Renormalization methods in truncated conformal space with applications

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Ph.D. Thesis

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Chapter 1

Introduction

Quantum field theory is one of the most successful and highly researched fields of modern physics. It was originally motivated to describe the physics of elementary particles. Later it turned out that field theoretical models can also be applied in condensed matter theory and in statistical physics. It is well known that statistical systems near the phase transition or critical point can be described by field theories, and the critical points are known to be conformal field theories (CFT). Although our physical world seems to be four-dimensional, there are many systems which can be described by low dimensional models. Such systems now have also become accessible to laboratory experiments, partly due to development in the field of trapped ultracold atoms, but there are also other materials (magnets, carbon nanotubes, superconducting macromolecules) that can be described by models in two-dimensional space-time. Two-dimensional systems are somewhat special, as in this case conformal symmetry allows us to solve these models at the critical point in many cases. In addition, a number of systems corresponding to integrable perturbations of the critical point as also exactly solvable.

However these methods are not applicable to non-integrable ones. This thesis is devoted to the renormalized version of the truncated conformal space approach (TCSA) which is originally developed by Yurov and Zamolodchikov in 1990 [YZ90]. Using this method both integrable and non-integrable theories can be studied in the vicinity of critical points.

Statistical systems are usually lattice models with local interactions. Therefore a natural length-scale appears, namely the lattice spacing $a$. Other relevant scale, on which the fluctuations are correlated is called correlation length $\xi$. These two length-scales might be very different. Indeed, from lessons of Monte Carlo simulations such systems show fractal behaviour and the correlation length diverges at the critical points. Therefore one expects that microscopic details do not really influence the global behaviour of statistical systems. This leads to the idea of the renormalization group.
Consider a $d$ dimensional statistical system with degrees of freedom (say spins) $s_i$ placed on the lattice sites. The Hamiltonian is $H(\{s_i\}, \{g_k\})$ where $g_k$s are coupling constants of various interactions between the spins. The correlation length depends on the couplings $\xi(\{g_k\})$. One can organize the spins in blocks $B_k$ of size $ba$ where $b > 1$. Every block contains $b^d$ of the original spins and one can assign a new spin to each block representing the effective spin. In this way we have a new spin lattice of spacing $d' = ba$ and the new correlation length is then $\xi' = \xi/b$. In the new system one can define new coupling constants $g_k^{(1)}$ in a way that the partition function does not change. One can keep going with this procedure, requiring that in each step the partition function remains the same. Then one can write

$$\left\{ g_k^{(n+1)} \right\} = \mathcal{R} \left( \left\{ g_k^{(n)} \right\} \right)$$

$$\xi \left( \left\{ g_k^{(n+1)} \right\} \right) = b^{-1} \xi \left( \left\{ g_k^{(n)} \right\} \right)$$

where $\mathcal{R}$ is called the renormalization group (RG) transformation which is usually a complicated nonlinear transformation. The RG equations describe RG flows in the space of the coupling constants. Renormalization group transformations can have fixed points where $g^* = \mathcal{R}(g^*)$, thus the correlation length can be either zero or infinity. The points of zero correlation length are called trivial fixed points, the interesting ones are where $\xi$ diverges. One can linearize the transformation in the vicinity of a fixed point as $\mathcal{R} \sim 1 + \mathcal{K}$, and use the eigenvectors $u_i$ of $\mathcal{K}$ to get

$$u'_i = b^{y_i} u_i$$

$u_i$s are called scaling variables of dimension $y_i$. For $y_i > 0$, the variable $u_i$ is called relevant, for $y_i < 0$, it is called irrelevant and for $y_i = 0$, it is marginal.

In the vicinity of the critical points the correlation length is much larger than the lattice spacing. It allows one to describe the system by a field theory with action

$$\mathcal{S} = \mathcal{S}^* + \sum \mu_i \int d^d x \phi_i(x)$$

where $\mathcal{S}^*$ is the action of the critical system. Under a scale transformation $x \rightarrow x/b$ the perturbing fields transform as $\phi_i \rightarrow b^{y_i} \phi_i$, where $x_i = d - y_i$ with $y_i$ being the dimension of the coupling $\mu_i$. The characterization of the couplings based on their dimension is the same as the scaling variables. From the RG point of view, relevant perturbations are important. Under an infinitesimal transformation with $b \simeq 1 + \delta l$ the coupling constants
change infinitesimally and the renormalization group transformation becomes

\[ \frac{d\mu_i}{dt} = \beta_i (\{\mu\}) \]

where \( \beta \)s are called \( \beta \)-functions. To connect this with the lattice discussion one can say that the \( y \)s are the derivatives of the \( \beta \)-functions at the fixed points. The field theory can be specified by listing the coupling constants in the vicinity of a given fixed point, at a certain fixed energy scale (inverse of the length scale). The RG equations then predict the behaviour of the system at different energy scales, e.g. when one says that the QCD is asymptotically free, it means that the coupling runs to zero at very high energies\(^1\).

One way to calculate the \( \beta \)-functions is perturbation theory. In the case of critical phenomena the fixed point theory is conformally invariant. To second order in the couplings, the perturbative \( \beta \)-function can be written in terms of the conformal dimensions of the perturbing scaling operators and their conformal operator product expansion coefficients as

\[ \beta_k \simeq (1 - \Delta_k) \mu_k - \sum_{i,j} C^k_{i,j} \mu_i \mu_j \]

However, the range of perturbative RG is limited, it cannot be applied in many interesting cases, such as to describe non-perturbative fixed points. In such cases one needs to include information about the non-perturbative behaviour of the system.

Perturbative renormalization first appeared in field theory as a method to eliminate divergences in Feynman diagrams. In its most efficient formulation it is implemented by adding counterterms to the Lagrangian, which correspond to a redefinition of parameters (such as masses and coupling constants). RG equations correspond to shifting the reference point for these counterterms, and effectively implement a partial resummation of perturbation theory.

This thesis is devoted to the renormalization procedures in the truncated conformal space approach. In two space-time dimensions the conformal symmetry is so restrictive that in many interesting cases one can solve the theory exactly\(^2\). If the theory is defined on a space-time cylinder the spectrum becomes discrete. Using the CFT machinery the conformal Hilbert space and matrix elements of operators can be constructed. Considering only conformal states with energy below a given cut-off one gets a finite dimensional problem and the off-critical Hamiltonian can be diagonalized numerically. The cut-off introduced this way plays the role of the energy scale discussed before. Although the main non-perturbative features of the system can be captured using this approach, the contri-

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1. The \( \beta \)-function of QCD is known only perturbatively.
2. In rational CFTs, every correlation function can in principle be obtained by solving some differential equations and imposing sewing relations to determine the OPE coefficients.
butions of omitted states are manifest since the results depend on the cut-off. To improve the accuracy of the method, one needs to use renormalization group to understand and eliminate the cut-off dependence. In conformal field theories with relevant perturbations the higher energy degrees of freedom affect the low energy behaviour weakly, therefore these contributions can be calculated perturbatively.

In this thesis we consider two ways of renormalization in TCSA. One is the counterterm formalism in which the effect of high energy degrees of freedom is modeled by adding counterterms to the naive TCSA results. The other way is to derive renormalization group equations. In the latter method one uses effective coupling constants in the truncated space which can be calculated by letting the couplings flow from the infinite energy scale to the cut-off scale, i.e. solve the RG equations with initial conditions fixed at infinite value of the cut-off.

In principle this method can be applied to any two-dimensional statistical system with known critical CFT. Our choice was to study various aspects of the family known as $q$-state Potts model.

The counterterm formalism was applied for the three-state Potts model away from criticality in the “thermal” direction in absence of magnetic field. This work was motivated by [RSTZ13] in which the authors found discrepancy in the two-particle phase shift between the predictions of integrable $S$-matrix theory and the results of density matrix renormalization group method for the quantum chain version of the model. During this work an excited state thermodynamic Bethe ansatz (TBA) system was proposed based on the integrability of the model which is expected to describe the spectrum in finite volume. Using the counterterm formalism we were able to verify the validity of the proposed TBA system and explain the origin of the phase shift problem in [LT14].

The RG method was applied both to the Ising model and the three-state Potts model in presence of magnetic field. The magnetic field breaks the integrability and leads to the interesting phenomenon of confinement. There is a growing interest in the confinement spectrum [DG08, FZ03, FZ06, Rut05, Rut09, Rut10a, Rut10b, Rut15]. Although these models are relatively simple, the lessons learned can lead to a deeper understanding of quark-confinement in QCD. The drawback of the counterterm formalism is that one has to construct counterterms for every state and the calculation becomes very complicated for higher energy states due to the degeneracies in the conformal spectrum. Unfortunately, mesons and baryons are located in the higher part of the spectrum. RG approach combined with extrapolation can take into account the effect of high energy degrees of freedom for all states simultaneously. It was applied in [LT15], where we used the Ising model as a benchmark of the method, since it is studied in details earlier. In the three-state case we could reproduce the recently predicted WKB meson spectrum in [Rut10a].
and the more recent results for baryon masses in [Rut15].

The thesis is organized as it follows. The TCSA algorithm and the renormalization procedures used in later chapters are presented in Chapter 2. In Chapter 3 we discuss the available analytical results for the $q$-state Potts model. After discussing general aspects we present the integrable scattering theory and the excited state thermodynamic Bethe ansatz equations in the three-state case, then the confinement phenomenon is discussed. The next two chapters contain the main results of this thesis. In Chapter 4 we present the detailed spectrum comparison between counterterm renormalized TCSA and excited state TBA predictions based on [LT14]. The results of [LT15] on the confinement phenomenon in the Ising model and in the three-state Potts model in presence of magnetic field are presented in Chapter 5. In Chapter 6 we conclude and give an outlook. Appendices contain the CFT results used in the thesis and additional tables on the numerical results.
Chapter 2

Truncated conformal space approach

In the Introduction we discussed that one can treat statistical physical systems as perturbed conformal field theories. In some cases such theories are integrable and bootstrap methods can be applied [Mus09]. For general perturbations only a few methods are available. This thesis concerns one of them, namely the truncated conformal space approach (TCSA). In the present chapter we introduce the method and the recently developed and applied renormalization procedures.

The TCSA is a variational method to study perturbations of conformal field theories. It is introduced by Yurov and Zamolodchikov in 1990 to the scaling Lee–Yang model [YZ90]. One can construct the matrix elements of local fields between conformal states. In finite volume, the spectrum of a CFT is discrete. Then treating only a subspace with energy or descendant level constrained below a given cut-off, one can numerically diagonalize a finite dimensional matrix. The authors of [YZ90] used 17 basis states and calculated the matrix elements by hand and got surprisingly good results. It turned out that the convergence properties depend on the conformal weight of the perturbing operators. The more relevant the perturbation is, the faster the convergence is. In the case of the Lee–Yang model the perturbation has negative dimension. For more complicated models and less relevant perturbations one has to take into account several thousand or even more states. Fortunately the TCSA construction can be automatized and done by computers.

After the first implementation, TCSA found various applications such as the spectrum of perturbations of more complicated minimal models [LM91, KTW97], perturbed $c = 1$ models [FR98] and very recently perturbations of Wess–Zumino–Novikov–Witten CFTs [BBL+13, KPTT15]. The method is also extended to models with boundary [DPTW98] and in the presence of defects [BH14]. Besides the spectrum, form factors can be also examined [PT08a, PT08b]. The extension to higher dimensions has been recently started [HRv15].
More complicated theories and less relevant perturbations require keeping a very large number of conformal basis states in order to produce reliable results, and the convergence can still be very slow. Renormalization procedures can help to manage this problem. Counterterms can be introduced to cancel the effect of high energy degrees of freedom [GW11] or renormalization group equations can be written for the couplings in order to get the infinite cut-off results [FGP’08]. Wilson’s numerical renormalization group, which is a non-perturbative way to treat the contributions of higher energy degrees of freedom, can be also applied, since the high energy degrees of freedom are weakly coupled to the low energy ones [KA07].

The main goal of this thesis is to improve the power of the renormalization processes. The starting point of the perturbative renormalization can be understood as the application of the Schrieffer–Wolff transformation. We generalized the counterterm calculation presented in [GW11] to higher orders in the inverse cut-off and also for excited states coming from degenerate descendant conformal levels in the UV limit. Renormalization group equations in [FGP’08 GW11] are also improved taking into account the observations in [HRv15 RV15].

In the first section of this chapter, we present the TCSA algorithm. Then we study the cut-off dependence by means of the Schrieffer–Wolff transformation. After that, the general scaling function counterterm calculation will be shown. Going beyond the counterterm formalism we introduce improved renormalization group equations for the coupling constants, then we comment on the effect of higher order corrections.

2.1 TCSA algorithm

This section is based on [Tak07]. We review the basic steps of the TCSA algorithm, namely the chiral basis and matrix element generation and sewing of the left and chiral vector spaces. The inputs of the algorithm are purely conformal field theory data. The conformal Ward identities [BPZ84 Gin91 DMS97] are used to construct the chiral metric and matrix elements. The structure of the Hilbert space is fixed by modular invariance [CIZ87]. Finally, the structure constants [DF85 PZ95 Run99 Run00] are used to construct the non-chiral matrix elements. Our notations and the relevant results of two-dimensional CFT are collected in Appendix A.

The chiral TCSA algorithm in its presented form was implemented by Gábor Takács, the Gramm–Schmidt orthogonalization and the sewing process were implemented by myself. In both cases we used Wolfram Mathematica for the implementation.

1 The author is grateful to J. Cserti for pointing out the possible relation between the Schrieffer–Wolff transformation and the TCSA renormalization.
2.1.1 TCSA Hamiltonian

Let us consider the action of a perturbed conformal field theory living on a cylinder of circumference $R$

$$S = S_{CFT} + \sum_i \mu_i \int_0^R dx \int_{-\infty}^\infty d\tau \Phi_i(x,\tau)$$

(2.1.1)

where $(x, \tau)$ are Euclidean space and time coordinates. $S_{CFT}$ is the action of the conformal field theory with central charge $c$, $\Phi_i$s are spinless primary fields of conformal weights $h_i = \bar{h}_i$ and $\mu_i$s are dimensionful couplings. The Hamiltonian in complex coordinates reads

$$H_{pl} = \frac{2\pi}{R} \left( L_0 + \bar{L}_0 - \frac{c}{12} \right) + \frac{2\pi}{R} \sum_i \mu_i \frac{R^{2-2h}}{(2\pi)^{2-2h}} \int_0^{2\pi} d\theta |\Phi_{i,pl}(e^{i\theta}, e^{-i\theta})|^2$$

(2.1.2)

The matrix element between two conformal states $|\Psi_a\rangle = |(h_a, \bar{h}_a), \{n_a, \bar{n}_a\}\rangle$ and $|\Psi_b\rangle = |(h_b, \bar{h}_b), \{n_b, \bar{n}_b\}\rangle$ is then

$$\tilde{H}_{ab} = \frac{2\pi}{R} \left( \Delta_a - \frac{c}{12} \right) \delta_{h_a,h_b} \delta_{\bar{h}_a,\bar{h}_b} \delta_{N_a,\bar{N}_a} \delta_{\bar{N}_a,\bar{N}_b} G_{ab} + \frac{2\pi}{R} \sum_i \mu_i \frac{R^{2-2h}}{(2\pi)^{1-2h}} \langle \Psi_a | \Phi_{pl}(1,1) | \Psi_b \rangle$$

(2.1.3)

where the integration is carried out since the perturbations are spinless, hence the two states have to have the same conformal spin, and

$$\Delta_a = N_a + \bar{N}_a + h_a + \bar{h}_a$$

(2.1.4)

The matrix $G_{ab}$ is the metric (Gramm matrix) at level $N_a$. For numerical calculations one has to use dimensionless Hamiltonian which can be easily done by dividing a mass scale $m$ (usually the lowest mass in the theory if it is known). One has to take into account that the states in the standard Virasoro representation are not orthonormal thus in numerical diagonalization it has to compensated by the inverse of the metric

$$h_{ab} = \frac{2\pi}{r} \left( N_a + \bar{N}_a + h_a + \bar{h}_a - \frac{c}{12} \right) \delta_{a,b} + \sum_i \lambda_i \left( G^{-1} B \right)_{ab} \delta_{s_a,s_b}$$

(2.1.5)

where $G$ is the metric and $B_{ab} = \langle \Psi_a | \Phi_{pl}(1,1) | \Psi_b \rangle$. We introduced the notations

$$r = mR$$

$$\lambda_i = \frac{\mu_i}{(2\pi)^{1-2h}}$$

(2.1.6)

Note that one can get rid of the inverse metric by orthogonalizing the basis.
In order to construct the TCSA Hamiltonian and diagonalize it one has to generate the conformal basis then calculate the metric and the matrix $B_{ab}$. It can be relatively easily done automatically by implementing the algorithm presented in the next sections.

### 2.1.2 Chiral basis generation

The Hilbert space of a CFT can be decomposed to tensor products of left and right Verma modules. A Verma module itself built in a highest weight vector, created by a primary field of the CFT. In this thesis only minimal CFTs are considered, where the decomposition is simply a direct sum. Therefore the complete Hilbert space can be written as

$$
\mathcal{H}_{CFT} = \bigoplus_{i,j} \mathcal{N}_{ij} M_i \otimes \bar{M}_j
$$

where the direct sum runs over the possible primary operators and the values of the integer numbers $\mathcal{N}_{ij}$ are determined by modular invariance \[CIZ87\].

Let us now turn our attention to the case of one simple chiral Verma module built on a highest weight state $|h\rangle$ which is created by a primary field of dimension $h$. It contains states of the form

$$
L_{-n_1}L_{-n_2} \ldots L_{-n_k} |h\rangle
$$

By means of the Virasoro algebra the order of the indices can be specified. We use the canonical ordering $n_1 \geq n_2 \geq \ldots \geq n_k$. The scalar product of such states can be calculated using the fact that

$$
L_n^\dagger = L_{-n}
$$

and the Virasoro commutation relations. Consider two vectors $\Psi$ and $\Psi'$ in Verma modules built on $|h\rangle$ and $|h'\rangle$. The scalar product vanishes if $h \neq h'$ or the descendant levels are different. The normalization is fixed by the highest-weight vectors as

$$
\langle h|h'\rangle = \delta_{h,h'}
$$

If the descendant level is non-zero one can say that $\Psi' = L_{-m} \Psi''$ with some $m > 0$. Then using (2.1.9) one can write

$$
\langle \Psi'|\Psi\rangle = \langle \Psi''|L_m \Psi\rangle
$$

Using the Virasoro commutation relations the generator $L_m$ can be sent to the left until it reaches the highest weight vector which is annihilated by it. Then the remaining terms are scalar products of lowest level vectors. This procedure can be implemented recursively with starting point (2.1.10).
Naively the basis vectors of a given descendant level \( N \) can be generated by listing all of the integer partitions of \( N \). However there are null vectors which have to factored out from the Hilbert space. The null vectors have zero length and they are perpendicular to all other vectors. At this point let us assume that we know the basis with the null vectors factored out up to level \( N - 1 \). Consider a CFT with central charge \( c \) and a Verma module built on the highest weight state \( h \). \( \text{Bas}_k(h, c) \) is the basis at descendant level \( k \) of this module. The basis at level \( N \) can be generated as

\[
\widetilde{\text{Bas}}_N(h, c) = \bigoplus_{k=1}^{N} L^{-k} \text{Bas}_{N-k}(h, c) \tag{2.1.12}
\]

where the tilde means that potential null vectors can be found in this set. To get rid of them, one can get the new basis elements one-by-one and calculate the Gramm matrix and its determinant after adding each new vector. If the Gramm matrix is turned to be degenerate the new vector can be dropped out. It can be proven that in this way the complete basis can be generated up to a given descendant level. This method can be continued to any descendant levels, only limited by memory and time.

### 2.1.3 Chiral matrix elements

Once one has the basis, the next step is to construct the TCSA matrix (2.1.5). First we can construct the chiral matrix elements and later the tensor product structure can be recovered from the chiral data.

In particular the quantities what we are looking for are of the form

\[
\langle \Phi | \mathcal{O} | \Psi \rangle
\]

where \( \mathcal{O} \) is a localized operator and \( \Phi \) and \( \Psi \) are elements of \( \mathcal{H}_{\text{CFT}} \). Here we consider only chiral operators and states. In general we can have three different chiral operators

\[
\mathcal{O}_a(z) = (L_{-n_1} L_{-n_2} \ldots L_{-n_k} \phi_h) (z) \\
\mathcal{O}_b(z) = (L_{-n'_1} L_{-n'_2} \ldots L_{-n'_k} \phi_h) (z) \\
\mathcal{O}_c(z) = (L_{-n''_1} L_{-n''_2} \ldots L_{-n''_k} \phi_h) (z) \tag{2.1.13}
\]

where \( \phi_h \) is a chiral primary field of conformal weight \( h \). The states corresponding to these operators can be denoted as

\[
| \Psi_a \rangle = \mathcal{O}_a(0) | 0 \rangle; \quad | \Psi_b \rangle = \mathcal{O}_b(0) | 0 \rangle; \quad | \Psi_c \rangle = \mathcal{O}_c(0) | 0 \rangle \tag{2.1.14}
\]
Then for constructing the TCSA matrix we need the following

\[ K_{ab}^c = \langle \Psi_a | O_b (1) | \Psi_c \rangle \]  

(2.1.15)

As for the metric, first we get rid of the generators on the left hand side, since the state \( |\Psi_a\rangle \) can written again as \( |\Psi'_a\rangle = L_{-m} |\Psi'_a\rangle \). Using the conformal Ward-identities one can write the action of a the generator as

\[ K_{ab}^c = - \left\langle \Psi'_a \left| \oint_{\infty} \frac{d\zeta}{2\pi i} \zeta^{m+1} T (\zeta) O_b (1) \right| \Psi'_c \right\rangle \]  

(2.1.16)

Then the integration contour can be deformed and it falls into two around the singularities at \( \zeta = 1 \) and \( \zeta = 0 \). Using radial ordering we have

\[ K_{ab}^c = \left\langle \Psi'_a \left| \oint_{0} \frac{d\zeta}{2\pi i} \zeta^{m+1} T (\zeta) O_b (1) \right| \Psi'_c \right\rangle + \left\langle \Psi'_a \left| O_b (1) \oint_{0} \frac{d\zeta}{2\pi i} \zeta^{m+1} T (\zeta) \right| \Psi'_c \right\rangle \]  

(2.1.17)

In the first term the function \( \zeta^{m+1} \) can be expanded to binomial series around 1, then recognizing the form of Virasoro generators in both terms we can write

\[ K_{ab}^c = \sum_{k=-1}^{\infty} \left( \frac{m+1}{k+1} \right) \langle \Psi'_a | (L_k O_b) (1) | \Psi'_c \rangle + \langle \Psi'_a | O_b (1) | L_m \Psi'_c \rangle \]  

(2.1.18)

Using now the algorithm discussed in the case of the scalar product the remaining generators can be eliminated in the second term. The remaining terms are already calculated if one applies recursion. The same procedure can be repeated until the left vector becomes primary. In the same way one can show that

\[ \langle \Psi_a | O_b (1) L_{-m} | \Psi'_c \rangle = \langle L_m \Psi_a | O_b (1) | \Psi'_c \rangle - \sum_{k=-1}^{\infty} \left( \frac{-m+1}{k+1} \right) \langle \Psi_a | (L_k O_b) (1) | \Psi'_c \rangle \]  

(2.1.19)

so the above procedure can be carried out for the right vector as well. One note: the generator \( L_{-1} \) bounces between the left and right vectors

\[ \langle \Psi_a | (L_{-1} O_b) (1) | \Psi'_c \rangle = - \langle \Psi_a | O_b (1) L_{-1} | \Psi'_c \rangle \]  

(2.1.20)

It is worth to collect them before the operator, since the action of \( L_{-1} \) is derivation with respect to the variable of the operator which on it acts. At the end there are primaries
on both sides and one can write that

$$\left\langle \phi_a \left| (L_{-1}^+ \phi_b) (1) \right| \phi_c \right\rangle = \left( \frac{\partial}{\partial z} \right)^k \left. \left\langle \phi_a \left| \phi_b (z) \right| \phi_c \right\rangle \right|_{z=1}^k$$

$$= \left( \frac{\partial}{\partial z} \right)^k \left. \frac{1}{z^{h_b-h_c+h_a}} \right|_{z=1}^k$$

$$= (h_a - h_b - h_c) \ldots (h_a - h_b - h_c - 1) \ldots (h_a - h_b - h_c - k + 1)$$

(2.1.21)

The normalization of the chiral three point functions is a matter of convention, since these so-called chiral vertex operators are non-physical. Of course when one sews together the two chiral algebras, the correct structure constants should be used as it is shown in the next section.

### 2.1.4 Left-right sewing and Gramm–Schmidt orthogonalization

The Hilbert space of a minimal model consists of direct sum of tensor products of Verma modules. In the previous section the construction of chiral matrix elements is presented. Since the perturbations are spinless, its matrix elements vanish when the conformal spins of the bra- and ket vectors are different. In this way the Verma modules can be decomposed into different spin sectors

$$\mathcal{V}_h \otimes \mathcal{V}_{\tilde{h}} = \bigoplus_s V_s \quad s = (h - \tilde{h}) \pm 0, \pm 1, \pm 2 \ldots$$

(2.1.22)

For our purposes only the spin zero sector is sufficient. Imposing descendant level cut-off $n$ the truncated space can be constructed as a direct sum

$$\mathcal{H}_{TCSA} = \bigoplus_{n, \tilde{n}} \text{Bas}_n (h, c) \otimes \text{Bas}_{\tilde{n}} (\tilde{h}, c) \quad \min (n, \tilde{n}) \leq k; \quad h + n = \tilde{h} + \tilde{n}$$

(2.1.23)

with taking into account the multiplicities in [2.1.7].

The non-chiral matrix elements can be constructed in the following way. Consider two non-chiral states $\Psi_a = \Psi_{a,L} \otimes \Psi_{a,R}$ and $\Psi_b = \Psi_{b,L} \otimes \Psi_{b,R}$ and the non-chiral operator $\mathcal{O}_{c, (h, \tilde{h})}$. Then the matrix element is written as

$$\left\langle \Psi_a \left| \mathcal{O}_{c, (h, \tilde{h})} (1, 1) \right| \Psi_b \right\rangle = C_{c,a,b}^c K_{c,a,b,L} K_{c,a,b,R}$$

(2.1.24)

where $C_{c,a,b}^c$ is the structure constant, $K_{c,a,b,L}^c$ and $K_{c,a,b,R}^c$ can be calculated in a way presented in the previous section.

A final remark is that though basis vectors of Verma modules built on different highest
weight states and the vectors from different levels are orthogonal to each other, in a certain level of a module the basis generated by the above procedure is not orthonormal. For the diagonalization it means that the numerical “action” matrix is the one which should be diagonalized, thus the inverse of the metric has to be taken into account as in (2.1.5). To calculate matrix elements (for example to measure form factors), the eigenvectors also should be rotated by the inverse metric. If one would like to implement the numerical renormalization group, in each step all the matrix parts should be carefully rotated. To avoid this problem, one can use Gramm–Schmidt orthogonalization. Since the fused Hilbert space can have huge dimension it can take much time. However, it is possible to carry out orthogonalization directly at the level of chiral basis states, which makes the process much more efficient.

2.2 Cut-off dependence

To get a finite dimensional Hilbert space we introduced an ultraviolet cutoff. Due to the structure of the conformal Hilbert space, there are two natural ways to perform it. One is to cut-off at a given conformal energy $E_{\text{cut}}$ the other is to truncate at a given descendant level $n$. The effect of these two differs when there are irrelevant primary fields in the CFT. Using energy cut-off is somewhat more natural but descendant level cut-off can be treated analytically in perturbation theory. In what follows the cut-off procedure is not specified, it is chosen to be general and denoted by $\Lambda$.

It is obvious that the results of the TCSA depend on the cut-off. If we can model the cut-off dependence we are in position to set up renormalization schemes and cancel the dependence. In this section we show how the cut-off dependence can be described by means of the Schrieffer–Wolff transformation.

2.2.1 The Schrieffer–Wolff transformation

Suppose that we have the following Hamiltonian

$$H = H_{\text{CFT}} + \lambda V$$  \hspace{1cm} (2.2.1)

acting on a Hilbert-space of the form

$$\mathcal{H} = \mathcal{H}_l \bigoplus \mathcal{H}_h$$  \hspace{1cm} (2.2.2)
with a low- and a high energy subspace (states below and above the cut-off). The Hamiltonian can be written in the block matrix form as

\[ H = \begin{pmatrix} H_{ll} & H_{lh} \\ H_{hl} & H_{hh} \end{pmatrix} \]

(2.2.3)

\[ H = H_{ll} + H_{lh} + H_{hl} + H_{hh} \]

(2.2.4)

\[ H_{low} = H_{CF T} + \lambda \Delta H \]

where \( H_{low} \) includes the diagonal part in the high-energy subspace, and \( \Delta H \) includes only interactions. In the renormalization process our aim is to have an effective Hamiltonian of the form

\[ H_{eff} = H_{low} + M \]

(2.2.5)

which acts only on the low energy subspace and have the same spectrum as the full Hamiltonian (at least for the low lying states). This can be done by the Schrieffer–Wolff transformation [SW66, BDL11].

Let \( \mathcal{H}_l \) be the Hilbert-space spanned by the eigenstates of the “unperturbed” Hamiltonian \( H_0 \), which can be either \( H_{CF T} \) or \( H_{low} \). The effective Hamiltonian can be written as an expansion in the coupling

\[ H_{eff} = H_0 P_0 + \lambda P_0 V P_0 + \sum_{n=2}^\infty \lambda^n H_{eff,n} \]

(2.2.6)

where \( P_0 \) is the projector onto \( \mathcal{H}_l \) and \( Q_0 = 1 - P_0 \) projects to the high-energy subspace \( \mathcal{H}_h \). The operations

\[ \mathcal{O} (X) = P_0 X Q_0 + Q_0 X P_0 \]

\[ \mathcal{D} (X) = P_0 X P_0 + Q_0 X Q_0 \]

(2.2.7)

perform the decomposition of an operator \( X \) into block-offdiagonal \( (X_{od}) \) and block-diagonal \( (X_d) \) parts with respect to the projector \( P_0 \). Let us define the following operation

\[ \mathcal{L} (X) = \sum_{i,j} \frac{\langle i | \mathcal{O} (X) | j \rangle}{E_i - E_j} | i \rangle \langle j | \]

(2.2.8)

where \( | i \rangle \) and \( | j \rangle \) are eigenstates of \( H_0 \) with eigenvalues \( E_i \) and \( E_j \) respectively. According to [BDL11] the leading term in (2.2.6) can be written as

\[ H_{eff,2} = \frac{1}{2} P_0 [\mathcal{L} (V_{od}), V_{od}] P_0 \]

(2.2.9)
Let us calculate the diagonal matrix element of this operator \( \langle \Psi | H_{\text{eff,2}} | \Psi \rangle \) where \( | \Psi \rangle \) is an eigenstate of \( H_0 \) living in \( \mathcal{H}_l \):

\[
\langle \Psi | H_{\text{eff,2}} | \Psi \rangle = \frac{1}{2} \langle \Psi | [\mathcal{L}(V_{\text{od}}), V_{\text{od}}] | \Psi \rangle \\
= \sum_{i,j} \frac{1}{2} \langle \Psi | \left[ \frac{i | V_{\text{od}} \rangle_j}{E_i - E_j} | i \rangle \langle j |, V_{\text{od}} \right] | \Psi \rangle \\
= \sum_{i,j} \frac{1}{2} \langle \Psi | \langle i | V_{\text{od}} | j \rangle E_i - E_j | i \rangle \langle j | V_{\text{od}} | i \rangle E_i - E_j | j \rangle | \Psi \rangle
\]

(2.2.10)

The first term in the summand can be rewritten as:

\[
\begin{align*}
\langle \Psi | \frac{i | V_{\text{od}} \rangle_j}{E_i - E_j} | i \rangle \langle j | V_{\text{od}} | \Psi \rangle &= \frac{i | V_{\text{od}} \rangle_j}{E_i - E_j} \langle \Psi | i \rangle \langle j | V_{\text{od}} | \Psi \rangle \\
&= \delta_{\Psi,i} \frac{i | V_{\text{od}} \rangle_j}{E_i - E_j} \langle j | V_{\text{od}} | \Psi \rangle
\end{align*}
\]

(2.2.11)

It is non-vanishing if \( |j \rangle \) lives in \( \mathcal{H}_h \). Similarly the second term

\[
- \langle \Psi | V_{\text{od}} | i \rangle \langle j | \frac{i | V_{\text{od}} \rangle_j}{E_i - E_j} | \Psi \rangle = - \frac{i | V_{\text{od}} \rangle_j}{E_i - E_j} \langle \Psi | V_{\text{od}} | i \rangle \langle j | \Psi \rangle \\
&= \delta_{\Psi,j} \frac{i | V_{\text{od}} \rangle_j}{E_i - E_j} \langle i | V_{\text{od}} | \Psi \rangle
\]

(2.2.12)

which is non-vanishing if \( |i \rangle \) lives in \( \mathcal{H}_h \). After performing the summation in the first term with respect to \( i \), and with respect to \( j \) in the second, and change the remaining summation index in the first from \( j \) to \( i \) we get the same form, so the \( \frac{1}{2} \) factor cancels and we get

\[
\langle \Psi | H_{\text{eff,2}} | \Psi \rangle = \sum_{i \in \mathcal{H}_h} \frac{\langle \Psi | V_{\text{od}} | i \rangle \langle i | V_{\text{od}} | \Psi \rangle}{E_\Psi - E_i}
\]

(2.2.13)

Using \( H_{\text{CFT}} \) the conformal Hamiltonian as \( H_0 \) and the conformal Hilbert state, it gives back the contribution used in [GW11, LT14], while choosing the interacting low energy Hamiltonian \( H_{\text{low}} \) and the its eigenstates gives the prescription used in [HRv15, RV15].

### 2.2.2 Cut-off dependence of energy levels

Once we have a Hamiltonian at a fixed cut-off \( \Lambda \), the question is how to calculate \( \langle 2.2.9 \rangle \). To do so we first calculate the contribution of the states at the shell \( (\Lambda, \Lambda + \Delta \Lambda) \). Recalling \( 2.2.9 \)

\[
H_{\text{eff,2}} = \frac{1}{2} P_0 [\mathcal{L}(W_{\text{od}}), W_{\text{od}}] P_0
\]

(2.2.14)
where
\[ \mathcal{L}(X) = \sum_{i,j} \frac{\langle i | \mathcal{O}(X) | j \rangle}{E_i - E_j} |i\rangle \langle j| \]  
(2.2.15)

and now \( P_0 \) projects to the low-energy subspace (below the cut off). The high energy states are now the states at the shell. It is obvious that in (2.2.15) the terms are non-vanishing if \( |i\rangle \in \mathcal{H}_l \) and \( |j\rangle \in \mathcal{H}_h \) or vice versa. So for calculating (2.2.14) one can split the summation into two parts
\[
\mathcal{L}(W_{od}) = \sum_{i \in \mathcal{H}_l, j \in \mathcal{H}_h} \frac{\langle i | W_{od} | j \rangle}{E_i - E_j} |i\rangle \langle j| + \sum_{i \in \mathcal{H}_h, j \in \mathcal{H}_l} \frac{\langle i | W_{od} | j \rangle}{E_i - E_j} |i\rangle \langle j| = \mathcal{L}^{(1)}(W_{od}) + \mathcal{L}^{(2)}(W_{od})
\]
(2.2.16)

where we used the fact that \( \mathcal{O}(W_{od}) = W_{od} \). Let us consider the first summation, where the denominator \( E_i - E_j < 0 \).
\[
\mathcal{L}^{(1)}(W_{od}) = \sum_{i \in \mathcal{H}_l, j \in \mathcal{H}_h} \frac{\langle i | W_{od} | j \rangle}{E_i - E_j} |i\rangle \langle j| = \sum_{i \in \mathcal{H}_l, j \in \mathcal{H}_h} |i\rangle \langle i| \frac{1}{E_i - E_j} W_{od} |j\rangle \langle j| \]
(2.2.17)

Schwinger’s proper time representation can be applied to get the contribution of this part to (2.2.14) and it leads to the following:
\[
\begin{align*}
\frac{1}{2} P_0 [\mathcal{L}^{(1)}(W_{od}) , W_{od}] P_0 &= \\
= \frac{1}{2} \sum_{i \in \mathcal{H}_l, j \in \mathcal{H}_h} P_0 |i\rangle \langle i| \frac{1}{E_i - E_j} W_{od} |j\rangle \langle j| W_{od} P_0 \\
- \frac{1}{2} \sum_{i \in \mathcal{H}_l, j \in \mathcal{H}_h} P_0 W_{od} |i\rangle \langle i| \frac{1}{E_i - E_j} W_{od} |j\rangle \langle j| P_0 \\
= - \frac{1}{2} \sum_{i \in \mathcal{H}_l, j \in \mathcal{H}_h} \int_0^\infty d\tau e^{(E_i - E_j)\tau} P_0 |i\rangle \langle i| W_{od} |j\rangle \langle j| W_{od} P_0 \\
+ \frac{1}{2} \sum_{i \in \mathcal{H}_l, j \in \mathcal{H}_h} \int_0^\infty d\tau e^{(E_i - E_j)\tau} P_0 W_{od} |i\rangle \langle i| W_{od} |j\rangle \langle j| P_0 \\
= - \frac{1}{2} \sum_{i \in \mathcal{H}_l, j \in \mathcal{H}_h} \int_0^\infty d\tau P_0 |i\rangle \langle i| e^{H_\tau} W_{od} e^{-H_\tau} |j\rangle \langle j| W_{od} P_0 \\
+ \frac{1}{2} \sum_{i \in \mathcal{H}_l, j \in \mathcal{H}_h} \int_0^\infty d\tau P_0 W_{od} |i\rangle \langle i| e^{H_\tau} W_{od} e^{-H_\tau} |j\rangle \langle j| P_0 \\
= - \frac{1}{2} \int_0^\infty d\tau P_0 P_0 e^{H_\tau} W_{od} e^{-H_\tau} Q_0 W_{od} P_0 \\
+ \frac{1}{2} \int_0^\infty d\tau P_0 W_{od} P_0 e^{H_\tau} W_{od} e^{-H_\tau} Q_0 P_0 \\
= - \frac{1}{2} \int_0^\infty d\tau P_0 e^{H_\tau} W_{od} e^{-H_\tau} Q_0 W_{od} P_0 \\
\end{align*}
\]
(2.2.18)
Similarly for the second term (note that now $E_i - E_j > 0$)

\[
\frac{1}{2} P_0 \left[ \mathcal{L}^{(2)}(W_{od}), W_{od} \right] P_0 = \\
= \frac{1}{2} \sum_{i \in H_h, j \in H_l} P_0 |i\rangle \langle i| \frac{1}{E_i - E_j} W_{od} |j\rangle \langle j| W_{od} P_0 \\
- \frac{1}{2} \sum_{i \in H_h, j \in H_l} P_0 W_{od} |i\rangle \langle i| \frac{1}{E_i - E_j} W_{od} |j\rangle \langle j| P_0 \\
= \frac{1}{2} \sum_{i \in H_h, j \in H_l} P_0 |j\rangle \langle j| \frac{1}{E_j - E_i} W_{od} |i\rangle \langle i| W_{od} P_0 \\
- \frac{1}{2} \sum_{i \in H_h, j \in H_l} P_0 W_{od} |j\rangle \langle j| \frac{1}{E_j - E_i} W_{od} |i\rangle \langle i| P_0 \\
= \frac{1}{2} \sum_{i \in H_h, j \in H_l} \int_0^\infty d\tau e^{(E_i - E_j)\tau} P_0 |j\rangle \langle j| W_{od} |i\rangle \langle i| W_{od} P_0 \\
- \frac{1}{2} \sum_{i \in H_h, j \in H_l} \int_0^\infty d\tau e^{(E_i - E_j)\tau} P_0 W_{od} |j\rangle \langle j| W_{od} |i\rangle \langle i| P_0 \\
= \frac{1}{2} \sum_{i \in H_h, j \in H_l} \int_0^\infty d\tau P_0 |j\rangle \langle j| e^{-H\tau} W_{od} e^{H\tau} |i\rangle \langle i| W_{od} P_0 \\
- \frac{1}{2} \sum_{i \in H_h, j \in H_l} \int_0^\infty d\tau P_0 W_{od} |j\rangle \langle j| e^{-H\tau} W_{od} e^{H\tau} |i\rangle \langle i| P_0 \\
= \frac{1}{2} \int_0^\infty d\tau P_0 Q_0 e^{-H\tau} W_{od} e^{H\tau} P_0 W_{od} P_0 \\
- \frac{1}{2} \int_0^\infty d\tau P_0 W_{od} Q_0 e^{-H\tau} W_{od} e^{H\tau} P_0 P_0 \\
= -\frac{1}{2} \int_0^\infty d\tau P_0 W_{od} Q_0 e^{-H\tau} W_{od} e^{H\tau} P_0 \\
= -\frac{1}{2} \int_{-\infty}^0 d\tau P_0 W_{od} Q_0 e^{H\tau} W_{od} e^{-H\tau} P_0 \\
= (2.2.19)
\]
Using that $W_{od} = W_{od}(0)$ and $e^{H\tau} W_{od}(0) e^{-H\tau} = W_{od}(\tau)$, and since the projectors $P_0$ and $Q_0$, the off-diagonal indices can be omitted

\[ H_{\text{eff,2}} = \frac{1}{2} P_0 [\mathcal{L} (W_{od}), W_{od}] P_0 = -\frac{1}{2} \int_{-\infty}^{0} d\tau P_0 W(0) Q_0 W_{od}(\tau) P_0 \]

\[ -\frac{1}{2} \int_{0}^{\infty} d\tau P_0 W(\tau) Q_0 W(0) P_0 \]

\[ = -\frac{1}{2} T \left\{ \int_{-\infty}^{\infty} d\tau P_0 W(0) Q_0 W(\tau) P_0 \right\} \quad (2.2.20) \]

One can eliminate the factor of $\frac{1}{2}$ using time reversal symmetry. Writing now $P_0 = P_\Lambda$ projects to the states below the cut-off and $Q_0 = \tilde{P}_{\Lambda + \Delta \Lambda}$ projects to the shell above the cut-off to get

\[ H_{\text{eff,2}} (\Lambda + \Delta \Lambda) = -\int_{0}^{\infty} d\tau P_\Lambda W(\tau) \tilde{P}_{\Lambda + \Delta \Lambda} W(0) P_\Lambda \quad (2.2.22) \]

\[ = -\sum_{i,j} \mu_i \mu_j \int_{0}^{\infty} dx_1 d\tau_1 \int_{0}^{R} dx_2 P_\Lambda \Phi_i (x_1, \tau_1) \tilde{P}_{\Lambda + \Delta \Lambda} \Phi_j (x_2, 0) P_\Lambda \]

where we used the form of the perturbation $W(\tau) = \sum_i \mu_i \int_{0}^{R} dx \Phi_i (x, \tau)$. Then we can model the contribution of the shell $(\Lambda, \Lambda + \Delta \Lambda)$ as

\[ H_{\text{eff}} (\Lambda, \Lambda + \Delta \Lambda) = \]

\[ = H_{\text{CFT}} P_\Lambda + \sum_i \mu_i \int_{0}^{R} dx P_\Lambda \Phi_i (x, 0) P_\Lambda \]

\[ -\sum_{i,j} \mu_i \mu_j \int_{0}^{\infty} dx_1 d\tau_1 \int_{0}^{R} dx_2 P_\Lambda \Phi_i (x_1, \tau_1) \tilde{P}_{\Lambda + \Delta \Lambda} \Phi_j (x_2, 0) P_\Lambda + \mathcal{O} (\{\mu^2\}) \]

\[ = H_{\text{TCFA}} + M (\Lambda) \quad (2.2.23) \]

This additional term can be interpreted as a non-local counterterm.

There is one subtlety here. The time evolution of the operators depends on the choice of the basis in the subspace used to construct $2.2.15$. In the conformal basis the local operators are conformal ones and CFT operator product expansion can be used. This choice is the one which is used in 2.3. Otherwise the phase factors should be compensated in order to have conformal operators, this compensation is discussed in 2.4.

In [LT14, GW11] this result is used to construct counterterms to energy levels. In these works the basis in the above calculation was chosen to be the conformal one. Ultraviolet limits of given states in the perturbed model are also conformal states. The cut-off
dependence of the levels than constructed as
\[ E_i (\Lambda + \Delta \Lambda) - E_i (\Lambda) = \text{UV} \langle i | M (\Lambda) | i \rangle_{\text{UV}} + \mathcal{O} (\lambda^2) \]  

(2.2.24)

where \( E_i (\Lambda) \) is the exact energy at truncation \( \Lambda \) of the \( i \)th state, while the state \( |i\rangle_{\text{UV}} \) is the UV limiting state.

In the works [HRv15, RV15] the authors used a different prescription; namely, they chose the low-energy basis as the eigensystem of the perturbed theory at cut-off \( \Lambda \) and used the corresponding eigenvectors in (2.2.24) instead of the conformal states.

In what follows we present the detailed calculation of (2.2.24) from [LT13]. Then the renormalization group interpretation of the level contributions is presented.

### 2.3 Counterterm formalism

In this section we present the energy counterterm construction used in [LT13] based on the result (2.2.24). This calculation is based on the one in [GW11], but it goes beyond the first order in the inverse cut-off and it is used for descendant and degenerate conformal states as well. Examples and applications in the thermally perturbed three-state Potts model can be found in Chapter 4.

Throughout this section we use descendant level truncation scheme as the abstract \( \Lambda \) cut-off of the previous section. The cut-off is chosen to be the level \( n - 1 \), the highest allowed descendant level in the truncated space, and the the \( n \)th level plays the role of the shell. Since the counterterm formalism is applied only for the thermally perturbed three-state Potts model, we consider only one perturbation, but the results can be easily generalized.

#### 2.3.1 General considerations

The TCSA provides a non-perturbative tool to handle perturbed conformal field theories, and the aim of the TCSA renormalization procedure is to speed up the convergence of the method and also to deal with ultraviolet divergences when necessary. Since the perturbation is supposed to be relevant, the running coupling flows to zero at high energies. As a result, the influence of the high energy degrees of freedom can be treated perturbatively. Suppose we consider a quantity \( Q \), for which TCSA with a cut-off at level \( n \) gives \( Q_{\text{TCSA}}(n) \) and let us write the exact value as follows:

\[ Q = Q_{\text{TCSA}}(n) + \delta Q(n) \]  

(2.3.1)
where $\delta Q(n)$ is a counterterm which can either go to zero (in the convergent case) or even be divergent when $n$ increases. The counterterm can be constructed by computing the contribution $Q_l$ of the $l$th level

$$Q = Q_{TCSA}(n) + \sum_{l=n+1}^{\infty} Q_l$$

$$= Q_{TCSA}(n) + \sum_{l=1}^{\infty} Q_l - \sum_{l=1}^{n} Q_l$$  \hspace{1cm} (2.3.2)

therefore the counterterm can be written as

$$\delta Q(n) = \sum_{l=1}^{\infty} Q_l - \sum_{l=1}^{n} Q_l$$  \hspace{1cm} (2.3.3)

Depending on the weight $h$ of the perturbation, the first sum on the second line can be either convergent or divergent. In the divergent case it is necessary to use an appropriate regularization method and renormalization scheme. For example, in integrable field theories we usually compare our results to predictions of the exact $S$-matrix or form factors resulting from the bootstrap, or to scaling functions predicted by the thermodynamic Bethe ansatz. In many cases, the model we investigate is just a member of a family of perturbed CFTs, where $h$ varies across the range of possible theories, and the exact predictions depend analytically on $h$. Therefore the relevant scheme is provided by analytically continuation from the range of parameter space where the theory is ultraviolet finite ($h < 1/2$).

### 2.3.1.1 Counterterms for scaling functions

If we choose our quantity $Q$ as a finite volume energy level $E_{\Psi,n}(R)$ and substitute

$$W(\tau) = \mu \int_0^R dx \Phi(\tau,x)$$  \hspace{1cm} (2.3.4)

we obtain (see (2.2.24))

$$E_{\Psi,n}(R) - E_{\Psi,n-1}(R) =$$

$$= -\mu^2 \int_0^R dx \int_0^R dx' \int_0^\infty d\tau \langle \Psi|\Phi(\tau,x)\hat{P}_n\Phi(0,x')|\Psi\rangle_{CFT} + O(\mu^3)$$

$$= -\mu^2 R \int_0^R dx \int_0^\infty d\tau \langle \Psi|\Phi(\tau,x)\hat{P}_n(\Psi)\Phi(0)|\Psi\rangle_{CFT} + O(\mu^3)$$  \hspace{1cm} (2.3.5)
where $\tilde{P}_n$ is the projector on states at level $n$, and we used translation invariance to eliminate one spatial integral, which in turn restricts the intermediate states to ones which have the same Lorentz momentum (or conformal spin) as $\Psi$; the corresponding restricted projector is denoted by $\hat{P}_n(\Psi)$. Passing to the scaling function we obtain

$$e_{\Psi,n}(r) - e_{\Psi,n-1}(r) = -\frac{\mu^2}{2\pi} \int_0^R dx \int_0^\infty d\tau \langle \Phi(\tau, x) \hat{P}_n(\Psi) \Phi(0, 0) | \Psi \rangle_{\text{CFT}} + O(\mu^3)$$

(2.3.6)

Mapping this expression on the conformal plane and using the definition [2.1.6] we finally get

$$e_{\Psi,n}(r) - e_{\Psi,n-1}(r) = -\frac{\lambda^2}{2\pi} \int_{|z|<1} d^2z (z\bar{z})^{h-1} \langle \Phi(1, 1) \hat{P}_n(\Psi) \Phi(z, \bar{z}) | \Psi \rangle_{\text{CFT}} + O(\lambda^3)$$

(2.3.7)

It is clear that in order to evaluate the counterterm it is necessary to construct the contribution of a given level $n$ to the conformal correlators. Note that here we used the conformal basis to construct [2.2.15]. Therefore the local operators are conformal operators, hence in what follows we can use CFT techniques in order to deal with the four-point function in (2.3.7).

2.3.1.2 Evaluating the level contributions

As pointed out in [BBL+13], the most systematic way to obtain the level contributions is by considering the Kallen–Lehmann spectral representation. In general the unperturbed state $|\Psi\rangle$ can be written as a linear combination of conformal states; this is necessary to allow for degenerate perturbation theory, which is relevant due to the high degeneracy in the conformal Hilbert space. Therefore we consider the two-point function of the perturbation between two conformal states $|i, \{n_i, \bar{n}_i\}, \alpha_i\rangle$ and $|j, \{n_j, \bar{n}_j\}, \alpha_j\rangle$, which are from conformal modules with conformal weights $(h_i, \bar{h}_i)$ and $(h_j, \bar{h}_j)$ and have descendant levels $(n_i, \bar{n}_i)$ and $(n_j, \bar{n}_j)$; the $\alpha_{i,j}$ index a basis in the conformal modules at the given level. Inserting a complete set of states we obtain

$$\langle i, \{n_i, \bar{n}_i\}, \alpha_i | \Phi(0, 0) \Phi(\tau, x) | j, \{n_j, \bar{n}_j\}, \alpha_j \rangle =$$

$$= \sum_{k,n,\alpha} \langle i, \{n_i, \bar{n}_i\}, \alpha_i | \Phi(0, 0) | k, \{n, \bar{n}\}, \alpha \rangle \langle k, \{n, \bar{n}\}, \alpha | \Phi(\tau, x) | j, \{n_j, \bar{n}_j\}, \alpha_j \rangle$$

(2.3.8)

where the states $|k, \{n, \bar{n}\}, \alpha\rangle$ form an orthonormal basis of the conformal module with conformal weight $(h_k, \bar{h}_k)$ at descendant level $(n, \bar{n})$. Note that translational invariance (via the spatial integrals) enforces

$$h_i - \bar{h}_i + n_i - \bar{n}_i = h_k - \bar{h}_k + n - \bar{n} = h_j - \bar{h}_j + n_j - \bar{n}_j$$

(2.3.9)
in the matrix elements that contribute to \([2.3.7]\). Using the space-time translation operator \(e^{-H\tau - iP_x}\) we can write:

\[
\langle i, \{n_i, \bar{n}_i\}, \alpha_i | \Phi(0,0) \Phi(\tau, x) | j, \{n_j, \bar{n}_j\}, \alpha_j \rangle = \\
= \sum_{k,n,\alpha} \langle i, \{n_i, \bar{n}_i\}, \alpha_i | \Phi(0,0) | k, \{n, \bar{n}\}, \alpha \rangle
\times \langle k, \{n, \bar{n}\}, \alpha | e^{H\tau + iP_x} \Phi(0,0) e^{-H\tau - iP_x} | j, \{n_j, \bar{n}_j\}, \alpha_j \rangle
\]

(2.3.10)

Mapping to the complex plane using \(z = e^{\frac{2\pi}{R}(\tau + ix)}\) one obtains

\[
\left(\frac{2\pi}{R}\right)^{2h} (z\bar{z})^h \Phi(z, \bar{z}) = \left(\frac{2\pi}{R}\right)^{2h} e^{\frac{2\pi}{R}(L_0 + L_0 - \frac{c}{24})} e^{\frac{2\pi}{R}(L_0 - L_0)\tau - i\frac{2\pi}{R}(L_0 - L_0)x}
\times \Phi(1, 1) e^{-\frac{2\pi}{R}(L_0 + L_0 - \frac{c}{24})} e^{\frac{2\pi}{R}(L_0 - L_0)\tau - i\frac{2\pi}{R}(L_0 - L_0)x}
\]

(2.3.11)

which gives

\[
\Phi(z, \bar{z}) = (z\bar{z})^{-h} z^{L_0} \bar{z}^{\bar{L}_0} \Phi(1, 1) z^{-L_0} \bar{z}^{-\bar{L}_0}
\]

(2.3.12)

Inserting this expression into (2.3.8) we obtain

\[
\langle i, \{n_i, \bar{n}_i\}, \alpha_i | \Phi(1, 1) \Phi(z, \bar{z}) | j, \{n_j, \bar{n}_j\}, \alpha_j \rangle = \\
= \sum_{k,n,\alpha} \langle i, \{n_i, \bar{n}_i\}, \alpha_i | \Phi(1, 1) | k, \{n, \bar{n}\}, \alpha \rangle \langle k, \{n, \bar{n}\}, \alpha | \Phi(1, 1) | j, \{n_j, \bar{n}_j\}, \alpha_j \rangle
\times z^{h_k + n - h_j - n_j - h_k + \bar{n} - \bar{n}_j - h}
\]

(2.3.13)

so the contribution of level \((n, \bar{n})\) from a given primary field with conformal dimensions \(h_k, \bar{h}_k\) can be found by first splitting the matrix element into conformal blocks

\[
\langle i, \{n_i, \bar{n}_i\}, \alpha_i | \Phi(1, 1) \Phi(z, \bar{z}) | j, \{n_j, \bar{n}_j\}, \alpha_j \rangle = \\
= \sum_k \langle i, \{n_i, \bar{n}_i\}, \alpha_i | \Phi(1, 1) \mathcal{P}_k \Phi(z, \bar{z}) | j, \{n_j, \bar{n}_j\}, \alpha_j \rangle
\]

(2.3.14)

with \(\mathcal{P}_k\) being the projector onto the conformal module \(\mathcal{V}_{h_k} \otimes \mathcal{V}_{\bar{h}_k}\), and then considering the coefficient of the term \(z^n \bar{z}^\bar{n}\) in the Taylor expansion of the function

\[
z^{-(h_k - h_j - n_j - h)} \bar{z}^{-(\bar{h}_k - \bar{h}_j - \bar{n}_j - h)} \langle i, \{n_i, \bar{n}_i\}, \alpha_i | \Phi(1, 1) \mathcal{P}_k \Phi(z, \bar{z}) | j, \{n_j, \bar{n}_j\}, \alpha_j \rangle
\]

(2.3.15)
2.3.2 Constructing counterterms to scaling functions

2.3.2.1 The ground state scaling function

For the ground state, the computations simplify considerably and one can write an explicit formula for the contributions up to level \( n \) in the form

\[
e_{0,n}(\lambda) = -\frac{c}{12} - \frac{\lambda^2}{2\pi} \int_{|z|<1} d^2z (z \bar{z})^{h-1} \langle 0|\Phi(1,1) P_n \Phi(z, \bar{z}) |0 \rangle + \mathcal{O}(\lambda^3)
\]  

(2.3.16)

Expanding the conformal two-point function into a binomial series one obtains

\[
\langle 0|\Phi(1,1) \Phi(z, \bar{z}) |0 \rangle = \frac{1}{(1-z)^{2h} (1-\bar{z})^{2\bar{h}}} = \sum_{m=0}^{\infty} \sum_{\bar{m}=0}^{\infty} \frac{\Gamma(2h+m)}{\Gamma(2h) \Gamma(m+1)} \frac{\Gamma(2\bar{h}+\bar{m})}{\Gamma(2\bar{h}) \Gamma(\bar{m}+1)}
\]

\[
\times z^m \bar{z}^{\bar{m}}
\]

(2.3.17)

Performing the angular integral selects the terms with \( m = \bar{m} \) and in addition gives a factor of \( 2\pi \). Now using the spectral expansion argument with \( h_i = \bar{h}_i = h_j = \bar{h}_j = 0 \), \( h_k = \bar{h}_k = h \) and \( n_i = \bar{n}_i = n_j = \bar{n}_j = 0 \), the second order TCSA contribution to the ground state scaling function coming from the \( m \)th level is the following

\[
\tilde{e}_{0,m} = -\int_0^1 dr r^{2h-1+2m} \left( \frac{\Gamma(2h+m)}{\Gamma(2h) \Gamma(m+1)} \right)^2
\]

\[
= -\frac{1}{2(h+m)} \left( \frac{\Gamma(2h+m)}{\Gamma(2h) \Gamma(m+1)} \right)^2
\]

(2.3.18)

With this the scaling function from TCSA truncated to level \( n \) is given by

\[
e_{0,n}(\lambda) = -\frac{c}{12} - \sum_{m=1}^{n} \tilde{e}_{0,m} \lambda^2 + \mathcal{O}(\lambda^3)
\]

(2.3.19)

The level \( m \) contribution \(2.3.18\) will be tested against TCSA in subsection 4.3.1.

2.3.2.2 Determining the counterterm

For large \( m \) one can expand

\[
\tilde{e}_{0,m} = -\frac{1}{2\Gamma(2h)^2} m^{4h-3} - \frac{4h^2 - 3h}{2\Gamma(2h)^2} m^{4h-4} - \frac{24h^4 - 44h^3 + 21h^2 - h}{6\Gamma(2h)^2} m^{4h-5}
\]

\[
- \frac{32h^6 - 104h^5 + 116h^4 - 49h^3 + 5h^2}{6\Gamma(2h)^2} m^{4h-6} + \mathcal{O}(m^{4h-7})
\]

(2.3.20)
The summation up to the TCSA cut-off level \( n \) can be performed using

\[
\sum_{m=1}^{n} m^\gamma = H_{n,-\gamma}
\]

where \( H_{n,-\gamma} \) is the so-called generalized harmonic number. For large \( n \) it has the expansion

\[
H_{n,-\gamma} = \zeta(-\gamma) + \frac{n^{\gamma+1}}{\gamma+1} + \frac{n^\gamma}{2} + \frac{\gamma n^{\gamma-1}}{12} + \frac{(-\gamma^3 + 3\gamma^2 - 2\gamma) n^{\gamma-3}}{720} + \ldots
\]

Now considering the construction of the counterterm as described in (2.3.3), the first term \( \zeta(-\gamma) \) cancels with the corresponding infinite sum term. So the counterterm for the ground state scaling function at level \( n \) is given by

\[
e_0(r) = e_{0,n}(r) + \delta e_{0,n}(r) + \mathcal{O}(\lambda^3)
\]

\[
\delta e_{0,n}(r) = \lambda^2 n^{4h-2} \left( \frac{1}{4(2h-1)} \frac{1}{\Gamma(2h)^2} + \lambda^2 n^{4h-3} \left( \frac{1 + 2h}{4\Gamma(2h)^2} \right) + \lambda^2 n^{4h-4} \left( \frac{24h^3 + 4h^2 - 13h - 4}{24\Gamma(2h)^2} \right) + \ldots \right)
\]

If the perturbing operator has dimension \( h > 1/2 \) the first correction to the counterterm is divergent; more terms (involving also ones which are of higher order in \( \lambda \)) become divergent as the weight of the perturbation increases. In this case a simple analytic continuation in \( h \) the above formula gives a prescription to renormalize the divergent TCSA result.

### 2.3.3 Excited states

#### 2.3.3.1 Construction of counterterms in general

The construction of the counterterm to the scaling function of excited states requires the level contributions of the correlator \( \langle i | \Phi (1, 1) \Phi (z, \bar{z}) | i \rangle \). For simplicity let us first suppose that the state \( |i\rangle \) is a highest weight vector; then the correlator can be written in terms of left and right chiral conformal blocks:

\[
\langle i | \Phi (1, 1) \Phi (z, \bar{z}) | i \rangle = \sum_{j} \left( C_{i\Phi}^j \right)^2 \mathcal{F}_{ii}^{\phi\phi} (j|z) \mathcal{F}_{ii}^{\bar{\phi}\bar{\phi}} (j|\bar{z})
\]

where

\[
\mathcal{F}_{ii}^{\phi\phi} (j|z) \mathcal{F}_{ii}^{\bar{\phi}\bar{\phi}} (j|\bar{z}) = \sum_{|k\rangle \in \mathcal{V}_h \otimes \mathcal{V}_h} \langle i | \Phi (1, 1) |k\rangle \langle k| \Phi (z, \bar{z}) |i\rangle
\]
the small $\phi$ refers to the chiral component (the perturbation has the same left and right moving weight, therefore they are identical), and

$$C_{\phi}^j = \langle j|\Phi(1,1)|i\rangle \quad \text{(2.3.26)}$$

is the CFT structure constant. From [2.3.1.2], the contribution of level $(n,\bar{n})$ comes from the coefficient of

$$z^{h_j+n-h_i-n_i-h}z^{\bar{h}_j+n-\bar{h}_i-\bar{n}_i-h} \quad \text{(2.3.27)}$$

In principle, this coefficient can be evaluated using the Virasoro symmetry for $F_{\Phi_i\Phi_j}(j|z)$: the lowest order coefficient is by convention normalized to one, and coefficients of subsequent powers can be computed using the conformal Ward identities [B.1.9] to evaluate descendant matrix elements in terms of primary ones. However, this gives a recursive method from which it is very hard to extract the large $n$ behaviour of the coefficients, which is necessary for the explicit construction of the counterterms.

An alternative method that leads to a systematic large $n$ expansion of the required coefficients is the following [GW11]. First we expand the conformal blocks in the dual channel (i.e. in terms of $1-z$) using the duality relations

$$F_{ij}^{kl}(p|z) = \sum_q F_{pq} \begin{bmatrix} k & l \\ i & j \end{bmatrix} F_{ij}^{kl}(q|1-z) \quad \text{(2.3.28)}$$

where the $F$s are the so-called fusion coefficients. With the following pictorial notation

$$F_{ij}^{kl}(p|z) = \begin{array}{c} k \hfill \\ \hline \hline i \hfill \\ \hline \hline j \hfill \end{array} \quad F_{ij}^{kl}(q|1-z) = \begin{array}{c} k \hfill \\ \hline \hline i \hfill \\ \hline \hline j \hfill \end{array} \quad \text{(2.3.29)}$$

one can write

$$F_{ij}^{kl}(p|z) = \sum_k F_{jk} \begin{bmatrix} \phi & \phi \\ i & i \end{bmatrix} \quad F_{ij}^{kl}(q|1-z) = \sum_k F_{ik} \begin{bmatrix} \phi & \phi \\ i & i \end{bmatrix} \quad \text{(2.3.30)}$$

The chiral conformal blocks in the dual channel can be expanded as

$$F_{ij}^{kl}(p|z) = (1-z)^{-h_k-h_l+h_p} \sum_{r=0}^{\infty} B_r \begin{bmatrix} k & l \\ i & j \end{bmatrix} (1-z)^r \quad \text{(2.3.31)}$$

26
with \( B_0 \left[ \begin{array}{c|c} k & l \\ \hline i & j \end{array} \right] = 1 \), and the rest of the coefficients \( B_r \) determined by Virasoro symmetry via the Ward identities (B.1.9). Introducing the shorthands \( \mathcal{F}_{jk}(i) = F_{jk} \left[ \begin{array}{c|c} \phi & \phi \\ \hline i & i \end{array} \right] \) and \( B_r(i, k) = B_r \left[ \begin{array}{c|c} \phi & \phi \\ \hline i & i & k \end{array} \right] \) we can write

\[
\langle i | \Phi (1, 1) \Phi (z, \bar{z}) | i \rangle = \sum_j \left( C_{\phi \phi}^j \right)^2 \left( \begin{array}{c|c} \phi & \phi \\ \hline i & i \\ \hline i & i \end{array} \right) \left( \begin{array}{c|c} \phi & \phi \\ \hline i & i \end{array} \right) \left( \begin{array}{c|c} \phi & \phi \\ \hline i & i \end{array} \right)
\]

\[
= \sum_j \left( C_{\phi \phi}^j \right)^2 \sum_k \mathcal{F}_{jk}(i) \left( \begin{array}{c|c} \phi & \phi \\ \hline i & i & k \end{array} \right) \times \sum_k \mathcal{F}_{jk'}(\bar{i}) \left( \begin{array}{c|c} \phi & \phi \\ \hline i & i & k' \end{array} \right)
\]

\[
= \sum_{j, k, k'} \left( C_{\phi \phi}^j \right)^2 \mathcal{F}_{jk}(i) \mathcal{F}_{jk'}(\bar{i}) \sum_{r, \bar{r} = 0}^{\infty} B_r(i, k) B_{\bar{r}}(\bar{i}, k') \times (1 - z)^{-2h + h_k + r} (1 - \bar{z})^{-2\bar{h} + \bar{h}_{k'} + \bar{r}}
\]

(2.3.32)

Reading off the coefficient of a required power of the form \( z^{n+\gamma} \), where \( n \) is the descendant level we are interested in, is then possible using the following consideration². Suppose we have a function \( f(z) \) that has singular points at 0, 1 and \( \infty \), and the following expansions around \( z = 0 \) and \( z = 1 \):

\[
f(z) = \sum_{n=0}^{\infty} C_n z^{n+\gamma} = \sum_{i=0}^{\infty} A_i (1 - z)^{-\alpha_i}
\]

(2.3.33)

where the exponents \( \alpha_i \) decrease with \( i \). Note that these properties are satisfied by the conformal blocks appearing in (2.3.32). Then

\[
C_n = \oint_{C_0} \frac{dz}{2\pi i} z^{-n-\gamma-1} f(z) = \oint_{C_1} \frac{dz}{2\pi i} z^{-n-\gamma-1} \sum_{i=0}^{\infty} A_i (1 - z)^{-\alpha_i}
\]

(2.3.34)

where we deformed the contour \( C_0 \) encircling \( z = 0 \) to \( C_1 \) enclosing the real line segment between \( z = 1 \) to \( z = \infty \). Exchanging the sum with the integration and substituting the

²We are very grateful to G. Watts for the idea underlying this consideration.
discontinuity of the \((1 - z)^{-\alpha_i}\) terms gives

\[
C_n = \sum_{i=0}^{\infty} A_i \frac{\sin \pi \alpha_i}{\pi} \int_{1}^{\infty} dt \, t^{-n-\gamma-1} (t - 1)^{-\alpha_i} = \sum_{i=0}^{\infty} \frac{\Gamma(\alpha_i + n + \gamma)}{\Gamma(\alpha_i) \Gamma(1 + n + \gamma)} A_i \tag{2.3.35}
\]

which provides the required coefficient as a series summed over \(i\). However, given that we aim at constructing the counterterm to finite order in \(1/n\), and in view of the behaviour

\[
\frac{\Gamma(\alpha + n + \gamma)}{\Gamma(\alpha) \Gamma(1 + n + \gamma)} = \frac{1}{\Gamma(\alpha)} \left( \frac{1}{n + \gamma} \right)^{-\alpha+1} \left( 1 + \frac{\alpha(\alpha - 1)}{2(n + \gamma)} + O \left( \frac{1}{(n + \gamma)^2} \right) \right) \tag{2.3.36}
\]

we only need to keep a finite number of terms from the \(i\) sum. The subsequent steps are the same as in subsection 2.3.2.1 Once the level \(n\) contribution to the matrix element has been extracted, the level \(n\) contribution to the scaling function of state \(i\) is given by

\[
e_{i,n}(r) - e_{i,n-1}(r) = -\frac{\lambda^2}{2\pi} \tilde{e}_{i,n} + O(\lambda^3) \tag{2.3.37}
\]

\[
\tilde{e}_{i,n} = \int_{|z|<1} d^2 z \, (z\bar{z})^{h-1} \langle i|\Phi(1,1)\tilde{P}_n|\Phi(z,\bar{z})|i\rangle
\]

where the integral can be performed the same way as in (2.3.18).

We remark that the step of exchanging the sum with the integral is only valid for terms in which \(n + \gamma + \alpha_i > 0\). Since the \(\alpha_i\) in general decrease without lower bound, for any finite \(n\) this only holds for finitely many terms in the sum. As a result, the \(1/n\) expansion gives an asymptotic series, as discussed later in subsection 4.3.1.2.

To construct the counterterm for the scaling function of descendant states, some modifications are needed. First of all, the descendant level of the state shifts the exponent of the wanted power of \(z\) and \(\bar{z}\), resulting in a shift in the dependence on the truncation level, as observed previously in \([GW11]\). In addition, the conformal blocks for the descendant states must be constructed from the primary ones, which can be accomplished using the Ward identities (B.1.9). We proceed to present two examples in 4.2, the first of which is a simple application of the method, while the second demonstrates both the treatment of degeneracies in the conformal Hilbert space and the procedure for descendant states.

### 2.3.4 Power counting

As discussed above, the leading large \(n\) behaviour is the same for all cases: \(\sim n^{h_2-2}\) with \(h = h_{2,1}\). We remark that this can be extracted from a simple power counting argument. The second order cut-off dependence is determined by the short-distance con-
tribution to the integrated correlator

\[ \int d^2 \vec{x} \langle \Psi | \Phi(\vec{x}) \Phi(0, 0) | \Psi \rangle \]

(2.3.38)

where \( \vec{x} = (\tau, x) \). In the scaling Potts model \( \Phi = \Phi_{2,1} \) which has the short-distance expansion

\[ \Phi_{2,1}(\vec{x}) \Phi_{2,1}(0, 0) \sim A \left( \frac{1}{r^{4h_{2,1}} + \text{descendants}} \right) + B \left( \frac{\Phi_{3,1}(0, 0)}{r^{4h_{2,1} - 2h_{3,1}} + \text{descendants}} \right) \]

(2.3.39)

where \( r = \sqrt{\tau^2 + x^2} \) and \( A \) and \( B \) are conformal OPE coefficients. The most singular term is the one coming from the identity (descendants always contribute terms that are less singular), and putting a short-distance cut-off \( r > 1/E_{\text{cut}} \) gives a leading dependence \( E_{\text{cut}}^{-2} \) by simple power counting. Since the TCSA cut-off for large \( n \) is

\[ E_{\text{cut}} = \frac{4\pi n}{L} + O(1) \]

(2.3.40)

the expected dependence is exactly \( n^{4h_{2,1} - 2} \).

### 2.4 Renormalization group

In the previous section, counterterms were constructed to eliminate the cut-off dependence of the energy levels. We observed, that the leading \( 1/n \) correction is the same for all states. Keeping only the leading term is equivalent to using the following effective Hamiltonian

\[ H_{\text{eff}} = H_{\text{TCSA}} + c(\Lambda) \mathbb{I} \]

(2.4.1)

where \( c(\Lambda) \) is the vacuum energy counterterm, universal for all states. Now let us recall the form of the effective Hamiltonian (2.2.23) in its simple form

\[ H_{\text{eff}}(\Lambda, \Lambda + \Delta\Lambda) = H_{\text{TCSA}} + M(\Lambda, \Delta\Lambda) \]

(2.4.2)

where \( M \) is expressed as an integral of bi-local operator products, which can approximated using the conformal OPE in terms of local operators. From the first order Schrieffer–Wolff transformation we have for the effective Hamiltonian (2.2.23)

\[ H_{\text{eff}}(\Lambda, \Lambda + \Delta\Lambda) = H_{\text{CFT}} P_A + \sum_i \mu_i \int_0^R dx P_A \Phi_i(x, 0) P_A \]

\[ - \sum_{i,j} \mu_i \mu_j \int_0^\infty \int_0^R dx_1 d\tau_1 \int_0^R dx_2 P_A \Phi_i(x_1, \tau_1) P_{\Lambda + \Delta\Lambda} \Phi_j(x_2, 0) P_A + O\left(\{\mu^2\}\right) \]

(2.4.3)
Introducing running couplings $\mu_i(\Lambda)$ which are fixed by the following renormalization condition:

$$H(\Lambda) = H_{eff}(\Lambda, \Lambda + \Delta \Lambda) \tag{2.4.4}$$

leads to

$$\sum_i \mu_i(\Lambda) \int_0^R dx P_{\Lambda+\Delta \Lambda} \Phi_i(x, 0) P_{\Lambda+\Delta \Lambda} =$$

$$\sum_i \mu_i(\Lambda + \Delta \Lambda) \int_0^R dx P_{\Lambda} \Phi_i(x, 0) P_{\Lambda} - \sum_{i,j} \mu_i(\Lambda + \Delta \Lambda) \mu_j(\Lambda + \Delta \Lambda) \int_0^\infty \int_0^R \int_0^R dx_1 d\tau_1 dx_2 P_{\Lambda} \Phi_i(x_1, \tau_1) \hat{P}_{\Lambda+\Delta \Lambda} \Phi_j(x_2, 0) P_{\Lambda}$$

$$+ O(\{\mu^2\}) \tag{2.4.5}$$

Using descendant level cut-off and mapping everything to the conformal plane and in terms of the redefined couplings (2.1.6) we have

$$\sum_i [\lambda_i(n) P_n \Phi_i(1, 1) P_n - \lambda_i(n-1) P_{n-1} \Phi_i(1, 1) P_{n-1}] =$$

$$= \frac{1}{2\pi} \sum_{i,j} \lambda_i(n-1) \lambda_j(n-1) \int_{|z|<1} d^2z (z\bar{z})^{h_j-1} P_{n-1} \Phi_i(1, 1) \hat{P}_n \Phi_j(z, \bar{z}) P_{n-1} \tag{2.4.6}$$

Projecting this equation by sandwiching $\langle \Phi_k | \ldots | 0 \rangle$ to get

$$\lambda_k(n) - \lambda_k(n-1) =$$

$$= \frac{1}{2\pi} \sum_{i,j} \lambda_i(n-1) \lambda_j(n-1) \int_{|z|<1} d^2z (z\bar{z})^{h_j-1} \langle \Phi_k | \Phi_i(1, 1) \hat{P}_n \Phi_j(z, \bar{z}) | 0 \rangle$$

$$+ O(\{\lambda^2\}) \tag{2.4.7}$$

where we used that

$$\langle \Phi_k | P_n = \langle \Phi_k |$$

$$P_n | 0 \rangle = |0\rangle \tag{2.4.8}$$

$$\langle \Phi_k | \Phi_i(1, 1) | 0 \rangle = \delta_{ki}$$

Then the renormalization group method is as follows. To get results for the coupling at infinite cut-off $\lambda_k(\infty)$ we can use this value as a boundary condition for (2.4.7) and use its solutions at the given cut-off in the TCSA program.
2.4.1 Leading order RG equations

Using the argument in 2.3.1.2 and the fact that
\[
\langle \Phi_k | \Phi_i (1, 1) \Phi_j (z, \bar{z}) | 0 \rangle = \frac{C^k_{ij}}{(1 - z)^{h_i + h_j - h_k} (1 - \bar{z})^{h_i + h_j - h_k}}
\]
(2.4.9)
with the binomial expansion used in 2.3.2, the integral in (2.4.7) can be performed, then we can write
\[
\lambda_k (n) - \lambda_k (n - 1) = \sum_{i,j} \lambda_i (n - 1) \lambda_j (n - 1) \frac{C^k_{ij}}{2(n + h_j)} \left( \frac{\Gamma (h_i + h_j - h_k + n)}{\Gamma (h_i + h_j - h_k) \Gamma (n + 1)} \right)^2
\]
+ \mathcal{O} (\{ \lambda^2 \})
(2.4.10)

where \( C^k_{ij} \) is the structure constant. Expanding this equation for large \( n \) to leading order, and approximating the difference on the left hand side with
\[
\lambda_k (n) - \lambda_k (n - 1) \simeq \frac{d\lambda (n)}{dn}
\]
(2.4.11)
one gets
\[
\frac{d\lambda_k (n)}{dn} = \sum_{i,j} \frac{C^k_{ij}}{2 \Gamma (h_i + h_j - h_k)^2} \lambda_i (n) \lambda_j (n) n^{2h_{ijk} - 3} (1 + \mathcal{O} (1/n)) + \mathcal{O} (\{ \lambda^2 \})
\]
(2.4.12)
where \( h_{ijk} = h_i + h_j - h_k \).

In the above calculation \( H_0 \) was chosen to the conformal Hamiltonian (see 2.2.2). In [HRv15, RV15] it is argued that the better choice is to take \( H_{low} \) as starting point, which is now the naive TCSA Hamiltonian (scaling function in this case). Therefore the exponent of \((z \bar{z})\) in (2.4.7) should be modified to the state dependent \( h_j - 1 - e_k \), where \( e_k \) is an eigenvalue of \( H_{low} \). For large \( r \), where the renormalization becomes important the of scaling functions behave as \( Br^2 + \) subleading terms, where \( B \) is the universal bulk energy constant (the Ising model is an exception since the bulk term is \( \sim r^2 \log r + \) subleading terms). For large \( r \) the leading contribution for any state is the same as for the ground state scaling function \( e_0 (r) \). From this argument we get improved RG equations in the form
\[
\frac{d\lambda_k (n)}{dn} = \frac{1}{2n - e_0 (r)} \sum_{i,j} \frac{C^k_{ij}}{2 \Gamma (h_i + h_j - h_k)^2} \lambda_i (n) \lambda_j (n) n^{2h_{ijk} - 2} (1 + \mathcal{O} (1/n)) + \mathcal{O} (\{ \lambda^2 \})
\]
(2.4.13)
where \( e_0 (r) \) can be estimated with the TCSA results at couplings fixed at infinite cut-off.
The difference between 2.4.12 and 2.4.13 is a partial summation of higher order corrections discussed in the next subsection.

2.4.2 Higher order contributions: extrapolation

The higher $1/n$ terms give state dependent corrections corresponding to non-local counterterms \cite{GW11, HRv15, RV15}. Their construction is quite involved and they are not known in a fully analytic form yet. However, there is an efficient shortcut that is sufficient for the purposes of the present work. After integration the leading terms already incorporated in the RG equations yield cut-off corrections with the scaling form

$$n^{2h_{ijk} - 2}$$

corresponding to the occurrence of $\Phi_k$ in the OPE $\Phi_i \times \Phi_j$. Further corrections can be dealt with by extrapolation. The RG improved TCSA results still show cut-off dependence, which allows us to fit functions in the form

$$f(n) = f_\infty + \sum_i c_in^{\alpha_i},$$  \hspace{1cm} (2.4.14)

where the exponents $\alpha_i$ depend on the above OPE and the conformal weights. In \S 3.2 we list the exponents are used in the Ising and the three-state Potts model.
Chapter 3

Analytical results for the \( q \)-state Potts model

The main goal of the present work was to improve the efficiency of the truncated conformal space approach with renormalization methods. It is necessary to check the validity of the procedures discussed in the last chapter in order to develop a method with reliable accuracy. For this purpose we studied the \( q \)-state Potts model, which is the generalization of the Ising model. In two dimensions for \( q \leq 4 \) the conformal field theory describing the critical point is well known and the action of the scaling field theory away from criticality can be written down since the physically important scaling operators (energy density, magnetic field) have been identified with the corresponding primary fields \[ \text{DF84, Nie84}. \]

The thermal perturbation is known to be integrable for all cases and the complete scattering theory is known \[ \text{CZ92, KS79, Smi91}. \] The \( q = 2 \) case (Ising model) is very well known since it can be identified with free massive Majorana fermions. Scattering theory for \( q = 3 \) is studied in details in \[ \text{RSTZ13} \] and the complete description of finite volume spectrum is given in \[ \text{LT14} \] via excited state thermodynamic Bethe ansatz equations. TBA predictions for the low energy states of the spectrum are compared to counterterm renormalized TCSA in \[ \text{LT14} \] and these results are presented in Chapter \[ 4 \].

Introducing a magnetic field breaks integrability, therefore exact results are not available anymore, however due to the phenomenon of confinement these systems have very rich structure. Many approximate methods have been developed such as conformal perturbation theory, form factor perturbation theory, Bethe–Salpeter equations and semi-classical methods. Comparison of these predictions to renormalization group improved TCSA has been published in \[ \text{LT15} \] and presented in Chapter \[ 5 \].

First we briefly review the general features of \( q \)-state Potts model then we present the results on the scattering theory and TBA equations in the thermally perturbed three-state
case, infrared and ultraviolet limit calculations for the TBA are also presented. In the last section we summarize the available results on the confinement phenomenon in the Ising and the three-state Potts case.

### 3.1 $q$-state Potts model

#### 3.1.1 General remarks

In this section we briefly overview the phenomenology of the $q$-state model based on [DG08]. The $q$-state Potts field theory is the scaling field theory of the $q$-state Potts model which is the generalization of the Ising model with $q$ different values (colours) of the lattice variables [PD52]. The lattice Hamiltonian can be written as

$$
\mathcal{H} = -\frac{1}{T} \sum_{(x,y)} \delta_{s(x),s(y)} - H \sum_x \delta_{s(x),q}
$$

(3.1.1)

where the first term is the standard nearest-neighbour interaction and second term is a generalized magnetic field in the “direction” of the $q$th colour. Without magnetic field ($H = 0$) the theory has $S_q$ permutation symmetry and has a critical temperature $T = T_c$, below which the system is in the ordered (ferromagnetic) phase while above $T_c$ the system is in the disordered (paramagnetic) phase.

The action of the scaling field theory of the system can be written as a perturbation of the conformal field theory corresponding to the critical point:

$$
S = S_{\text{CFT}}^{(q)} + \tau \int d^2x \varepsilon(x) + h \int d^2x \sigma(x)
$$

(3.1.2)

where the couplings $\tau$ and $h$ are related to the lattice couplings$^1$

$$
\tau \sim T - T_c
$$

(3.1.3)

$$
h \sim H
$$

(3.1.4)

and $x = (x_1, x_2)$ are Euclidean time and space coordinates. The corresponding CFT [BPZ84] is defined for $q \leq 4$ and has the central charge [DFS84]

$$
c(q) = 1 - \frac{6}{t(t + 1)}
$$

(3.1.5)

$^1$Note that our convention for the coupling $h$ differs by a minus sign from [DG08]
where the parameter $t$ is related to $q$ via the relation

$$\sqrt{q} = 2 \sin \frac{\pi (t - 1)}{2(t + 1)}$$

(3.1.6)

and the thermal and magnetic fields $\varepsilon$ and $\sigma$ are identified with spinless relevant primaries $\Phi_{2,1}$ and $\Phi_{(t-1)/2,(t+1)/2}$ of the CFT and have scaling dimensions \[DF84, Nie84\]

$$2h_{\varepsilon}^{(q)} = X_{\varepsilon}^{(q)} = \frac{1}{2} \left( 1 + \frac{3}{t} \right)$$

(3.1.7)

$$2h_{\sigma}^{(q)} = X_{\sigma}^{(q)} = \frac{(t - 1)(t + 3)}{8t(t + 1)}$$

(3.1.8)

The thermal operator preserves the $S_q$ symmetry while adding the magnetic field breaks the permutation symmetry according to $S_q \rightarrow S_{q-1}$.

In absence of the magnetic field the theory is integrable and the complete scattering theory is known \[CZ92, KS79\]. In the paramagnetic phase there is only one ground state and there are quasi-particle excitations of mass $m$ over the unique vacuum. In the ferromagnetic phase there exist $q$ degenerate ground states and the elementary excitations are kinks interpolating between them, with their mass $m$ related to the coupling $\tau$ via the mass gap relation \[Fat94\]

$$\tau = \kappa^{(q)} m^{2-2h_{\varepsilon}^{(q)}}$$

(3.1.9)

where

$$\kappa^{(2)} = \frac{1}{2\pi}$$

(3.1.10)

$$\kappa^{(3)} = \frac{\Gamma \left( \frac{4}{21} \right) \Gamma \left( \frac{2}{7} \right) \Gamma \left( \frac{5}{7} \right)}{4 \times 2^{1/5} \pi^{8/5} \Gamma \left( \frac{1}{10} \right)} \sqrt{\frac{\Gamma \left( -\frac{1}{7} \right) \Gamma \left( \frac{2}{7} \right)}{\Gamma \left( -\frac{2}{5} \right) \Gamma \left( \frac{6}{5} \right)}} = 0.1643033 \ldots$$

(3.1.11)

As it is not necessary to understand the results of this thesis, we do not present here the general scattering theory therefore we comment on specific values of $q$. In the Ising case ($q = 2$) both in the ordered and in the ordered phase the excitations can be identified with free massive fermions, therefore the scattering amplitudes are simply $-1$, then the finite volume spectrum can be given by using box quantization. In the case of the three-state Potts model ($q = 3$) in the disordered phase there exist a particle and its anti-particle while in the ordered phase there are two types of kinks (kink and anti-kink). The (anti-)kinks can be identified with the (anti-)particles. The scattering amplitudes in both in the ordered and the disordered phases are presented in details in \[3.2.1\]. The finite volume spectrum can be described by excited state thermodynamic Bethe ansatz equations which can be found in \[3.2.2\].

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Let us now consider the models in the ferromagnetic phase. In a magnetic field $h$ in the direction of the $q$th colour the degeneracy between the $q$ ground states is lifted. For $h < 0$ there is a single true vacuum and $q - 1$ metastable ones, while for $h > 0$ their roles are exchanged. The expectation values of the magnetic field operator in the direction of the $\gamma$th colour is in the $\alpha$th ground state of the zero field theory is written as

$$\langle \sigma_\gamma \rangle_\alpha \equiv \langle 0_\alpha | \sigma_\gamma (x) | 0_\alpha \rangle = \frac{v^{(q)}}{q - 1} (q \delta_{\gamma,\alpha} - 1)$$ (3.1.12)

where $v^{(q)}$ can be calculated using the results of [FLZZ98]:

$$v^{(2)} = 1.3578383417 \cdots \times m^{1/8}$$ (3.1.13)
$$v^{(3)} = 1.9382577836 \cdots \times m^{2/15}$$ (3.1.14)

As a result, the difference of the energy density between the false and the true vacua is given by [DG08]

$$\Delta \varepsilon = \delta \varepsilon_\alpha - \delta \varepsilon_q \simeq h \left( \langle \sigma_q \rangle_\alpha - \langle \sigma_q \rangle_q \right) = - \frac{v^{(q)} q}{q - 1} h \quad \alpha \neq q$$ (3.1.15)

The combination $v^{(q)} q / (q - 1)$ is denoted by $\beta^{(q)}$ and called the string tension; $\beta^{(q)} |h|$ gives the slope of confining linear potential induced by the magnetic field between two kinks separated by false vacuum configuration. Furthermore one expects that in finite volume there are states with energy linearly increasing with the volume corresponding to false vacuum configurations. The detailed description of the confinement phenomena can be found in 333.

### 3.1.2 CFT for Ising and three-state Potts critical points

#### 3.1.2.1 Ising model

For the Ising model with central charge $c = 1/2$, the full Hilbert space is

$$\mathcal{H}^{(2)} = S_{0,0} \oplus S_{\frac{1}{2}, \frac{1}{2}} \oplus S_{\frac{1}{16}, \frac{1}{16}}$$ (3.1.16)

where the sectors on the first line are even, the ones on the second line are odd under $S_2$. The Hamiltonian is

$$H = H_{CFT}^{(2)} + \tau \int dx \varepsilon + h \int dx \sigma$$ (3.1.17)

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where

$$\varepsilon = \Phi_{1/2,1/2}, \quad \sigma = \Phi_{1/10,1/10} \quad (3.1.18)$$

Note that for this model the values $h$ and $-h$ are physically equivalent since they are related by the $\mathbb{Z}_2$ symmetry of the conformal field theory.

### 3.1.2.2 Three-state Potts

The scaling limit of Potts model at the critical point is a minimal conformal field theory with central charge

$$c = \frac{4}{5} \quad (3.1.19)$$

[BPZ84] [Dot84]. The Hilbert space of the Potts model is the $D_4$ modular invariant [CIZ87]

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_1 \quad (3.1.20)$$

where

$$\mathcal{H}_0 = S_{0,0} \oplus S_{\frac{1}{2},\frac{1}{2}} \oplus S_{\frac{1}{2},\frac{1}{2}} \oplus S_{3,3}$$

$$\mathcal{H}_\pm = S_{\frac{1}{10},\frac{1}{10}} \oplus S_{\frac{1}{10},\frac{1}{10}}$$

$$\mathcal{H}_1 = S_{\frac{2}{5},\frac{2}{5}} \oplus S_{\frac{2}{5},\frac{2}{5}} \oplus S_{0,0} \oplus S_{3,0} \quad (3.1.21)$$

The $D_4$ conformal field theory is invariant under the permutation group $S_3$ generated by two elements $Z$ and $C$ with the relations

$$Z^3 = 1 \quad C^2 = 1 \quad CZC = Z^{-1} \quad (3.1.22)$$

which have the signatures

$$\text{sign } Z = +1 \quad \text{sign } C = -1 \quad (3.1.23)$$

The sectors in $\mathcal{H}_0$ of (3.1.20) are invariant under the action of the permutation group $S_3$, the ones in $\mathcal{H}_\pm$ form the two-dimensional irreducible representation, which is characterized by the following action of the generators:

$$C |\pm\rangle = \pm |\mp\rangle$$

$$Z |\pm\rangle = \cos \left( \frac{2\pi}{3} \right) |\pm\rangle \pm \sin \left( \frac{2\pi}{3} \right) |\mp\rangle \quad (3.1.24)$$

while those in $\mathcal{H}_1$ transform according to the signature representation of $S_3$.

The Hamiltonian is given by
\[ H = H_{\text{CFT}}^{(3)} + \tau \int dx \varepsilon + h \int dx \sigma \]  
\text{(3.1.25)}

where
\[ \varepsilon = \Phi_{\frac{2}{5}, 2}^\dagger \quad \sigma = \Phi_{\frac{2}{5}, -\frac{2}{5}}^\dagger \]  
\text{(3.1.26)}

### 3.2 Thermally perturbed three-state Potts model

#### 3.2.1 Infinite volume scattering theory

As we mentioned in the previous section, thermal perturbations of the \( q \)-state Potts model critical point are integrable, since in all minimal models perturbation with primary field \( \Phi_{2,1} \) is integrable [Zam89]. Integrability has serious consequences on the scattering theory [ZZ79]. In two dimensions all scattering processes factorize to product of two-particle scatterings. The knowledge of the two-particle scattering amplitudes therefore allows us to describe any scattering processes. In this section we present the particle spectrum and the scattering amplitudes in the case of the thermally perturbed three-state Potts model both in the paramagnetic and the ferromagnetic phases in infinite volume.

In the paramagnetic phase, the vacuum is non-degenerate and the spectrum consists of a pair of particles \( A \) and \( \bar{A} \) of mass \( m \) which form a doublet under \( S_3 \) [Smi91]:

\[ C|A(\theta)\rangle = |\bar{A}(\theta)\rangle \quad Z|A(\theta)\rangle = e^{\frac{2\pi i}{3}}|A(\theta)\rangle \]

\[ C|\bar{A}(\theta)\rangle = |A(\theta)\rangle \quad Z|\bar{A}(\theta)\rangle = e^{-\frac{2\pi i}{3}}|\bar{A}(\theta)\rangle \]  
\text{(3.2.1)}

For \( q = 3 \) the conformal weight of the thermal operator is \( h_\varepsilon^{(3)} = 2/5 \) thus the mass gap relation equation \text{(3.1.9)} reads

\[ \tau = \kappa^{(3)}(m)^{6/5} \]

\[ \kappa^{(3)} = \frac{\Gamma \left( \frac{3}{10} \right) \Gamma \left( \frac{2}{3} \right) \Gamma \left( \frac{5}{6} \right)}{4 \times 2^{1/5} \pi^{8/5} \Gamma \left( \frac{7}{10} \right)} \left( \frac{\Gamma \left( \frac{1}{5} \right) \Gamma \left( \frac{7}{5} \right)}{\Gamma \left( \frac{2}{5} \right) \Gamma \left( \frac{6}{5} \right)} \right)^{6/5} = 0.1643033 \ldots \]  
\text{(3.2.2)}

The generator \( C \) is identical to charge conjugation (\( \bar{A} \) is the antiparticle of \( A \)). The two-particle scattering amplitudes are

\[ S_{AA}(\theta_{12}) = S_{\bar{A}A}(\theta_{12}) = \frac{\sinh \left( \frac{\theta_{12}}{2} + \frac{\theta}{5} \right)}{\sinh \left( \frac{\theta_{12}}{2} - \frac{\theta}{5} \right)} \]

\[ S_{A\bar{A}}(\theta_{12}) = S_{\bar{A}A}(\theta_{12}) = -\frac{\sinh \left( \frac{\theta_{12}}{2} + \frac{\theta}{5} \right)}{\sinh \left( \frac{\theta_{12}}{2} - \frac{\theta}{5} \right)} \]  
\text{(3.2.3)}
where $\theta_{12} = \theta_1 - \theta_2$ is the rapidity difference of the incoming particles. This $S$ matrix was confirmed by thermodynamic Bethe ansatz \cite{Zam90}. We remark that the pole in the $S_{AA} = S_{\bar{A}A}$ amplitudes at

$$\theta_{12} = \frac{2\pi i}{3}$$

(3.2.4)

corresponds to the interpretation of particle $\bar{A}$ as a bound state of two particles $A$ and similarly $A$ as a bound state of two $\bar{A}$s, under the bootstrap principle (a.k.a. “nuclear democracy”). Accordingly, the above amplitudes satisfy the bootstrap relations

$$S_{AA}(\theta) = S_{AA}(\theta + \pi i/3)S_{AA}(\theta - \pi i/3)$$
$$S_{AA}(\theta) = S_{\bar{A}A}(\theta + \pi i/3)S_{\bar{A}A}(\theta - \pi i/3)$$

(3.2.5)

The pole in $S_{\bar{A}A} = S_{\bar{A}\bar{A}}$ amplitudes at

$$\theta_{12} = \frac{\pi i}{3}$$

(3.2.6)

has the same interpretation, but in the crossed channel.

The excitations in the ferromagnetic phase are topologically charged \cite{CZ92}. The vacuum is three-fold degenerate

$$|0\rangle_a = -1, 0, 1$$

(3.2.7)

where the action of $S_3$ is

$$Z|0\rangle_a = |0\rangle_{a+1 \mod 3} \quad C|0\rangle_a = |0\rangle_{-a}$$

(3.2.8)

and the excitations are kinks of mass $m$ interpolating between adjacent vacua. The kink of rapidity $\theta$, interpolating from $a$ to $b$ is denoted by

$$K_{ab}(\theta) \quad a - b = \pm 1 \mod 3$$

(3.2.9)

and can be interpreted as a spin flip up/down (depending on the sign). The scattering processes of the kinks are of the form

$$K_{ab}(\theta_1) + K_{bc}(\theta_2) \rightarrow K_{ad}(\theta_1) + K_{dc}(\theta_2)$$

(3.2.10)
with the scattering amplitudes equal to

\[ S \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) (\theta_{12}) = \begin{cases} S_{AA}(\theta_{12}) & \text{if } b = d \\ S_{A\bar{A}}(\theta_{12}) & \text{if } a = c \end{cases} \]  \quad (3.2.11)  

This essentially means that apart from the restriction of kink succession dictated by the vacuum indices (adjacency rules) the following identifications can be made

\[ K_{ab}(\theta) \equiv A(\theta) \quad a - b = +1 \mod 3 \]
\[ K_{ab}(\theta) \equiv \bar{A}(\theta) \quad a - b = -1 \mod 3 \]  \quad (3.2.12)  

in all other relevant physical aspects (such as the bound state interpretation given above).

### 3.2.2 Excited state TBA equations

#### 3.2.2.1 The TBA equations in the paramagnetic phase

Since the Potts $S$-matrices in the high-temperature (paramagnetic) phase equation (3.2.3) are diagonal, the ground state TBA can be written down in a straightforward manner [Zam90]:

\[
\begin{align*}
\epsilon_1(\theta) &= mR \cosh \theta - \phi_1 \ast L_1(\theta) - \phi_2 \ast L_2(\theta) \\
\epsilon_2(\theta) &= mR \cosh \theta - \phi_2 \ast L_2(\theta) - \phi_1 \ast L_1(\theta)
\end{align*}
\]  \quad (3.2.13)  

where the kernels are given by the derivatives of the phase-shift

\[
\begin{align*}
\phi_1(\theta) &= -i \frac{d}{d\theta} \log S_1(\theta) = -\frac{\sqrt{3}}{1 + 2 \cosh \theta} \\
\phi_2(\theta) &= -i \frac{d}{d\theta} \log S_2(\theta) = \frac{\sqrt{3}}{1 - 2 \cosh \theta}
\end{align*}
\]  \quad (3.2.14)  

and we introduced the notations

\[
L_i(\theta) = \log(1 + e^{-\epsilon_i(\theta)}) \quad A \ast B(\theta) = \int \frac{d\lambda}{2\pi} A(\theta - \lambda)B(\lambda)
\]  \quad (3.2.15)  

The ground state energy can be obtained as

\[
E_0(R) = - \int \frac{d\theta}{2\pi} m \cosh \theta L_1(\theta) - \int \frac{d\theta}{2\pi} m \cosh \theta L_2(\theta)
\]  \quad (3.2.16)  

The two pseudo-energy functions $\epsilon_{1,2}(\theta)$ correspond to the two particles $A$ and $\bar{A}$. Since the ground state is charge neutral, one has $\epsilon_1(\theta) = \epsilon_2(\theta) = \epsilon(\theta)$ and the equation for $\epsilon(\theta)$ turns out to be identical to the TBA for the scaling Lee–Yang model, with the ground
state energy differing by a factor of 2 [Zam90].

Following the argument of analytic continuation as described in [DT96, DT98] leads to the following general form of the excited TBA equations:

\[
\begin{align*}
\epsilon_1(\theta) &= mR \cosh \theta + \sum_k \log \frac{S_1(\theta - \theta_k^+)}{S_2(\theta - \theta_k^-)} + \sum_l \log \frac{S_2(\theta - \bar{\theta}_l^+)}{S_1(\theta - \bar{\theta}_l^-)} - \phi_1 * L_1(\theta) - \phi_2 * L_2(\theta) \\
\epsilon_2(\theta) &= mR \cosh \theta + \sum_k \log \frac{S_2(\theta - \theta_k^+)}{S_1(\theta - \theta_k^-)} + \sum_l \log \frac{S_1(\theta - \bar{\theta}_l^+)}{S_2(\theta - \bar{\theta}_l^-)} - \phi_1 * L_2(\theta) - \phi_2 * L_1(\theta)
\end{align*}
\]

with the energy expressed as

\[
E(R) = -im \sum_k (\sinh \theta_k^+ - \sinh \bar{\theta}_k^+) - im \sum_l (\sinh \theta_l^- - \sinh \bar{\theta}_l^-) - \int \frac{d\theta}{2\pi} m \cosh \theta (L_1(\theta) + L_2(\theta))
\]

where \(\theta_k^\pm\) and \(\bar{\theta}_k^\pm\) are positions of singularities picked up during the continuation. Reality of the finite volume energy \(E(R)\) requires that \(\epsilon_2(\theta) = \epsilon_1(\theta)^*\) for real \(\theta\), which in turn suggests

\[
\bar{\theta}_k^\pm = (\theta_k^\pm)^*
\]

Then the independent relations for the singularity positions can be written as

\[
\epsilon_1(\theta_k^+) = \pi i (2n_k^+ + 1) \quad \epsilon_2(\theta_k^-) = \pi i (2n_k^- + 1)
\]

Indeed, the analysis of the infrared limit below shows that this is the correct choice. However, when continuing to small volumes, some branching transitions may occur for specific levels, just as observed for the scaling Lee–Yang model in [DT96, BLZ97].

3.2.2.2 The infrared limit of the TBA

In the infrared limit, the convolution terms can be neglected in (3.2.17). Writing

\[
\begin{align*}
\theta_k^+ &= \lambda_k^+ + i\rho_k^+ \\
\theta_k^- &= \lambda_k^- + i\rho_k^- \\
\bar{\theta}_k^+ &= \lambda_k^+ - i\rho_k^+ \\
\bar{\theta}_k^- &= \lambda_k^- - i\rho_k^-
\end{align*}
\]
the real part of the relations (3.2.20) read

\[ 0 = mR \cosh \lambda^r \cos \rho^r + \sum_k \text{Re} \log \frac{S_1(\lambda^r - \lambda_k^r + i(\rho^r - \rho_k^r))}{S_2(\lambda^r - \lambda_k^r + i(\rho^r + \rho_k^r))} + \sum_k \text{Re} \log \frac{S_2(\lambda^r + \lambda_k^r + i(\rho^r + \rho_k^r))}{S_1(\lambda^r + \lambda_k^r + i(\rho^r - \rho_k^r))} \]  

(3.2.22)

\[ 0 = mR \cosh \lambda^- \cos \rho^- + \sum_k \text{Re} \log \frac{S_1(\lambda^- - \lambda_k^- + i(\rho^- - \rho_k^-))}{S_2(\lambda^- - \lambda_k^- + i(\rho^- + \rho_k^-))} + \sum_k \text{Re} \log \frac{S_2(\lambda^- + \lambda_k^- + i(\rho^- + \rho_k^-))}{S_1(\lambda^- + \lambda_k^- + i(\rho^- - \rho_k^-))} \]  

(3.2.23)

For large \( R \) the first term grows arbitrarily large, therefore one of the \( S \)-matrix terms must approach a pole. Now the singularity positions with upper index \( + \) (corresponding to particle species \( A \)) are expected to be pairwise different, and similarly for upper index \( - \) (particle species \( \bar{A} \)) due to the effective exclusion statistics of the particles resulting from \( S_1(0) = -1 \). In addition, the singularity positions of the two species must vary independently, as they describe the momenta of different particles. Therefore in both equations the singularity of the \( S \)-matrix closest to the real axis comes from the \( S_2 \) in the \( k = r \) term of the first sums. This forces the asymptotic behaviour

\[ \rho_k^r \to \frac{\pi}{6} \quad \text{for} \quad mR \to \infty \]  

(3.2.24)

Now we can put

\[ \rho_k^+ = \frac{\pi}{6} + \tilde{\delta}_k^+ \]

\[ \rho_k^- = \frac{\pi}{6} + \tilde{\delta}_k^- \]  

(3.2.25)

and keeping only the dominant terms gives

\[ 0 = \frac{\sqrt{3}}{2} mR \cosh \lambda^r + \text{Re} \log \left( -S_2 \left( \frac{i\pi}{3} + 2i\tilde{\delta}_r^+ \right) \right) + \ldots \]  

\[ 0 = \frac{\sqrt{3}}{2} mR \cosh \lambda^- + \text{Re} \log \left( -S_2 \left( \frac{i\pi}{3} + 2i\tilde{\delta}_r^- \right) \right) + \ldots \]  

(3.2.26)

Using

\[ S_2 \left( \frac{i\pi}{3} + 2i\tilde{\delta}_r^\pm \right) = -\frac{\sqrt{3}}{2\tilde{\delta}_r^\pm} + O(1) \]  

(3.2.27)
we get the leading behavior

\[ |\delta_r^\pm| \sim C \exp \left( -\frac{\sqrt{3}}{2} mR \cosh \lambda_r^\pm \right) \]

(3.2.28)

where the constant \( C \) depends on the \( \lambda_k^\pm \) with \( k \neq r \).

Turning now to the imaginary part of relations (3.2.20), the \( \delta_k^\pm \) can be safely put to zero:

\[
\pi(2n_r^+ + 1) = \frac{1}{2} mR \sinh \lambda_r^+ + \sigma_r^+ + \sum_{k \neq r} \text{Im} \log \frac{S_1(\lambda_r^+ - \lambda_k^+)}{S_2(\lambda_r^+ - \lambda_k^+ + i \frac{\pi}{3})} \\
+ \sum_l \text{Im} \log \frac{S_2(\lambda_r^- - \lambda_l^-)}{S_1(\lambda_r^- - \lambda_l^- + i \frac{\pi}{3})}
\]

(3.2.29)

\[
\pi(2n_r^- + 1) = \frac{1}{2} mR \sinh \lambda_r^- + \sigma_r^- + \sum_{k} \text{Im} \log \frac{S_2(\lambda_r^- - \lambda_k^-)}{S_1(\lambda_r^- - \lambda_k^- + i \frac{\pi}{3})} \\
+ \sum_{l \neq r} \text{Im} \log \frac{S_1(\lambda_r^- - \lambda_l^-)}{S_2(\lambda_r^- - \lambda_l^- + i \frac{\pi}{3})}
\]

(3.2.30)

where

\[
\sigma_r^\pm = \text{Im} \log \left( -S_2 \left( \frac{i\pi}{3} + 2i\delta_r^\pm \right) \right) = \begin{cases} 
0 & \delta_r^\pm > 0 \\
\pi & \delta_r^\pm < 0 
\end{cases}
\]

(3.2.31)

Now for real \( \lambda \)

\[
\text{Im} \log \frac{S_1(\lambda)}{S_2(\lambda + i \frac{\pi}{3})} = -\frac{i}{2} \log S_1(\lambda) - \pi \text{sign}(\lambda)
\]

\[
\text{Im} \log \frac{S_2(\lambda)}{S_1(\lambda + i \frac{\pi}{3})} = -\frac{i}{2} \log S_2(\lambda) + \pi \text{sign}(\lambda)
\]

(3.2.32)

which leads to

\[
2\pi I_r^+ = mR \sinh \lambda_r^+ + \sum_{k \neq r} -i \log S_1(\lambda_r^+ - \lambda_k^+) + \sum_l -i \log S_2(\lambda_r^+ - \lambda_l^-)
\]

\[
2\pi I_r^- = mR \sinh \lambda_r^- + \sum_{k} -i \log S_2(\lambda_r^- - \lambda_k^+) + \sum_{l \neq r} -i \log S_1(\lambda_r^- - \lambda_l^-)
\]

(3.2.33)

where the quantum numbers are

\[
I_r^\pm = 4n_r^\pm + 2 - \sigma_r^\pm - 2(\pi \text{ terms from eqn. } (3.2.32))
\]

(3.2.34)

Equations (3.2.33) describe the correct asymptotic quantization conditions for particle rapidities in the paramagnetic phase of the scaling Potts model, and the asymptotic form
of the energy of the state (3.2.18) also turns out to be the correct one:

\[ E(R) = m \sum_k \cosh \lambda_k^+ + m \sum_l \cosh \lambda_l^- \]  
(3.2.35)

### 3.2.2.3 Relation to the excited state TBA of the scaling Lee–Yang model

For special states where the number and rapidities of the \( A \) and \( \bar{A} \) particles are identical

\[ \{ \theta_k^+ \} = \{ \theta_k^- \} \]  
(3.2.36)

the two pseudo-energy functions are identical \( \epsilon_1(\theta) = \epsilon_2(\theta) =: \epsilon(\theta) \), and the TBA equations (3.2.17) and (3.2.18) reduce to

\[ \epsilon(\theta) = mR \cosh \theta + \sum_k \log \frac{S_{LY}(\theta - \theta_k)}{S_{LY}(\theta - \bar{\theta}_k)} - \phi_{LY} \times L(\theta) \]
\[ e^{\epsilon(\theta_k)} = -1 \]  
(3.2.37)
\[ E(R) = 2 \left\{ -im \sum_k (\sinh \theta_k - \sinh \bar{\theta}_k) - \int \frac{d\theta}{2\pi} m \cosh \theta L(\theta) \right\} \]

where

\[ \theta_k = \theta_k^+ = \theta_k^- \quad \bar{\theta}_k = \bar{\theta}_k^+ = \bar{\theta}_k^- \]
\[ \phi_{LY}(\theta) = -i \frac{d}{d\theta} \log S_{LY}(\theta) \]  
(3.2.38)

and

\[ S_{LY}(\theta) = \frac{\sinh \theta + i \sin \frac{2\pi}{3}}{\sinh \theta - i \sin \frac{2\pi}{3}} \]  
(3.2.39)

is the well-known \( S \)-matrix of the scaling Lee–Yang model [CM89]. The system (3.2.37) is just the excited TBA of the scaling Lee–Yang model [DT96, BLZ97], with the energy expression multiplied by a factor of two. This correspondence is a generalization of the relation between the ground state TBAs, which was originally noted by Zamolodchikov [Zam89].

### 3.2.2.4 The TBA equations in the ferromagnetic phase

Due to the invariance of sector \( \mathcal{H}_0 \) under Kramers–Wannier duality \( \mu \to -\mu \), the ground state TBA in the ferromagnetic phase is the same as in the paramagnetic one. However, there appear two additional vacuum states in the \( \mathcal{H}_\pm \) sectors, which are obtained by inserting a twist operator \( \mathcal{Z}^{\pm 1} \) in the partition function, where \( \mathcal{Z} \) is the cyclic permutation in \( S_3 \) introduced in 3.1.2.2. The general vacuum TBA can be written as
\[ \epsilon_1(\theta) = i\omega + mR \cosh \theta - \phi_1 \star L_1(\theta) - \phi_2 \star L_2(\theta) \]
\[ \epsilon_2(\theta) = -i\omega + mR \cosh \theta - \phi_1 \star L_2(\theta) - \phi_2 \star L_1(\theta) \] (3.2.40)

where the vacuum states in \( \mathcal{H}_0 \) and \( \mathcal{H}_\pm \) correspond to \( \omega = 0 \) and \( \omega = \pm 2\pi/3 \), respectively. The excited state TBA can be found by the same argument as in the other phase, with the result

\[ \epsilon_1(\theta) = i\omega + mR \cosh \theta + \sum_{k=1}^{N_+} \log \frac{S_1(\theta - \theta_k^+)}{S_2(\theta - \theta_k^-)} + \sum_{l=1}^{N_-} \log \frac{S_2(\theta - \theta_l^-)}{S_1(\theta - \theta_l^+)} - \phi_1 \star L_1(\theta) - \phi_2 \star L_2(\theta) \] (3.2.41)
\[ \epsilon_2(\theta) = -i\omega + mR \cosh \theta + \sum_{k=1}^{N_+} \log \frac{S_2(\theta - \theta_k^+)}{S_1(\theta - \theta_k^-)} + \sum_{l=1}^{N_-} \log \frac{S_1(\theta - \theta_l^-)}{S_2(\theta - \theta_l^+)} - \phi_1 \star L_2(\theta) - \phi_2 \star L_1(\theta) \] (3.2.42)

\[ e^{\epsilon_1(\theta)} = e^{\epsilon_1(\bar{\theta})} = -1 \]
\[ e^{\epsilon_2(\theta)} = e^{\epsilon_2(\bar{\theta})} = -1 \] (3.2.43)

\[ E(R) = -im \sum_k (\sinh \theta_k^+ - \sinh \bar{\theta}_k^-) - im \sum_l (\sinh \theta_l^- - \sinh \bar{\theta}_l^+) \]
\[ - \int \frac{d\theta}{2\pi} m \cosh \theta (L_1(\theta) + L_2(\theta)) \] (3.2.44)

Another difference from the paramagnetic phase is that the excitations are now kinks mediating between neighboring vacua. Due to periodic boundary conditions the total number of kink steps must be divisible by three, so there is the constraint

\[ N_+ = N_- \mod 3 \] (3.2.45)

In general, sectors \( \mathcal{H}_\pm \) contain states with twists \( \pm 2\pi/3 \), while sectors \( \mathcal{H}_0/\mathcal{H}_1 \) contain untwisted states that are \( \mathcal{C} \)-even/odd. As discussed in 3.2.1, the kink stepping in forward direction will be identified with \( A \), while the one stepping in reverse direction with \( \bar{A} \), as they can be considered to be in one-to-one correspondence with the particle species in the paramagnetic phase.

In the ferromagnetic case, the infrared limiting quantization conditions (3.2.33) are also modified by the presence of the twist
\[
2\pi I_+^r = \omega + mR \sinh \lambda_+^r - \sum_{k \neq r} i \log S_1(\lambda_+^r - \lambda_+^k) - \sum_i i \log S_2(\lambda_+^r - \lambda_+^i) \quad (3.2.46)
\]
\[
2\pi I_-^r = -\omega + mR \sinh \lambda_-^r - \sum_k i \log S_1(\lambda_-^r - \lambda_-^k) - \sum_{l \neq r} i \log S_1(\lambda_-^r - \lambda_-^l)
\]

### 3.2.3 Ultraviolet limit of the TBA

Let us introduce a short-hand notation for the source terms

\[
g(\theta|\theta^+, \theta^-) = \sum_{k=1}^{N^+} \log \frac{S_1(\theta - \theta_+^k)}{S_2(\theta - \theta^-_k)} + \sum_{l=1}^{N^-} \log \frac{S_2(\theta - \theta^-_l)}{S_1(\theta - \theta_+^l)}
\]

\[
\bar{g}(\theta|\theta^+, \theta^-) = \sum_{k=1}^{N^+} \log \frac{S_2(\theta - \theta_+^k)}{S_1(\theta - \theta^-_k)} + \sum_{l=1}^{N^-} \log \frac{S_1(\theta - \theta^-_l)}{S_2(\theta - \theta_+^l)} \quad (3.2.47)
\]

so that we can write the TBA equations in the form

\[
\epsilon_1(\theta) = i\omega + mR \cosh \theta + g(\theta|\theta^+, \theta^-) - \phi_1 \star L_1(\theta) - \phi_2 \star L_2(\theta)
\]

\[
\epsilon_2(\theta) = -i\omega + mR \cosh \theta + \bar{g}(\theta|\theta^+, \theta^-) - \phi_1 \star L_2(\theta) - \phi_2 \star L_1(\theta)
\]

\[
e^{\epsilon_1(\theta_+^k)} = e^{\epsilon_1(\theta^-_k)} = -1
\]

\[
e^{\epsilon_2(\theta_+^k)} = e^{\epsilon_2(\theta^-_k)} = -1
\]

\[
E(R) = -im \sum_k (\sinh \theta_+^k - \sinh \theta^-_k) - im \sum_l (\sinh \theta^-_l - \sinh \theta_+^l) - \int \frac{d\theta}{2\pi} m \cosh \theta \left( L_1(\theta) + L_2(\theta) \right)
\]

\[
\quad - \int \frac{d\theta}{2\pi} m \cosh \theta \left( L_1(\theta) + L_2(\theta) \right) \quad (3.2.48)
\]

where the twist parameter can take the values where

\[
\omega = \frac{2\pi}{3} n_\omega \quad n_\omega = -1, 0, +1 \quad (3.2.49)
\]

We only derive the right-moving conformal behaviour; the left-moving part can be obtained in a similar way. For \( mR \ll 1 \) the right kink limit of the TBA is obtained by redefining

\[
\theta \to \theta - \log \frac{1}{mR} \quad (3.2.50)
\]

and similarly for the positions of the sources

\[
\theta^+_k \to \theta^+_k - \log \frac{1}{mR} \quad \bar{\theta}^+_k \to \bar{\theta}^+_k - \log \frac{1}{mR} \quad (3.2.51)
\]
Those sources whose positions remain finite in the limit are called right movers. To obtain the limit of the source terms, one can compute

\[
\begin{align*}
\lim_{\theta \to +\infty} \log \frac{S_1(\theta - \theta_k^+)}{S_2(\theta - \theta_k^+)} &= \frac{2\pi}{3} i \\
\lim_{\theta \to +\infty} \log \frac{S_2(\theta - \theta_k^+)}{S_1(\theta - \theta_k^+)} &= \frac{2\pi}{3} i \\
\lim_{\theta \to -\infty} \log \frac{S_1(\theta - \theta_k^-)}{S_2(\theta - \theta_k^-)} &= \frac{2\pi}{3} i \\
\lim_{\theta \to -\infty} \log \frac{S_2(\theta - \theta_k^-)}{S_1(\theta - \theta_k^-)} &= -\frac{2\pi}{3} i
\end{align*}
\]

(3.2.52)

Taking the limit \(mR \to 0\) we get that the right kink limiting functions

\[
\epsilon_i^R(\theta) = \lim_{R \to 0} \epsilon_i(\theta - \log mR)
\]

(3.2.53)
satisfy the equations

\[
\begin{align*}
\epsilon_1^R(\theta) &= \frac{1}{2} e^\theta + i \omega_R + g_R(\theta) \theta^+ - \phi_1 \ast L_1(\theta) - \phi_2 \ast L_2(\theta) \\
\epsilon_2^R(\theta) &= \frac{1}{2} e^\theta - i \omega_R + \bar{g}_R(\theta) \theta^- - \phi_1 \ast L_2(\theta) - \phi_2 \ast L_1(\theta) \\
\epsilon_1^R(\theta_k^+) &= i\pi (2n_k^+ + 1) \\
\epsilon_2^R(\theta_k^-) &= i\pi (2n_l^- + 1)
\end{align*}
\]

(3.2.54)

where

\[
\begin{align*}
g_R(\theta|\theta^+,\theta^-) &= \sum_{k=1}^{N_R^+} \log \frac{S_1(\theta - \theta_k^+)}{S_2(\theta - \theta_k^+)} + \sum_{l=1}^{N_R^-} \log \frac{S_2(\theta - \theta_l^-)}{S_1(\theta - \theta_l^-)} \\
\bar{g}_R(\theta|\theta^+,\theta^-) &= \sum_{k=1}^{N_R^+} \log \frac{S_2(\theta - \theta_k^+)}{S_1(\theta - \theta_k^+)} + \sum_{l=1}^{N_R^-} \log \frac{S_1(\theta - \theta_l^-)}{S_2(\theta - \theta_l^-)}
\end{align*}
\]

(3.2.55)

with sums only over the right-movers, and the effective right twist is

\[
\omega_R = \omega + \frac{2\pi}{3} \left[ (N^- - N_R^-) - (N^+ - N_R^+) \right]
\]

(3.2.56)

The right handed component of the effective central charge can be written as

\[
c_R = \frac{6}{2\pi} \sum_{k=1}^{N_R^+} \left( e^{\theta_k^+} - e^{\theta_k^-} \right) + \frac{6}{2\pi} \sum_{l=1}^{N_R^-} \left( e^{\theta_l^-} - e^{\theta_l^+} \right) + \frac{3}{\pi^2} \int d\theta \frac{e^\theta}{2} \left( L_1(\theta) + L_2(\theta) \right)
\]

(3.2.57)
We can rewrite these equations in the following form

\[
\begin{align*}
\epsilon_1^R(\theta) &= \frac{1}{2}e^\theta + i\omega_R - 2i\pi m_R + g_R(\theta|\theta^+, \theta^-) - \phi_1 \ast L_1(\theta) - \phi_2 \ast L_2(\theta) \\
\epsilon_2^R(\theta) &= \frac{1}{2}e^\theta - i\omega_R + 2i\pi m_R + \tilde{g}_R(\theta|\theta^+, \theta^-) - \phi_1 \ast L_2(\theta) - \phi_2 \ast L_1(\theta) \\
\epsilon_1^R(\theta^+_k) &= i\pi \left(2n^+_k - 2m_R + 1\right) \\
\epsilon_2^R(\theta^-_l) &= i\pi \left(2n^-_l - 2m_R + 1\right)
\end{align*}
\] (3.2.58)

where \(m_R\) is defined from

\[
\begin{align*}
i\omega_R + \lim_{\theta \to -\infty} g_R(\theta|\theta^+, \theta^-) &= \frac{2\pi}{3} \left[n_\omega + (N^- - N^-_R) - (N^+ - N^+_R) - N^-_R + N^+_R\right] \\
&= \frac{2\pi}{3} (3m_R + \tilde{n}_\omega)
\end{align*}
\] (3.2.59)

where \(m_R\) is an integer and \(\tilde{n}_\omega = 0\) or \(\pm 1\) is the remainder. One can then use the standard dilogarithm trick [Zam90, DT96] to write

\[
\begin{align*}
\frac{1}{2}e^\theta &= \epsilon_1^R(\theta) - i\omega_R + 2i\pi m_R - g_R(\theta|\theta^+, \theta^-) + \phi_1 \ast L_1(\theta) + \phi_2 \ast L_2(\theta) \\
\frac{1}{2}e^\theta &= \epsilon_2^R(\theta) + i\omega_R - 2i\pi m_R - \tilde{g}_R(\theta|\theta^+, \theta^-) + \phi_1 \ast L_2(\theta) + \phi_2 \ast L_1(\theta)
\end{align*}
\] (3.2.60)

Differentiating the two sides

\[
\begin{align*}
\frac{1}{2}e^\theta &= \frac{d}{d\theta} \left\{ \epsilon_1^R(\theta) - g_R(\theta|\theta^+, \theta^-) + \phi_1 \ast L_1(\theta) + \phi_2 \ast L_2(\theta) \right\} \\
\frac{1}{2}e^\theta &= \frac{d}{d\theta} \left\{ \epsilon_2^R(\theta) - \tilde{g}_R(\theta|\theta^+, \theta^-) + \phi_1 \ast L_2(\theta) + \phi_2 \ast L_1(\theta) \right\}
\end{align*}
\] (3.2.61)

and substituting into the expression (3.2.57) we obtain

\[
\begin{align*}
\epsilon_R &= \frac{6}{2\pi} \sum_{k=1}^{N^+_R} \left( e^{\theta^+_k} - e^{\tilde{\theta}^+_k} \right) + \frac{6}{2\pi} \sum_{l=1}^{N^-_R} \left( e^{\theta^-_l} - e^{\tilde{\theta}^-_l} \right) \\
&+ \frac{3}{\pi^2} \int d\theta \frac{d}{d\theta} \left\{ \epsilon_1^R(\theta) - g_R(\theta|\theta^+, \theta^-) + \phi_1 \ast L_1(\theta) + \phi_2 \ast L_2(\theta) \right\} L_1(\theta) \\
&+ \frac{3}{\pi^2} \int d\theta \frac{d}{d\theta} \left\{ \epsilon_2^R(\theta) - \tilde{g}_R(\theta|\theta^+, \theta^-) + \phi_1 \ast L_2(\theta) + \phi_2 \ast L_1(\theta) \right\} L_2(\theta)
\end{align*}
\] (3.2.62)

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\[ c_R = \frac{6}{2\pi} \sum_{k=1}^{N_R^+} \left( e^{\theta^+_k} - e^{\bar{\theta}^+_k} \right) + \frac{6}{2\pi} \sum_{l=1}^{N_R^-} \left( e^{\theta^-_l} - e^{\bar{\theta}^-_l} \right) \]

\[ + \frac{3}{\pi^2} \int_{\epsilon^R_1(-\infty)}^{+\infty} d\epsilon \log(1 + e^{-\epsilon}) + \frac{6}{\pi^2} \int_{\epsilon^R_1(-\infty)}^{+\infty} d\epsilon \log(1 + e^{-\epsilon}) \]

\[ + \frac{3}{\pi^2} \int d\theta \frac{d}{d\theta} \left\{ -g_R(\theta, \theta^+, \theta^-) + \phi_1 \ast L_1(\theta) + \phi_2 \ast L_2(\theta) \right\} L_1(\theta) \]

\[ + \frac{3}{\pi^2} \int d\theta \frac{d}{d\theta} \left\{ -\tilde{g}_R(\theta, \theta^+, \theta^-) + \phi_1 \ast L_2(\theta) + \phi_2 \ast L_1(\theta) \right\} L_2(\theta) \quad (3.2.63) \]

where the integrals over \( \epsilon \) must be taken over an appropriate contour in the \( \epsilon \) plane which is analytically equivalent the curves \( \epsilon^R_i(\theta) \) as \( \theta \) runs over the real line. In the next step, we can treat the \( \theta \) integrals using partial integration:

\[ \int d\theta \frac{d}{d\theta} \left\{ -g_R(\theta, \theta^+, \theta^-) + \phi_1 \ast L_1(\theta) + \phi_2 \ast L_2(\theta) \right\} L_1(\theta) \]

\[ + \int d\theta \frac{d}{d\theta} \left\{ -\tilde{g}_R(\theta, \theta^+, \theta^-) + \phi_1 \ast L_2(\theta) + \phi_2 \ast L_1(\theta) \right\} L_2(\theta) \]

\[ = \int d\theta \left\{ -g_R(\theta, \theta^+, \theta^-) L_1(\theta) - \tilde{g}_R(\theta, \theta^+, \theta^-) L_2(\theta) \right\} \]

\[ - \int d\theta \left\{ \phi_1 \ast L_1(\theta) + \phi_2 \ast L_2(\theta) \right\} L'_1(\theta) \]

\[ - \int d\theta \left\{ \phi_1 \ast L_2(\theta) + \phi_2 \ast L_1(\theta) \right\} L'_2(\theta) \]

\[ + \left[ \left\{ \phi_1 \ast L_1(\theta) + \phi_2 \ast L_2(\theta) \right\} L_1(\theta) \right]_{-\infty}^{\infty} \]

\[ + \left[ \left\{ \phi_1 \ast L_2(\theta) + \phi_2 \ast L_1(\theta) \right\} L_2(\theta) \right]_{-\infty}^{\infty} \quad (3.2.64) \]

and the fact that \( L_{1,2}(\infty) = 0 \) to obtain

\[ c_R = \frac{6}{2\pi} \sum_{k=1}^{N_R^+} \left( e^{\theta^+_k} - e^{\bar{\theta}^+_k} \right) + \frac{6}{2\pi} \sum_{l=1}^{N_R^-} \left( e^{\theta^-_l} - e^{\bar{\theta}^-_l} \right) \]

\[ + \frac{3}{\pi^2} \int_{\epsilon^R_1(-\infty)}^{+\infty} d\epsilon \log(1 + e^{-\epsilon}) + \frac{3}{\pi^2} \int_{\epsilon^R_1(-\infty)}^{+\infty} d\epsilon \log(1 + e^{-\epsilon}) \]

\[ + \frac{3}{\pi^2} \int d\theta \left\{ -g'_R(\theta, \theta^+, \theta^-) L_1(\theta) - \tilde{g}'_R(\theta, \theta^+, \theta^-) L_2(\theta) \right\} \]

\[ - \frac{3}{\pi^2} \left[ \phi_1 \ast L_1(-\infty) + \phi_2 \ast L_2(-\infty) \right] L_1(-\infty) \]

\[ - \frac{3}{\pi^2} \left[ \phi_1 \ast L_2(-\infty) + \phi_2 \ast L_2(-\infty) \right] L_2(-\infty) \quad (3.2.65) \]
The remaining integrals can be expressed using the kink TBA equations for \( \theta \to -\infty \):

\[
\phi_1 \ast L_1(-\infty) + \phi_2 \ast L_2(-\infty) = -\epsilon_1^R(-\infty) + i\omega_R - 2i\pi m_R + g_R(-\infty|\theta^+, \theta^-) \\
\phi_1 \ast L_2(-\infty) + \phi_2 \ast L_1(-\infty) = -\epsilon_2^R(-\infty) - i\omega_R + 2i\pi m_R + \bar{g}_R(-\infty|\theta^+, \theta^-)
\]

(3.2.66)

Using the definition of \( m_R \) leads to

\[
\phi_1 \ast L_1(-\infty) + \phi_2 \ast L_2(-\infty) = -\epsilon_1^R(-\infty) + i\frac{2\pi}{3} \tilde{n}_\omega \\
\phi_1 \ast L_2(-\infty) + \phi_2 \ast L_1(-\infty) = -\epsilon_2^R(-\infty) - i\frac{2\pi}{3} \tilde{n}_\omega 
\]

(3.2.67)

For the terms involving \( g'_R \) and \( \bar{g}'_R \) we can write

\[
g'_R(\theta|\theta^+, \theta^-) = \sum_{k=1}^{N_R^+} i\phi_1(\theta - \theta^+_k) - i\phi_2(\theta - \bar{\theta}^+_k) + \sum_{l=1}^{N_R^-} i\phi_2(\theta - \theta^-_l) - i\phi_1(\theta - \bar{\theta}^-_l) \\
\bar{g}'_R(\theta|\theta^+, \theta^-) = \sum_{k=1}^{N_R^+} i\phi_2(\theta - \theta^+_k) - i\phi_1(\theta - \bar{\theta}^+_k) + \sum_{l=1}^{N_R^-} i\phi_1(\theta - \theta^-_l) - i\phi_2(\theta - \bar{\theta}^-_l)
\]

(3.2.68)

and so

\[
\int d\theta \left\{ -g'_R(\theta|\theta^+, \theta^-) L_1(\theta) - \bar{g}'_R(\theta|\theta^+, \theta^-) L_2(\theta) \right\} \\
= -2\pi i \sum_{k=1}^{N_R^+} \left\{ \phi_1 \ast L_1(\theta^+_k) + \phi_2 \ast L_2(\bar{\theta}^+_k) - \phi_2 \ast L_1(\bar{\theta}^+_k) - \phi_1 \ast L_2(\theta^+_k) \right\} \\
\quad - 2\pi i \sum_{l=1}^{N_R^-} \left\{ -\phi_1 \ast L_1(\bar{\theta}^-_l) - \phi_2 \ast L_2(\bar{\theta}^-_l) + \phi_2 \ast L_1(\theta^-_l) + \phi_1 \ast L_2(\theta^-_l) \right\} 
\]

(3.2.69)

Now we can eliminate the convolution terms using the equations determining the singularity positions

\[
i\pi \left( 2n^+_k + 1 \right) = \frac{1}{2} e^{\theta^+_k} + i\omega_R + g_R(\theta^+_k|\theta^+, \theta^-) - \phi_1 \ast L_1(\theta^+_k) - \phi_2 \ast L_2(\theta^+_k) \\
-i\pi \left( 2n^+_k + 1 \right) = \frac{1}{2} e^{\bar{\theta}^+_k} - i\omega_R + \bar{g}_R(\theta^+_k|\theta^+, \theta^-) - \phi_1 \ast L_2(\bar{\theta}^+_k) - \phi_2 \ast L_1(\theta^+_k)
\]

(3.2.70)
\[-i\pi \left(2n_i^- + 1\right) = \frac{1}{2} e^{\theta_i^-} + i\omega_R + g_R(\theta_i^- | \theta^+, \theta^-) - \phi_1 \ast L_1(\theta_i^-) - \phi_2 \ast L_2(\theta_i^-)\]
\[i\pi \left(2n_i^- + 1\right) = \frac{1}{2} e^{\theta_i^-} - i\omega_R + \bar{g}_R(\theta_i^- | \theta^+, \theta^-) - \phi_1 \ast L_2(\theta_i^-) - \phi_2 \ast L_1(\theta_i^-)\]  
(3.2.71)

The end result is

\[c_R = \frac{3}{\pi^2} \left\{ \int_{e^R(-\infty)}^{+\infty} de \log(1 + e^{-\epsilon}) \right\} + \frac{3}{\pi^2} \left\{ \int_{e^R(-\infty)}^{+\infty} de \log(1 + e^{-\epsilon}) \right\} \]
\[-12 \sum_{k=1}^{N_R^+} (2n_k^+ + 1) - 12 \sum_{l=1}^{N_R^-} (2n_l^- + 1) + \frac{12}{\pi} \omega_R \left( N_R^+ - N_R^- \right) \]
\[-\frac{6}{\pi} \sum_{k=1}^{N_R^+} \left\{ g_R(\theta_k^+ | \theta^+, \theta^-) - \bar{g}_R(\theta_k^- | \theta^+, \theta^-) \right\} \]
\[-\frac{6}{\pi} \sum_{l=1}^{N_R^-} \left\{ \bar{g}_R(\theta_l^- | \theta^+, \theta^-) - g_R(\theta_l^+ | \theta^+, \theta^-) \right\} \]
\[-\frac{3}{\pi^2} \frac{1}{2} \left[ -\log Y_1 + i \frac{2\pi}{3} \tilde{n}_\omega \right] \log \left(1 + Y_1^{-1}\right) \]
\[-\frac{3}{\pi^2} \frac{1}{2} \left[ -\log Y_2 - i \frac{2\pi}{3} \tilde{n}_\omega \right] \log \left(1 + Y_2^{-1}\right) \]

where \(Y_i = \epsilon_i (-\infty)\) are solutions of the plateau equation

\[\log Y_1 = i \frac{2\pi}{3} \tilde{n}_\omega + \frac{1}{3} \log \left(1 + Y_1^{-1}\right) + \frac{2}{3} \log \left(1 + Y_2^{-1}\right)\]
\[\log Y_2 = -i \frac{2\pi}{3} \tilde{n}_\omega + \frac{2}{3} \log \left(1 + Y_1^{-1}\right) + \frac{1}{3} \log \left(1 + Y_2^{-1}\right) \]  
(3.2.73)

From [Mar91, Fen92], the solutions of these equations are known, together with the values of the dilogarithm integrals:

\[\frac{3}{\pi^2} \left\{ \int_{e^R(-\infty)}^{+\infty} de \log(1 + e^{-\epsilon}) \right\} + \frac{3}{\pi^2} \left\{ \int_{e^R(-\infty)}^{+\infty} de \log(1 + e^{-\epsilon}) \right\} \]
\[-\frac{3}{\pi^2} \frac{1}{2} \left[ -\log Y_1 + i \frac{2\pi}{3} \tilde{n}_\omega \right] \log \left(1 + Y_1^{-1}\right) - \frac{3}{\pi^2} \frac{1}{2} \left[ -\log Y_2 - i \frac{2\pi}{3} \tilde{n}_\omega \right] \log \left(1 + Y_2^{-1}\right) \]
\[= \begin{cases} 
\frac{2}{3} \\ 
\tilde{n}_\omega = 0 \\
-\frac{2}{3} \\
\tilde{n}_\omega = \pm 1 
\end{cases} \]  
(3.2.74)

Using standard identities for the logarithm of products, the contributions containing the sums of \(g_R\) and \(\bar{g}_R\) terms in (3.2.72) naively evaluate to zero. However, this result is
changed by taking care of the branch cuts of the logarithms. Using the notations of subsection 3.2.2.2, the contribution depends on the signs of $\delta_i^\pm$ of the corresponding singularities and can quickly be evaluated individually for every state considered.

3.3 Confinement in magnetic field

As we have seen in 3.1.1 in presence of magnetic field one of the vacua is preferred or punished in the ferromagnetic phase. The relative energy density is given in 3.1.15. Depending on the sign of the magnetic fields and the type of the kinks, they can feel linear potential between each other, which can confine them. In the Ising case there is only two vacua and two types of kinks, here the confinement spectrum does not depend on the sign of the magnetic field: both cases it contains two-kink bound-states called “mesons”. As we will see the Potts case is more complicated, the spectrum depends on the direction of the magnetic field and since there are three vacua and two types of kinks, besides the mesons it contains three-kink bound-states, “baryons”. In the following we recall the results can be found in the literature for the meson and baryon spectrum.

3.3.1 Meson masses in the Ising model

The first mass estimation is the one obtained by McCoy and Wu [MW78]

$$m_n^{Ai} = m(2 + \lambda^{2/3}z_n) \quad (3.3.1)$$

where $-z_n$ is the $n$th zero of the Airy function and $\lambda$ is the dimensionless ratio

$$\lambda = \frac{\beta^{(2)}|h|}{m^2} \quad (3.3.2)$$

This solution can be derived from the quantum mechanical system of two kinks in a linear potential. The quantum mechanical system also allows solutions corresponding to the zeros for the derivative of the Airy function; however, the corresponding wave-functions are symmetric and so forbidden due to the fermionic nature of the kinks.

The WKB mass spectrum can be obtained by solving the quantization condition

$$\frac{\sinh (2\varphi_n) - 2\varphi_n}{\lambda} = \frac{2\pi (n - 1/4)}{m_{n}^{WKB}} = 2m \cosh (\varphi_n) \quad (3.3.3)$$
It can be improved further by adding higher corrections in $\lambda$:

$$\sinh (2 \vartheta_n) - 2 \vartheta_n = 2\pi (n - 1/4) \lambda + \sum_{k=1}^{\infty} \lambda^{k+1} S_k(\vartheta_n)$$

$$m_n^{WKB} = 2m \cosh (\vartheta_n)$$  \hspace{1cm} (3.3.4)

The first term in this expansion is given in [FZ06] and it is

$$S_1 (\vartheta) = \frac{1}{\sinh (2 \vartheta)} \left( -\frac{1}{6} \sinh^2 (\vartheta) + \frac{5}{24} \sinh^2 (\vartheta) + \frac{1}{4} \cosh^2 (\vartheta) - \frac{1}{12} \right)$$  \hspace{1cm} (3.3.5)

The Bethe–Salpeter equation (with various improvements) leads to a low energy expansion of the form

$$\frac{(m_n^{le})^2}{4m^2} = 1 + \sum_{k=1}^{\infty} \mu_k t^k$$  \hspace{1cm} (3.3.6)

with the parameter $t = \lambda^{2/3}$. Different approximations of the $\mu_k$ coefficients taking into account multi-quark corrections (such as quark mass renormalization and renormalization of the short range quark-antiquark interaction), and string tension renormalization can be found in [FZ03, Rut05, FZ06, Rut09]. The low-energy expansion for the meson mass $\tilde{m}_n$ from the Bethe–Salpeter equation has the form [FZ06]

$$\frac{\tilde{m}_n^2}{4m^2} = 1 + z_n t^2 + \frac{z_n^2}{5} t^4 - \left( \frac{3 z_n^3}{175} + \frac{57}{280} \right) t^6 + \left( \frac{23 z_n^4}{7875} + \frac{1543 z_n}{12600} \right) t^8$$

$$+ \left( \frac{13}{1120\pi} - \frac{1894 z_n^5}{3031875} - \frac{23983 z_n^2}{242550} \right) t^{10} + \frac{3313 z_n}{10080} t^{11} + \ldots$$  \hspace{1cm} (3.3.7)

while the radiative corrections modify this expression according to

$$\frac{m_n^{le} - \tilde{m}_n}{m} = a_2 t^2 + \frac{z_n}{6} (4c_2 - a_2) t^8 - \frac{B_2}{4} t^9 + O(t^{10})$$  \hspace{1cm} (3.3.8)

where

$$a_2 = 0.0710809 \ldots \quad c_2 = -0.003889 \ldots$$  \hspace{1cm} (3.3.9)

are the leading order quark mass and string tension renormalization corrections computed in [FZ06] and

$$B_2 = 0.8$$  \hspace{1cm} (3.3.10)

is the leading order interaction renormalization correction obtained in [Rut09, Rut10b].
3.3.2 Mesonic states in the three state Potts model

For this latter model, the Bethe–Salpeter has not been carried out (albeit the setting was established in [Rut10a]), so we quote the linear potential quantum mechanics and the WKB results.

3.3.2.1 \( h < 0 \)

For this sign of the magnetic field there is a single stable vacuum and two metastable ones. The two-kink configurations are

\[
K_{3\alpha}(\vartheta_1)K_{\alpha 3}(\vartheta_2)
\]

where 3 is the stable vacuum, while \( \alpha = 1, 2 \) are metastable vacua. Due to the presence of this degree of freedom allowed by \( \alpha \) both antisymmetric and symmetric solutions are allowed. From the simple quantum mechanical picture of two-kink configurations it is clear that in the sector of zero total momentum the charge conjugation (C) parity of kink-antikink and therefore also meson states is equal to their parity under spatial reflections.

The spectrum predicted by quantum mechanics in the linear potential is

\[
m_{\pm,n}^{(Ai)} = m(2 + \lambda^{2/3}z_n) + O(\lambda^{4/3}) \\
m_{\pm,n}^{(Ai)} = m(2 + \lambda^{2/3}z'_n) + O(\lambda^{4/3})
\]

where \( \lambda \) is the dimensionless ratio

\[
\lambda = \beta^{(3)}|h| \frac{m^2}{|m|}
\]

and \( -z_n \) is the \( n \)th zero of the Airy function, while \( -z'_n \) is the \( n \)th zero of its derivative.

For this case, the WKB quantization is given by [Rut10a]

\[
\frac{\sinh (2\vartheta_n)}{\lambda} - 2\vartheta_n = 2\pi \left(n - \frac{1}{4}\right) + 2\arctan \left(\frac{\tanh 2\vartheta_n}{\sqrt{3}}\right) + i\mathcal{A}(2\vartheta_n) + O(\lambda) \quad \text{odd} \\
\frac{\sinh (2\vartheta_n)}{\lambda} - 2\vartheta_n = 2\pi \left(n - \frac{3}{4}\right) + 2\arctan \left(\frac{\tanh 2\vartheta_n}{\sqrt{3}}\right) + i\mathcal{A}(2\vartheta_n) + O(\lambda) \quad \text{even}
\]

\[
\mathcal{A}(\vartheta) = \log \left(\frac{\sinh(i\pi/3 + \vartheta)}{\sinh(i\pi/3 - \vartheta)}\right)
\]

\[
m_{\pm,n}^{WKB} = 2m \cosh (\vartheta_n)
\]

This includes effects of nontrivial kink-antikink scattering, and therefore despite being semi-classical it goes beyond the simple quantum mechanical result above.
3.3.2.2 \( h > 0 \)

For this sign of the magnetic field there are two stable vacua and one metastable. The allowed two-kink configurations are

\[ K_{\alpha\beta}(\vartheta_1)K_{3\alpha}(\vartheta_2) \]

Charged meson states correspond to \( \alpha \neq \beta \), and their spectrum is given by the WKB quantization condition \cite{Rut10a} is given by

\[
\frac{\sinh (2\vartheta_n) - 2\vartheta_n}{\lambda} = 2\pi \left( n - \frac{1}{4} \right) + iA(2\vartheta_n) + O(\lambda) \tag{3.3.18}
\]

Note that charged meson single-particle states do not satisfy periodic boundary conditions on a circle due to \( \alpha \neq \beta \), therefore they cannot be observed in TCSA.

On the other hand, neutral meson states do not exist, as they easily decompose under the process

\[ K_{\alpha\beta}(\vartheta_1)K_{3\alpha}(\vartheta_2) \rightarrow K_{\alpha\beta}(\vartheta_2)K_{\beta\alpha}(\vartheta_1) \]

where for \( \alpha = 1, 2 \) one has \( \beta = 2, 1 \), respectively. This process is allowed by the Chim–Zamolodchikov kink scattering amplitudes \cite{CZ92}; also note that the kinks mediating between the stable vacua 1, 2 are not confined.

3.3.3 Baryon masses in the three state Potts model

For the case \( h < 0 \) all the kinks are confined. As a result, one may have three-kink bound states of the form

\[ K_{31}(\vartheta_1)K_{12}(\vartheta_2)K_{23}(\vartheta_3) \]

\[ K_{32}(\vartheta_1)K_{21}(\vartheta_2)K_{13}(\vartheta_3) \]

corresponding to baryons and antibaryons. Both of these particles have the same spectrum due to charge-conjugation symmetry, and can be modeled in the form of a quantum mechanical three-body system. The low energy estimates for the baryon masses were recently obtained by Rutkevich \cite{Rut15} with the result

\[
M_n^\pm = m(3 + \left( \beta^{(3)}|h|/m^2 \right)^{2/3}c_n^\pm) + O(|h|^{4/3}) \tag{3.3.19}
\]
where the ± correspond to parity under space reflection, with the following numerical values of $\epsilon$ for the first three states:

\[
\begin{align*}
\epsilon_1^+ &= 4.602 & \epsilon_2^+ &= 5.912 & \epsilon_3^+ &= 7.098 \\
\epsilon_1^- &= 6.650 & \epsilon_2^- &= 7.734 & \epsilon_3^- &= 8.753
\end{align*}
\]

There are no baryons for $h > 0$ as the kinks between the two stable vacua are not confined.
Chapter 4

Counterterm renormalization in the Potts model

In this chapter we present the numerical results of [LT14] on the low energy spectrum of the thermally perturbed three-state Potts model. This work was motivated by the discrepancy between the phase shift calculations on the chain version of the model and the scattering theory results found in [RSTZ13]. We found complete agreement between the excited state thermodynamic Bethe ansatz predictions and counterterm renormalized truncated conformal space approach results for the lowest lying states, both in the paramagnetic and in the ferromagnetic phase. The two-particle phase shift has been studied and we found that the above mentioned discrepancy could be understand by the effect of irrelevant operators coming from the operator product expansion of the perturbing fields.

The operator product expansion of the thermal perturbation with itself only contains other operators, therefore the RG equations are trivial in the leading order. To study a few low-lying states, the construction of counterterms can be done, therefore subleading $1/n$ terms can be taken into account.

In the first part of this section we identify the limiting conformal states of certain states in finite volume using the UV limit of the TBA equations. In order to give examples of the construction of TCSA counterterms, explicit level contribution and counterterm calculations are presented based on Section 2.3. In the second part the numerical results are presented. First we analyze the theoretical predictions for the level contributions then present our results for the renormalized TCSA spectrum compared to the TBA predictions. Finally the behaviour of the two-particle phase shifts after renormalization is presented.
4.1 TBA UV limits for specific states

4.1.1 Vacuum states

For the vacuum state in $\mathcal{H}_0$ (see 3.1.2.2) one obtains \[ c_R = c_L = \frac{2}{5} \] (4.1.1)

In the ferromagnetic phase, the vacuum states in $\mathcal{H}_\pm$, corresponding to $\omega = \pm 2\pi/3$ satisfy \[ c_R = c_L = -\frac{2}{5} \] (4.1.2)

The corresponding conformal weights can be computed from

\[ c_{R,L} = c - 24\Delta_{R,L} \] (4.1.3)

and give $\Delta_{R,L} = 1/15$.

4.1.2 One-particle states

In the paramagnetic state, the lowest energy levels in a given momentum sector of $\mathcal{H}_\pm$ are expected to correspond to one-particle states. Considering a one-particle $A$ state with a singularity located at $\theta^+$ such that

\[ \epsilon_1(\theta^+) = i\pi(2n^+ + 1) \quad \text{Im} \theta^+ = \frac{\pi}{6} + \delta^+ \] (4.1.4)

the following result is obtained for $n^+ > 0$

\[ c_R = -\frac{2}{5} \left( 2n^+ - \sigma^+ \right) \quad \sigma^+ = \begin{cases} 0 & \delta^+ > 0 \\ 1 & \delta^+ < 0 \end{cases} \] (4.1.5)

which corresponds to a right descendant of $\Phi_{1/2}^+$ with momentum quantum number $2n^+ - \sigma^+$; for $n^+ < 0$, the result is similar, but it is a left descendant instead.

For a stationary particle, a numerical analysis of the TBA equation in the infrared shows that the relevant quantum numbers are $n^+ = 0$ and $\delta^+ > 0$; in such a case $\theta^+$ is purely imaginary. When decreasing the volume, the position of the singularity at a
critical value \( mR = r_c \) reaches the line

\[
\delta^+ = \frac{\pi}{3} \quad (4.1.6)
\]

Similarly to the Lee–Yang case, for \( mR < r_c \) the equation requires analytic continuation. We do not go into the details here; the relevant methods can be found in [DT96, BLZ97]. The UV limit can be computed simply by noticing that because the singularity is stuck in the middle, the two kink systems become identical to the twisted ground system (with opposite values of twists on the two sides), therefore

\[
c_R = c_L = -\frac{2}{5} \quad (4.1.7)
\]

corresponding to the primary state created by \( \Phi_{\frac{1}{5}, \frac{1}{5}} \).

### 4.1.3 Untwisted two-particle states \( AA^- \)

Supposing that one of the particles is right moving (\( \theta^+ > 0 \)), while the other one is left-moving (\( \theta^- < 0 \)) with

\[
e_1(\theta^+) = i\pi(2n^+ + 1) \quad \text{Im}\theta^+ = \frac{\pi}{6} + \delta^+
\]

\[
e_2(\theta^-) = -i\pi(2n^- + 1) \quad \text{Im}\theta^- = \frac{\pi}{6} + \delta^-
\]

the result is

\[
c_R = \frac{2}{5} - 12 \left( \frac{2}{5} + 2n^+ - \sigma^+ \right)
\]

\[
c_L = \frac{2}{5} - 12 \left( \frac{2}{5} + 2n^- - \sigma^- \right) \quad (4.1.9)
\]

\[
\sigma^\pm = \begin{cases} 
0 & \delta^\pm > 0 \\
1 & \delta^\pm < 0 
\end{cases}
\quad (4.1.10)
\]

corresponding to descendants of either \( \Phi_{\frac{2}{5}, \frac{2}{5}} \) \( / \Phi_{\frac{2}{5}, \frac{2}{5}} \) (in \( \mathcal{H}_0 \)), or either of \( \Phi_{\frac{2}{5}, \frac{2}{5}} \) \( / \Phi_{\frac{2}{5}, \frac{2}{5}} \) (in \( \mathcal{H}_1 \)).

In fact there are in general two degenerate states, because charge conjugation leaves the TBA result invariant:

\[
\frac{1}{\sqrt{2}} \left( |A(\lambda^+), \bar{A}(\lambda^-)\rangle + |A(\lambda^-), \bar{A}(\lambda^+)\rangle \right) \in \mathcal{H}_0
\]

\[
\frac{1}{\sqrt{2}} \left( |A(\lambda^+), \bar{A}(\lambda^-)\rangle - |A(\lambda^-), \bar{A}(\lambda^+)\rangle \right) \in \mathcal{H}_1 \quad (4.1.11)
\]
with $\lambda^{\pm} = \text{Re}\theta^{\pm}$. These two states are completely degenerate, which is indeed valid in TCSA up to the numerical precision that can be attained.

The only exception is when the state is composed of two zero momentum particles

$$|A(0)\bar{A}(0)\rangle$$

(4.1.12)

with

$$\theta^+ = \theta^- = i\left(\frac{\pi}{6} + \delta\right) \quad \delta > 0$$

(4.1.13)

This state is non-degenerate and in $\mathcal{H}_0$; its scaling function is just twice the stationary one-particle scaling function in the Lee–Yang model, as discussed in 3.2.2.3.

### 4.1.4 Twisted $A\bar{A}$ states and $AA/\bar{A}\bar{A}$ states

In the ferromagnetic phase, the lowest excited states in $\mathcal{H}_{\pm}$ are $A\bar{A}$ states with non-zero twists

$$\omega = \pm \frac{2\pi}{3}$$

(4.1.14)

Using the results in 3.2.3, it turns out that these states correspond to descendants of $\Phi_{\frac{3}{5},\frac{3}{5}}$. In the paramagnetic phase, the same levels are described in TBA as two-particle $AA/\bar{A}\bar{A}$ states for $\mathcal{H}_-/\mathcal{H}_+$, respectively.

### 4.2 Example counterterm calculations

#### 4.2.1 The first $A\bar{A}$ two-particle state in the Potts model

In the scaling three-state Potts model the first excited state in sector $\mathcal{H}_0$ a two-particle state which in the scattering picture consists of two stationary particles, one of which is of species $A$ and the other is $\bar{A}$. The UV limit of this excited state level corresponds to the highest weight vector in the conformal module

$$S_{\frac{2}{5},\frac{2}{5}}$$

(4.2.1)

this can be seen either from TCSA or using the excited TBA equation introduced later. Therefore the excited state scaling function has the limiting value (4.1.9)

$$e_1(0) = -\frac{1}{12} \cdot \frac{4}{5} + 2 \cdot \frac{2}{5} = \frac{11}{15}$$

(4.2.2)
All fields that occur in the calculation below have identical left and right conformal weights, so it is useful to introduce the shorter notation
\[ \Phi_{r,s} = \Phi_{h_r,s h_{r,s}} \] (4.2.3)

The conformal state is created by the primary field \( \Phi_{2,1} \):
\[ |\Phi_{2,1}⟩ = |2/5, 2/5⟩ = \Phi_{2,1}(0, 0)|0⟩ \] (4.2.4)

Using the conformal fusion rules
\[ \Phi_{2,1} \times \Phi_{2,1} = I + \Phi_{3,1} \] (4.2.5)

the relevant two-point function can be expanded into conformal blocks as follows
\[ \langle \Phi_{2,1} | \Phi_{2,1} (1, 1) \Phi_{2,1} (z, \bar{z}) | \Phi_{2,1}⟩ = \]
\[ = \left( \mathcal{C}_{\phi_{2,1}^{\Phi_{2,1}}} \right)^2 \left[ \begin{array}{c|c} \phi_{2,1} & \phi_{2,1} \\ \hline \phi_{2,1} & \phi_{2,1} \end{array} \right]^2 + \left( \mathcal{C}_{\phi_{2,1}^{\Phi_{2,1}}} \right)^2 \left[ \begin{array}{c|c} \phi_{2,1} & \phi_{2,1} \\ \hline \phi_{2,1} & \phi_{2,1} \end{array} \right]^2 \] (4.2.6)

where the operation of taking the modulus squared corresponds to the product of holomorphic (\( z \)-dependent) and antiholomorphic (\( \bar{z} \)-dependent) factors.

The level \( n \) contribution can be constructed from the coefficients of \( (z \bar{z})^{n-2h_{2,1}} \) in the first term and of \( (z \bar{z})^{h_{3,1}+n-2h_{2,1}} \) in the second term of the correlator, respectively. Using the duality relations (2.3.30) we can rewrite the two terms as

\[ \phi_{2,1} |\phi_{2,1}⟩ = \mathcal{F}_{\phi_{2,1}[\phi_{2,1}]} \left[ \begin{array}{c|c} \phi_{2,1} & \phi_{2,1} \\ \hline \phi_{2,1} & \phi_{2,1} \end{array} \right] + \mathcal{F}_{\phi_{2,1}[\phi_{2,1}]} \left[ \begin{array}{c|c} \phi_{2,1} & \phi_{2,1} \\ \hline \phi_{2,1} & \phi_{2,1} \end{array} \right] \]
\[ \phi_{3,1} |\phi_{2,1}⟩ = \mathcal{F}_{\phi_{3,1}[\phi_{2,1}]} \left[ \begin{array}{c|c} \phi_{2,1} & \phi_{2,1} \\ \hline \phi_{2,1} & \phi_{2,1} \end{array} \right] + \mathcal{F}_{\phi_{3,1}[\phi_{2,1}]} \left[ \begin{array}{c|c} \phi_{2,1} & \phi_{2,1} \\ \hline \phi_{2,1} & \phi_{2,1} \end{array} \right] \] (4.2.7)
From (B.1.12), the series expansions of the dual channel conformal blocks are the following

\[
\begin{align*}
\phi_{2,1} & \quad \phi_{2,1} \\
\phi_{1,1} & \\
\phi_{2,1} & \quad \phi_{2,1} \\
\phi_{3,1} & \quad = (1 - z)^{-2h_{2,1}} \left( 1 + \frac{2h_{2,1}^2}{c} (1 - z)^2 + \mathcal{O}((1 - z)^3) \right) \\
\phi_{2,1} & \\
\phi_{2,1} & \quad \phi_{2,1} \\
\phi_{2,1} & \quad \phi_{2,1} \\
\phi_{3,1} & \quad = (1 - z)^{-2h_{2,1} + h_{3,1}} \left( 1 + \frac{h_{3,1}}{2} (1 - z) + \mathcal{O}((1 - z)^2) \right)
\end{align*}
\]

(4.2.8)

Keeping only the leading terms and applying (2.3.35), we can rewrite the conformal blocks in (4.2.6) in the following way:

\[
\begin{align*}
\phi_{2,1} & \quad \phi_{2,1} \\
\phi_{1,1} & \\
\phi_{2,1} & \quad \phi_{2,1} \\
\phi_{3,1} & \quad = \sum_{n=0}^{\infty} \left\{ \mathcal{F}_{\phi_{2,1}} \left[ \phi_{2,1} \right] \left( \frac{\Gamma(n)}{\Gamma(2h_{2,1}) \Gamma(n + 1 - 2h_{2,1})} + \ldots \right) \\
& + \mathcal{F}_{\phi_{3,1}} \left[ \phi_{2,1} \right] \left( \frac{\Gamma(n - h_{3,1})}{\Gamma(2h_{2,1} - h_{3,1}) \Gamma(n + 1 - 2h_{2,1})} + \ldots \right) \right\} z^{n - 2h_{2,1}} \quad \text{(4.2.9)}
\end{align*}
\]

\[
\begin{align*}
\phi_{2,1} & \quad \phi_{3,1} \\
\phi_{2,1} & \quad \phi_{2,1} \\
\phi_{2,1} & \quad = \sum_{n=0}^{\infty} \left\{ \mathcal{F}_{\phi_{3,1}} \left[ \phi_{2,1} \right] \left( \frac{\Gamma(n + h_{3,1})}{\Gamma(2h_{2,1}) \Gamma(n + 1 + h_{3,1} - 2h_{2,1})} + \ldots \right) \\
& + \mathcal{F}_{\phi_{3,1}} \left[ \phi_{2,1} \right] \left( \frac{\Gamma(n)}{\Gamma(2h_{2,1} - h_{3,1}) \Gamma(n + 1 + h_{3,1} - 2h_{2,1})} + \ldots \right) \right\} z^{h_{3,1} + n - 2h_{2,1}} \quad \text{(4.2.10)}
\end{align*}
\]

where the ellipsis indicate terms which are subleading for large \( n \), resulting from the subleading terms in (4.2.8). Putting together the left and right moving parts, and performing
the integral \([2.3.37]\), the level \(n\) contribution to the coefficient of \(\lambda^2\) can be expressed as

\[
\tilde{e}_{1,n} = -\frac{\left(C_{\Phi_2,1}^2\right)^2}{2h_{2,1} + 2(n - 2h_{2,1})} \left(\mathcal{F}_{\Pi}[\phi_{2,1}] \Gamma(n) \Gamma(n + 1 - 2h_{2,1}) + \ldots\right)
\]

\[
+ \mathcal{F}_{\phi_3,1}[\phi_{2,1}] \Gamma(n - h_{3,1}) \Gamma(n + 1 - 2h_{2,1}) + \ldots \right)^2
\]

\[
- \left(\mathcal{C}_{\Phi_2,1}^2\right)^2 \left(\mathcal{F}_{\phi_3,1}[\phi_{2,1}] \Gamma(n + h_{3,1}) \Gamma(n - 2h_{2,1} + h_{3,1}) + \ldots\right)^2 \right)
\]

\[
+ \mathcal{F}_{\phi_3,1}[\phi_{2,1}] \Gamma(n - 2h_{2,1} + h_{3,1}) \Gamma(n - 2h_{2,1} + h_{3,1}) + \ldots\right)^2 \right) (4.2.11)
\]

From this expression one can construct the large \(n\) counterterm as for the ground state case in \([2.3.2.1]\). The leading behaviour of the counterterm is

\[
\delta e_{1,n}(r) = \lambda^2 n^{4h - 2} \frac{1}{4(2h - 1) \Gamma(2h)^2} + \ldots \quad (4.2.12)
\]

\((h = h_{2,1})\) which is the same as for the ground state. The reason is that this comes from the identity operator in the operator product expansion of the perturbing operator with itself, and the matrix elements of this term are independent of the state considered, so this term is universal.

### 4.2.2 The second \(A\bar{A}\) two-particle state and the first AAA three-particle state

Both from TCSA and excited states TBA, the ultraviolet limit of the scaling function of the second \(A\bar{A}\) state is

\[
e_2(0) = -\frac{1}{125} + 2 \cdot \frac{2}{5} + 2 = \frac{41}{15} \quad (4.2.13)
\]

The zero-momentum part of the Hilbert space of the \(M_{5,6}\) minimal model at this level is doubly degenerate: it is spanned by \(L_{-1}\bar{L}_{-1} \left| \frac{2}{5}, \frac{2}{5} \right\rangle \) and \(\left| \frac{7}{5}, \frac{2}{5} \right\rangle \). So one has to use degenerate perturbation theory and diagonalize the perturbing operator in this subspace, which leads to the two eigenstates

\[
|\pm\rangle = \frac{1}{\sqrt{2}} \left( \left| \frac{7}{5}, \frac{7}{5} \right\rangle \pm \frac{1}{2h_{2,1}} L_{-1}\bar{L}_{-1} \left| \frac{2}{5}, \frac{2}{5} \right\rangle \right) \quad (4.2.14)
\]

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From TCSA one can see that $|+\rangle$ corresponds to the first $AAA$ three-particle state and $|−\rangle$ to the second $A\bar{A}$ two-particle state. For the evaluation of the counterterm we therefore need to consider the following conformal four-point functions:

- $⟨\Phi_{3,1}|\Phi_{2,1} (1, 1) \Phi_{2,1} (z, \bar{z}) |\Phi_{3,1}⟩$
- $⟨\Phi_{2,1}|L_1\tilde{L}_1\Phi_{2,1} (1, 1) \Phi_{2,1} (z, \bar{z}) L_{-1}\tilde{L}_{-1}|\Phi_{2,1}⟩ = D⟨\Phi_{2,1}|\Phi_{2,1} (1, 1) \Phi_{2,1} (z, \bar{z}) |\Phi_{2,1}⟩$, where $D$ is some differential operator constructing the descendant matrix element
- $⟨\Phi_{3,1}|\Phi_{2,1} (1, 1) \Phi_{2,1} (z, \bar{z}) L_{-1}\tilde{L}_{-1}|\Phi_{2,1}⟩$

Due to the fusion rules, the last two are eventually zero. Therefore to order $λ^2$, the counterterm for both the two-particle state and the three-particle state is

$$\delta e_{2,n}(r) = \frac{1}{2} \left( \delta e_{|\tau,\bar{\tau}⟩,n}(r) + \frac{1}{4h_{2,1}^2} \delta e_{L_{-1}\tilde{L}_{-1}|\tau,\bar{\tau},n}(r) \right) \quad (4.2.15)$$

where the indices indicate the contributing matrix element.

The first contribution can be calculated following the procedure in subsection 4.2.1: it is necessary to compute the level contributions for $⟨\Phi_{3,1}|\Phi_{2,1} (1, 1) \Phi_{2,1} (z, \bar{z}) |\Phi_{3,1}⟩$. To obtain it one needs the following OPEs:

$$\Phi_{2,1} \times \Phi_{2,1} = 1 + \Phi_{3,1}$$
$$\Phi_{2,1} \times \Phi_{3,1} = \Phi_{2,1} + \Phi_{4,1}$$
$$\Phi_{3,1} \times \Phi_{3,1} = 1 + \Phi_{3,1} \quad (4.2.16)$$

which lead to

$$⟨\Phi_{3,1}|\Phi_{2,1} (1, 1) \Phi_{2,1} (z, \bar{z}) |\Phi_{3,1}⟩ = \left( \frac{C_{\Phi_{2,1}\Phi_{3,1}}}{\phi_{\Phi_{2,1}\Phi_{3,1}}^2} \right)^2 \begin{vmatrix} \phi_{2,1} & \phi_{2,1} \\ \phi_{3,1} & \phi_{3,1} \end{vmatrix}^2 + \left( \frac{C_{\Phi_{4,1}\Phi_{3,1}}}{\phi_{\Phi_{4,1}\Phi_{3,1}}^2} \right)^2 \begin{vmatrix} \phi_{2,1} & \phi_{2,1} \\ \phi_{3,1} & \phi_{3,1} \end{vmatrix}^2 \quad (4.2.17)$$

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For the level $n$ contribution one needs the coefficient of $z^{n-h_{3,1}}$ in the first term and $z^{h_{4,1}+n-h_{2,1}-h_{3,1}}$ in the second term. Rewriting the conformal blocks in the dual channel

\[
\phi_{3,1} \phi_{3,1} \phi_{2,1} \phi_{2,1} = \mathcal{F}_{\phi_{2,1}[\phi_{3,1}]} \phi_{3,1} \phi_{3,1} \phi_{1,1} + \mathcal{F}_{\phi_{1,1}[\phi_{3,1}]} \phi_{3,1} \phi_{3,1} \phi_{2,1} \phi_{2,1}
\]

and using the expansions

\[
\phi_{2,1} \phi_{2,1} \phi_{3,1} \phi_{3,1} = (1-z)^{-2h_{2,1}} \left( 1 + \frac{2h_{2,1} h_{3,1}}{c} (1-z)^2 + \mathcal{O} \left( (1-z)^3 \right) \right)
\]

\[
\phi_{2,1} \phi_{2,1} \phi_{3,1} \phi_{3,1} = (1-z)^{-2h_{2,1}+h_{3,1}} \left( 1 + \frac{h_{3,1}}{2} (1-z) + \mathcal{O} \left( (1-z)^2 \right) \right)
\]

one can determine the necessary coefficients. Keeping only the leading terms in the $(1-z)$ expansion and using (2.3.35) yields:

\[
\phi_{2,1} \phi_{2,1} \phi_{3,1} \phi_{3,1} = \sum_{n=0}^{\infty} \left\{ \mathcal{F}_{\phi_{2,1}[\phi_{3,1}]} \left( \frac{\Gamma(n+2h_{2,1}-h_{3,1})}{\Gamma(2h_{2,1}) \Gamma(n+1-h_{3,1}) + \ldots} \right) \right. \\
\left. + \mathcal{F}_{\phi_{1,1}[\phi_{3,1}]} \left( \frac{\Gamma(n+2h_{2,1}-2h_{3,1})}{\Gamma(2h_{2,1}-h_{3,1}) \Gamma(n+1-h_{3,1}) + \ldots} \right) \right\} z^{n-h_{3,1}}
\]
where the ellipsis indicate terms which are subleading for large \( n \), resulting from the subleading terms in \((4.2.19)\). Putting together the left and right moving parts, and performing the integral \((2.3.37)\), the level contribution from this channel is then

\[
\tilde{\epsilon}_{\mid \bar{z} \bar{z} \rangle, n} = -\frac{(c_{\Phi_2,1}^\phi_{2,1} c_{\Phi_3,1}^\phi_{3,1})^2}{2 h_{2,1} + 2 (n - h_{3,1})} \\
\times \left( F_{\phi_2,1}[\phi_3,1] \Gamma (n + 2 h_{2,1} - h_{3,1}) \frac{\Gamma (n + 1 - h_{3,1})}{\Gamma (2 h_{2,1}) \Gamma (n + 1 - h_{3,1})} + \ldots \right)^2 \\
+ F_{\phi_2,1}[\phi_3,1] \frac{\Gamma (n + h_{2,1} - 2 h_{3,1})}{\Gamma (2 h_{2,1} - h_{3,1}) \Gamma (n + 1 - h_{3,1})} + \ldots \right)^2 \\
- \frac{(c_{\Phi_4,1}^\phi_{4,1} c_{\Phi_3,1}^\phi_{3,1})^2}{2 h_{2,1} + 2 (n + h_{4,1} - h_{2,1} - h_{3,1})} \\
\times \left( F_{\phi_4,1}[\phi_3,1] \frac{\Gamma (n + h_{2,1} + h_{4,1} - h_{3,1})}{\Gamma (2 h_{2,1}) \Gamma (n + 1 + h_{4,1} - h_{2,1} - h_{3,1})} + \ldots \right)^2 \\
+ F_{\phi_4,1}[\phi_3,1] \frac{\Gamma (n + h_{2,1} - h_{3,1} + h_{4,1})}{\Gamma (2 h_{2,1} - h_{3,1}) \Gamma (n + 1 + h_{4,1} - h_{2,1} - h_{3,1})} + \ldots \right)^2.
\]

These can be used to determine the counterterm \( \tilde{\epsilon}_{\mid \bar{z} \bar{z} \rangle, n} (r) \) following the steps in \[2.3.2.2\]; we omit the explicit form as it is quite long and not really illuminating.

For the second term one needs to repeat the computation in subsection \[4.2.1\] but replacing all objects with those pertaining to the descendant conformal block given in \[B.1.13\] and \[B.1.14\] and \[B.1.15\].
4.3 Numerical comparison

The evaluation of the TCSA spectrum consists of several steps:

1. First the numerical “raw” TCSA spectrum is determined by diagonalizing the TCSA Hamiltonian (2.1.5). We used cutoffs $n = 6, 7, 8, 9, 10, 11, 12$ (the highest ones corresponding to several thousand states kept in each sector), and restricted our analysis to states with total momentum zero.

2. Next for any given energy level, the level contributions are constructed analytically. For the vacuum we know the $n$-dependence of order $\lambda^2$ contributions in a closed form. For more general excited states, the procedure in subsection 2.3.3.1 gives the level contributions as a series in inverse powers of $n$. Given these level contributions, one can check whether the TCSA results are reproduced to a sufficient precision.

3. Constructing the counterterms one eliminates the cut-off dependence of the TCSA to order $\lambda^2$. A useful check on this method is to evaluate the residual order $\lambda^2$ cut-off dependence of the renormalized TCSA results, which must be sufficiently close to zero. Note that this does not eliminate all the cut-off dependence, as it may also come from higher order in $\lambda$.

4. Finally, one can compare the renormalized TCSA data to the TBA results.

4.3.1 Level contributions and accuracy of counterterms

For the first and second steps listed above, we can look at the level contributions to scaling functions $e(r)$ before and after subtraction. We have done this analysis for all the energy levels that are considered for the comparison to TBA in subsection 4.3.2. Below we show and comment on the examples of

- the ground state in $\mathcal{H}_0$, for which the exact level contributions are known;

- the first excited state in $\mathcal{H}_0$, which illustrates the use of the expansion in the dual channel for a primary state;

- the second and third excited states in $\mathcal{H}_0$, which include two novelties: the contribution of a descendant state, and degenerate perturbation theory.

For the other states, the picture is the same; we omit the detailed results as they would add nothing substantial to the demonstration of the method.
4.3.1.1 Ground state

For the ground state which is the lowest level in sector $\mathcal{H}_0$, the $O(\lambda^2)$ level contributions are known exactly for any $n$ and are given in (2.3.18). From the TCSA data, the difference between two subsequent values of the cut-off $n$ can be fitted with a function $a + b\lambda^2 + c\lambda^4$ and the coefficient $b$ extracted. This was performed in the volume range $0 \leq r \leq 1$ which under (2.1.6) corresponds to $0 \leq \lambda \lesssim 0.113765$. To see whether the counterterm (2.3.23) really removes the cut-off dependence, one can repeat the same procedure for the subtracted TCSA data. The results, shown in Table 4.1 demonstrate how efficient the renormalization procedure is.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Exact</th>
<th>TCSA</th>
<th>Subtracted TCSA</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>$-0.0160116158$</td>
<td>$-0.0160116106$</td>
<td>$7.36449 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>10</td>
<td>$-0.013898947$</td>
<td>$-0.013898948$</td>
<td>$2.83545 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>11</td>
<td>$-0.0122228492$</td>
<td>$-0.0122228377$</td>
<td>$2.91101 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>12</td>
<td>$-0.0108656858$</td>
<td>$-0.0108656810$</td>
<td>$1.46659 \cdot 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 4.1: Level contribution of the coefficient of $\lambda^2$ in the perturbative series for the ground state scaling function.

4.3.1.2 Stationary $A\bar{A}$ pair

For the first excited state in $\mathcal{H}_0$ which is contains a pair of particles, both with zero momentum, one can use the counterterm constructed in subsection 4.2.1. In contrast to the ground state, the exact $n$-dependence of the level contributions is not available, and we use the approximation constructed from the expansion (B.1.8) of the conformal block in the dual channel, to obtain an approximation in powers of $1/n$, the leading term of which is presented in (4.2.12). Although the expansion (B.1.8) is convergent, the $1/n$ expansion of the level contribution resulting after the application of the integral formula (2.3.35) is only asymptotic. This means that for any $n$, including more terms from the conformal block in the dual channel at first improves the result, but then the error starts to grow. On the other hand, for higher $n$ (and therefore lower $1/n$) the series starts to diverge at higher order. This can be manifestly seen in Table 4.2 where contributions resulting from the inclusion of the conformal block expansion to order $n$ is labeled $B_n$. It turns out that for $n = 12$ the $B_5$ and $B_6$ approximations give essentially exact results, so they can be used to construct the counterterm. The effect of this counterterm is demonstrated in Table 4.3 which again shows that to order $\lambda^2$ the truncation dependence is almost totally eliminated.
Table 4.2: Level contributions for the lowest A\(\bar{A}\) level.

<table>
<thead>
<tr>
<th>(n)</th>
<th>TCSA</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>B4</th>
<th>B5</th>
<th>B6</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>-0.0159516</td>
<td>-0.0159007</td>
<td>-0.0159229</td>
<td>-0.0159573</td>
<td>-0.0159517</td>
<td>-0.0159470</td>
<td>-0.0159695</td>
</tr>
<tr>
<td>10</td>
<td>-0.0138470</td>
<td>-0.0138127</td>
<td>-0.0138255</td>
<td>-0.0138513</td>
<td>-0.0138462</td>
<td>-0.0138462</td>
<td>-0.0138501</td>
</tr>
<tr>
<td>11</td>
<td>-0.0121777</td>
<td>-0.0121536</td>
<td>-0.0121615</td>
<td>-0.0121809</td>
<td>-0.0121769</td>
<td>-0.0121777</td>
<td>-0.0121783</td>
</tr>
<tr>
<td>12</td>
<td>-0.0108263</td>
<td>-0.0108087</td>
<td>-0.0108138</td>
<td>-0.0108286</td>
<td>-0.0108257</td>
<td>-0.0108264</td>
<td>-0.0108264</td>
</tr>
</tbody>
</table>

Table 4.3: \(O(\lambda^2)\) level contributions for the lowest A\(\bar{A}\) level after subtraction, where the approximations B5 and B6 were used.

<table>
<thead>
<tr>
<th>(n)</th>
<th>B5</th>
<th>B6</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>(-4.55659 \cdot 10^{-6})</td>
<td>(1.54134 \cdot 10^{-5})</td>
</tr>
<tr>
<td>10</td>
<td>(-7.53545 \cdot 10^{-7})</td>
<td>(2.90049 \cdot 10^{-6})</td>
</tr>
<tr>
<td>11</td>
<td>(-2.94012 \cdot 10^{-6})</td>
<td>(5.70248 \cdot 10^{-7})</td>
</tr>
<tr>
<td>12</td>
<td>(1.07722 \cdot 10^{-6})</td>
<td>(9.84199 \cdot 10^{-8})</td>
</tr>
</tbody>
</table>

4.3.1.3 Second \(A\bar{A}\) and first \(AAA\) levels

The second excited level is degenerate at the fixed point with the third one. This is the pair of states described in subsection 4.2.2. The level contributions for these states are shown in Table 4.4, while Table 4.5 shows the residuals after subtraction. The perturbative results are somewhat less accurate for these states; however these states are higher up on the spectrum and therefore are more affected by higher-order terms in the cut-off and \(\lambda\). Still, as we demonstrate later these counterterms result in a spectacular improvement in the agreement between TCSA and the TBA predictions.

Table 4.4: \(O(\lambda^2)\) level contributions for the second \(A\bar{A}\) and the first \(AAA\) state

<table>
<thead>
<tr>
<th>(n)</th>
<th>(A\bar{A})</th>
<th>(AAA)</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>B4</th>
<th>B5</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>-0.0180702</td>
<td>-0.0180669</td>
<td>-0.0178814</td>
<td>-0.0179488</td>
<td>-0.0180169</td>
<td>-0.0180169</td>
<td>-0.0180553</td>
</tr>
<tr>
<td>10</td>
<td>-0.0154775</td>
<td>-0.0154641</td>
<td>-0.0153561</td>
<td>-0.0153936</td>
<td>-0.0155160</td>
<td>-0.0154957</td>
<td>-0.0154819</td>
</tr>
<tr>
<td>11</td>
<td>-0.0134662</td>
<td>-0.0134641</td>
<td>-0.0133835</td>
<td>-0.0134058</td>
<td>-0.0134953</td>
<td>-0.0134780</td>
<td>-0.0134791</td>
</tr>
<tr>
<td>12</td>
<td>-0.0118654</td>
<td>-0.0118636</td>
<td>-0.0118217</td>
<td>-0.0118877</td>
<td>-0.0118881</td>
<td>-0.0118752</td>
<td>-0.0118782</td>
</tr>
</tbody>
</table>

4.3.2 Comparing the renormalized TCSA to the TBA results

The third step listed in the beginning of section 4.3 is the actual construction of counterterms. This was described in Section 2.3 and is straightforward given the level contributions tested above.

The last step is to compare the renormalized TCSA data to the TBA predictions. We must take into account that the TBA and the perturbed conformal field theory (TCSA)
Table 4.5: \( O(\lambda^2) \) level contributions for second \( AA \) and the first \( AAA \) state after subtraction

<table>
<thead>
<tr>
<th>( n )</th>
<th>( B5\ AA )</th>
<th>( B5\ AAA )</th>
<th>( B6\ AA )</th>
<th>( B6\ AAA )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>(-5.63362 \cdot 10^{-5})</td>
<td>(-5.30678 \cdot 10^{-5})</td>
<td>(3.32776 \cdot 10^{-4})</td>
<td>(3.36045 \cdot 10^{-4})</td>
</tr>
<tr>
<td>10</td>
<td>(5.13623 \cdot 10^{-6})</td>
<td>(7.80905 \cdot 10^{-6})</td>
<td>(7.29807 \cdot 10^{-5})</td>
<td>(7.56536 \cdot 10^{-5})</td>
</tr>
<tr>
<td>11</td>
<td>(1.30180 \cdot 10^{-5})</td>
<td>(1.50275 \cdot 10^{-5})</td>
<td>(2.58255 \cdot 10^{-5})</td>
<td>(2.78350 \cdot 10^{-5})</td>
</tr>
<tr>
<td>12</td>
<td>(1.29015 \cdot 10^{-5})</td>
<td>(1.46115 \cdot 10^{-5})</td>
<td>(1.48606 \cdot 10^{-5})</td>
<td>(1.65706 \cdot 10^{-5})</td>
</tr>
</tbody>
</table>

Energy levels differ by the so-called universal bulk energy term \[ Zam90 \]

\[
E_{TBA}(R) = E_{TCSA}(R) - BR \tag{4.3.1}
\]

where

\[
B = \frac{1}{2\sqrt{3}} m^2 \tag{4.3.2}
\]

Therefore following \[ 4.3.1 \] we compare the TBA data to TCSA data with the predicted bulk energy contribution subtracted (with the exception of figure \[ 4.3.1 \]). Some numbers are given in tables in Appendix C; here we only show a few plots for illustration.

Energy levels Figure \[ 4.3.1 \] shows the comparison for the ground state, comparing raw TCSA data for several values of the cut-off, the renormalized TCSA and the TBA data. The renormalization is so efficient in removing the cut-off dependence that we only show the renormalized TCSA data for the highest cut-off, as the others would not be discernible on the plot. The comparison for excited states is shown in figure \[ 4.3.2 \]; it has essentially the same features.

Two-particle phase shifts A more sensitive test is provided by examining the phase-shift extracted from the various two-particle states. Using the Bethe–Yang equations \[ 3.2.33, 3.2.46 \] one can extract phase-shift data from the TCSA spectrum to compare with theoretical predictions. Because the effect of the phase-shift is subleading compared to the momentum quantum number, it is much more sensitive to the accuracy of the numerics. From \[ 3.2.3 \], we define the following phase-shift functions

\[
\delta_{AA}(\theta) = -i \log S_{AA}(\theta)
\]
\[
\delta_{A\bar{A}}(\theta) = -i \log S_{A\bar{A}}(\theta) \tag{4.3.3}
\]

Note that the identification of the ferromagnetic phase kink states with the paramagnetic phase particles defined in \[ 3.2.12 \] makes these definitions applicable in the ferromagnetic phase as well.
Figure 4.3.1: Comparing TCSA and TBA for the ground state. The slow convergence of the TCSA is apparent from the raw data; renormalized data are only presented for level 12, as the others would not be discernible on the plot. In this plot we kept the whole energy without submitting the bulk term (corresponding to the linear behaviour for large volumes) to show that the renormalization also gives back the right value for the universal bulk energy term.

The phase shifts are extracted from two-particle states with zero total-momentum, consisting with a pair of particles with opposite rapidities $\theta$ and $-\theta$. The “experimental” value for the rapidity is determined from

$$E_\Psi(R) - E_0(R) = 2m \cosh \theta \quad (4.3.4)$$

where $E_\Psi(R)$ and $E_0(R)$ are the two-particle and vacuum levels, while the value of the phase shift at $2\theta$ is determined from the quantization conditions (3.2.33, 3.2.46) which reduce to a single equation

$$2\pi I = \omega + mR \sinh \theta + \delta(2\theta) \quad (4.3.5)$$

where the twist $\omega$ is always zero in the paramagnetic phase which contains both neutral ($AA$) and charged ($AA, \bar{A}A$) two-particle levels. In the ferromagnetic phase it can take the values $\omega = 0, \pm 2\pi/3$; however, in this case there are only $A\bar{A}$ levels.
Figure 4.3.2: Comparing TCSA and TBA for excited states. PM stands for paramagnetic, FM for ferromagnetic phase, GS means ground state (twisted in the ferromagnetic phase). The paramagnetic $AA$ and twisted ferromagnetic $\bar{A}A$ are so close numerically that they eventually overlap at this resolution.

The phase shifts extracted from the TCSA data can be compared to the predictions of the infinite volume scattering amplitudes (3.2.3, 3.2.11). For large volumes, corresponding to small $\theta$ we expect truncation effects to dominate. For small volumes the finite size corrections decaying exponentially in the volume make up most of the deviation. To demonstrate that, we also compare the TCSA phase shift to a “effective finite volume phase shift” obtained by substituting the exact TBA energy levels into (4.3.4, 4.3.5). In contrast with the true infinite volume scattering amplitudes, the effective finite volume phase shift is state-dependent. These comparisons are presented for $\delta_{A\bar{A}}$ in figures 4.3.3, 4.3.4 and for $\delta_{AA}$ in figure 4.3.5.

Note that the deviation of the TCSA phase-shift in the high energy (small volume) regime is fully explained by TBA, which is not very surprising in view of the excellent agreement between TCSA and TBA demonstrated in Appendix C. For low energies (large volumes) the agreement is very much improved by the renormalization procedure. We also demonstrate that the residual cut-off dependence is practically nonexistent except for very
low energies; the remaining deviation in that regime is expected to be due to $O(\lambda^3)$ cut-off effects, the elimination of which would necessitate the extension of the renormalization procedure to higher order.

Figure 4.3.3: Comparing $\delta_{A\bar{A}}(\theta)$ extracted from the third excited TCSA level in sector $\mathcal{H}_0$ (lowest lying moving $A\bar{A}$ state) to the scattering theory predictions (3.2.33,3.2.46) and to TBA.

Figure 4.3.4: Comparing $\delta_{A\bar{A}}(\theta)$ extracted from the first excited TCSA level in sectors $\mathcal{H}_\pm$ in the ferromagnetic phase (lowest lying twisted $A\bar{A}$ state) to the scattering theory predictions (3.2.46) and to TBA.
Figure 4.3.5: Comparing $\delta_{AA}(\theta)$ extracted from the first excited TCSA level in sectors $H_{\pm}$ in the paramagnetic phase (lowest lying $AA/\bar{A}\bar{A}$ state) to the scattering theory predictions (3.2.33) and to TBA.
Chapter 5

Renormalization group for confinement

Switching on the magnetic field leads to the confinement of the kinks. The confined mesonic and baryonic states are in the higher part of the spectrum, therefore the counterterm construction is very complicated. Furthermore various operator product expansions of the perturbations contain the perturbing operators as well. Therefore the couplings run in the leading order in contrast to the thermal three-state Potts case.

In the present chapter the results of [LT15] are presented on the confinement spectrum. After discussing some details on the truncated space for the Ising and the three-state Potts model the RG equations and the extrapolating exponents are presented. Finally the numerical results are discussed.

5.1 TCSA details for the Ising and Potts field theories

5.1.1 Scaling Ising model

The Hilbert space of any conformal field theory can be decomposed into products of irreducible representations of the left and right moving Virasoro algebras, which can be specified by giving their left and right conformal weights as

\[ S_{h,\bar{h}} = \mathcal{V}_h \otimes \mathcal{V}_{\bar{h}} \]  \hspace{1cm} (5.1.1)

and every such sector corresponds to a primary field \( \Phi_{h,\bar{h}} \). For the Ising model with central charge \( c = 1/2 \) the full Hilbert space is

\[ \mathcal{H}^{(2)} = S_{0,0} \oplus S_{1/2,1/2} \oplus S_{1/2,-1/2} \]  \hspace{1cm} (5.1.2)
where the sectors on the first line are even, the ones on the second line are odd. The Hamiltonian is

\[ H = H^{(2)}_{\text{CFT}} + \tau \int dx \varepsilon + h \int dx \sigma \]  

(5.1.3)

where

\[ \varepsilon = \Phi_{\frac{1}{2}, \frac{1}{2}} \quad \sigma = \Phi_{\frac{1}{16}, \frac{1}{16}} \]  

(5.1.4)

Note that for this model the values \( h \) and \( -h \) are physically equivalent since they are related by the \( \mathbb{Z}_2 \) symmetry of the conformal field theory.

For the Ising model we used the following level cut-offs with the dimensions of the truncated Hilbert space indicated below:

<table>
<thead>
<tr>
<th>( n )</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{dim} )</td>
<td>77</td>
<td>127</td>
<td>213</td>
<td>338</td>
<td>551</td>
<td>840</td>
<td>1330</td>
<td>1994</td>
<td>3023</td>
<td>4476</td>
</tr>
</tbody>
</table>

5.1.2 Scaling 3-state Potts model

The scaling limit of Potts model at the critical point is a minimal conformal field theory with central charge

\[ c = \frac{4}{5} \]  

[BPZ84, Dot84]. The Hilbert space of the Potts model is the \( D_4 \) modular invariant [CIZ87]

\[ \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \mathcal{H}_1 \]  

(5.1.6)

where

\[ \mathcal{H}_0 = S_{0,0} \oplus S_{\frac{2}{5}, \frac{2}{5}} \oplus S_{\frac{4}{5}, \frac{4}{5}} \oplus S_{3,3} \]

\[ \mathcal{H}_\pm = S_{\frac{1}{16}, \frac{1}{16}}^\pm \oplus S_{\frac{3}{16}, \frac{3}{16}}^\pm \]

\[ \mathcal{H}_1 = S_{\frac{4}{5}, \frac{4}{5}} \oplus S_{\frac{2}{5}, \frac{2}{5}} \oplus S_{0,3} \oplus S_{3,0} \]  

(5.1.7)

The \( D_4 \) conformal field theory is invariant under the permutation group \( S_3 \) generated by two elements \( Z \) and \( C \) with the relations

\[ Z^3 = 1 \quad C^2 = 1 \quad C Z C = Z^{-1} \]  

(5.1.8)

which have the signatures

\[ \text{sign } Z = +1 \quad \text{sign } C = -1 \]  

(5.1.9)
The sectors in $H_0$ of (5.1.6) are invariant under the action of the permutation group $S_3$, the ones in $H_{\pm}$ form the two-dimensional irreducible representation, which is characterized by the following action of the generators:

\[ C|\pm\rangle = \pm|\mp\rangle \]
\[ Z|\pm\rangle = \cos\left(\frac{2\pi}{3}\right)|\pm\rangle \pm \sin\left(\frac{2\pi}{3}\right)|\mp\rangle \] (5.1.10)

while those in $H_1$ transform according to the signature representation of $S_3$.

The Hamiltonian is given by

\[ H = H_{CFT}^{(3)} + \tau \int dx \epsilon + h \int dx \sigma \] (5.1.11)

where

\[ \epsilon = \Phi_{\frac{4}{5} \frac{2}{5}} \quad \sigma = \Phi_{\frac{1}{15} \frac{1}{15}} \] (5.1.12)

In our considerations an important role will be played by charge conjugation parity: the even sector under $C$ consists of $H_0$ and $H_{++}$, while the odd sector consists of $H_1$ and $H_{--}$, with the dimensions as a function of the level cut-off $n$ given below:

<table>
<thead>
<tr>
<th>$n$</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim, even sector</td>
<td>634</td>
<td>1210</td>
<td>2426</td>
<td>4437</td>
<td>8258</td>
<td>14545</td>
</tr>
<tr>
<td>dim, odd sector</td>
<td>816</td>
<td>1572</td>
<td>3039</td>
<td>5592</td>
<td>10121</td>
<td>(17904)</td>
</tr>
</tbody>
</table>

In the case of the odd sector the left/right descendant levels must be different for some of the fields the right in order to get spinless fields. In our convention, the truncation level is chosen to agree with the smaller of the descendant levels. In the extrapolations we used level cut-offs from 6 to 11 in the even sector and from 6 to 10 in the odd sector.

### 5.2 Renormalization group equations and extrapolation details

To study the confinement spectrum one needs the relative energy levels from which the leading cut-off dependence cancels. Therefore the running of the identity coupling can be omitted to leading order.

In the Ising model, the leading exponents can be summarized as:

\[
\begin{array}{c|ccc}
\Phi_a \Phi_b \backslash \Phi_c & 1 & \sigma & \epsilon \\
\hline
\sigma \sigma & -\frac{7}{4} & - & -\frac{11}{4} \\
\sigma \epsilon & - & -1 & - \\
\epsilon \epsilon & 0 & - & - \\
\end{array}
\]
The exponent 0 corresponds to a logarithmic divergence in the ground state energy, which cancels from the relative energy levels. The exponent $-7/4$ also corresponds to ground state renormalization and therefore also cancels. Thus the leading improved RG equations read

$$\frac{d\lambda_\sigma(n)}{dn} = \frac{1}{2n-e_0(r)} C_{\sigma\sigma} \lambda_\sigma^2(n)n^{-\frac{7}{4}}$$

$$\frac{d\lambda_\epsilon(n)}{dn} = \frac{1}{2n-e_0(r)} C_{\sigma\sigma} \lambda_\sigma^2(n)n^{-\frac{11}{4}}$$

(5.2.1)

The two other exponents are taken care of by the running couplings. Therefore it is only necessary to take into account the highest subleading 1/n corrections, which lead to a residual cut-off dependence of the form

$$\tilde{e}_i^{(n)}(r) = e_i(r) + \frac{A_i(r)}{n} + \frac{B_i(r)}{n^2} + O(n^{-11/4})$$

(5.2.2)

where both subleading terms come from 1/n corrections to the $\epsilon\epsilon 1$ term.

In the Potts model, the leading exponents are

<table>
<thead>
<tr>
<th>$\Phi_\alpha \Phi_\beta \Phi_c$</th>
<th>1</th>
<th>$\sigma$</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma\sigma$</td>
<td>$-\frac{26}{15}$</td>
<td>$-\frac{28}{15}$</td>
<td>$-\frac{28}{15}$</td>
</tr>
<tr>
<td>$\sigma\epsilon$</td>
<td>$-\frac{6}{5}$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\epsilon\epsilon$</td>
<td>$-\frac{2}{5}$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

(where some fields contained in the OPE which are even more suppressed) and the RG equations read

$$\frac{d\lambda_\sigma(n)}{dn} = \frac{1}{2n-e_0(r)} C_{\sigma\sigma} \lambda_\sigma^2(n)n^{-\frac{26}{15}} + \frac{1}{2n-e_0(r)} C_{\sigma\sigma} \lambda_\epsilon^2(n)n^{-\frac{28}{15}}$$

$$\frac{d\lambda_\epsilon(n)}{dn} = \frac{1}{2n-e_0(r)} C_{\sigma\sigma} \lambda_\epsilon^2(n)n^{-\frac{38}{15}}$$

(5.2.3)

and the residual cut-off dependence is

$$e_i^{(n)}(r) = e_i(r) + \frac{A_i(r)}{n^{7/5}} + \frac{B_i(r)}{n^{11/5}} + O(n^{-12/5})$$

(5.2.4)

Our prescription for the RG-TCSA is as follows. Using units of $m$ (the kink mass at $\tilde{h} = 0$, see (3.1.9)), $\tau$ is just a fixed dimensionless number given by $k^{(q)}$. Therefore the TCSA has two dimensionless parameters, one of which is given by the value of $\tilde{h}$ in units of $m$, i.e. the ratio

$$\tilde{h} = \frac{\tilde{h}}{m^{2-2h_\sigma}}$$

(5.2.5)

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and the other is dimensionless volume parameter \( r = mR \). For any value of \( r \) and \( \tilde{h} \) the physical values of the perturbed CFT couplings are

\[
\lambda_c = -\frac{\kappa(q) r^{2-2h_c}}{(2\pi)^{1-2h_c}} \\
\lambda_\sigma = \frac{\tilde{h}}{m^{2-2\sigma}} r^{2-2h_\sigma} (2\pi)^{1-2h_\sigma}
\]

(5.2.6)

Taking these as initial conditions at \( n = \infty \), the couplings can be run according to the RG equations (5.2.1, 5.2.3) to determine their value at the given cut-off \( n \). We remark that in all our calculations the couplings ran very little so this has practically no effect, but as a matter of principle it must be done before we proceed to extrapolation. Once the RG eliminated all the leading cut-off dependencies, the renormalized TCSA Hamiltonian can be numerically diagonalized and then the residual cut-off dependence eliminated by fitting (5.2.2) for the Ising and (5.2.4) for the Potts case. In the case of the Ising model it turns out that the residual cut-off dependence alternates in sign between odd and even cut-offs, so the data for even and odd values of \( n \) were fitted separately, as demonstrated in figure 5.2.1. For the case of the Potts model no such alternation was observed, and the data could be reliably extrapolated including both even and odd values of the level cut-off \( n \) as illustrated in 5.2.2. We also remark that in the Potts case we only took into account the two exponents indicated in (5.2.4); the 11/5 and 12/5 exponents are too close together, and their effect is too small compared to the leading 7/5 to include both in the fit.

5.3 Results

5.3.1 Testing ground: Ising model with magnetic field

5.3.1.1 False vacuum

From the TCSA data it is possible to evaluate the energy density of the false vacuum relative to the stable one. For smaller volume, the TCSA converges fast, but for greater volumes the efficiency of the extrapolation procedure is apparent. The theoretical predictions can be calculated using (3.1.15) where the renormalized string tension given in [FZ06] has also been taken into account. However, at the present precision the two predictions cannot be distinguished. Our results are illustrated in Figure 5.3.1.

Note that the false vacuum level is not a continuous level in the volume, therefore its linear rise does not in fact contradict the statement that all relative energy levels \( \tilde{e}_i(r) \) go to a constant for \( r \to \infty \). The metastable states are seen as level avoidances in finite
Figure 5.2.1: Extrapolation fits of the relative energy levels in the Ising model in the ferromagnetic phase with magnetic field $\tilde{h} = 0.008$ for the first four excited states at dimensionless volume $mR = 10$.

Meson masses

The meson masses can be estimated by first extrapolating the levels separately for even and odd cut-offs, and taking the average of the two results. For a more precise measurement the numerical procedure was slightly modified by keeping only the $n = 10 \ldots 15$ data and using only the $1/n$ term in (5.2.2). The reason for this is that the meson level data do not allow fitting the $1/n^2$ term with a sufficient precision.

Selecting a given extrapolated one-particle level, one then finds the volume where the level is the most flat. The data still contain exponential finite size effects, which can be suppressed by fitting the part of the extrapolated $n$th meson level just before the flat
portion by an exponential function

\[ \tilde{e}_n(r) = m_n + A_n e^{-B_n r} \] (5.3.1)

and taking \( m_n \) as the estimated mass of the \( n \)th meson. The results are shown in Table 5.1 (we note that the data did not permit the volume extrapolation for \( \tilde{h} = 0.4 \), so the TCSA numbers quoted there are just the value of the level at the point where it is most flat). The theoretical prediction “Airy” is given by eqn. (3.3.1), while BS and WKB given by (3.3.3) and (3.3.8), respectively; the last line (iWKB) corresponds to taking into account the first correction in (3.3.4). One can see that the prediction from simple quantum mechanics in a linear potential (“Airy”) is generally only precise to a percent level or even worse for large magnetic fields, while WKB is an order of magnitude better. The theoretically rather involved Bethe–Salpeter approach only improves on WKB for very low mass mesons, while for higher masses the WKB is generally better. It is also clear that for \( \tilde{h} \gtrsim 0.2 \) the only working theoretical framework is the WKB method.
Figure 5.3.1: Energy of the false vacuum for different values of the magnetic field in the Ising model. Lines are the theoretical predictions: blue is for (3.1.15), while purple one takes into account the string tension renormalization (the two lines are almost indistinguishable in the graphs). Dotted lines for different cut-off data, red dots are extrapolated (average of even and odd cut-off extrapolations)
Figure 5.3.2: Energy of the false vacuum for different values of the magnetic field in the three state Potts model. Continuous lines are the theoretical predictions from (3.1.15), dashed lines are data with coupling constant renormalization before extrapolation for some values of the cut-off. Red dots are extrapolated data in the $C$-even sector, while black squares are extrapolated data from the $C$-odd sector.
<table>
<thead>
<tr>
<th>$h$</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0175</td>
<td>TCSA 2.303</td>
<td>2.526</td>
<td>2.7175</td>
</tr>
<tr>
<td>0.0175</td>
<td>Airy 2.3068</td>
<td>2.5364</td>
<td>2.7243</td>
</tr>
<tr>
<td>0.0175</td>
<td>BS 2.3021</td>
<td>2.5228</td>
<td>2.7005</td>
</tr>
<tr>
<td>0.0175</td>
<td>WKB 2.3000</td>
<td>2.5223</td>
<td>2.7003</td>
</tr>
<tr>
<td>0.0175</td>
<td>iWKB 2.3011</td>
<td>2.5225</td>
<td>2.7004</td>
</tr>
<tr>
<td>0.0250</td>
<td>TCSA 2.382</td>
<td>2.659</td>
<td>2.8975</td>
</tr>
<tr>
<td>0.0250</td>
<td>Airy 2.3891</td>
<td>2.6803</td>
<td>2.9187</td>
</tr>
<tr>
<td>0.0250</td>
<td>BS 2.3816</td>
<td>2.6590</td>
<td>2.8813</td>
</tr>
<tr>
<td>0.0250</td>
<td>WKB 2.3791</td>
<td>2.6583</td>
<td>2.8812</td>
</tr>
<tr>
<td>0.0250</td>
<td>iWKB 2.3803</td>
<td>2.6586</td>
<td>2.8812</td>
</tr>
<tr>
<td>0.0375</td>
<td>TCSA 2.497</td>
<td>2.855</td>
<td>3.1575</td>
</tr>
<tr>
<td>0.0375</td>
<td>Airy 2.5099</td>
<td>2.8915</td>
<td>3.2039</td>
</tr>
<tr>
<td>0.0375</td>
<td>BS 2.4969</td>
<td>2.8556</td>
<td>3.1414</td>
</tr>
<tr>
<td>0.0375</td>
<td>WKB 2.4941</td>
<td>2.8552</td>
<td>3.1418</td>
</tr>
<tr>
<td>0.0375</td>
<td>iWKB 2.4954</td>
<td>2.8553</td>
<td>3.1417</td>
</tr>
<tr>
<td>0.05</td>
<td>TCSA 2.597</td>
<td>3.027</td>
<td>3.3775</td>
</tr>
<tr>
<td>0.05</td>
<td>Airy 2.6177</td>
<td>3.0800</td>
<td>3.4584</td>
</tr>
<tr>
<td>0.05</td>
<td>BS 2.5987</td>
<td>3.0281</td>
<td>3.3684</td>
</tr>
<tr>
<td>0.05</td>
<td>WKB 2.5958</td>
<td>3.0283</td>
<td>3.3701</td>
</tr>
<tr>
<td>0.05</td>
<td>iWKB 2.5971</td>
<td>3.0282</td>
<td>3.3698</td>
</tr>
<tr>
<td>0.1</td>
<td>TCSA 2.933</td>
<td>3.588</td>
<td>4.1175</td>
</tr>
<tr>
<td>0.1</td>
<td>Airy 2.9805</td>
<td>3.7143</td>
<td>4.3151</td>
</tr>
<tr>
<td>0.1</td>
<td>BS 2.9312</td>
<td>3.5851</td>
<td>4.0919</td>
</tr>
<tr>
<td>0.1</td>
<td>WKB 2.9317</td>
<td>3.5946</td>
<td>4.1130</td>
</tr>
<tr>
<td>0.1</td>
<td>iWKB 2.9320</td>
<td>3.5933</td>
<td>4.1114</td>
</tr>
<tr>
<td>0.2</td>
<td>TCSA 3.447</td>
<td>4.451</td>
<td>5.2275</td>
</tr>
<tr>
<td>0.2</td>
<td>Airy 3.5565</td>
<td>4.7213</td>
<td>5.6750</td>
</tr>
<tr>
<td>0.2</td>
<td>BS 3.4115</td>
<td>4.3556</td>
<td>5.0251</td>
</tr>
<tr>
<td>0.2</td>
<td>WKB 3.4474</td>
<td>4.4508</td>
<td>5.2257</td>
</tr>
<tr>
<td>0.2</td>
<td>iWKB 3.4426</td>
<td>4.4455</td>
<td>5.2209</td>
</tr>
<tr>
<td>0.4</td>
<td>TCSA 4.222</td>
<td>5.717</td>
<td>6.8575</td>
</tr>
<tr>
<td>0.4</td>
<td>Airy 4.4707</td>
<td>6.3198</td>
<td>7.8337</td>
</tr>
<tr>
<td>0.4</td>
<td>BS 3.8964</td>
<td>4.8679</td>
<td>4.9503</td>
</tr>
<tr>
<td>0.4</td>
<td>WKB 4.2295</td>
<td>5.7277</td>
<td>6.8706</td>
</tr>
<tr>
<td>0.4</td>
<td>iWKB 4.2074</td>
<td>5.7102</td>
<td>6.8559</td>
</tr>
</tbody>
</table>

Table 5.1: Meson masses in the Ising model. Theoretical predictions are shown with 4 digits accuracy, while for TCSA we show the digits that can be reliably extracted (with the last digit having an estimated precision of order 1).
The upshot is that for all practical purposes WKB can be taken as the most reliable description of the spectrum over all the parameter range: it gives an approximation within $10^{-3}$ relative precision. This is an important lesson given that there are no Bethe–Salpeter predictions available in the three-state Potts case yet; however, we can take the WKB as an accurate prediction for comparison with the three-state Potts meson data.

### 5.3.2 Three-state Potts model

For the three-state Potts model we only consider the domain $h < 0$, since according to the discussion in subsection 3.3.2 there are no meson one-particle levels for $h > 0$, and reading off meson masses from two-particle states is difficult both for numerical reasons (the spectrum is dense, so level identification is difficult) and for theoretical reasons (extraction of masses with any precision requires modeling the meson-meson scattering). In addition, as discussed in subsection 3.3.3 there are no baryon states for $h > 0$.

#### 5.3.2.1 False vacuum

The relative energy of the false vacuum against the volume is shown in Figure 5.3.2. Notice that the extrapolation is again very effective.

#### 5.3.2.2 Meson masses

In contrast to the Ising case here we can only use the method of extracting the value at the flattest portion of curve to estimate meson masses, due to the presence of the “false meson” resonance corresponding to a kink-anti-kink bound state configuration starting and ending in one of the false vacua. The “wavy” feature these resonance plateaus introduce in the spectrum prevent application of the exponential fit (5.3.1) to eliminate finite size effects in the meson mass. This effect can be seen in the plot 5.3.3, which also demonstrates the efficiency of the numerical extrapolation procedure. The mesonic spectrum against the absolute values of the magnetic field can be seen on figure 5.3.4. The deviations between the WKB prediction and the numerically determined masses are typically of the order of a few times $10^{-3}$, except in a few cases when a larger deviation of order $10^{-2}$ is observed. These are cases when the flattest portion of the meson level contains a level crossing with the false vacuum, which makes the truncation level extrapolation less precise.

#### 5.3.2.3 Baryon masses

The baryonic states are in the higher part of the spectrum. As a result, because of the many level crossings the state must be carefully identified for each value of the volume
and cut-off in order to carry out the extrapolation. The masses are extracted as the value of the extrapolated energy levels at its flattest point. As noted in subsection 3.3.3, baryons and antibaryons have the same spectra in infinite volume. In finite volume the eigenstates are the charge conjugation ($C$) even and odd combinations, as can be seen from the results shown in Figure 5.3.3. The deviations between the theoretical prediction and the numerically determined masses are typically of the order of a few times $10^{-3}$, except in a few cases when a larger deviation of order a few times $10^{-2}$ is observed, which in this case is mostly due to difficulties of locating the level in the dense part of the spectrum.

![Figure 5.3.3](image)

Figure 5.3.3: Effect of the extrapolation for the first three mesonic state in the $C$-even and odd sectors with magnetic field $\tilde{h} = -0.02$. Dashed lines for data with running coupling for some values of the cut off, while large dots with dotted lines are the extrapolated data. We also plotted the energy of the even sector false vacuum (both before and after extrapolation, the latter marked with green) shifted up by $2m$ to demonstrate that the wavy feature corresponds to the “false meson” resonance (meson configuration over the false vacuum).
Figure 5.3.4: Meson masses against the magnetic field in the three state Potts model. Black dots show TCSA results from the $C$-even sector while red is for $C$-odd. The green and blue lines show the WKB predicted for parity even and odd states. As expected, for mesons the two parities coincide.

Figure 5.3.5: The three lowest baryon masses against the magnetic field in the three state Potts model. Black dots show TCSA results from the $C$-even sector while red is for $C$-odd. The green and blue lines show the predictions (3.3.19, 3.3.20) for space parity even and odd states, respectively. It can be seen that for each spatial parity there are two states, corresponding to the $C$-even/odd combinations of the baryon with its antiparticle.
Chapter 6

Discussion and outlook

In this thesis we reviewed two different renormalization methods for the truncated conformal space approach with applications. The counterterm formalism which is applied to the thermally perturbed three-state Potts model is published in the work [LT14]. The renormalization group method is applied to the confinement spectrum in the Ising and the three-state Potts models with magnetic field, which can be found in [LT15].

In the counterterm approach we construct cut-off dependent energy counterterms for each states in order to eliminate the truncation dependence of the TCSA energy levels. The drawback of this method is that the computation becomes progressively more complicated when higher energy states are considered. In contrast, the renormalization group approach combined with extrapolation can renormalize all the states simultaneously, however at present it can only treat the leading order term in $1/n$.

We have seen that using the counterterm approach the cut-off dependence can be eliminated effectively for low lying states and the results are in perfect correspondence with thermodynamic Bethe ansatz calculations. The idea of this approach goes back to [GW11], our improvements are the construction of higher order terms in the inverse cut-off and the extension to descendant and degenerate conformal states.

The results of [RZ06] and [RSTZ13] show that scattering phase of quasi-particles from the long distance limit of the spin chain differs from factorized scattering theory. Our results show that the low energy behaviour of the phase-shift extracted from TCSA has strong cut-off dependence and shows similar features as on the spin chain. With our counterterm approach it is clear that this behaviour can be described by presence of irrelevant operators. Since the phase-shift is calculated from relative energy levels the leading $1/n$ correction from the identity operator is absent. From the OPE $\Phi_{2,1} \times \Phi_{2,1} = \mathbb{I} + \Phi_{3,1}$ one sees that the next correction is due to operator $\Phi_{3,1}$ which is an irrelevant operator with conformal weight $h_{3,1} = 7/5$. From the renormalization group point of view, it means that the coupling of the irrelevant operator should be non-zero at finite
cut-off and it flows to zero when the cut-off is removed.

Turning to the confinement in the $q$-state Potts model, the application of the counterterms are complicated since mesons and baryons are in the higher energy part of the spectrum, furthermore the couplings run in the leading order of the inverse cut-off. These can be solved by applying the improved renormalization group equations, which works universally for all states. Higher order $1/n$ corrections can then be taken into account by numerical extrapolation. This approach works universally for all states. As a benchmark we used the Ising model where we found agreement between the TCSA and the theoretical predictions for the meson spectrum known in the literature. For the case of stronger magnetic field we found that the WKB meson spectrum describes the confinement very well. Then we extended the same method to the three-state Potts model and found that the meson spectrum can again be described by WKB, and we also verified the validity of the recently proposed baryon spectrum. We discussed the correspondence of the spatial and the charge parity of the confined particles, and we showed that for the masons spatial and charge parity are always the same, while for the baryons they can appear in any combinations.

The results of the thesis lead to some important conclusions from a quantitative viewpoint. The renormalized TCSA results show that the thermodynamic Bethe ansatz equations describe the finite volume spectrum in the thermally perturbed three-state Potts model. It is also found that for the $q$-state Potts model in weak magnetic field the meson spectrum can be described very precisely by WKB methods, while the baryon spectrum can be modeled efficiently by the three particle quantum mechanical problem. These findings are expected to be relevant in further investigations of these models as descriptions of statistical systems, as the theoretical predictions provide input for such calculations.

The improvements of the renormalized truncated space approach also have relevance from the technical viewpoint. As we pointed out, in the original work of Yurov and Zamolodchikov the truncated space gave high accuracy results with a few conformal states. For more complicated models the convergence can be very slow. Since the conformal space grows exponentially with the cut-off the dimension of the truncated space can be huge. The renormalization is extremely important in the case of asymptotically free and WZNW conformal field theories \cite{BBL, KPTT15}, which appear in bosonized strongly correlated electronic systems. In such models the available cut-offs with today’s computers are very low, thus the raw TCSA results are far from the exact ones. Numerical renormalization group (NRG) methods can be used to get results from higher cut-offs \cite{KA07}, but the range and the accuracy of NRG is limited as well. Combining the NRG improvement with a perturbative RG extrapolation, however, leads to sufficiently precise spectra to test
theoretical predictions and also allows to extract further nontrivial information about the dynamics of the model [KPTT15].

An interesting open problem is to compute analytically the sub-leading cut-off corrections. This can be done by generalizing the calculations of [RV15]. The complete “package” of the combination of NRG and analytic RG with subleading corrections is currently under development and it is expected to be done in the near future. This opens the way to investigate a wide range of two-dimensional field theories previously inaccessible for the TCSA method.

There is also a growing interest in applying TCSA to theories beyond two space-time dimensions\(^1\). The work [HRv15] treats the perturbations of the free boson theory in 2.5 dimensions\(^2\), where renormalization is applied as well. They found reassuring results for this specific case. In contrast in two dimensions, non-trivial fixed points have been studied as well. The characterization of CFTs and the bootstrap program in higher dimensions are subjects of intensive ongoing research [EPP+12, EPP+14, GR14], even in presence of boundary [GLMR15]. The conformal data sufficient for set up the TCSA to study theories in the vicinity of strongly interacting fixed points in higher dimensions hopefully will be available soon.

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\(^2\)CFT can be defined in fractional dimensions. The reason of this choice is purely technical, namely the absence of null-vectors.
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Appendix A

Two-dimensional CFT

A.1 Conformal transformations

Let us consider first a $d$ dimensional space-time with metric tensor $g_{\mu\nu}$. The $x \rightarrow x'$ coordinate transformation is called conformal if it leaves the metric invariant up to a scale factor

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \quad (A.1.1)$$

It can be easily seen that such transformations leave invariant the angles between curves crossing each other in a point. Considering an infinitesimal transformation $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(x)$ the condition [A.1.1] leads to the differential equation

$$(d-1) \partial^2 f = 0 \quad (A.1.2)$$

where $\partial^2 = \partial_\mu \partial^\mu$ and $f = \frac{2}{d} \partial_\mu \epsilon^\mu$. Thus in one dimension every smooth coordinate transformation is conformal. In general the above constraint leads to the finite transformations which form the (global) conformal group

$$
\begin{align*}
x'^\mu &= x^\mu + a^\mu & \text{translation} \\
x'^\mu &= \alpha x^\mu & \text{dilation} \\
x'^\mu &= M_{\nu}^\mu x^\nu & \text{rigid rotation} \\
x'^\mu &= \frac{x^\mu - b^\mu x^2}{1-2b^\mu x^\mu + b^2 x^2} & \text{special conformal transformation}
\end{align*} \quad (A.1.3)
$$

Let’s turn now our attention to the two dimensional case with coordinates $(z^0, z^1)$ and consider a coordinate transformation $z^\mu \rightarrow w^\mu(z)$ under which the metric transform as

$$g^{\mu\nu} \rightarrow \left( \frac{\partial w^\mu}{\partial z^\alpha} \right) \left( \frac{\partial w^\nu}{\partial z^\beta} \right) g^{\alpha\beta} \quad (A.1.4)$$
To satisfy the condition [A.1.1] the transformation has to satisfy
\[
\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \tag{A.1.5}
\]
or
\[
\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \quad \text{and} \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \tag{A.1.6}
\]

The system of equations [A.1.5] is known from complex analysis as the Cauchy–Riemann equations which hold for holomorphic functions, while [A.1.6] holds for antiholomorphic functions. Therefore it is convenient to use complex coordinates:
\[
z = z^0 + iz^1 \quad \overline{z} = z^0 + iz^1 \quad z^0 = \frac{1}{2} (z + \overline{z}) \quad z^1 = \frac{1}{2i} (z - \overline{z})
\]
\[
\frac{\partial}{\partial z} = \frac{\partial}{\partial z^0} - i \frac{\partial}{\partial z^1} \quad \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial z^0} + i \frac{\partial}{\partial z^1}
\]

in these coordinates one has
\[
\overline{\partial} w (z, \overline{z}) = 0 \quad \partial \overline{w} (z, \overline{z}) = 0 \tag{A.1.8}
\]

The complex coordinates are treated as independent coordinates, and the physical space itself is the so-called real surface where \(z^* = \overline{z}\). Since [A.1.8] infinitesimal transformations of \(z\) and \(\overline{z}\) can be treated separately. The holomorphic one reads
\[
z' = z + \epsilon (z) \quad \epsilon (z) = \sum_{-\infty}^{\infty} c_n z^{n+1} \tag{A.1.9}
\]

and the antiholomorphic can be written in the same way using bars. From the transformation of a spinless and dimensionless field one can read that the generators of the conformal transformations are
\[
\ell_n = -z^{n+1} \partial \quad \overline{\ell}_n = -\overline{z}^{n+1} \overline{\partial} \tag{A.1.10}
\]

and satisfy the Witt algebra commutation relations
\[
[\ell_n, \ell_m] = (n - m) \ell_{n+m} \quad [\overline{\ell}_n, \overline{\ell}_m] = (n - m) \overline{\ell}_{n+m} \quad [\ell_n, \overline{\ell}_m] = 0 \tag{A.1.11}
\]
It is worth to mention that the generators $\ell_{-1} = -\partial$, $\ell_0 = -z\partial$ and $\ell_1 = -z^2\partial$ form a finite subalgebra, which is associated to the global conformal group. They generate translations, dilations and special conformal transformations on the complex plane. The generators that preserve the real surface are

$$\ell_n + \bar{\ell}_n \quad i(\ell_n - \bar{\ell}_n)$$

namely $\ell_0 + \bar{\ell}_0$ generates dilations and $i(\ell_0 - \bar{\ell}_0)$ rotations on the real surface.

A.1.1 Primary fields, radial quantization and Ward identities

Primary fields and energy momentum tensor Recalling the results presented in [A.1] in the two dimensional case local conformal transformations are holomorphic and antiholomorphic mappings $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$ of the complex coordinates $z$ and $\bar{z}$. The primary fields are defined by their transformation properties under the local conformal transformations

$$\Phi_{h,\bar{h}}(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^\bar{h} \Phi_{h,\bar{h}}(f(z), \bar{f}(\bar{z}))$$

where $h$ and $\bar{h}$ are real numbers and called right and left conformal weights. Under infinitesimal transformations $z \to z + \epsilon(z)$ and $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$ the primary fields transform as

$$\delta_{\epsilon,\bar{\epsilon}}\Phi_{h,\bar{h}}(z, \bar{z}) = \left((h\partial\epsilon + \epsilon\partial\bar{h}) + (\bar{h}\partial\bar{\epsilon} + \bar{\epsilon}\partial h)\right) \Phi_{h,\bar{h}}(z, \bar{z})$$

In two dimensions and in complex coordinates [A.1] the matrix elements of the stress energy tensor read

$$T_{zz} \equiv T(z, \bar{z}) = \frac{1}{4}(T_{00} - T_{11} + 2iT_{01})$$

$$T_{\bar{z}\bar{z}} \equiv \bar{T}(z, \bar{z}) = \frac{1}{4}(T_{00} - T_{11} - 2iT_{01})$$

$$T_{zz} = T_{\bar{z}\bar{z}} = \frac{1}{4}\Theta(z, \bar{z}) = \frac{1}{4}(T_{00} + T_{11}) = \frac{1}{4}T^\mu_\mu$$

and the continuity equation reads

$$\bar{\partial}T + \frac{1}{4}\partial\Theta = 0$$

$$\partial\bar{T} + \frac{1}{4}\bar{\partial}\Theta = 0$$

since in a conformal invariant theory the trace of the energy momentum tensor vanishes
\[ T(z, \bar{z}) = T(z) \quad \text{and} \quad \bar{T}(z, \bar{z}) = \bar{T}(|z|) \]

Radial quantization and conformal Ward identities

Let us denote the Euclidean time and space coordinates are \( \tau \) and \( x \) and introduce periodic boundary condition in the space direction i.e. \( x + \frac{\lambda}{\tau} = x \), therefore our theory is defined on a space-time cylinder. As in Eq. (A.1), we can define complex coordinates \( \zeta, \bar{\zeta} = \tau \pm ix \). Under the exponential mapping

\[
\zeta \rightarrow z = \exp \frac{2\pi}{R} \zeta \\
\bar{\zeta} \rightarrow \bar{z} = \exp \frac{2\pi}{R} \bar{\zeta}
\]

(A.1.16)

the cylinder maps to the complex plane where the origin and the complex infinity correspond to the infinite past and future, equal \( \tau \) time slices on the cylinder can be identified by circles around the origin of radius \( e^{\frac{2\pi}{R}} \). Therefore the usual time-ordering on the cylinder becomes radial ordering on the conformal plane. The \( T \)-ordered operator products become \( R \)-ordered products

\[
R(\phi_1(z) \phi_2(w)) = \begin{cases} \\
\phi_1(z) \phi_2(w); & |z| < |w| \\
\pm \phi_2(w) \phi_1(z); & |w| < |z|
\end{cases}
\]

(A.1.17)

where in the second line the minus sign corresponds to fermionic operators. Then the spatial integral of density becomes angular integral on the plane. The conserved charge corresponding to conformal transformations thus can be written as

\[
Q = \frac{1}{2\pi i} \oint \left( dz T(z) \epsilon(z) + d\bar{z} \bar{T}(\bar{z}) \bar{\epsilon}(\bar{z}) \right)
\]

(A.1.18)

Using this definition the equal time commutator of an operator with a spatial integral of a local operator on the cylinder becomes the contour integration of the radial ordered product on the plane

\[
\left[ \int dx B, A \right]_{E.T.} \rightarrow \oint dz R(B(z) A(w))
\]

(A.1.19)

From now on all operator products are understood as radial ordered product, therefore we shall drop the \( R \) notation. Since the change of an operator under an infinitesimal transformation can be expressed by commutator of the charge and the operator itself the infinitesimal transformation rule of the primary fields can be satisfied if the operator product of the energy-momentum tensor and a primary field has the following short
distance singularity structure:

\[ T(z) \Phi_{h,\bar{h}}(w, \bar{w}) = \frac{h}{(z-w)^2} \Phi_{h,\bar{h}}(w, \bar{w}) + \frac{1}{z-w} \partial \Phi_{h,\bar{h}}(w, \bar{w}) + \text{reg.} \]

\[ \bar{T}(\bar{z}) \Phi_{h,\bar{h}}(w, \bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2} \Phi_{h,\bar{h}}(w, \bar{w}) + \frac{1}{\bar{z}-\bar{w}} \bar{\partial} \Phi_{h,\bar{h}}(w, \bar{w}) + \text{reg.} \quad (A.1.20) \]

The conformal Ward identities can be written as

\[ \langle T(z) \Phi_{h_1,\bar{h}_1}(z_1, \bar{z}_1) \ldots \rangle = \frac{1}{2\pi i} \oint_C dz \epsilon(z) \left[ \sum_{i=1}^n \left( \frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i} \partial_i \right) \langle \Phi_{h_1,\bar{h}_1}(z_1, \bar{z}_1) \ldots \rangle \right] \]

(A.1.21)

and

\[ \langle \bar{T}(\bar{z}) \Phi_{h_1,\bar{h}_1}(z_1, \bar{z}_1) \ldots \rangle = \frac{1}{2\pi i} \oint_{\bar{C}} d\bar{z} \epsilon(\bar{z}) \left[ \sum_{i=1}^n \left( \frac{\bar{h}_i}{(\bar{z}-\bar{z}_i)^2} + \frac{1}{\bar{z}-\bar{z}_i} \bar{\partial}_i \right) \langle \Phi_{h_1,\bar{h}_1}(z_1, \bar{z}_1) \ldots \rangle \right] \]

(A.1.22)

where the contour \( C \) encircles all the coordinates in the product.

### A.1.2 Virasoro representations

**Virasoro algebra** The stress-energy tensor is not a primary field since the operator product of it with itself can be written as

\[ T(z) T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w) \]

\[ \bar{T}(\bar{z}) \bar{T}(\bar{w}) = \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2} \bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}} \bar{\partial} \bar{T}(\bar{w}) \quad (A.1.23) \]

where the constants \( c \) and \( \bar{c} \) are the central charges of the theory and they generally coincide, such as the models considered in the present thesis.

The stress-energy tensor can be expanded to Taylor–Laurent series (say around the origin)

\[ T(z) = \sum_{-\infty}^{\infty} \frac{L_n}{z^{n+2}} \quad \bar{T}(\bar{z}) = \sum_{-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}} \]

The action of the mode \( L_n \) on a local field can be written as

\[ L_n A(z, \bar{z}) = \frac{1}{2\pi i} \oint d\zeta (\zeta - z)^{n+1} T(\zeta) A(z, \bar{z}) \]

\[ \bar{L}_n A(z, \bar{z}) = \frac{1}{2\pi i} \oint d\bar{\zeta} (\bar{\zeta} - \bar{z})^{n+1} \bar{T}(\bar{\zeta}) A(z, \bar{z}) \quad (A.1.24) \]
The modes of the stress-energy tensor satisfy the commutation relation of the Virasoro algebra

\[
[L_m, L_n] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}
\]

\[
[\bar{L}_m, L_n] = (n - m) \bar{L}_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}
\]

\[
[L_n, \bar{L}_m] = 0
\]

(A.1.25)

Which is the central extension of the Witt algebra with central charge \( c \). This is the symmetry algebra of two dimensional conformal field theory. In order to classify the universality classes of critical system one has to examine and classify the representations of the Virasoro algebra.

**Operator state correspondence** Let us define the conformal vacuum state |0\rangle, which is invariant under conformal transformations. One can act then on a vacuum state by a field. The state is called “in” state if

\[
|A\rangle_{in} \equiv \lim_{z, \bar{z} \to 0} A(z, \bar{z}) |0\rangle
\]

(A.1.26)

Recalling the form of the exponential mapping from the cylinder to the plane this state corresponds to a state which is created in the infinite past. Similarly the “out” state should be created in the infinite future, therefore it corresponds to a field evaluated in \( z, \bar{z} \to \infty \) which can implemented by a conformal transformation \( z = \frac{1}{w} \). In this way we have

\[
|A\rangle_{out} \equiv \lim_{w, \bar{w} \to 0} \langle 0| \tilde{A}(w, \bar{w})
\]

(A.1.27)

Then the adjoint operator \( \tilde{A} \) is related to \( A \) as

\[
\tilde{A}(w, \bar{w}) = A\left(\frac{1}{w}, \frac{1}{\bar{w}}\right) (-w^{-2})^h (-\bar{w}^{-2})^{\bar{h}}
\]

(A.1.28)

Finally we have

\[
|A\rangle_{out} = \lim_{z, \bar{z} \to \infty} \langle 0| A(z, \bar{z}) z^{2h} \bar{z}^{2\bar{h}}
\]

(A.1.29)

and it can be easily shown that \( |A\rangle_{out} = |A\rangle_{in} \)†.

**Verma modules** As it is presented in the previous section primary fields are the basic objects of a conformal invariant field theory. Indeed they play a crucial role in the representations of the Virasoro algebra. Highest weight states are created by primary fields
\[ |h, \bar{h}\rangle = \Phi_{h, \bar{h}}(0,0)|0\rangle \] and using equation (A.1.24) they satisfy the following

\[
\begin{align*}
L_0 |h, \bar{h}\rangle &= h |h, \bar{h}\rangle \\
\bar{L}_0 |h, \bar{h}\rangle &= \bar{h} |h, \bar{h}\rangle \\
L_n |h, \bar{h}\rangle &= \bar{L}_n |h, \bar{h}\rangle = 0 \quad n \geq 1
\end{align*}
\]

Using the Virasoro commutation relations equation (A.1.25) one can show that if one acts by a generator of negative index the new state is also \( L_0 \) eigenstate

\[
\begin{align*}
L_0 L_{-n} |h, \bar{h}\rangle &= (n + h) |h, \bar{h}\rangle \\
\bar{L}_0 \bar{L}_{-n} |h, \bar{h}\rangle &= (\bar{n} + \bar{h}) |h, \bar{h}\rangle
\end{align*}
\] (A.1.31)

The states in a representation therefore can be generated by acting generators of negative index. Such representation is called Verma module.

\[ \mathcal{V}_{h, \bar{h}} = \text{span} \left\{ L_{-n_1} L_{-n_2} \cdots L_{-n_k} \bar{L}_{-\bar{n}_1} \bar{L}_{-\bar{n}_2} \cdots \bar{L}_{-\bar{n}_l} |h, \bar{h}\rangle |k, l, n_i, \bar{n}_j \in \mathbb{Z}^+ \right\} \] (A.1.32)

The \( L_0 \) eigenvalues of such a state \( |\psi\rangle = L_{-n_1} L_{-n_2} \cdots L_{-n_k} \bar{L}_{-\bar{n}_1} \bar{L}_{-\bar{n}_2} \cdots \bar{L}_{-\bar{n}_l} |h, \bar{h}\rangle \) is

\[
\begin{align*}
L_0 |\psi\rangle &= (n_1 + n_2 + \cdots + n_k + h) |\psi\rangle \\
\bar{L}_0 |\psi\rangle &= (\bar{n}_1 + \bar{n}_2 + \cdots + \bar{n}_l + \bar{h}) |\psi\rangle
\end{align*}
\] (A.1.33)

the sum of the indices \( n_1 + n_2 + \cdots + n_k \) is called left descendant level. Recalling the adjoint of the energy momentum tensor it is easy to see that the adjoint states can be created by generators of positive index acting on the “out” vacuum. For the same reason generators of negative indices annihilate the “out” vacuum. Then the norm and overlap of such states can be calculated using the commutation relations of the Virasoro algebra.

### A.1.3 Minimal models

Since the left and right Virasoro generators commute the whole theory can be factorized into left and right chiral parts and the whole theory can be reconstructed using the tensor product of the two commuting symmetry algebras. Let us now consider only the right chiral part of the representation, all the following considerations can be applied for the left chiral representation. The level \( N \) descendant vectors of a chiral Verma module built on the highest weight state \( |h\rangle = \phi_h(0)|0\rangle \) and has the form

\[ |h, \{n\}\rangle = L_{-n_1} L_{-n_2} \cdots L_{-n_l} |h\rangle \] (A.1.34)
where \( N = \sum n_i \). The convenient normalization is
\[
\langle h | h' \rangle = \delta_{h,h'} \quad (A.1.35)
\]
Applying the definition of the adjoint to the mode expansion of the stress-energy tensor it can be shown that
\[
L_n^\dagger = L_{-n} \quad (A.1.36)
\]
then using Virasoro commutation relation one has
\[
\langle h, \{n\} | h', \{n'\} \rangle = \delta_{h,h'} \delta_{N,N'} G(N)_{\{n\},\{n'\}} \quad (A.1.37)
\]
where \( G(N) \) is called the level \( N \) Gram matrix. In order to get unitary theories it is necessary to ensure that the Hilbert space does not contain negative norm states. To do this it is necessary to analyze the determinant of the Gram matrix which can be written explicitly due to the work of Kac it is know as Kac determinant
\[
\det G(N) = A_N \prod_{r,s \geq 1; \; rs < N} [h - h_{r,s}]^{P(N - rs)} \quad (A.1.38)
\]
where \( P(N - rs) \) is the number of integer partitions of \( N - rs \). In order to study critical behaviour it is enough study models with \( c \leq 1 \). It turned out as well that the most relevant conformal field theories in this point of view are the so-called minimal models \( \mathcal{M}_{p,q} \) with
\[
c = 1 - 6 \frac{(p - q)^2}{pq} \quad (p, q) = 1
\]
\[
h_{r,s} = \frac{(pr - qs)^2 - (p - q)^2}{4pq} \quad 1 \leq r \leq q - 1; 1 \leq s \leq p - 1 \quad (A.1.39)
\]
where \( p \) and \( q \) are co-prime integers. The unitary minimal models are those with \( p = q + 1 \). The matrix made of the numbers \( h_{r,s} \) is called the Kac table with symmetry
\[
h_{r,s} = h_{q-r,p-s} \quad (A.1.40)
\]
and the primary operators in a minimal model can have these conformal weights. Such operator will be denoted as \( \phi_{r,s} \) and it is identified with \( \phi_{q-r,p-s} \). It is important to note that in the Verma module built on \( \phi_{r,s} \) the Kac determinant vanishes at level \( rs \) and \( (q - r)(p - s) \). It means that the naive states form a linearly dependent system e.g. there can be vectors of zero length which are called null vectors.
From equation (A.1.20) it can be seen that the singular part of the operator product expansion of a primary field with the stress energy tensor contains the field itself and its derivative. In general the operator product expansion (OPE) contains whole conformal families

\[ \Phi_a(z, \bar{z}) \Phi_b(w, \bar{w}) = \sum_p \sum_{\{k, \bar{k}\}} C_{ab}^{p\{k, \bar{k}\}} (z - w)^{h_p - h_A - h_B + K} (\bar{z} - \bar{w})^{\bar{h}_p - \bar{h}_A - \bar{h}_B + \bar{K}} \Phi_p^{\{k, \bar{k}\}} (w, \bar{w}) \]  

where the summation over \( p \) runs over the primary fields of the theory, the set \( \{k, \bar{k}\} \) are the set of right and left descendant indices and \( K = \sum_i k_i, \bar{K} = \sum_i \bar{k}_i \) are the right and left descendant level. The constants \( C_{ab}^{p\{k, \bar{k}\}} \) can be written as

\[ C_{ab}^{p\{k, \bar{k}\}} = C_{ab}^p \beta_{ab}^{p\{k, \bar{k}\}} \]  

where the structure constants \( C_{ab}^p \) are defined via the three point function of the primary fields \( a, b \) and \( p \):

\[ \langle \Phi_a | \Phi_p (1, 1) | \Phi_b \rangle = C_{ab}^p \]  

and they play a crucial role in the solution of a conformal field theory: in principle every correlation function can be reduced to them using the OPE successively. In minimal models the structure constants are known exactly.

For an important example consider now the four point function

\[ \langle \Phi_i (z_1, \bar{z}_1) \Phi_j (z_2, \bar{z}_2) \Phi_k (z_3, \bar{z}_3) \Phi_l (z_4, \bar{z}_4) \rangle \]

Taking now the OPE in two different way one gets the crossing symmetry equation

\[ \langle \Phi_l | \Phi_j (1, 1) | \Phi_k (z, \bar{z}) | \Phi_i \rangle = \sum_p C_{ij}^p C_{kl}^{pm} F_{ij}^{lm} (p|z) \bar{F}_{il}^{jm} (p|\bar{z}) \]

where the function \( F_{ij}^{kl} (p|z) \) and \( F_{ij}^{kl} (q|1 - z) \) are related via the linear relation

\[ F_{ij}^{kl} (p|z) = \sum_q F_{pq}^{k} \left[ \begin{array}{cc} k & l \\ i & j \end{array} \right] F_{ij}^{kl} (q|1 - z) \]

where the function \( F_{ij}^{kl} (p|z) \) and \( F_{ij}^{kl} (q|1 - z) \) are related via the linear relation

\[ F_{ij}^{kl} (p|z) = \sum_q F_{pq}^{k} \left[ \begin{array}{cc} k & l \\ i & j \end{array} \right] F_{ij}^{kl} (q|1 - z) \]

where \( F_s \) are called fusion coefficients.
Operator algebra of primary fields in minimal models  In the minimal model \( \mathcal{M}_{p,q} \) there are finite number of primary fields indexed by the Kac table indices. It can be written which families are contained in the operator product of two primary fields, i.e. the operator algebra:

\[
\phi_{r,s} \times \phi_{m,n} = \sum_{k=1+|r-m| \mod 2}^{k_{\text{max}}} \sum_{l=1+|s-n| \mod 2}^{l_{\text{max}}} \phi_{k,l} \tag{A.1.46}
\]

where

\[
k_{\text{max}} = \min (r + m - 1, 2q - 1 - r - m)
\]
\[
l_{\text{max}} = \min (s + n - 1, 2p - 1 - s - n) \tag{A.1.47}
\]
Appendix B

CFT Data

B.1 Three-state Potts

B.1.1 Conformal blocks

The conformal blocks needed in this work are known in a closed form for the Potts model [Dot84]. Below we summarize the necessary data for the renormalization computations. Considering the following correlators

\[
\langle \Phi_{r,s} | \Phi_{2,1}(1) \Phi_{2,1}(z) | \Phi_{r,s} \rangle
\]  

(B.1.1)

the conformal blocks forming a basis for their chiral part around \( z = 0 \) are\(^1\)

\[
\begin{pmatrix}
\phi_{2,1}(1) \\
\phi_{r-1,s} \\
\phi_{r,s}(\infty)
\end{pmatrix}
\begin{pmatrix}
\phi_{2,1}(z) \\
\phi_{r,s}(0)
\end{pmatrix} = (1 - z)^{3/5} z^{(-1+6r-5s)/10} \binom{6}{5} (1 - r + s, 6; 1 - \frac{6}{5} r + s | z)
\]

(B.1.2)

where the small \( \phi \)s refer to the chiral components, and

\[
\binom{6}{5} (\alpha, \beta; \gamma | z)
\]

(B.1.3)

\(^1\)The paper [Dot84] gives the correlator in a different basis, and so their conformal blocks must be transformed by an appropriate conformal mapping to obtain the ones used here. One can also obtain the blocks in our basis directly from the results of section 8.3.3 in the monograph [DMS97].
denotes the standard hypergeometric function. The basis for the conformal blocks around 
\( z = 1 \) is given by

\[
\phi_{2,1}(1) \phi_{2,1}(z) = (1 - z)^{-4/5} z^{(-1+6r-5s)/10} \binom{6}{5} \binom{6}{5} (1 + r) + s, -\frac{1}{5}; -\frac{2}{5}|1 - z)
\]

\[
\phi_{2,1}(1) \phi_{2,1}(z) = (1 - z)^{3/5} z^{(-1+6r-5s)/10} \binom{6}{5} \binom{6}{5} (1 + r) - s; \frac{12}{5}|1 - z)
\]

For clarity of conventions, the insertion points of the fields were displayed above; in the following considerations they are suppressed. Denoting \( \Phi = \Phi_{2,1} \) and its chiral part by \( \phi = \phi_{2,1} \), the two bases are related by the following duality relations

\[
\phi \bigg|_i j \bigg|_i = \sum_k \mathcal{F}_{jk}[i] \phi \bigg|_i k \bigg|_i
\]

where

\[
\mathcal{F}_{jk}[i] = F_{jk} \begin{bmatrix} \phi & \phi \\ i & i \end{bmatrix}
\]

are the so-called fusion coefficients. These can be easily obtained using the transformation formulas obeyed by the hypergeometric functions. The ones relevant for our calculations
Another necessary ingredient is the expansion of the blocks (B.1.4) around $z = 1$. One way this can be calculated is computing the series expansion of the conformal blocks using the Taylor series of the hypergeometric function and the binomial series.

On the other hand, a model independent way to obtain the expansion is provided by Virasoro symmetry. Recalling the notations in (2.3.31)

$$[L_n(z), \Phi(w, \bar{w})] = h(n + 1)(w - z)^n \Phi(w, \bar{w}) + (w - z)^{n+1} \partial_w \Phi(w, \bar{w})$$

$$[\bar{L}_n(\bar{z}), \Phi(w, \bar{w})] = \bar{h}(n + 1)(\bar{w} - \bar{z})^n \Phi(w, \bar{w}) + (\bar{w} - \bar{z})^{n+1} \partial_{\bar{w}} \Phi(w, \bar{w})$$

(B.1.9)

the first few coefficients can be easily obtained using the conformal Ward identities, which give the following commutation relations between the Virasoro generators and the primary fields:

$$L_n(z) = \oint_z \frac{d\zeta}{2\pi i} (\zeta - z)^{n+1} T(\zeta)$$

$$\bar{L}_n(\bar{z}) = \oint_{\bar{z}} \frac{d\bar{\zeta}}{2\pi i} (\bar{\zeta} - \bar{z})^{n+1} \bar{T}(\bar{\zeta})$$

(B.1.10)

are the modes of the conformal energy momentum tensor located at $(z, \bar{z})$; the modes located at $z = \infty$ are given by

$$L_n(\infty) = -\oint_{\infty} \frac{d\zeta}{2\pi i} \zeta^{-n+1} T(\zeta) = L_{-n}(0)$$

$$\bar{L}_n(\infty) = -\oint_{\infty} \frac{d\bar{\zeta}}{2\pi i} \bar{\zeta}^{-n+1} \bar{T}(\bar{\zeta}) = \bar{L}_{-n}(0)$$

(B.1.11)
We computed the block coefficients up to $r = 5$, but for the sake of brevity we only give the first three cases:

\[
B_0 \left[ \begin{array}{ccc} j & j & k \\ i & i & \end{array} \right] = 1 \tag{B.1.12}
\]

\[
B_1 \left[ \begin{array}{ccc} j & j & k \\ i & i & \end{array} \right] = \frac{h_k}{2}
\]

\[
B_2 \left[ \begin{array}{ccc} j & j & k \\ i & i & \end{array} \right] = \frac{h_k [c + 8] h_k^2 + 2(c - 4) h_k + c + 4 h_j (h_k - 1) + 8 h_k^3]}{4 (2c - 5) h_k + c + 16 h_k^2} + \frac{4 h_i [h_j (4 h_k + 2) + (h_k - 1) h_k]}{4 (2c - 5) h_k + c + 16 h_k^2}
\]

where $h_i, h_j$ and $h_k$ are the conformal weights of the respective fields.

For descendant state calculations we consider the first level only, as this is all we need in the main text. The duality relations have the same fusion coefficients

\[
\phi \big| \begin{array}{ccc} j \\ L_{-1} i \end{array} \big| \phi = \sum_k F_{jk}[i] \big| \begin{array}{ccc} \phi \\ L_{-1} i \end{array} \big| \phi \big| \begin{array}{ccc} k \\ L_{-1} i \end{array} \big| \tag{B.1.13}
\]

and the conformal blocks in the dual channel can be expanded as

\[
\left(1 - z\right)^{-2 h_i + h_k} \sum_{r=0}^{\infty} B_r \left[ \begin{array}{ccc} j \\ L_{-1} i \end{array} \bigg| \begin{array}{ccc} j \\ L_{-1} i \end{array} \right] k \bigg] \left(1 - z\right)^r \tag{B.1.14}
\]

where we computed the coefficients up to $r = 5$. The first three of them are

\[
B_0 \left[ \begin{array}{ccc} j & j & k \\ L_{-1} i & L_{-1} i & \end{array} \right] = 2 h_i + h_k^2 - h_k \tag{B.1.15}
\]

\[
B_1 \left[ \begin{array}{ccc} j & j & k \\ L_{-1} i & L_{-1} i & \end{array} \right] = \frac{1}{2} h_k \left(2 h_i + h_k^2 - h_k\right)
\]

\[
B_2 \left[ \begin{array}{ccc} j & j & k \\ L_{-1} i & L_{-1} i & \end{array} \right] = \frac{h_k \left(h_k^2 - 1\right)}{4 \left(2 c - 5\right) h_k + c + 16 h_k^2} \frac{\left(h_j \left(ch_k + c + 8 h_k^2 - 4\right) + 4 h_j \left(h_k + 2\right)\right)}{4 \left(2 c - 5\right) h_k + c + 16 h_k^2} \frac{8 h_i^2 \left(h_j \left(4 h_k + 2\right) + \left(h_k - 1\right) h_k\right)}{4 \left(2 c - 5\right) h_k + c + 16 h_k^2} \frac{h_i h_k \left((c + 12) h_k^2 + 2 (c - 5) h_k + c + 10 h_k^3 - 4\right)}{2 \left(2 c - 5\right) h_k + c + 16 h_k^2}
\]
\[ + \frac{8h_i h_j (h_k^3 + 4h_k^2 + 3h_k + 1)}{2 (2c - 5)h_k + c + 16h_k^2} \]

\section*{B.1.2 Structure constants}

For reference, here we list the matrix elements of the field \( \Phi_{2,5} \) between the primary states of the Hilbert space (3.1.20). These can be arranged by the four sectors (3.1.21), as matrix elements between different sectors vanish. The full set of structure constants can be obtained from \([\text{Run00}]\); here we present them in a basis of states which is orthonormal.

In the sector \( \mathcal{H}_0 = S_{0,0} \oplus S_{1,0} \oplus S_{2,5} \oplus S_{3,0} \), the matrix of \( \Phi_{2,5}(1,1) \) on the basis of primary states (ordered the same way as the modules) is

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0.9363044884 & 0 \\
0 & 0.9363044884 & 0 & 0.8076923077 \\
0 & 0 & 0.8076923077 & 0
\end{pmatrix}
\]  (B.1.16)

In both sectors \( \mathcal{H}_\pm = S_{1,0} \oplus S_{1,5} \oplus S_{2,5} \oplus S_{3,3} \), one has

\[
\begin{pmatrix}
0.5461776182 & 2/3 \\
2/3 & 0
\end{pmatrix}
\]  (B.1.17)

while in \( \mathcal{H}_1 = S_{1,0} \oplus S_{1,5} \oplus S_{2,5} \oplus S_{3,0} \) the matrix elements are

\[
\begin{pmatrix}
0 & 0.9363044884 & 0.8987170343 & 0 \\
0.9363044884 & 0 & 0 & 0.8987170343 \\
0.8987170343 & 0 & 0 & 0 \\
0 & 0.8987170343 & 0 & 0
\end{pmatrix}
\]  (B.1.18)

The above structure constants are in one-to-one correspondence with the operator product coefficients involving the field \( \Phi = \Phi_{2,5} \). Writing the operator product expansion in the form

\[
\Phi_{2,5}(z, \bar{z}) A(0,0) = \sum_B C_B^{\Phi A, B} \frac{B(0,0)}{z^{h_A + 2/5 - h_B} \bar{z}^{h_A + 2/5 - h_B}}
\]  (B.1.19)

the OPE coefficients are

\[
C_B^{\Phi A} = \langle B| \Phi_{2,5}(1,1)|A \rangle
\]  (B.1.20)

For primary states, these coefficients are given above in \([\text{B.1.16}, \text{B.1.17}, \text{B.1.18}]\); for descendant states they be constructed from the primary ones by a recursive application of the conformal Ward identities \([\text{B.1.9}]\).
In presence of magnetic field, the symmetry is broken to $S_2$, therefore the Hilbert space decomposes into $H_{\text{even}} = S_{0,0} \oplus S_{1/16,1/16} \oplus S_{1/8,1/8} \oplus S_{7/16,7/16} \oplus S_{3/4,3/4}$ and $H_{\text{odd}} = S_{-1/16,1/16} \oplus S_{-1/8,1/8} \oplus S_{-7/16,7/16} \oplus S_{-3/4,3/4} \oplus S_{0,3} \oplus S_{3,0}$. The OPE coefficients of the magnetic fields in $H_{\text{even}}$ are the following

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0.772412 & 0.546178 & 0.235702 & 0.0260085 & 0.000474834 \\
0 & 0.546178 & 0 & 0.666667 & 0 & 0 \\
0 & 0.235702 & 0.666667 & 0 & 0.388889 & 0 \\
0 & 0.0260085 & 0 & 0.388889 & 0 & 0 \\
0 & 0.000474834 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

while in $H_{\text{odd}}$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -0.119186i & 0.509175i \\
0 & 0 & 0 & 0 & 0 & -0.119186i & 0.509175i \\
0 & 0 & 0 & 0 & 0 & -0.0217907i & 0 \\
0 & 0 & 0 & 0 & 0 & -0.0217907i & 0 \\
0.119186i & 0.119186i & 0.0217907i & 0.0217907i & -0.772412 & -0.235702 \\
-0.509175i & -0.509175i & 0 & 0 & -0.235702 & 0
\end{pmatrix}
$$

B.2 Ising model

B.2.1 Structure constants

The Hilbert space has the form $H = S_{0,0} \oplus S_{1/16,1/16} \oplus S_{1/8,1/8}$. The structure constants for the thermal perturbation are

$$
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0
\end{pmatrix}
$$

and for the magnetic field

$$
\begin{pmatrix}
0 & 0 & 1 \\
0 & \frac{1}{2} & 0 \\
1 & 0 & 0
\end{pmatrix}
$$
Appendix C

Tables for the comparison between renormalized TCSA numerics and TBA predictions

<table>
<thead>
<tr>
<th></th>
<th>( r = 0.1 )</th>
<th>( r = 1 )</th>
<th>( r = 3 )</th>
<th>( r = 5 )</th>
<th>( r = 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TBA</td>
<td>-4.1958706705</td>
<td>-0.595088</td>
<td>-0.8907</td>
<td>-1.446</td>
<td>-2.021</td>
</tr>
<tr>
<td>raw TCSA level 12</td>
<td>-4.1947973491</td>
<td>-0.568125</td>
<td>-0.7649</td>
<td>-1.188</td>
<td>-1.603</td>
</tr>
<tr>
<td>renormalized TCSA level 8</td>
<td>-4.1958706700</td>
<td>-0.595083</td>
<td>-0.8903</td>
<td>-1.443</td>
<td>-2.011</td>
</tr>
<tr>
<td>renormalized TCSA level 12</td>
<td>-4.1958706700</td>
<td>-0.595085</td>
<td>-0.8905</td>
<td>-1.444</td>
<td>-2.014</td>
</tr>
</tbody>
</table>

Ground state in \( \mathcal{H}_0 \) sector

<table>
<thead>
<tr>
<th></th>
<th>( r = 0.1 )</th>
<th>( r = 1 )</th>
<th>( r = 3 )</th>
<th>( r = 5 )</th>
<th>( r = 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TBA</td>
<td>46.1055595046</td>
<td>4.8947517</td>
<td>2.32864</td>
<td>2.05871</td>
<td>2.008</td>
</tr>
<tr>
<td>raw TCSA level 12</td>
<td>46.1066313406</td>
<td>4.9216798</td>
<td>2.45431</td>
<td>2.317</td>
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<td>ren. TCSA level 8</td>
<td>46.1055594445</td>
<td>4.8947579</td>
<td>2.32914</td>
<td>2.062</td>
<td>2.022</td>
</tr>
<tr>
<td>ren. TCSA level 12</td>
<td>46.1055595057</td>
<td>4.8947566</td>
<td>2.32897</td>
<td>2.061</td>
<td>2.018</td>
</tr>
</tbody>
</table>

First excited state in \( \mathcal{H}_0 \) sector: stationary \( \bar{A}A \)

In the above data for small volumes, instead of analytically continuing the TBA we simply used the correspondence with the scaling Lee–Yang model, as the Lee–Yang TCSA is much easier to implement and numerically precise enough for the present comparison.
Second excited state in $\mathcal{H}_0$ sector: moving $A\bar{A}$

<table>
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<tr>
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<tr>
<td>TBA</td>
<td>57.0847781</td>
<td>17.09738</td>
<td>5.9176</td>
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<td>3.064</td>
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<td>raw TCSA level 12</td>
<td>57.0898932</td>
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<td>57.0847809</td>
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<td>57.0847788</td>
<td>17.09740</td>
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Third excited state in $\mathcal{H}_0$ sector: $AAA$ three-particle state

<table>
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<tbody>
<tr>
<td>renormalized TCSA level 8</td>
<td>11.4023746</td>
<td>9.340648</td>
<td>6.66667</td>
<td>4.685</td>
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Stationary one particle state (ground state in $\mathcal{H}_\pm$ in the paramagnetic phase)

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<td>TBA</td>
<td>0.951783</td>
<td>0.930075</td>
<td>0.95363</td>
<td>0.989</td>
<td>0.997</td>
</tr>
<tr>
<td>raw TCSA level 12</td>
<td>0.986768</td>
<td>1.00174</td>
<td>1.08041</td>
<td>1.249</td>
<td>1.422</td>
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<tr>
<td>renormalized TCSA level 8</td>
<td>0.951786</td>
<td>0.930273</td>
<td>0.95457</td>
<td>0.993</td>
<td>1.015</td>
</tr>
<tr>
<td>renormalized TCSA level 12</td>
<td>0.951776</td>
<td>0.930195</td>
<td>0.95420</td>
<td>0.991</td>
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</table>

Twisted vacuum (ground state in $\mathcal{H}_\pm$ in the ferromagnetic phase)

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</thead>
<tbody>
<tr>
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<td>3.9667856906</td>
<td>0.204269</td>
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<td>0.000145</td>
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<tr>
<td>raw TCSA level 12</td>
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Twisted vacuum (ground state in $\mathcal{H}_\pm$ in the ferromagnetic phase)

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<td>79.613697179</td>
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<td>3.26417</td>
<td>2.4947</td>
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<tr>
<td>raw TCSA level 12</td>
<td>79.614806090</td>
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<td>3.39399</td>
<td>2.7611</td>
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First $AA$ two-particle state (first excited state in $\mathcal{H}_\\pm$ in the paramagnetic phase)

<table>
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</tr>
<tr>
<td>raw TCSA level 12</td>
<td>79.614803694</td>
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<td>3.38117</td>
<td>2.7324</td>
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<td>8.191922</td>
<td>3.25158</td>
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<td>renormalized TCSA level 12</td>
<td>79.613694773</td>
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First twisted $A\bar{A}$ two-particle state (first excited state in $\mathcal{H}_\\pm$ in the ferromagnetic phase)
Bibliography


Renormalization methods in truncated conformal space with applications

Máté Lencsés

Summary

The PhD dissertation is concerned with the renormalization methods of the truncated conformal space approach (TCSA). TCSA is a variational method which was developed by Yurov and Zamolodchikov to the investigation of the spectrum of the scaling Lee–Yang model. Since then, it has been applied in various perturbed conformal field theories to the examination of the spectrum and form factors as well.

The basis of the approach is that the conformal Hilbert space can be constructed and operator matrix elements can be determined in the conformal basis. In finite volume the spectrum becomes discrete and introducing an ultraviolet cut-off makes the Hilbert space finite dimensional and the Hamilton operator can be diagonalized numerically. The TCSA results depend on the cut-off. The speed of convergence depends on the conformal dimensions of the perturbing operators. In case of irrelevant perturbation the results diverge. The aim of the renormalization is to cancel the cut-off dependence, speed-up the convergence and deal with the divergences.

In this thesis I present the perturbative approach to the cut-off dependence based on the Schrieffer–Wolff transformation, implemented in two different ways. The first is the counterterm method. I have constructed cut-off dependent counterterms generalizing the results of Giokas and Watts to higher order terms in the inverse cut-off and to degenerate conformal states. Adding the counterterms to the raw TCSA data the cut-off dependence can be removed to the order of the counterterm construction, leading to a much better approximation of the exact (infinite cut-off) spectrum. This method is applied in the thermally perturbed three-state Potts model, where I compare the renormalized TCSA spectrum of the low-lying states to thermodynamic Bethe ansatz (TBA) predictions and find excellent agreement. The phase-shift problem pointed out in the work of Rapp et al. is understood by the presence of an irrelevant operator in the operator product expansion with itself. To construct the counterterms the UV limit of the TBA equation is carried out.

In the second method, renormalization group (RG) equations are written to the coupling constants. Solving the RG equations one gets cut-off dependent running couplings which can be used in the TCSA program in order to eliminate the leading order cut-off dependence. The advantage of this method is that it can be applied for all states simultaneously. Higher orders can be dealt with extrapolation in the remaining cut-off dependence. This combined method is applied to the Ising model and the three-state Potts model in presence of magnetic field. The available predictions in the literature for the mesonic and baryonic masses are verified.
Renormálási módszerek a csonkolt konform állapottérben alkalmazásokkal

Lencsés Máté

Összefoglaló


A dolgozatban ismertetem a levágásfüggőség perturbatív megközelítését a Shcreiffer–Wolff transzformáció alapján, majd bemutatok két módszert a kezelésére. Az első megközelítés az ellentagos renormálás. A TCSA eredményekhez levágásfüggő ellentagokat konstruáltam, Giokas és Watts eredményeit általánosítva a levágás inverzében magasabb korrekciókat figyelmebe véve, illetve degenerált leszármaztatott konform állapotok esetére. Az így generált ellentagokat a nyers adatokból levonva kapjuk a levágásfüggetlen renormált spektrumot, mely visszaadja az egzakt (végelen levágást tartozó) eredményeket. Az ellentagos renormálást a háromállapotú Potts-modellre alkalmaztam, ahol a spektrum alján fekvő állapotok energiáját hasonlítottam össze gerjesztett állapot termodinamikai Bethe ansatz (TBA) eredményekkel, meggyőző egyezést találva. A korábban, Rapp és tsai. munkájában talált kétrészesce fázisítás problémát sikertelen magmagyarázni egy irreleváns operátor jelenlétével a perturbáció önmagával vett operátorszorzat kifejtésében. Az ellentagok konstrukciójahez meg kellett határoznia a TBA egyenletek zérus térfogatú (ultraibolya) határesetét.

A második renormálási megközelítésben renormálási csoport egyenleteket írunk fel a csatolásokra, melyeket megoldva levágásfüggő effektív csatolásokat kapunk, melyeket a TCSA programban használva a levágásfüggés vezető rendje eltávolítható. Nagy előnye ennek a módszernek, hogy állapotfüggetlenül alkalmazható. A magasabb rendeket a maradék levágásfüggésben extrapolálva kezelhetjük. Ezt a kombinált módszert az Ising- és a háromállapotú Potts-modellben használtam a bezárás jelenségének vizsgálatára mágneses tér jelenlétében. Itt sikerült igazolni az irodalomban található mezon- és bariontőmeg jóslatokat.
a doktori értekezés szerzőjének aláírása
