panel (b)], are also shown in a separate plot. The examples plotted in panels (a) and (b) are seen to be qualitatively very similar.

A nonzero $\mathcal{A}_\mu(\delta_\tau^{(n)}(t))$, $n = 1, 2$, only appears in Fig. 3.16 in the climate change period$^9$, indicating nonergodicity. In this period, accordingly, the component $\mathcal{A}_\mu(\mathcal{A}_\tau(y(t)))$ [or $\mathcal{A}_\mu(\mathcal{C}_\tau^{(2)}(x(t)))$ in panel (b)], arising from SRT statistics, can be seen to deviate from the instantaneous ensemble average $\mathcal{A}_\mu(y(t))$ [or the second cumulant $\mathcal{C}_\mu^{(2)}(x(t))$ in panel (b)]. The SRT component is much smoother than the ensemble average [or second cumulant$^{10}$]. The strong fluctuations observable in $\mathcal{A}_\mu(\delta_\tau(t))$ [or $\mathcal{A}_\mu(\delta_\tau^{(2)}(t))$ in panel (b)] thus originate in the ensemble average $\mathcal{A}_\mu(y(t))$ [or the second cumulant $\mathcal{C}_\mu^{(2)}(x(t))$ in panel (b)]. The presence of fluctuations

$^9$More precisely, a nonzero value is present after $-\tau/2$, even before the beginning of the climate change in $t = 0$, because the SRT statistics are ordered to the window centers. The use of lagging windows (when the SRT statistics are ordered to the endpoints of the windows) would restrict the nonzero values to the climate change period strictly.

$^{10}$From Fig. 3.16 one might guess that $\mathcal{A}_\mu(\mathcal{C}_\tau^{(2)}(x(t)))$ is a moving average of $\mathcal{C}_\mu^{(2)}(x(t))$. We numerically verified, however, that

$$\mathcal{A}_\mu(\mathcal{C}_\tau^{(2)}(\varphi(t))) - \mathcal{A}_\tau(\mathcal{C}_\mu^{(2)}(\varphi(t))) \neq 0.$$  

(3.8)
Figure 3.16: The time-dependence of a few characteristic statistical measures of the natural measure as indicated in the panels (derived from a numerical ensemble of $10^6$ trajectories). The red line of the upper plot [$A(\delta_\tau(t))$ and $A(\delta_\tau^{(2)}(t))$ in panels (a) and (b), respectively] is the difference of the magenta and the black lines (the two lines of the lower plot) in view of (3.5). Panel (a) concerns the average in the variable $y$, and panel (b) concerns the second cumulant in the variable $x$. The window length is $\tau = 72$ years. Observe that $\sigma(\delta_\tau^{(n)}(t))$ and $A(|\delta_\tau^{(n)}(t)|)$, $n = 1, 2$, lie very close to each other for $t < 0$.

in the latter is a characteristic of the temporal evolution of the snapshot attractor and of its natural measure in our particular model. Due to the fluctuations in $A(\delta_\tau^{(n)}(t))$, $n = 1, 2$, the ensemble average of the signed ergodicity deficit, as a function of $t$, repeatedly crosses the value of zero, with a fixed $\tau$. These crossings compose, however, a set of measure zero on the time axis $t$, and the set would change for other choices of $\tau$. Therefore, $A(\delta_\tau^{(n)}(t))$, $n = 1, 2$, indicates faithfully the nonergodic property either if one takes finite intervals for the time of observation $t$ or if one investigates multiple choices for the window length $\tau$. As for the standard deviation $\sigma(\delta_\tau^{(n)}(t))$, $n = 1, 2$, it is surprisingly close in Fig. 3.16 to a constant during both the stationary and the changing climate period. This means that the spread of the individual realizations has, when the underlying probability measure changes in time, approximately the same importance in SRT statistics as when it does not change. The expected absolute ergodicity deficit $A(|\delta_\tau^{(n)}(t)|)$, $n = 1, 2$ lies away from zero during its full time evolution, which indicates that
a single realization can never be used for extracting relevant information about the natural measure (illustrated in Fig. 3.16 for our particular choice of $\tau$). One sees that $\mathcal{A}_\mu(\{\delta^{(n)}_t(t)\})$ is very close to the larger out of the values of $\sigma_\mu(\delta^{(n)}_t(t))$ and $|\mathcal{A}_\mu(\delta^{(n)}_t(t))|$. This is in harmony with the behavior of $\mathcal{A}_\mu(\{\delta^{(1)}_t(t)\})$ in Fig. 3.13 corresponding to particular time instants but showing different possible values for $\tau$.

#### 3.5 Analyzing an alternative concept of ergodicity [T8]

Ergodicity is generally defined via temporal statistics along a single realization. In the literature, however, an alternative definition showed up recently [89], relying on an unusual way of taking temporal averages and with an ensemble of trajectories. The members of this ensemble are initiated at different time instants. This definition corresponds in effect to an average taken over the endpoints of these members, although the averaging is formally expressed by an integral over time.

More precisely, let a value of the observable $\varphi$ at time $t$ be denoted by $\varphi(t; t_0, x_0)$ that emerges from an (arbitrarily chosen) initial position $x_0$ of the phase space at time $t_0$. This notation expresses the need of the use of the two-time evolution operator of general nonautonomous dissipative dynamical system, as mentioned in Section 1.3.1. An artificial time average over a window of length $\tau$ (ordered to the final instant $t$) can be evaluated [89] by considering different trajectories of temporal length $t - t'$ which start from the same initial position $x_0$ at time $t'$ and yield the value $\varphi(t; t', x_0)$ at the final instant $t$, then integrating over the time instants $t'$:

$$\frac{1}{\tau} \int_{t-\tau}^{t} \varphi(t; t', x_0) dt'. \tag{3.9}$$

We call this average artificial because the integration is taken over the initial time instants $t'$, and this is impossible to carry out in any single time series. In fact, this integration defines an averaging over the endpoints of trajectories initiated in different time instants. This is why this average is essentially also an ensemble average in itself.

Reference [89] claims that it is usual among nonautonomous dynamical systems that for almost every initial position $x_0$ in the phase space the following ergodic property is satisfied:

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_{t-\tau}^{t} \varphi(t; t', x_0) dt' = \mathcal{A}_\mu(\varphi(t)), \tag{3.10}$$

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for any sufficiently smooth observable \( \varphi \). Relation (3.10) was originally formulated for a random dynamical system in [89]; we apply it here to a deterministic nonautonomous case. Qualitatively speaking, the left hand side of (3.10) is an average of points on the snapshot attractor belonging to time instant \( t \) since trajectories rapidly forget their initial conditions (after time \( t_c \)), due to dissipation. The endpoints of only those trajectories are not on the attractor that started at \( t' > t - t_c \), but they form a negligible proportion of all the trajectories taken, with the exception of small values of the window length \( \tau \). One thus expects that for \( \tau \gg t_c \), the endpoints populate the snapshot attractor representing faithfully its natural measure, and the two sides of (3.10) are then practically equivalent. It is intuitively appealing that such a relation, generalizing Birkhoff’s ergodic theorem for general nonautonomous systems, might exist, but one must not forget that the time average on the left hand side is by far not the natural time average over a time series.

For any particular initial value \( x_0 \) we define the signed ergodicity deficit \( d_\tau \) with this artificial time average on a window length \( \tau \) as\(^{11}\)

\[
d_\tau(t) = \frac{1}{\tau} \int_{t-\tau}^t \varphi(t' ; t', x_0) \, dt' - \mathcal{A}_\mu(\varphi(t)). \tag{3.12}
\]

Similarly to \( \delta_\tau(t) \), \( d_\tau(t) \) has a probability distribution, numerically obtainable from different initial positions \( x_0 \). The observable \( \varphi \) may be a dynamical variable, we choose \( \varphi = y \) in the Lorenz model.

We plot the numerically obtained distribution \( P(d_\tau(t)) \) of signed ergodicity deficits \( d_\tau(t) \) [calculated with a yearly sum instead of an integral in (3.12)] for the variable \( y \) in Fig. 3.17 for the same three window lengths \( \tau \) and two time instants \( t \) (one in the stationary and one in the changing climate) as in Fig. 3.10. In both kinds of climate, the distributions appear to be symmetric with respect to zero, and their width shrinks with increasing \( \tau \). This indicates that a changing climate is similar to a stationary one from the point of view of the ergodicity concept in (3.10).

For a more detailed investigation, we plot in Fig. 3.18 the dependence on \( \tau \) of the average signed ergodicity deficit \( \mathcal{A}_\mu(d_\tau) \), of the standard deviation \( \sigma_\mu(d_\tau) \) and of the expected absolute

\[^{11}\text{The signed ergodicity deficit } \delta_\tau \text{ based on a lagging window, like here, would read, instead of (3.5), as}
\]

\[
\delta_\tau(t) = \frac{1}{\tau} \int_{t-\tau}^t \varphi(t') \, dt' - \mathcal{A}_\mu(\varphi(t)). \tag{3.11}
\]
Figure 3.17: Distribution $P(d_\tau(t))$ of signed ergodicity deficits $d_\tau(t)$ (3.12) based on the variable $\varphi = y$, calculated over an ensemble of $N = 10,000$ trajectories. The bin size is 0.025. In each panel we compare 3 different values of the window length $\tau$. The time instants $t$ of observation are as in Fig. 3.10. Unlike in Fig. 3.10, no significant difference can be observed between the plots of the stationary and the changing climate.

Figure 3.18: $A_\mu(d_\tau), \sigma_\mu(d_\tau)$ and $A_\mu(|d_\tau|)$ as a “continuous” function of the window length $\tau$, calculated for a numerical ensemble of 10,000 trajectories, where the signed ergodicity deficit $d_\tau$ is based on the variable $y$, as in Fig. 3.17. The time instants $t$ of observation are indicated in the panels. Similarly as in Fig. 3.17, the stationary and the changing climate produce similar plots.

ergodicity deficit $A_\mu(|d_\tau|)$ for the same two time instants $t$ as in Fig. 3.17. As mentioned, the endpoints of trajectories initiated at time instants $t' > t - t_c$ are not yet on the snapshot attractor. As long as such trajectories dominate the “integral” in (3.12), $A_\mu(d_\tau)$ differs from zero considerably, which is a feature that is not common with $A_\mu(\delta_\tau)$. This is seen up to $\tau < t_c \approx 5$ years in Fig. 3.18. For larger $\tau$, however, $A_\mu(d_\tau)$ takes on the value of zero. The standard deviation $\sigma_\mu(d_\tau)$, of course, converges to zero as well (we find that it follows the same power law as the standard deviation $\sigma_\mu(\delta_\tau)$ in Fig. 3.13). As a consequence, the expected absolute ergodicity deficit $A_\mu(|d_\tau|)$ also converges to zero with increasing $\tau$, which
means that the difference between “temporal” and ensemble averages disappears. We conclude that the alternative ergodicity concept of [89] is trivially fulfilled in our deterministic system, and presumably also in any typical nonautonomous system.

This alternative ergodicity concept, as one relying on the use of an ensemble of trajectories, might, however, be relevant only in problems where different realizations are practically accessible. In the real Earth System we have only one historically registered realization, evaluating “temporal” averages as defined in (3.10) is thus not possible. In a sense, this alternative view artificially eliminates the striking difference between ergodic and nonergodic cases, expressed e.g. by Fig. 3.11.

3.6 Discussion

We argued that the only appropriate probabilities, reflecting the internal variability of the dynamics, for an instant of time are described by the natural measure of the snapshot attractor corresponding to this time instant. The temporal change of this probability measure leads to the forced response of the dynamics. Any probability distribution depending on the way of its generation (e.g. on the initial conditions) would be very specific to the particular procedure. The natural measure of the snapshot attractor is, however, unique, as shown in Section 3.3.3.

We also showed that due to the time-dependence of the natural measure under a non-stationary forcing temporal averages evaluated along single trajectories deviate from ensemble averages. Since the natural measure is the unique probability measure that describes appropriately the probabilities according to which the dynamical variables take on particular values, single-realization temporal averages, deviating from the former, lack any probabilistic interpretation. This is why temporal averages, in contrast to what is widely believed [8], are principally not suitable for characterizing the climate and its changes.

For characterizing the difference between single-realization temporal statistics and ensemble statistics, two relevant sets of quantities have been introduced: (i) We numerically find the set $\mathcal{A}_\mu(\delta_r^{(n)}(t))$, $n = 1, 2$, to qualify a dynamical system to be ergodic or nonergodic at time $t$ by taking on the value 0 or not, respectively, with any generic window length $\tau$. $|\mathcal{A}_\mu(\delta_r^{(n)}(t))|$ is proposed as a measure of nonergodicity. (ii) $\mathcal{A}_\mu(|\delta_r^{(n)}(t)|)$, on the other hand, stands for the expected distance of a single-realization temporal statistics, taken over a given window length $\tau$, from the ensemble statistics in an arbitrarily chosen time $t$. 85
Note that even if $A_\mu(\delta_\tau) = 0$ under a nonautonomous dynamics, observations over finite window lengths $\tau$ typically exhibit nonzero ergodicity deficit, $A_\mu(|\delta_\tau|) > 0$, due to the spread between different realizations which is characterized by $\sigma_\mu(\delta_\tau) > 0$. If this was to be catered for by increasing $\tau$, it might well be that precious little is achieved due to the slow scale-free decay of $\sigma_\mu(\delta_\tau)$ with $\tau$. Since aperiodically forced nonautonomous systems are generic in nature, it can be expected that by increasing $\tau$ there will always be a value beyond which $A_\mu(\delta_\tau)$ becomes nonzero, also yielding a positive contribution to the expected ergodicity deficit.

In our particular example we defined ergodic and nonergodic cases according to the constraint in Section 3.4.3, based on the autonomous or nonautonomous nature of the dynamics in a time window of length $\tau$. This constraint ensures that all of our findings can be generalized for systems that are either autonomous or nonautonomous over the entire time axis. Nevertheless, our model system is autonomous only for $t < 0$. The success of studying autonomous dynamics and of demonstrating ergodicity-related properties in our model system indicates that it may be useful to talk about ergodic regimes in time even if the complete dynamical system is not ergodic. We suggest ergodic regimes to be recognized by $A(\delta^{(n)}_\tau(t)) = 0$, for $n = 1, 2$, with any window length $\tau > 0$ (below a crossover to $A(\delta^{(n)}_\tau(t)) \neq 0$).

The advantage of knowing whether a system is ergodic or nonergodic lies in the fact that in the former case the currently available theories of dynamical systems, e.g. the one based on periodic orbits (see e.g. [26]) are likely to be applicable, while otherwise only the snapshot attractor approach remains. This branch of research is currently rapidly evolving and sheds new light on phenomena not only in the climatic context (see e.g. [140]), but also on general aspects of dynamical systems, like e.g. transitions in many degree of freedom chaotic systems [141]. In what follows, we discuss some practical issues concerning the applicability of the snapshot attractor framework.

So far we have discussed only instantaneous values of any statistics. In a climatic context, however, one is often interested in the weather of a time interval (e.g. a season in the year, or a few decades). In such situations we first evaluate the quantity of interest\(^ {12}\) on the time interval of interest along single realizations, and then calculate the statistics of this quantity over the ensemble of the realizations (representing the probabilities faithfully). We call such statistics interval-wise-taken E (ensemble) statistics to avoid the use of the more technically sounding E statistics of SRT data being consistent with our terminology. The forced response

\(^ {12}\)This can be any arbitrary transformation of the trajectory values, including e.g. SRT (single-realization temporal) statistics.
of the dynamics is naturally reflected also in the time-dependence of interval-wise-taken E statistics. We emphasize that the observation of this time-dependence is conceptually different from observing the temporal change in some SRT statistics [by which a climate change is defined in [8]].

A natural question concerning the relevance of snapshot attractors and of their natural measures is the robustness against noisy perturbations. Experience shows in any dynamical system that as long as the noise is weak, only the small scale structure changes: fractality is washed out to a space-filling pattern but only below a short length scale in the phase space. This implies that averages can change only slightly in the presence of weak noise. A detailed investigation of noise effects from the point of view of snapshot attractors is given by [95].

Another practical issue is the size of the ensemble needed to see the basic difference between single-realization and snapshot-type results. Results in [139] show that a 40-member ensemble may suffice practical purposes in an intermediate-complexity climate model, the Planet Simulator of the University of Hamburg, the dynamics of which is mainly governed by the atmosphere.

In order to incorporate more and more processes of the Earth System, there is a natural need for the investigation of models with higher and higher complexities. A relevant question is the extent of the practical applicability of the snapshot framework to such models. In principle, all nonautonomous dynamical systems have snapshot attractors with a natural measure on them. A case of particular interest is that when correlation times along trajectories are much longer in certain variables than in others (i.e., the case of time scale separation). One might then wish to calculate the natural distribution restricted to the variables with short correlation times, keeping those with long correlation times approximately constant. This way one avoids the trajectories to visit those parts of the attractor that cannot be reached in some restricted time from a particular initial condition on the attractor (e.g. people are never interested in ice ages when thinking about the 21st century). To this end we propose to follow an ensemble of trajectories such that their initial conditions are distributed widely in the fast variables and are chosen according to given values in the slow variables. Although we propose to let also the slow variables evolve, they can be expected to remain approximately constant in the time interval of interest, which implies that the spread between different members of the ensemble remains small in these variables. Such an ensemble may be expected to converge to a “conditional” snapshot attractor in the fast variables after a certain convergence time. This construction might provide
well-defined conditional probability distributions in situations where the correlation times in
different variables are well separated. As for more general cases, we might hope that one can
find in most of the models a reasonable cutoff in time for separating the degrees of freedom
that are treated as frozen-in on the time scales of interest from those that are not.
Chapter 4

Outlook

In view of the results detailed in this thesis work, we may conclude that numerous uncovered phenomena may be found even in the better-studied areas of environmental fluid dynamics (such as in the dynamics of vorticity), and that finding the appropriate framework (such as that of snapshot attractors) may be essential for describing environmental systems that are not yet understood (such as the climate system). Still, we have been applying the theory of dynamical systems for all our purposes, and the success of this approach demonstrates the power of this theory. We are applying it currently and are planning to continue its application also in further environmental topics which are briefly described below.

First, we are continuing our work on modulated vortex pairs. The motion of the modulated vortex pairs is never chaotic, which suggests the existence of a conserved quantity beyond kinetic energy. This is, of course, related to the invariance of the dynamics in the longitudinal coordinate $\lambda$. What is more interesting is the opportunity of introducing an effective one-dimensional potential for the dynamics of $\varphi$ (the latitudinal coordinate of the vortex pair’s center of mass) in which this new conserved quantity appears as a parameter. This effective potential has a closed form for dipoles, and its calculation needs only a numerical quadrature for finite-sized vortex pairs. Knowing the potential one can easily uncover the full range of the possible trajectory forms, even without actually solving the equations of motion.

We are also studying the applicability of the snapshot attractor picture to high-dimensional models. Our testbed is the intermediate-complexity climate model called Planet Simulator, developed at the University of Hamburg. In [139] we show that this model, regardless of its high dimensionality, also has a time-dependent snapshot attractor if we apply an aperiodic forcing via changing the CO$_2$ concentration of the atmosphere. In particular, the convergence of an
ensemble of different initial conditions to the snapshot attractor and its natural distribution is shown to be exponential. The temporal change, induced by the changing CO\textsubscript{2} concentration, in the statistics of several variables (such as the surface temperature on small geographical scale, or the so-called North Atlantic Oscillation index) proves to be quantifiable in the snapshot attractor picture only, due to the presence of large fluctuations along single realizations. In a “double ramp” scenario (with an increasing and a decreasing ramp in the CO\textsubscript{2} concentration) we find a dynamical hysteresis which underlines that states of the climate during a climate change cannot be understood by analyzing stationary climatic states.

A very natural application of the snapshot attractor approach is the advection of particles in an aperiodic fluid flow. In fact, the original setup in which Romeiras, Grebogi and Ott introduced the snapshot attractor concept [22] was an advection problem. For us, it is natural to continue our previous work on a conceptual cloud model [40] by including a chaotically driven fluid flow. We find [142] that the main characteristics of the advection of inertial particles (raindrops) remain the same as in a time-periodic fluid flow. What is more, the properties of the spreading of a localized set of initial conditions turn out [142] to be governed by a snapshot attractor.

Finally, we mention a rather different field within dynamical systems, that of chaotic scattering [68, 26, 29]. This is not strictly related to environmental physics, but may be basic for a large range of phenomena, mostly in fluid flows and the associated advection problems, in classical molecular dynamics and in celestial mechanics. In a scattering problem a set of asymptotic incoming variables, usually describing free motion, is mapped by the scattering map to a set of asymptotic outgoing variables, describing a similar motion to that for the incoming variables, via a Hamiltonian dynamical system in which a scattering region is defined by a scattering potential. While scattering systems and chaotic scattering in particular are well-understood for two degrees of freedom, the behavior of systems with more degrees of freedom remains a topic of research. A starting point to address the problem is the so-called stack idea [143]. It first considers a three-degrees-of-freedom system which has some symmetry and a corresponding conserved quantity so that the phase space is a stack of those of a two-degrees-of-freedom systems. The chaotic set of such a system is usually a so-called normally hyperbolic invariant manifold (NHIM). Then a perturbation is applied to destroy the symmetry. In [144] we investigate a case in which the NHIM continues to exist for arbitrarily large perturbed cases. In this system we numerically point out a one-to-one correspondence between the strength of the
perturbation and the size of the envelope covering the so-called scattering singularities in the initial asymptotes. We can thus measure the deviation from the symmetric case by asymptotic observations. In [145] we take an example when the NHIM decays with increasing perturbation and turns to a fully hyperbolic chaotic set (i.e., a saddle), and we give the details of this decay. A main objective of the study of scattering systems is solving the inverse scattering problem, which, in the case of chaotic scattering, means the reconstruction of the chaotic set from cross section data. In [146] we do this for a fully hyperbolic system with the help of the so-called rainbow singularities of the doubly differential cross section which are caustics of the scattering map and are shown to be closely related to the periodic orbits of the chaotic set such that the symbolic dynamics and the scaling factors can be deduced.

We emphasize that all research that we do concerns only a tiny portion of the application range of dynamical systems theory. As mentioned in the Introduction (in Section 1.1.4), basically anything is a dynamical system that has an equation of motion. This makes this theory relevant across almost the full range of classical physical sciences: it has proven to be especially useful in celestial mechanics (for an early review see [147]) and in fluid dynamics [148], but it is also applied to e.g. the investigation of granular materials [149]. The methods of time series analysis [150, 151] are mainly based on dynamical systems theory. This theory appears widely also outside of physics: the foundations of chemical reaction kinetics [152] and biological population dynamics [153] rely on it at their very basics, but other fields also make profit of dynamical systems theory [154]. With regard to the large number of active research topics in which it is useful, dynamical systems theory is clearly part of modern physics.
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Appendix

Derivation of the full spherical dipole equations

The dimensionless equations of motion (1.33) are written for an infinitesimally small vortex pair as:

\[
\begin{align*}
\frac{d}{dt} (\varphi \pm d\varphi) &= \frac{\Gamma'(\varphi \mp d\varphi)}{2\pi D'} \cos(\varphi \pm d\varphi) \sin(\pm 2d\lambda) / 2\pi D'^2, \\
\frac{d}{dt} (\lambda \pm d\lambda) &= \frac{\Gamma'(\varphi \mp d\varphi)}{2\pi D'} \\
&\quad \times \frac{\cos(\varphi \pm d\varphi) \sin(\varphi \mp d\varphi) - \sin(\varphi \pm d\varphi) \cos(\varphi \pm d\varphi) \cos(\pm 2d\lambda)}{\sin(\varphi \pm d\varphi)}. 
\end{align*}
\]

(4.1a)

(4.1b)

Note that these are in fact four equations. As a first step, we determine the right hand side of this system of equations up to first order. Since \( D'^2 \), appearing in the denominator, is a quantity of second order, one can see, after expanding the trigonometric expressions up to first order, that the circulations (in the numerators on the right hand side) should be calculated up to second order. Therefore, the result (2.1) for the circulation of an element of a dipole can be substituted. As a next step, we express \( \dot{\varphi}, \dot{d\varphi}, \dot{\lambda} \) and \( \dot{d\lambda} \):

\[
\begin{align*}
\dot{\varphi} &= -\frac{1}{\pi D'^2} \Gamma' \cos \varphi d\lambda, \\
\dot{d\varphi} &= -\frac{1}{\pi D'^2} [\Gamma' d\varphi + a^2 \pi \delta(\varphi) \cos \varphi] \sin \varphi d\lambda, \\
\dot{\lambda} &= \frac{1}{\cos \varphi \pi D'^2} \Gamma' d\varphi, \\
\dot{d\lambda} &= \frac{1}{\pi D'^2} [\Gamma' \left( \frac{d\varphi^2}{\cos^2 \varphi} - d\lambda^2 \right) + a^2 \pi \delta(\varphi) \frac{d\varphi}{\cos \varphi}] \sin \varphi, 
\end{align*}
\]

(4.2a)

(4.2b)

(4.2c)

(4.2d)

where

\[
\delta(\varphi) = 1 - \frac{\sin \varphi_t}{\sin \varphi}. 
\]

(4.3)
(Note that $\lim_{\varphi \to 0} \delta(\varphi) = \pm \infty$, the right hand sides of (4.2b), (4.2d), however, always remain finite.) One can see that the left and the right hand side of any of Eqs. (4.2a)-(4.2d) are of the same order.

Taking the first temporal derivative of (4.2a) and (4.2c) leads to:

\[
\frac{d}{dt} \dot{\varphi} = \frac{1}{\pi D' \Gamma'} \sin \varphi \dot{\varphi} d\lambda - \frac{1}{\pi D'^2} \Gamma' \cos \varphi \dot{\lambda}, \quad (4.4a)
\]
\[
\frac{d}{dt} \left( \cos \varphi \dot{\lambda} \right) = \frac{1}{\pi D'^2} \Gamma' \dot{\varphi}. \quad (4.4b)
\]

Expressing the derivatives $\dot{\varphi}$, $\dot{\lambda}$ and $\dot{\varphi}$ on the right hand side from the system (4.2), we arrive at (2.2).

In view of (2.5), the velocity components $(u, v)$ of the center of mass of the dipole are obtained from (4.2a), (4.2c) as

\[
u = \frac{1}{\pi D'^2} \Gamma' \cos \varphi \dot{\lambda}, \quad (4.5b)\]

The velocity modulus is thus

\[
|u| \equiv \sqrt{u^2 + v^2} = \frac{\Gamma'}{\pi D'^2} \sqrt{\dot{\varphi}^2 + \cos^2 \varphi \dot{\lambda}^2} = \frac{\Gamma'}{2\pi D'}, \quad (4.6)
\]

since

\[
D'^2 = 2(1 - \cos \gamma_{12}) = 4 \left( \dot{\varphi}^2 + \cos^2 \varphi \dot{\lambda}^2 \right). \quad (4.7)
\]
Bibliography


Summary

Within environmental fluid dynamics, phenomena on a rotating sphere still need a more precise understanding. Our research, aimed at investigating the motion of vortex pairs and the chaotic advection in their velocity fields, and providing a testbed for a generally used approximation, the so-called $\beta$-plane approximation, contributes to this issue. The description of climate dynamics, in turn, is not at all matured, this poses in fact one of the most important tasks of our age. Its probabilistic approach is based on the snapshot attractor picture. Our aim was to work out in detail the method of applying snapshot attractors, and to quantify their deviation from the attractors of time-independent systems.

On a rotating sphere, the dynamics of modulated point vortex pairs is investigated, where modulation is chosen to reflect the conservation of angular momentum (potential vorticity). The advection dynamics generated by vortex pairs, which is chaotic on a rotating sphere, may be closed and open on short time scales. On long times, due to spherical topology, even the open advection patterns are found to gradually cross over to that corresponding to a homogeneous closed mixing.

In the model of modulated point vortex pairs we point out a qualitative difference in the trajectories obtained in a $\beta$-plane approximation from those of the full spherical dynamics. This is so in spite of the fact that the deviations from the initial latitude remain small. This is a consequence of the mathematical inconsistency of the traditional $\beta$-plane approximation the details of which we also point out analytically. The advection patterns in the velocity field of vortex pairs are also found to considerably deviate from each other under the two treatments, and quantities characterizing transport properties strongly differ as well.

In the topic of climate, we argue that the concepts of snapshot attractors and of their natural probability measures are the only available tools by means of which mathematically sound statements can be made about averages, variances or other statistics for a given time instant in a changing climate. We emphasize that the natural measure is independent of the ensemble used, provided it is initiated in the past earlier than a convergence time. To illustrate these concepts, we take a low-order, conceptual circulation model in which we introduce a linear forcing. We claim that the internal variability of the climate can be quantified by the natural measure since it characterizes the chaotic motion on the snapshot attractor, and that the forced response is the temporal change of the natural measure.

In nonautonomous dynamical systems, like in climate dynamics, temporal averages taken along a single trajectory would differ from the corresponding ensemble averages even in the infinite-time limit: ergodicity does not hold. It is worth considering the difference of the two kinds of averages, which we call the signed ergodicity deficit, by taking time windows of finite length for temporal averaging. We point out that the probability distribution of the signed ergodicity deficits is qualitatively different in ergodic and nonergodic cases: its ensemble average is zero and nonzero, respectively. The standard deviation of the signed ergodicity deficit, characterizing the spread between different realizations, exhibits a power-law decrease with increasing window length. Furthermore, we emphasize the importance of the modulus of the signed ergodicity deficit, the ensemble average of which describes the expected deviation from fulfilling the ergodic property. With a fixed window length, the time-dependence of the ensemble average of a signed ergodicity deficit, and consequently that of its modulus, are non-trivial in nonergodic cases, in contrast to ergodic cases when these quantities do not change in time. This nonergodic snapshot framework may be useful in climate dynamics, within which we propose the measures of nonergodicity as new quantifiers of climate change.

We also investigate the signed ergodicity deficit with a non-conventional definition of time averages, as recently appeared in the literature. Within this framework no qualitative difference is found between ergodic and nonergodic cases.
Összefoglaló

A környezeti hidrodinamikában a forgó gömbi jelenségek még ma is pontosabb megértésre szorulnak. Ehhez járul hozzá az örvénypárok mozgásának és a sebességterüknél lejátszódó kaotikus sodródásnak a vizsgálat megelőzése, egyúttal egy általánosan használt közelítésnek, az ún. \( \beta \)-sík közelítésnek teszkörnyezetet adó kutatásunk. A klímadinamika leírása pedig még korántsem kiforrott, ez valójában korunk egyik legfontosabb feladatát adja. Ennek a valószínűségi megközelítését alapozza meg a snapshot attraktoros kép. A célunk az volt, hogy a snapshot attraktorok alkalmazási módját részletesen kidolgozzuk, és az időben állandó rendszerek attraktoraitól mutatkozó eltérésséket kvantifikáljuk.

A forgó gömbön modulált ponttörvény-párok dinamikáját vizsgáljuk, amelyben a moduláció az impulzusmomentum (potenciális örvényesség) megmaradását tükrözi. Az örvénypárok által generált advekciós dinamika, amely egy forgó gömbön kaotikus, lehet zárt és nyitott rövid időskálán. Hosszú időkre, a gömbi topológiának köszönhetően, a nyitott advekciós mintázatokról is azt találjuk, hogy fokozatos átmenné a homogén zárt keverés nemzetközi mintázatba.

A modulált ponttörvény-párok modelljében kimutatunk egy kvalitatív különbséget a \( \beta \)-sík közelítésben kapott trajektoriák és a teljes gömbi dinamika trajektoriái között. Ez annak ellenére van így, hogy a kezdeti szélességtől mért eltérés kicsi marad. Ez a hagyományos \( \beta \)-sík közelítés matematikai inkonzisztenciájának a következménye, amelynek a részleteire analitikusan is rámutatunk. Az örvénypárok sebességterében kialakuló advekciós mintázatokról is azt találjuk, hogy lényegesen eltérnek egymástól a kétféle kezelési módon, és a transzport-tulajdonságokat jellemző mennyiségek szintén erősen különböznak.

A klíma témakörében amellett érvelünk, hogy a snapshot attraktoroknak és ezek természeti mértékeinek a fogalma az egyetlen rendelkezésre álló eszköz, amellyel matematikailag értelmes állítások tehetők átlagokra anonktizálható a változó klímában. Hangsúlyozzuk, hogy a természetes mérték független a használt trajektória-együttestől, feltéve, hogy azt egy konvergenciaidőnél régebbi múltban indítjuk. Ezen fogalmak illusztrálására egy alacsonyrendű, konceptiós áramlási modellel veszünk, amelyben egy lineáris gerjesztést vezetünk be. Azt állítjuk, hogy a klíma belső változkozásainak a természetes mértékkel kvantifikálható, mivel az a snapshot attraktor lejátszódó kaotikus mozgást jellemzi, és hogy a gerjesztett változás a természetes mérték időbeli megváltozása.


Az előjelű ergodicitás-deficitten az időablakoknak egy nemkonvencionális, az irodalomban nemrég megjelent definíciójával is megvizsgáljuk. Ebben a leírásmodban nem találunk kvalitatív különbséget ergodikus és nemergodikus esetek között.
I. A doktori értekezés adatai
A szerzõ neve: Drótos Gábor
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A témavezető munkahelye: ELTE TTK Elméleti Fizikai Tanszék

II. Nyilatkozatok
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a) hozzájárulok, hogy a doktori fokozat megszerzését követõen a doktori értekezésen és a tézisek nyilvánosságára kerüljenek az ELTE Digitális Intézményi Tudástárban. Felhatalmazom a Természettudományi Károly Tudományoskereszt és Egyetemközi Kapcsolatok Osztályának ügyintézőjét, Breti Éva, hogy az értekezést és a téziseket feltöltse az ELTE Digitális Intézményi Tudástárba, és ennek során kitöltse a feltöltéshez szükséges nyilatkozatokat.
b) kérem, hogy a mellékelhető kérelmekben részletezett szabadalmi, illetõleg oltalmi bejelentés közzétételéig a doktori értekezést ne bocsássák nyilvánosságra az Egyetemi Könyvtában és az ELTE Digitális Intézményi Tudástárban;
c) kérem, hogy a nemzetbiztonsági okból minõsített adatot tartalmazó doktori értekezést a minõsítés (dátum)-ig tartó idõtartama alatt ne bocsássák nyilvánosságra az Egyetemi Könyvtában és az ELTE Digitális Intézményi Tudástárban;
d) kérem, hogy a mû kiadására vonatkozó mellékelhetõ kiadó szerződésre tekintettel a doktori értekezést a könyv megjelenéséig ne bocsássák nyilvánosságra az Egyetemi Könyvtában, és az ELTE Digitális Intézményi Tudástárban csak a könyv bibliográfiai adatait tegyék közzé. Ha a könyv a fokozatszerzést követõen egy évig nem jelenik meg, hozzájárulok, hogy a doktori értekezésen és a tézisek nyilvánosságára kerüljenek az Egyetemi Könyvtában és az ELTE Digitális Intézményi Tudástárban.

2. A doktori értekezés szerzőjeként kijelentem, hogy
a) az ELTE Digitális Intézményi Tudástárba feltöltendõ doktori értekezés és a tézisek saját eredeti, önálló szellemi munkám és legjobb tudomáson szerint nem sértet vele senki szerzői jogait;
b) a doktori értekezés és a tézisek nyomtatott változatai és az elektronikus adathordozón benyújtott tartalmak (szöveg és ábrák) mindenben megtegyeznek.
3. A doktori értekezés szerzőjeként hozzájárulok a doktori értekezés és a tézisek szövegének plégiumkeresõ adatbázisba helyezéséhez és plégiumellenõrzõ vizsgálatok lefuttatásához.

Kelt: Budapest, 2015.10.22.

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a doktori értekezés szerzőjének aláírása

38 Beiktatta az Egyetemi Doktori Szabályzat módosításáról szóló CXXXIX/2014. (VI. 30.) Szen. sz. határozat.
40 A kérdés alapján összeapróval tért ki.
41 A megfelelõ szöveg aláírandó.
42 A doktori értekezés benyújtásával egyidejûleg be kell adni a tudományágú doktori tanácsot és az adatok megadására vonatkozó okiratot.
43 A doktori értekezés benyújtásával egyidejûleg be kell nyújtani a minõsített adatokat vonatkozó közokiratot.
44 A doktori értekezés benyújtásával egyidejûleg be kell nyújtani a mû kiadásáról szóló kiadói szerződést.