

ON THE WEIGHTED LEBESGUE FUNCTION OF FOURIER–JACOBI SERIES

Ágnes Chripkó (Budapest, Hungary)

Dedicated to Professor Antal Járαι on his 60th birthday

Abstract. S.A. Agahanov and G.I. Natanson [1] established lower and upper bounds for the Lebesgue functions $L_n^{(\alpha,\beta)}(x)$ of Fourier–Jacobi series on the interval $[-1, 1]$. The bounds differ from each other only in a constant factor depending on Jacobi parameters α and β , so their result is of final character. The aim of this paper is to extend their estimation for the weighted Lebesgue functions $L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$ using Jacobi weights with parameters γ and δ . We shall also give sufficient conditions with respect to α, β, γ and δ for which the order of the weighted Lebesgue functions is $\log(n+1)$ on the whole interval $[-1, 1]$.

1. Introduction

It is known that the Lebesgue functions of an approximation process play an important role in the convergence of that process. The Lebesgue functions $L_n^{(\alpha,\beta)}(x)$ (see (2.1)) of Fourier–Jacobi series have been studied by many authors.

2010 Mathematics Subject Classification: 41A10, 42C10.

Key words and phrases: Lebesgue function, Fourier–Jacobi series.

The Project is supported by the European Union and co-financed by the European Social Fund (grant agreement no. TAMOP 4.2.1/B-09/1/KMR-2010-0003).

G. Szegő [10, 9.3.] showed that for every fixed number $\varepsilon \in (0, 1)$

$$\max_{x \in [-1+\varepsilon, 1-\varepsilon]} L_n^{(\alpha, \beta)}(x) \sim \log(n+1)$$

$$(n \in \mathbb{N} := \{1, 2, \dots\}).$$

Here and in what follows for the positive functions $a_n, b_n : I \rightarrow \mathbb{R}$ (I is an interval of \mathbb{R}) the notation

$$a_n(x) \sim b_n(x) \quad (x \in I, n \in \mathbb{N})$$

means that there exist positive constants c_1, c_2 independent of x and n such that

$$c_1 \leq \frac{a_n(x)}{b_n(x)} \leq c_2 \quad (x \in I, n \in \mathbb{N}).$$

H. Rau [7] showed that the order of the Lebesgue functions at the points -1 and 1 is $n^{\sigma+\frac{1}{2}}$, where $\sigma = \max\{\alpha, \beta\}$.

S. A. Agahanov and G. I. Natanson [1] proved the following result: if $\alpha, \beta > -\frac{1}{2}$ then

$$L_n^{(\alpha, \beta)}(x) \sim \log\left(n(1-x)^{\varepsilon(\alpha)}(1+x)^{\varepsilon(\beta)} + 1\right) + \sqrt{n} \left(|P_n^{(\alpha, \beta)}(x)| + |P_{n+1}^{(\alpha, \beta)}(x)|\right)$$

$$(x \in [-1, 1], n \in \mathbb{N}),$$

where

$$\varepsilon(t) = \begin{cases} \frac{1}{2}, & \text{if } t \in \mathbb{R} \setminus \{\frac{1}{2}\} \\ 0, & \text{if } t = \frac{1}{2} \end{cases}$$

and $P_n^{(\alpha, \beta)}(x)$ is the n th Jacobi polynomial.

The aim of this paper is to extend this estimation by using suitable Jacobi weights. We will give conditions for the weight parameters γ and δ such that the order of the weighted Lebesgue functions $L_n^{(\alpha, \beta), (\gamma, \delta)}(x)$ is $\log(n+1)$ on the whole interval $[-1, 1]$.

2. Pointwise estimate of the weighted Lebesgue function

For parameters $\alpha, \beta > -1$ we shall denote by $P_n^{(\alpha, \beta)}$ the n th Jacobi polynomial with the normalization

$$P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n} \quad (n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}).$$

They are orthogonal with respect to the Jacobi weight function

$$w^{(\alpha,\beta)}(x) := (1-x)^\alpha(1+x)^\beta \quad (x \in (-1, 1)).$$

The n th *Lebesgue function of Fourier–Jacobi series* is defined by

$$(2.1) \quad L_n^{(\alpha,\beta)}(x) := \int_{-1}^1 |K_n^{(\alpha,\beta)}(x, y)| w^{(\alpha,\beta)}(y) dy \\ (n \in \mathbb{N}, x \in [-1, 1]),$$

where the kernel function $K_n^{(\alpha,\beta)}(x, y)$ can be expressed as

$$(2.2) \quad K_n^{(\alpha,\beta)}(x, y) = \sum_{k=0}^n \{h_k^{(\alpha,\beta)}\}^{-1} P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(y) = \\ = \lambda_n^{(\alpha,\beta)} \frac{P_{n+1}^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) - P_n^{(\alpha,\beta)}(x) P_{n+1}^{(\alpha,\beta)}(y)}{x - y}.$$

Here

$$(2.3) \quad h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2k + \alpha + \beta + 1} \frac{\Gamma(k + \alpha + 1) \Gamma(k + \beta + 1)}{\Gamma(k + 1) \Gamma(k + \alpha + \beta + 1)},$$

and

$$(2.4) \quad \lambda_n^{(\alpha,\beta)} = \frac{2^{-\alpha-\beta}}{2n + \alpha + \beta + 2} \frac{\Gamma(n + 2) \Gamma(n + \alpha + \beta + 2)}{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}$$

(see [10, (4.3.3) and (4.5.2)]), where $\Gamma(p)$ ($p > 0$) is the Gamma function.

For $\gamma, \delta \geq 0$ we define the n th *weighted Lebesgue function of Fourier–Jacobi series* by

$$(2.5) \quad L_n^{(\alpha,\beta),(\gamma,\delta)}(x) := w^{(\gamma,\delta)}(x) \int_{-1}^1 |K_n^{(\alpha,\beta)}(x, y)| w^{(\alpha-\gamma,\beta-\delta)}(y) dy \\ (n \in \mathbb{N}, x \in [-1, 1]).$$

For the existence of this integral, we shall assume that the parameters γ, δ satisfy the inequalities

$$(2.6) \quad \gamma < \alpha + 1, \quad \delta < \beta + 1.$$

Theorem. *Suppose that $\alpha, \beta > -\frac{1}{2}$ and $\gamma, \delta \geq 0$ satisfy the inequalities*

$$(2.7) \quad \frac{\alpha}{2} + \frac{1}{4} < \gamma < \frac{\alpha}{2} + \frac{3}{4} \quad \text{and} \quad \frac{\beta}{2} + \frac{1}{4} < \delta < \frac{\beta}{2} + \frac{3}{4}.$$

Then we have for all $n \in \mathbb{N}$ and $x \in [-1, 1]$ that

$$(2.8) \quad c_1 w^{(\gamma, \delta)}(x) \phi_n^{(\alpha, \beta)}(x) \leq L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \leq c_2 \tilde{w}_n^{(\gamma, \delta)}(x) \phi_n^{(\alpha, \beta)}(x)$$

with the constants $c_1, c_2 > 0$ independent of x and n , where

$$\begin{aligned} \phi_n^{(\alpha, \beta)}(x) &:= \log \left(n \sqrt{1-x^2} + 1 \right) + \\ &+ \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} \left(\sqrt{1+x} + \frac{1}{n} \right)^{\beta + \frac{1}{2}} \left(|P_n^{(\alpha, \beta)}(x)| + |P_{n+1}^{(\alpha, \beta)}(x)| \right), \end{aligned}$$

and

$$\tilde{w}_n^{(\gamma, \delta)}(x) := \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left(\frac{\sqrt{1+x}}{\sqrt{1+x} + \frac{1}{n}} \right)^{2\delta}.$$

We note that the conditions for the parameters $\alpha, \beta, \gamma, \delta$ in Theorem imply the inequalities in (2.6).

Corollary. *Suppose that $\alpha, \beta > -\frac{1}{2}$ and $\gamma, \delta \geq 0$ satisfy the inequalities (2.7). Then we have*

$$\max_{x \in [-1, 1]} L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \sim \log(n+1) \quad (n \in \mathbb{N}).$$

Remark. A result similar to this Corollary was proved by U. Luther and G. Mastroianni [5]. This paper does not contain a pointwise estimation (cf. (2.8)).

3. Preliminaries

In what follows for the functions $a_n, b_n : I \rightarrow \mathbb{R}$ (I is an interval of \mathbb{R}) the notation

$$a_n(x) = O(b_n(x)) \quad (x \in I, n \in \mathbb{N})$$

means that there exists a positive constant c independent of x and n such that

$$|a_n(x)| \leq c b_n(x) \quad (x \in I, n \in \mathbb{N}).$$

3.1. Formulas for Jacobi polynomials. Here we list those well known formulas which we shall use throughout the paper.

If $\alpha, \beta > -1$ then for every $x \in [-1, 1]$ and $n \in \mathbb{N}$ we have

$$(3.1) \quad P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x)$$

(see [10, (4.1.3)]) and

$$(3.2) \quad \frac{d}{dx} \left\{ P_n^{(\alpha, \beta)}(x) \right\} = \frac{1}{2}(n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1, \beta+1)}(x)$$

(see [10, (4.21.7)]).

An important bound for Jacobi polynomials can be given in this form: if $\alpha, \beta > -1$ then

$$(3.3) \quad \left| P_n^{(\alpha, \beta)}(x) \right| = O\left(n^{-\frac{1}{2}}\right) \left(\sqrt{1-x} + \frac{1}{n} \right)^{-\alpha - \frac{1}{2}} \\ (0 \leq x \leq 1, n \in \mathbb{N})$$

(see [6, 2.3.22]).

A more precise formula is the following. Let $\alpha, \beta > -1$. Then we have

$$(3.4) \quad P_n^{(\alpha, \beta)}(\cos s) = n^{-\frac{1}{2}} k(s) \left(\cos(Ns + \nu) + \frac{O(1)}{n \sin s} \right),$$

where

$$\frac{c}{n} \leq s \leq \pi - \frac{c}{n}, \quad k(s) = k^{(\alpha, \beta)}(s) = \pi^{-\frac{1}{2}} \left(\sin \frac{s}{2} \right)^{-\alpha - \frac{1}{2}} \left(\cos \frac{s}{2} \right)^{-\beta - \frac{1}{2}}, \\ N = n + \frac{1}{2}(\alpha + \beta + 1), \quad \nu = -\left(\alpha + \frac{1}{2}\right) \frac{\pi}{2}.$$

Here c is a fixed positive number and the bound for the error term holds uniformly in the interval $\left[\frac{c}{n}, \pi - \frac{c}{n}\right]$ (see [10, (8.21.18)]).

If $\alpha, \beta, \mu > -1$ then we have uniformly in $n \in \mathbb{N}$ that

$$(3.5) \quad \int_0^1 |P_n^{(\alpha, \beta)}(y)|(1-y)^\mu dy \sim \begin{cases} n^{\alpha-2\mu-2}, & \text{if } 2\mu < \alpha - \frac{3}{2} \\ n^{-\frac{1}{2}} \log n, & \text{if } 2\mu = \alpha - \frac{3}{2} \\ n^{-\frac{1}{2}}, & \text{if } 2\mu > \alpha - \frac{3}{2} \end{cases}$$

(see [10, (7.34.1)]).

Let $p > 0$ be a fixed real number. Then

$$\frac{\Gamma(n+p)}{\Gamma(n)} \sim n^p \quad (n \in \mathbb{N})$$

(see [8, p. 166]). Thus for the numbers (2.3) and (2.4) we have

$$(3.6) \quad h_n^{(\alpha, \beta)} \sim \frac{1}{n} \quad (n \in \mathbb{N}), \\ \lambda_n^{(\alpha, \beta)} \sim n \quad (n \in \mathbb{N}).$$

We introduce the notations

$$\begin{aligned}\overline{P}_n(x) &:= P_n^{(\alpha+1, \beta)}(x), \\ \widetilde{P}_n(x) &:= P_n^{(\alpha+1, \beta+1)}(x).\end{aligned}$$

Using the formulas [10, (4.5.7)] we obtain that

$$(3.7) \quad \frac{1}{2}(1-x^2)\widetilde{P}_{n-1}(x) = \left(x + \frac{\alpha - \beta}{2n + \alpha + \beta + 2}\right)P_n^{(\alpha, \beta)}(x) - \frac{2n + 2}{2n + \alpha + \beta + 2}P_{n+1}^{(\alpha, \beta)}(x).$$

Moreover, by [10, (4.5.4)] we have

$$(3.8) \quad \left(1 + \frac{\alpha + \beta}{2n + 2}\right)(1-x)\overline{P}_n(x) = \frac{n + \alpha + 1}{n + 1}P_n^{(\alpha, \beta)}(x) - P_{n+1}^{(\alpha, \beta)}(x).$$

3.2. Auxiliary results.

Lemma 1. *Suppose that $R \geq 1$ and $A < 0$ are fixed real numbers. Then with a suitable index $N \in \mathbb{N}$ we have*

$$(3.9) \quad \int_{s + \frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t - s} dt \sim \left(s + \frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R} + 1\right) + 1\right]$$

uniformly in $s \in [0, \frac{\pi}{2}]$ and $n \in \mathbb{N}$, $n > N$.

Proof. Let us introduce the following notation

$$I := I(n, s, A, R) := \int_{s + \frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t - s} dt$$

$$(n \in \mathbb{N}, s \in [0, \frac{\pi}{2}], A < 0, R \geq 1).$$

In order to prove the statement, we split the interval $[0, \frac{\pi}{2}]$ into three parts:

$$[0, \frac{\pi}{2}] = [0, \frac{R}{n}] \cup \left(\frac{R}{n}, \frac{2\pi}{9}\right) \cup \left[\frac{2\pi}{9}, \frac{\pi}{2}\right].$$

CASE 1. Let $0 \leq s \leq \frac{R}{n}$ and $t \in [s + \frac{R}{n}, \frac{2\pi}{3}]$. From $2s \leq s + \frac{R}{n} \leq t$ it follows that

$$\frac{1}{2}t \leq t - s \leq t.$$

Therefore we have

$$(3.10) \quad \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} t^{A-1} dt \leq \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s} dt \leq 2 \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} t^{A-1} dt.$$

Since

$$(3.11) \quad \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} t^{A-1} dt = \frac{1}{|A|} \left[\left(s + \frac{R}{n} \right)^A - \left(\frac{2\pi}{3} \right)^A \right],$$

we obtain the following upper estimation of I :

$$(3.12) \quad I \leq \frac{2}{|A|} \left(s + \frac{R}{n} \right)^A \left[\log \left(\frac{ns}{R} + 1 \right) + 1 \right].$$

Now, let us consider the lower estimation. If $n \geq \frac{6R}{\pi}$ and $A < 0$, then $\left(\frac{n\pi}{3R} \right)^A \leq 2^A$. Therefore using (3.10) and (3.11) we get

$$\begin{aligned} I &\geq \frac{1}{|A|} \left[\left(s + \frac{R}{n} \right)^A - \left(\frac{2\pi}{3} \right)^A \right] = \frac{1}{|A|} \left(s + \frac{R}{n} \right)^A \left[1 - \left(\frac{\frac{2\pi}{3}}{s + \frac{R}{n}} \right)^A \right] \geq \\ &\geq \frac{1}{|A|} \left(s + \frac{R}{n} \right)^A \left[1 - \left(\frac{\frac{2\pi}{3}}{\frac{2R}{n}} \right)^A \right] = \frac{1}{|A|} \left(s + \frac{R}{n} \right)^A \left[1 - \left(\frac{n\pi}{3R} \right)^A \right] \geq \\ &\geq \frac{1-2^A}{|A|} \left(s + \frac{R}{n} \right)^A = \frac{1-2^A}{|A|} \left(s + \frac{R}{n} \right)^A \frac{1+\log 2}{1+\log 2} \geq \\ &\geq \frac{1-2^A}{|A|(1+\log 2)} \left(s + \frac{R}{n} \right)^A \left[1 + \log \left(\frac{ns}{R} + 1 \right) \right], \end{aligned}$$

where we used the fact that from $\frac{ns}{R} \leq 1$ it follows that $\log 2 \geq \log \left(\frac{ns}{R} + 1 \right)$.

Consequently,

$$\begin{aligned} I &\geq c \left(s + \frac{R}{n} \right)^A \left[\log \left(\frac{ns}{R} + 1 \right) + 1 \right] \\ &\left(s \in \left[0, \frac{R}{n} \right], A < 0, R \geq 1, n \geq \frac{6R}{\pi} \right), \end{aligned}$$

with a constant $c > 0$ independent of s and n .

This inequality together with (3.12) prove (3.9), if $0 \leq s \leq \frac{R}{n}$.

CASE 2. Let $\frac{R}{n} < s < \frac{2\pi}{9}$. Then $s + \frac{R}{n} < 2s < 3s < \frac{2\pi}{3}$. Now we split the integral I into two parts:

$$I = \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s} dt = \int_{s+\frac{R}{n}}^{3s} \frac{t^A}{t-s} dt + \int_{3s}^{\frac{2\pi}{3}} \frac{t^A}{t-s} dt =: I_1 + I_2.$$

For I_1 we have

$$\begin{aligned} I_1 &= \int_{s+\frac{R}{n}}^{3s} \frac{t^A}{t-s} dt \leq \left(s + \frac{R}{n}\right)^A \int_{s+\frac{R}{n}}^{3s} \frac{1}{t-s} dt = \\ &= \left(s + \frac{R}{n}\right)^A \left[\log(2s) - \log \frac{R}{n} \right] = \left(s + \frac{R}{n}\right)^A \log \left(\frac{2ns}{R} \right) = \\ &= \left(s + \frac{R}{n}\right)^A \left[\log 2 + \log \frac{ns}{R} \right] \leq \left(s + \frac{R}{n}\right)^A \left[\log \left(\frac{ns}{R} + 1 \right) + 1 \right]. \end{aligned}$$

If $3s \leq t$ then $s \leq \frac{1}{3}t$, i.e. $s + \frac{2}{3}t \leq t$. Thus

$$\frac{2}{3}t \leq t - s \leq t.$$

Therefore for I_2 we get

$$\begin{aligned} I_2 &= \int_{3s}^{\frac{2\pi}{3}} \frac{t^A}{t-s} dt \leq \frac{3}{2} \int_{3s}^{\frac{2\pi}{3}} t^{A-1} dt = \frac{3}{2|A|} \left[(3s)^A - \left(\frac{2\pi}{3}\right)^A \right] \leq \\ &\leq \frac{3}{2|A|} (2s)^A \leq \frac{3}{2|A|} \left(s + \frac{R}{n}\right)^A. \end{aligned}$$

Summarizing the above formulas we obtain that there exists a constant $c > 0$ independent of n and s such that

$$(3.13) \quad \begin{aligned} I &\leq c \left(s + \frac{R}{n}\right)^A \left[\log \left(\frac{ns}{R} + 1 \right) + 1 \right] \\ &\left(s \in \left(\frac{R}{n}, \frac{2\pi}{9} \right), A < 0, R \geq 1, n \geq \frac{6R}{\pi} \right). \end{aligned}$$

For the lower estimation of I it is enough to consider the integral I_1 . Since

$s + \frac{R}{n} \leq t \leq 3s \leq 3\left(s + \frac{R}{n}\right)$, thus by $A < 0$ we get that

$$\begin{aligned}
 I_1 &= \int_{s+\frac{R}{n}}^{3s} \frac{t^A}{t-s} dt \geq 3^A \left(s + \frac{R}{n}\right)^A \int_{s+\frac{R}{n}}^{3s} \frac{1}{t-s} dt = \\
 (3.14) \qquad &= 3^A \left(s + \frac{R}{n}\right)^A \left(\log(2s) - \log \frac{R}{n}\right) = \\
 &= 3^A \left(s + \frac{R}{n}\right)^A \log\left(2\frac{ns}{R}\right).
 \end{aligned}$$

The following inequality holds:

$$(3.15) \qquad \frac{\log(2x)}{\log(x+1)+1} > \frac{\log 2}{1+\log 2} \quad (x \geq 1).$$

Indeed, if $x \geq 1$ then

$$\begin{aligned}
 \frac{\log(2x)}{\log(x+1)+1} &\geq \frac{\log(2x)}{\log(2x)+1} = 1 - \frac{1}{\log(2x)+1} \geq \\
 &\geq 1 - \frac{1}{1+\log 2} = \frac{\log 2}{1+\log 2}.
 \end{aligned}$$

Since $\frac{ns}{R} \geq 1$ we obtain from (3.14) and (3.15) that

$$I \geq I_1 \geq \frac{3^A \log 2}{1+\log 2} \left(s + \frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R} + 1\right) + 1\right],$$

which together with (3.13) prove (3.9), if $\frac{R}{n} < s < \frac{2\pi}{9}$.

CASE 3. Let $\frac{2\pi}{9} \leq s \leq \frac{\pi}{2}$ and $t \in [s + \frac{R}{n}, \frac{2\pi}{3}]$. Then

$$(3.16) \qquad s + \frac{R}{n} \leq t \leq \frac{2\pi}{3} \leq 3s \leq 3\left(s + \frac{R}{n}\right),$$

so we have the following upper estimation of I :

$$\begin{aligned}
 I &= \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s} dt \leq \left(s + \frac{R}{n}\right)^A \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{1}{t-s} dt = \\
 (3.17) \qquad &= \left(s + \frac{R}{n}\right)^A \left[\log\left(\frac{2\pi}{3} - s\right) - \log \frac{R}{n}\right] = \\
 &= \left(s + \frac{R}{n}\right)^A \log\left[\frac{n}{R}\left(\frac{2\pi}{3} - s\right)\right] \leq \left(s + \frac{R}{n}\right)^A \log\left(\frac{2ns}{R}\right) = \\
 &= \left(s + \frac{R}{n}\right)^A \left[\log \frac{ns}{R} + \log 2\right] \leq \left(s + \frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R} + 1\right) + 1\right].
 \end{aligned}$$

For the lower estimation of I we use the condition $A < 0$ and (3.16). Then we have

$$\begin{aligned}
 (3.18) \quad I &= \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^A}{t-s} dt \geq 3^A \left(s + \frac{R}{n}\right)^A \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{1}{t-s} dt = \\
 &= 3^A \left(s + \frac{R}{n}\right)^A \log \left[\frac{n}{R} \left(\frac{2\pi}{3} - s \right) \right] = \\
 &= 3^A \left(s + \frac{R}{n}\right)^A \log \left(\frac{\pi}{2} \cdot \frac{4}{3} \frac{n}{R} - \frac{ns}{R} \right) \geq \\
 &\geq 3^A \left(s + \frac{R}{n}\right)^A \log \left(\frac{1}{3} \frac{ns}{R} \right).
 \end{aligned}$$

The following inequality is true:

$$(3.19) \quad \frac{\log\left(\frac{1}{3}x\right)}{\log(x+1)+1} > \frac{\log\frac{4}{3}}{\log(8e)} \quad (x \geq 4).$$

Indeed, if $x \geq 4$, then

$$\begin{aligned}
 &\frac{\log\left(\frac{1}{3}x\right)}{\log(x+1)+1} > \frac{\log\left(\frac{1}{3}x\right)}{\log(2x)+1} = \frac{\log\left(\frac{1}{3}x\right)}{\log\left(\frac{1}{3}x\right)+\log 6+1} = \\
 &= 1 - \frac{\log(6e)}{\log\left(\frac{1}{3}x\right)+\log(6e)} \geq 1 - \frac{\log(6e)}{\log\frac{4}{3}+\log(6e)} = \frac{\log\frac{4}{3}}{\log(8e)}.
 \end{aligned}$$

Let $n \geq \frac{18R}{\pi}$. Then $\frac{ns}{R} \geq \frac{n}{R} \frac{2\pi}{9} \geq 4$. Thus using (3.18) and (3.19) we obtain

$$I \geq 3^A \frac{\log\frac{4}{3}}{\log(8e)} \left(s + \frac{R}{n}\right)^A \left[\log\left(\frac{ns}{R} + 1\right) + 1 \right],$$

which together with (3.17) prove (3.9), if $\frac{2\pi}{9} \leq s \leq \frac{\pi}{2}$.

Lemma 1 is proved. ■

Lemma 2. *If $A > -1$, $n \in \mathbb{N}$ and $s \in \left(\frac{1}{n}, \frac{\pi}{2}\right]$, then there exists a constant $c > 0$ independent from s and n such that*

$$\int_0^{s-\frac{1}{n}} \frac{t^A}{s-t} dt \leq c \left(s + \frac{1}{n}\right)^A \log(ns+1).$$

Proof. Consider the following identity:

$$\begin{aligned} \int_0^{s-\frac{1}{n}} \frac{t^A}{s-t} dt &= \frac{1}{s} \int_0^{s-\frac{1}{n}} \frac{t^A[(s-t)+t]}{s-t} dt = \\ &= \frac{1}{s} \int_0^{s-\frac{1}{n}} t^A dt + \frac{1}{s} \int_0^{s-\frac{1}{n}} \frac{t^{A+1}}{s-t} dt =: I_1 + I_2. \end{aligned}$$

For I_1 we have

$$I_1 = \frac{1}{s} \int_0^{s-\frac{1}{n}} t^A dt = \frac{1}{s} \frac{\left(s - \frac{1}{n}\right)^{A+1}}{A+1} \leq c s^A,$$

where $c > 0$ is independent of s and n . From $A+1 > 0$ it follows that

$$I_2 = \frac{1}{s} \int_0^{s-\frac{1}{n}} \frac{t^{A+1}}{s-t} dt \leq s^A \int_0^{s-\frac{1}{n}} \frac{1}{s-t} dt = s^A \log(ns),$$

therefore

$$I_1 + I_2 \leq c s^A (1 + \log(ns)) \leq c s^A \log(ns + 1).$$

Since

$$\frac{1}{2} \leq \frac{s}{s + \frac{1}{n}} = 1 - \frac{1}{ns + 1} \leq 1,$$

we have that there exists a $c > 0$ independent of s and n such that

$$s^A \leq c \left(s + \frac{1}{n}\right)^A,$$

which proves our statement. ■

4. Proof of Theorem

In this section we shall use the following notations:

$$P_n(x) := P_n^{(\alpha, \beta)}(x), \quad \lambda_n := \lambda_n^{(\alpha, \beta)}.$$

By (3.1) we have the following symmetry property of the kernel function (2.2)

$$K_n^{(\alpha,\beta)}(x,y) = K_n^{(\beta,\alpha)}(-x,-y) \\ (x,y \in [-1,1], \quad n \in \mathbb{N}, \quad \alpha,\beta > -1).$$

Using this we obtain the symmetry property of the weighted Lebesgue function:

$$(4.1) \quad L_n^{(\alpha,\beta),(\gamma,\delta)}(-x) = L_n^{(\beta,\alpha),(\delta,\gamma)}(x) \\ (x,y \in [-1,1], \quad n \in \mathbb{N}, \quad \alpha,\beta > -1, \quad \gamma,\delta \geq 0),$$

which means that it is enough to prove (2.8) for $x \in [0,1]$ only.

From now on we will assume that $x \in [0,1]$.

In what follows, C or c (or $C_1, C_2, \dots, c_1, c_2, \dots$) will always denote a positive constant (not necessarily the same at different occurrences) independent of n and x . Also, N will always denote a fixed natural number, not necessarily the same at different occurrences.

4.1. Upper estimation of $L_n^{(\alpha,\beta),(\gamma,\delta)}(x)$. In order to estimate (2.5) we split the integral into two parts:

$$\int_{-1}^1 |K_n^{(\alpha,\beta)}(x,y)| w^{(\alpha-\gamma,\beta-\delta)}(y) dy = \int_{-1}^{-\frac{1}{2}} \dots dy + \int_{-\frac{1}{2}}^1 \dots dy.$$

In the second integral we use the substitutions

$$y = \cos t \quad (0 \leq t \leq \frac{2\pi}{3}) \quad \text{and} \quad x = \cos s \quad (0 \leq s \leq \frac{\pi}{2}),$$

and consider the following two cases:

$$(i) \quad \frac{1}{n} \leq s \leq \frac{\pi}{2} \quad \text{and} \quad (ii) \quad 0 \leq s \leq \frac{1}{n}.$$

In the first case we split the second integral into three parts:

$$\int_{-\frac{1}{2}}^1 \dots dy = \int_0^{\frac{2\pi}{3}} \dots dt = \int_0^{s-\frac{1}{n}} \dots dt + \int_{s-\frac{1}{n}}^{s+\frac{1}{n}} \dots dt + \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \dots dt.$$

Thus we have

$$L_n^{(\alpha,\beta),(\gamma,\delta)}(x) =: \sum_{k=1}^4 J_k,$$

where

$$\begin{aligned}
 J_1 &= w^{(\gamma, \delta)}(x) \int_{-1}^{-\frac{1}{2}} |K_n^{(\alpha, \beta)}(x, y)| w^{(\alpha - \gamma, \beta - \delta)}(y) \, dy, \\
 J_2 &= w^{(\gamma, \delta)}(x) \int_{s + \frac{1}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha, \beta)}(x, \cos t)| w^{(\alpha - \gamma, \beta - \delta)}(\cos t) \sin t \, dt, \\
 J_3 &= w^{(\gamma, \delta)}(x) \int_{s - \frac{1}{n}}^{s + \frac{1}{n}} |K_n^{(\alpha, \beta)}(x, \cos t)| w^{(\alpha - \gamma, \beta - \delta)}(\cos t) \sin t \, dt, \\
 J_4 &= w^{(\gamma, \delta)}(x) \int_0^{s - \frac{1}{n}} |K_n^{(\alpha, \beta)}(x, \cos t)| w^{(\alpha - \gamma, \beta - \delta)}(\cos t) \sin t \, dt.
 \end{aligned}$$

In the second case the lower bound in J_3 is 0 and $J_4 := 0$.

4.1.1. *Estimation of J_1 .* Here we use the formula (2.2). Since $x \geq 0$ we have $|x - y| \geq \frac{1}{2}$ ($-1 \leq y \leq -\frac{1}{2}$). Consequently,

$$\begin{aligned}
 J_1 &= w^{(\gamma, \delta)}(x) \int_{-1}^{-\frac{1}{2}} \lambda_n \frac{|P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)|}{|x - y|} w^{(\alpha - \gamma, \beta - \delta)}(y) \, dy \leq \\
 &\leq 2\lambda_n w^{(\gamma, \delta)}(x) |P_n(x)| \int_{-1}^{-\frac{1}{2}} |P_{n+1}(y)| w^{(\alpha - \gamma, \beta - \delta)}(y) \, dy + \\
 &+ 2\lambda_n w^{(\gamma, \delta)}(x) |P_{n+1}(x)| \int_{-1}^{-\frac{1}{2}} |P_n(y)| w^{(\alpha - \gamma, \beta - \delta)}(y) \, dy.
 \end{aligned}$$

By (3.1) we have

$$\begin{aligned}
 &\int_{-1}^{-\frac{1}{2}} |P_n(y)| w^{(\alpha - \gamma, \beta - \delta)}(y) \, dy = \int_{-1}^{-\frac{1}{2}} |P_n^{(\alpha, \beta)}(y)| (1 - y)^{\alpha - \gamma} (1 + y)^{\beta - \delta} \, dy \leq \\
 &\leq c \int_{-1}^{-\frac{1}{2}} |P_n^{(\alpha, \beta)}(y)| (1 + y)^{\beta - \delta} \, dy = c \int_{\frac{1}{2}}^1 |P_n^{(\beta, \alpha)}(y)| (1 - y)^{\beta - \delta} \, dy \leq
 \end{aligned}$$

$$\leq c \int_0^1 |P_n^{(\beta, \alpha)}(y)|(1-y)^{\beta-\delta} dy.$$

Since $\delta < \frac{\beta}{2} + \frac{3}{4}$, i.e. $2(\beta - \delta) > \beta - \frac{3}{2}$ it follows by (3.5) that the last integral has the upper bound $cn^{-\frac{1}{2}}$. Consequently,

$$\int_{-1}^{-\frac{1}{2}} |P_n(y)|w^{(\alpha-\gamma, \beta-\delta)}(y) dy = O(n^{-\frac{1}{2}}) \quad (n \in \mathbb{N}).$$

Collecting the above formulas and using (3.6) we obtain

$$(4.2) \quad J_1 = O(\sqrt{n})w^{(\gamma, \delta)}(x) \left(|P_n^{(\alpha, \beta)}(x)| + |P_{n+1}^{(\alpha, \beta)}(x)| \right) \\ (x \in [0, 1], n \in \mathbb{N}).$$

4.1.2. *Estimation of J_2 .* The expression

$$J_2 = w^{(\gamma, \delta)}(x) \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha, \beta)}(x, \cos t)|w^{(\alpha-\gamma, \beta-\delta)}(\cos t) \sin t dt$$

may be simplified by using the following formulas:

$$w^{(\gamma, \delta)}(x) = (1-x)^\gamma(1+x)^\delta \sim (1-x)^\gamma \quad (x \in [0, 1]),$$

$$w^{(\alpha-\gamma, \beta-\delta)}(\cos t) \sin t = (1-\cos t)^{\alpha-\gamma}(1+\cos t)^{\beta-\delta} \sin t \sim t^{2(\alpha-\gamma)+1} \\ (t \in [0, \frac{2\pi}{3}]),$$

$$x - y = \cos s - \cos t = 2 \sin \frac{t+s}{2} \sin \frac{t-s}{2} \sim t^2 - s^2 \sim t(t-s) \\ (s \in [0, \frac{\pi}{2}], t \in [s, \frac{2\pi}{3}]).$$

Thus by (2.2) and (3.6) we have uniformly in $x \in [0, 1]$ and $n \in \mathbb{N}$ that

$$J_2 \sim (1-x)^\gamma \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha, \beta)}(x, \cos t)| t^{2(\alpha-\gamma)+1} dt \sim \\ \sim n(1-x)^\gamma \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \left| P_{n+1}(x)P_n(\cos t) - P_n(x)P_{n+1}(\cos t) \right| \frac{t^{2(\alpha-\gamma)}}{t-s} dt.$$

Following the idea of [1, p. 15] we use the identity

$$(4.3) \quad \begin{aligned} & P_{n+1}(y)P_n(x) - P_n(y)P_{n+1}(x) = \\ & = \left(1 + \frac{\alpha + \beta}{2n + 2}\right) [(1-x)\bar{P}_n(x)P_n(y) - (1-y)\bar{P}_n(y)P_n(x)], \end{aligned}$$

which may be verified by using (3.8).

Thus we have uniformly in $x \in [0, 1]$ and $n \in \mathbb{N}$ that

$$\begin{aligned} J_2 &= O(n)(1-x)^{\gamma+1}|\bar{P}_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |P_n(\cos t)| \frac{t^{2(\alpha-\gamma)}}{t-s} dt + \\ &+ O(n)(1-x)^\gamma|P_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} |\bar{P}_n(\cos t)| \frac{t^{2(\alpha-\gamma)+2}}{t-s} dt = \\ &= O(\sqrt{n})(1-x)^{\gamma+1}|\bar{P}_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt + \\ &+ O(\sqrt{n})(1-x)^\gamma|P_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{t-s} dt =: J_{21} + J_{22}, \end{aligned}$$

where we used (3.3) and $\sqrt{1-\cos t} \sim t$ ($t \in [0, \frac{2\pi}{3}]$).

From the condition $\frac{\alpha}{2} + \frac{1}{4} < \gamma$ it follows that $\alpha - 2\gamma - \frac{1}{2} < -1$, so by Lemma 1, $s \sim \sqrt{1-x}$ ($\cos s = x \in [0, 1]$) and (3.3) we obtain

$$\begin{aligned} J_{21} &= O(\sqrt{n})(1-x)^{\gamma+1}|\bar{P}_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt = \\ &= O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma+2} (\log(n\sqrt{1-x} + 1) + 1). \end{aligned}$$

Similarly, for J_{22} we have (since $\alpha - 2\gamma + \frac{1}{2} \in (-1, 0)$)

$$J_{22} = O(\sqrt{n})(1-x)^\gamma|P_n(x)| \int_{s+\frac{1}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{t-s} dt =$$

$$= O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left(\log(n\sqrt{1-x} + 1) + \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} |P_n(x)| \right).$$

Finally we obtain the estimate

$$(4.4) \quad J_2 = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left(\log(n\sqrt{1-x} + 1) + \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} (|P_n(x)| + |P_{n+1}(x)|) + 1 \right),$$

which holds uniformly in $x \in [0, 1]$ and $n \in \mathbb{N}$, $n > N$.

4.1.3. Estimation of J_3 . The expression J_3 may be simplified (see the estimate of J_2):

$$J_3 \sim (1-x)^\gamma \int_{s-\frac{1}{n}}^{s+\frac{1}{n}} |K_n^{(\alpha, \beta)}(x, \cos t)| t^{2(\alpha-\gamma)+1} dt$$

$$(x \in [0, 1], s \in [0, \frac{\pi}{2}]),$$

if $s \geq \frac{1}{n}$ (the lower bound of the integral is 0 if $0 \leq s \leq \frac{1}{n}$). For the kernel function we shall use the following estimates (see (3.3) and (3.6))

$$\begin{aligned} |K_n^{(\alpha, \beta)}(x, \cos t)| &= \left| \sum_{k=0}^n \frac{1}{h_k} P_k(x) P_k(\cos t) \right| = \left| \frac{1}{h_0} + \sum_{k=1}^n \frac{1}{h_k} P_k(x) P_k(\cos t) \right| = \\ &= O(1) \left(1 + \sum_{k=1}^n k |P_k(x)| |P_k(\cos t)| \right) = \\ &= O(1) \left(1 + \sum_{k=1}^n k k^{-\frac{1}{2}} \left(\sqrt{1-x} + \frac{1}{k} \right)^{-\alpha - \frac{1}{2}} k^{-\frac{1}{2}} \left(t + \frac{1}{k} \right)^{-\alpha - \frac{1}{2}} \right) = \\ &= O(1) \left(1 + n \left(\sqrt{1-x} + \frac{1}{n} \right)^{-\alpha - \frac{1}{2}} t^{-\alpha - \frac{1}{2}} \right) \\ &(x \in [0, 1], t \in [0, \frac{2\pi}{3}]). \end{aligned}$$

If $\frac{1}{n} < s \leq \frac{\pi}{2}$ then we have uniformly in $x = \cos s$ that

$$J_3 = O(1) (1-x)^\gamma \left\{ \int_{s-\frac{1}{n}}^{s+\frac{1}{n}} t^{2(\alpha-\gamma)+1} dt + \frac{n}{\left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}}} \int_{s-\frac{1}{n}}^{s+\frac{1}{n}} t^{\alpha-2\gamma+\frac{1}{2}} dt \right\}.$$

Since

$$\int_{s-\frac{1}{n}}^{s+\frac{1}{n}} t^A \sim \frac{s^A}{n} \quad \left(\frac{1}{n} \leq s \leq \pi, n \in \mathbb{N}, A > -1 \right),$$

we obtain by $s \sim \sqrt{1-x}$ that

$$\begin{aligned} J_3 &= O(1)(1-x)^\gamma \left\{ \frac{s^{2(\alpha-\gamma)+1}}{n} + \frac{s^{\alpha-2\gamma+\frac{1}{2}}}{(\sqrt{1-x} + \frac{1}{n})^{\alpha+\frac{1}{2}}} \right\} = \\ &= O(1)(1-x)^\gamma \left\{ s^{2(\alpha-\gamma)+1} + \frac{1}{(\sqrt{1-x} + \frac{1}{n})^{2\gamma}} \right\} = \\ &= O(1)(1-x)^\gamma \frac{1}{(\sqrt{1-x} + \frac{1}{n})^{2\gamma}} = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma}. \end{aligned}$$

If $0 \leq s \leq \frac{1}{n}$ then (see the definition of J_3 in Section 4.1) we get

$$J_3 = O(1)(1-x)^\gamma \left\{ \int_0^{s+\frac{1}{n}} t^{2(\alpha-\gamma)+1} dt + \frac{n}{(\sqrt{1-x} + \frac{1}{n})^{\alpha+\frac{1}{2}}} \int_0^{s+\frac{1}{n}} t^{\alpha-2\gamma+\frac{1}{2}} dt \right\}.$$

Since $\gamma < \alpha + 1$ and $\gamma < \frac{\alpha}{2} + \frac{3}{4}$ we have $2(\alpha-\gamma)+1 > -1$ and $\alpha-2\gamma+\frac{1}{2} > -1$. So by

$$\int_0^{s+\frac{1}{n}} t^A dt \sim \left(s + \frac{1}{n} \right)^{A+1} \quad (s \geq 0, A > -1)$$

we obtain

$$\begin{aligned} J_3 &= O(1)(1-x)^\gamma \left\{ \left(s + \frac{1}{n} \right)^{2(\alpha-\gamma)+2} + \frac{n \left(s + \frac{1}{n} \right)^{\alpha-2\gamma+\frac{3}{2}}}{(\sqrt{1-x} + \frac{1}{n})^{\alpha+\frac{1}{2}}} \right\} = \\ &= O(1)(1-x)^\gamma \left\{ \frac{1}{n^{2(\alpha+1-\gamma)}} + n(\sqrt{1-x} + \frac{1}{n}) \frac{1}{(\sqrt{1-x} + \frac{1}{n})^{2\gamma}} \right\} = \\ &= O(1)(1-x)^\gamma \left\{ 1 + \frac{1}{(\sqrt{1-x} + \frac{1}{n})^{2\gamma}} \right\} = \\ &= O(1)(1-x)^\gamma \frac{1}{(\sqrt{1-x} + \frac{1}{n})^{2\gamma}} = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma}. \end{aligned}$$

Finally we get the estimate

$$(4.5) \quad J_3 = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma},$$

which holds uniformly in $x \in [0, 1]$ and $n \in \mathbb{N}$.

4.1.4. Estimation of J_4 . First we remark that $J_4 = 0$ if $0 \leq s \leq \frac{1}{n}$, so we suppose that $s \in [\frac{1}{n}, \frac{\pi}{2}]$, i.e. $x = \cos s \in [0, 1 - \frac{c}{n^2}] =: I_n$. The expression J_4 may be simplified (see the estimation of J_2) by using the relation

$$|x - y| \sim |t^2 - s^2| \sim s|t - s| \sim \sqrt{1-x}|t - s| \\ \left(\frac{1}{n} \leq s \leq \frac{\pi}{2}, \quad t \in [0, s - \frac{1}{n}] \right).$$

Namely, we have (uniformly in $x \in I_n$ and $n \in \mathbb{N}$)

$$J_4 = w^{(\gamma, \delta)}(x) \int_0^{s - \frac{1}{n}} |K_n^{(\alpha, \beta)}(x, \cos t)| w^{(\alpha - \gamma, \beta - \delta)}(\cos t) \sin t \, dt \sim \\ \sim n(1-x)^{\gamma - \frac{1}{2}} \int_0^{s - \frac{1}{n}} |P_{n+1}(x)P_n(\cos t) - P_n(x)P_{n+1}(\cos t)| \frac{t^{2(\alpha - \gamma) + 1}}{s - t} \, dt.$$

Using the identity (4.3) and the estimate (3.3) we obtain

$$J_4 = O(n)(1-x)^{\gamma - \frac{1}{2}} \left\{ (1-x)|\bar{P}_n(x)| \int_0^{s - \frac{1}{n}} |P_n(\cos t)| \frac{t^{2(\alpha - \gamma) + 1}}{s - t} \, dt + \right. \\ \left. + |P_n(x)| \int_0^{s - \frac{1}{n}} t^2 |\bar{P}_n(\cos t)| \frac{t^{2(\alpha - \gamma) + 1}}{s - t} \, dt \right\} = \\ = O(\sqrt{n})(1-x)^{\gamma + \frac{1}{2}} |\bar{P}_n(x)| \int_0^{s - \frac{1}{n}} \frac{t^{\alpha - 2\gamma + \frac{1}{2}}}{s - t} \, dt + \\ + O(\sqrt{n})(1-x)^{\gamma - \frac{1}{2}} |P_n(x)| \int_0^{s - \frac{1}{n}} \frac{t^{\alpha - 2\gamma + \frac{3}{2}}}{s - t} \, dt =: J_{41} + J_{42} \\ \left(\frac{1}{n} \leq s = \arccos x \leq \frac{\pi}{2}, \quad n \in \mathbb{N} \right).$$

Since $\gamma < \frac{\alpha}{2} + \frac{3}{4}$, thus $\alpha - 2\gamma + \frac{1}{2} > -1$ we have by using Lemma 2 and $s \sim \sqrt{1-x}$ that

$$\begin{aligned} J_{41} &= O(\sqrt{n})(1-x)^{\gamma+\frac{1}{2}} |\overline{P}_n(x)| \left(s + \frac{1}{n}\right)^{\alpha-2\gamma+\frac{1}{2}} \log(ns+1) = \\ &= O(\sqrt{n}) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}}\right)^{2\gamma} |\overline{P}_n(x)| \left(\sqrt{1-x} + \frac{1}{n}\right)^{\alpha+\frac{3}{2}} \log(ns+1) = \\ &= O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}}\right)^{2\gamma} \log(n\sqrt{1-x} + 1) \\ &\quad (x \in I_n, \quad n \in \mathbb{N}). \end{aligned}$$

Similarly,

$$\begin{aligned} J_{42} &= O(\sqrt{n})(1-x)^{\gamma-\frac{1}{2}} |P_n(x)| \left(s + \frac{1}{n}\right)^{\alpha-2\gamma+\frac{3}{2}} \log(ns+1) = \\ &= O(\sqrt{n}) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}}\right)^{2\gamma} |P_n(x)| \frac{\left(\sqrt{1-x} + \frac{1}{n}\right)^{\alpha+\frac{3}{2}}}{\sqrt{1-x}} \log(ns+1) = \\ &= O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}}\right)^{2\gamma} \log(n\sqrt{1-x} + 1) \\ &\quad (x \in I_n, \quad n \in \mathbb{N}). \end{aligned}$$

Summarizing the above formulas we obtain

$$(4.6) \quad J_4 = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}}\right)^{2\gamma} \log(n\sqrt{1-x} + 1) \\ (x \in I_n, \quad n \in \mathbb{N}).$$

4.1.5. *The final upper estimate.* Using (4.2), (4.4), (4.5) and (4.6) we have

$$\begin{aligned} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) &= O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}}\right)^{2\gamma} \left(\log(n\sqrt{1-x} + 1) + \right. \\ &\quad \left. \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n}\right)^{\alpha+\frac{1}{2}} (|P_n(x)| + |P_{n+1}(x)|) + 1\right) \\ &\quad (x \in [0, 1], \quad n \in \mathbb{N}, \quad n > N). \end{aligned}$$

Let $\bar{x} \in (0, 1)$ be the closest number to 1 for which

$$P_n(\bar{x}) = \frac{1}{2}P_n(1) \sim n^\alpha$$

holds. If $x \in [0, \bar{x}]$ then

$$(4.7) \quad 1 - x \geq 1 - \bar{x} = \frac{P_n(1) - P_n(\bar{x})}{P'_n(\xi)} \sim \frac{1}{n^2} \quad (\xi \in (\bar{x}, 1))$$

(see (3.2)). Thus

$$\log(n\sqrt{1-x} + 1) \geq c.$$

If $x \in (\bar{x}, 1]$ then $P_n(x) \sim n^\alpha$, so

$$\sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} (|P_n(x)| + |P_{n+1}(x)|) \geq c.$$

This means that also

$$\begin{aligned} L_n^{(\alpha, \beta), (\gamma, \delta)}(x) &= O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left(\log(n\sqrt{1-x} + 1) + \right. \\ &\quad \left. \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} (|P_n(x)| + |P_{n+1}(x)|) \right) \\ &\quad (x \in [0, 1], n \in \mathbb{N}, n > N) \end{aligned}$$

is true.

From this we have uniformly in $x \in [-1, 1]$ and $n \in \mathbb{N}$, $n > N$ that

$$L_n^{(\alpha, \beta), (\gamma, \delta)}(x) = O(1) \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left(\frac{\sqrt{1+x}}{\sqrt{1+x} + \frac{1}{n}} \right)^{2\delta} \phi_n^{(\alpha, \beta)}(x),$$

where

$$\begin{aligned} \phi_n^{(\alpha, \beta)}(x) &= \log \left(n\sqrt{1-x^2} + 1 \right) + \\ &+ \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} \left(\sqrt{1+x} + \frac{1}{n} \right)^{\beta + \frac{1}{2}} \left(|P_n^{(\alpha, \beta)}(x)| + |P_{n+1}^{(\alpha, \beta)}(x)| \right). \end{aligned}$$

Thus the upper estimation in (2.8) is proved.

4.2. Lower estimation of $L_n^{(\alpha, \beta), (\gamma, \delta)}(x)$. Because of symmetry, it is enough to consider $x \in [0, 1]$. We shall give three different lower estimations for the weighted Lebesgue function.

4.2.1. The first lower estimation. If $\alpha, \beta > -1$ and $\gamma, \delta \geq 0$, then there exists a constant $c > 0$ independent of x and n such that

$$(4.8) \quad L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \geq c w^{(\gamma, \delta)}(x) \quad (x \in [0, 1], n \in \mathbb{N}).$$

Indeed, using the orthogonality of Jacobi polynomials we have

$$\int_{-1}^1 K_n^{(\alpha,\beta)}(x,y)w^{(\alpha,\beta)}(y) dy = 1 \quad (x \in [0, 1], n \in \mathbb{N}).$$

Therefore

$$\begin{aligned} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) &= w^{(\gamma,\delta)}(x) \int_{-1}^1 |K_n^{(\alpha,\beta)}(x,y)| \frac{w^{(\alpha,\beta)}(y)}{(1-y)^\gamma(1+y)^\delta} dy \geq \\ &\geq c w^{(\gamma,\delta)}(x) \int_{-1}^1 |K_n^{(\alpha,\beta)}(x,y)| w^{(\alpha,\beta)}(y) dy \geq \\ &\geq c w^{(\gamma,\delta)}(x) \int_{-1}^1 K_n^{(\alpha,\beta)}(x,y)w^{(\alpha,\beta)}(y) dy = c w^{(\gamma,\delta)}(x). \end{aligned}$$

4.2.2. *The second lower estimation.* If $\alpha, \beta > -1$ and $\gamma, \delta \geq 0$, then there exists a constant $c > 0$ independent of x and n such that

$$(4.9) \quad \begin{aligned} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) &\geq c w^{(\gamma,\delta)}(x) \sqrt{n} (|P_n(x)| + |P_{n+1}(x)|) \\ &(x \in [0, 1], n \in \mathbb{N}). \end{aligned}$$

In [1, p. 18] it was proven that

$$\begin{aligned} \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} |K_n^{(\alpha,\beta)}(x, \cos t)| dt &\geq c \sqrt{n} (|P_n(x)| + |P_{n+1}(x)|), \\ &(x \in [0, 1], n \in \mathbb{N}), \end{aligned}$$

from which (4.9) follows immediately.

4.2.3. *The third lower estimation.* It is clear that

$$(4.10) \quad \begin{aligned} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) &\geq \\ &\geq w^{(\gamma,\delta)}(x) \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha,\beta)}(x, \cos t)| w^{(\alpha-\gamma,\beta-\delta)}(\cos t) \sin t dt \end{aligned}$$

for all $x = \cos s \in [0, 1]$ and $R > 0$. Using the ideas of [1], we shall give a lower estimation for the right hand side of (4.10) with a suitable number $R > 1$.

Since

$$w^{(\alpha-\gamma, \beta-\delta)}(\cos t) \sin t \sim t^{2\alpha-2\gamma+1} \\ (s \in [0, \frac{\pi}{2}], \quad t \in [s, \frac{2\pi}{3}]),$$

we obtain from (4.10) that

$$(4.11) \quad L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \geq c(1-x)^\gamma \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha, \beta)}(x, \cos t)| \cdot t^{2\alpha-2\gamma+1} dt.$$

The estimation the above integral is performed in several steps.

STEP 1. From (3.7) it follows that

$$F_n(x, y) := P_{n+1}(y)P_n(x) - P_n(y)P_{n+1}(x) = \frac{2n + \alpha + \beta + 2}{4(n+1)} \times \\ \times \left\{ (1-x^2)\tilde{P}_{n-1}(x)P_n(y) - (1-y^2)\tilde{P}_{n-1}(y)P_n(x) + (y-x)P_n(x)P_n(y) \right\},$$

so by (3.6) we have uniformly for all $x \in [0, 1]$ and $n \in \mathbb{N}$ that

$$\begin{aligned} |K_n^{(\alpha, \beta)}(x, y)| &= \lambda_n^{(\alpha, \beta)} \left| \frac{F_n(x, y)}{x-y} \right| \geq \\ &\geq c_n \left| \frac{(1-x^2)\tilde{P}_{n-1}(x)P_n(y) - (1-y^2)\tilde{P}_{n-1}(y)P_n(x)}{x-y} - P_n(x)P_n(y) \right| \geq \\ &\geq c_1 n \left| \frac{(1-x^2)\tilde{P}_{n-1}(x)P_n(y) - (1-y^2)\tilde{P}_{n-1}(y)P_n(x)}{x-y} \right| - c_2 n |P_n(x)| |P_n(y)|. \end{aligned}$$

Since $|x-y| = |\cos s - \cos t| \sim t(t-s)$ we have

$$\begin{aligned} &\int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |K_n^{(\alpha, \beta)}(x, \cos t)| \cdot t^{2\alpha-2\gamma+1} dt \geq \\ &\geq c_1 \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left| (1-x^2)\tilde{P}_{n-1}(x)P_n(y) - (1-y^2)\tilde{P}_{n-1}(y)P_n(x) \right| \frac{t^{2\alpha-2\gamma}}{t-s} dt - \\ &\quad - c_2 n |P_n(x)| \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |P_n(\cos t)| t^{2\alpha-2\gamma+1} dt. \end{aligned}$$

Therefore by (3.5) we get uniformly for all $x \in [0, 1]$ and $n \in \mathbb{N}$ that

$$(4.12) \quad \begin{aligned} & L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \geq \\ & \geq c_1 n (1-x)^\gamma \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left| (1-x^2)\tilde{P}_{n-1}(x)P_n(y) - (1-y^2)\tilde{P}_{n-1}(y)P_n(x) \right| \frac{t^{2\alpha-2\gamma}}{t-s} dt - \\ & \quad - c_2 \sqrt{n}(1-x)^\gamma |P_n(x)|. \end{aligned}$$

STEP 2. For the estimation of the integral

$$I := \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left| (1-x^2)\tilde{P}_{n-1}(x)P_n(y) - (1-y^2)\tilde{P}_{n-1}(y)P_n(x) \right| \frac{t^{2\alpha-2\gamma}}{t-s} dt$$

we use the asymptotic formula (3.4) for the Jacobi polynomials

$$P_n(y) = P_n^{(\alpha, \beta)}(y) \quad \text{and} \quad \tilde{P}_{n-1}(y) = P_{n-1}^{(\alpha+1, \beta+1)}(y),$$

which gives

$$\begin{aligned} P_n^{(\alpha, \beta)}(\cos t) &= \frac{k^{(\alpha, \beta)}(t)}{\sqrt{n}} \left(\cos(Nt + \nu) + \frac{O(1)}{n \sin t} \right), \\ P_{n-1}^{(\alpha+1, \beta+1)}(\cos t) &= \frac{k^{(\alpha+1, \beta+1)}(t)}{\sqrt{n-1}} \left(\cos(\bar{N}t + \bar{\nu}) + \frac{O(1)}{n \sin t} \right) = \\ &= \frac{2k^{(\alpha, \beta)}(t)}{\sqrt{n-1} \sin t} \left(\cos(\bar{N}t + \bar{\nu}) + \frac{O(1)}{(n-1) \sin t} \right), \end{aligned}$$

where

$$\bar{N} = n-1 + \frac{(\alpha+1) + (\beta+1) + 1}{2} = N$$

and

$$\bar{\nu} = -\frac{2(\alpha+1) + 1}{4} \pi = \nu - \frac{\pi}{2}.$$

We have

$$\begin{aligned} & (1-x^2)\tilde{P}_{n-1}(x)P_n(y) - (1-y^2)\tilde{P}_{n-1}(y)P_n(x) = \\ & = \frac{k^{(\alpha, \beta)}(t)}{\sqrt{n}} \left\{ (1-x^2)\tilde{P}_{n-1}(x) \cos(Nt + \nu) - 2\sqrt{\frac{n}{n-1}} P_n(x) \sin t \cdot \sin(Nt + \nu) \right\} + \\ & + O\left(\frac{1}{n^{3/2}}\right) (1-x^2)\tilde{P}_{n-1}(x) \cdot \frac{k^{(\alpha, \beta)}(t)}{\sin t} + O\left(\frac{1}{(n-1)^{3/2}}\right) P_n(x) \cdot k^{(\alpha, \beta)}(t). \end{aligned}$$

If $0 < s + \frac{R}{n} \leq t \leq \frac{2\pi}{3}$, then

$$k^{(\alpha,\beta)}(t) = \frac{1}{\sqrt{\pi}} \left(\sin \frac{t}{2} \right)^{-\alpha-\frac{1}{2}} \left(\cos \frac{t}{2} \right)^{-\beta-\frac{1}{2}} \sim t^{-\alpha-\frac{1}{2}}.$$

Therefore

$$\begin{aligned} I &\geq \frac{c_1}{\sqrt{n}} \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left| (1-x^2) \tilde{P}_{n-1}(x) \cos(Nt+\nu) - \right. \\ &\quad \left. - 2\sqrt{\frac{n}{n-1}} P_n(x) \sin t \cdot \sin(Nt+\nu) \right| \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt - \\ &\quad - \frac{c_2}{n^{3/2}} \left\{ (1-x^2) |\tilde{P}_{n-1}(x)| \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{3}{2}}}{t-s} dt + |P_n(x)| \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt \right\}. \end{aligned}$$

STEP 3. Using the above inequality and (4.12) we have

$$\begin{aligned} (4.13) \quad L_n^{(\alpha,\beta),(\gamma,\delta)}(x) &\geq c_1 \sqrt{n} (1-x)^\gamma \times \\ &\times \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left| (1-x^2) \tilde{P}_{n-1}(x) \cos(Nt+\nu) - 2\sqrt{\frac{n}{n-1}} P_n(x) \sin t \cdot \sin(Nt+\nu) \right| \times \\ &\quad \times \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt - c_2 \sqrt{n} (1-x)^\gamma |P_n(x)| - c_3 \varrho_1(n, x), \end{aligned}$$

where

$$\begin{aligned} \varrho_1(n, x) &= \frac{(1-x)^\gamma}{\sqrt{n}} \times \\ &\times \left\{ (1-x^2) |\tilde{P}_{n-1}(x)| \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{3}{2}}}{t-s} dt + |P_n(x)| \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt \right\}. \end{aligned}$$

Since $t \geq \frac{R}{n}$ we have

$$\begin{aligned} \varrho_1(n, x) &\leq c \frac{\sqrt{n}}{R} (1-x)^\gamma \times \\ &\times \left\{ (1-x^2) |\tilde{P}_{n-1}(x)| \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt + |P_n(x)| \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma+\frac{1}{2}}}{t-s} dt \right\}. \end{aligned}$$

Using Lemma 1, $s \sim \sqrt{1-x}$ and (3.3) we get uniformly for all $x \in [0, 1]$ and $n \in \mathbb{N}$ that

$$\varrho_1(n, x) \leq c \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \times \\ \times \left\{ \frac{1}{R} \left[\log(n\sqrt{1-x} + 1) + 1 \right] + \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} |P_n(x)| \right\}.$$

STEP 4. Now, we consider the integral in (4.13) and write $\sin s = \sqrt{1-x^2}$ instead of $\sin t$. Then by the Lagrange mean value theorem we have

$$\sin t = \sin s + \tau = \sqrt{1-x^2} + \tau$$

with $|\tau| \leq t - s$. Thus we obtain an error term in the integral, which we shall denote by $\varrho_2(n, x)$. Therefore we have uniformly in $x \in [0, 1]$ and $n \in \mathbb{N}$ that

$$L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \geq c_1 \sqrt{n} (1-x)^\gamma \sqrt{1-x^2} \times \\ \times \int_{s + \frac{R}{n}}^{\frac{2\pi}{3}} \left| \sqrt{1-x^2} \tilde{P}_{n-1}(x) \cos(Nt + \nu) - 2\sqrt{\frac{n}{n-1}} P_n(x) \sin(Nt + \nu) \right| \times \\ \times \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt - c_2 \varrho_2(n, x) - c_3 \varrho_1(n, x) - c_4 \sqrt{n} (1-x)^\gamma |P_n(x)|,$$

where

$$\varrho_2(n, x) = 2\sqrt{n} (1-x)^\gamma \frac{n}{n-1} |P_n(x)| \int_{s + \frac{R}{n}}^{\frac{2\pi}{3}} |\sin(Nt + \nu)| t^{\alpha-2\gamma-\frac{1}{2}} dt \leq \\ \leq c \sqrt{n} (1-x)^\gamma |P_n(x)| \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha-2\gamma+\frac{1}{2}} \leq c \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma}$$

(using $s \sim \sqrt{1-x}$ and (3.3)).

Let

$$\psi := \arg \left(\sqrt{1-x^2} \tilde{P}_{n-1}(x) + i2\sqrt{\frac{n}{n-1}} P_n(x) \right).$$

Then we have uniformly in $x \in [0, 1]$ and $n \in \mathbb{N}$ that

$$L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \geq c_1 (1-x)^\gamma \times \\ \times \left(n(1-x^2) \left((1-x^2) \tilde{P}_{n-1}^2(x) + \frac{4n}{n-1} P_n^2(x) \right) \right)^{\frac{1}{2}} \times$$

$$\begin{aligned} & \times \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |\cos(Nt + \nu + \psi)| \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt - \\ & - c_2 \varrho_2(n, x) - c_3 \varrho_1(n, x) - c_4 \sqrt{n}(1-x)^\gamma |P_n(x)|. \end{aligned}$$

STEP 5. Now we will estimate the integral

$$B := \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} |\cos(Nt + \nu + \psi)| \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt.$$

Since $|\cos t| \geq \cos^2 t = \frac{1+\cos(2t)}{2}$ it follows that

$$B \geq \frac{1}{2} \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \left(1 + \cos 2(Nt + \nu + \psi)\right) \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt.$$

Using Lemma 1 we have

$$\begin{aligned} & \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt \geq c \left(s + \frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \left[\log\left(\frac{ns}{R} + 1\right) + 1\right] \geq \\ & \geq c \left(s + \frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \left[\log(ns + 1) + 1 - \log R\right], \end{aligned}$$

and by the second mean value theorem

$$\begin{aligned} & \int_{s+\frac{R}{n}}^{\frac{2\pi}{3}} \cos 2(Nt + \nu + \psi) \frac{t^{\alpha-2\gamma-\frac{1}{2}}}{t-s} dt = \frac{\left(s + \frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}}}{R/n} \times \\ & \times \int_{s+\frac{R}{n}}^{\xi} \cos 2(Nt + \nu + \psi) dt \leq c \left(s + \frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \left(\xi \in \left(s + \frac{R}{n}, \frac{2\pi}{3}\right)\right). \end{aligned}$$

Then we get

$$B \geq c_1 \left(s + \frac{R}{n}\right)^{\alpha-2\gamma-\frac{1}{2}} \left[\log(ns + 1) + 1 - c_2\right].$$

STEP 6. From this we obtain

$$\begin{aligned} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) &\geq c_1 (1-x)^\gamma \left(n(1-x^2) \left((1-x^2) \tilde{P}_{n-1}^2(x) + \frac{4n}{n-1} P_n^2(x) \right) \right)^{\frac{1}{2}} \times \\ &\quad \times \left(s + \frac{R}{n} \right)^{\alpha-2\gamma-\frac{1}{2}} \left[\log(ns+1) + 1 - c_2 \right] - \\ &\quad - c_3 \varrho_2(n, x) - c_4 \varrho_1(n, x) - c_5 \sqrt{n}(1-x)^\gamma |P_n(x)|. \\ &\quad (x \in [0, 1], n \in \mathbb{N}, n > N). \end{aligned}$$

By (3.3) and $s \sim \sqrt{1-x}$ we have

$$\begin{aligned} C(x) &:= (1-x)^\gamma \left(s + \frac{R}{n} \right)^{\alpha-2\gamma-\frac{1}{2}} \times \\ &\quad \times \left\{ n(1-x^2) \left((1-x^2) \tilde{P}_{n-1}^2(x) + \frac{4n}{n-1} P_n^2(x) \right) \right\}^{\frac{1}{2}} \leq \\ &\quad \leq c_1 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \leq c_2, \end{aligned}$$

which means that

$$\begin{aligned} L_n^{(\alpha,\beta),(\gamma,\delta)}(x) &\geq c_1 C(x) \left[\log(n\sqrt{1-x}+1) + 1 \right] - \\ &\quad - c_2 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left[\frac{1}{R} (\log(n\sqrt{1-x}+1) + 1) + \right. \\ &\quad \left. + \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha+\frac{1}{2}} |P_n(x)| + 1 \right] - c_3 \sqrt{n}(1-x)^\gamma |P_n(x)| \\ &\quad (x \in [0, 1], n \in \mathbb{N}, n > N). \end{aligned}$$

Let $\bar{x} \in (0, 1)$ be the closest number to 1 for which

$$P_n(\bar{x}) = \frac{1}{2} P_n(1) \sim n^\alpha$$

holds. If $x \in [0, \bar{x}]$ then by (4.7) we have

$$s \sim \sqrt{1-x} \geq \sqrt{1-\bar{x}} \geq \frac{c}{n},$$

thus

$$\left(s + \frac{R}{n} \right)^{\alpha-2\gamma-\frac{1}{2}} \geq c s^{\alpha-2\gamma-\frac{1}{2}},$$

which means that

$$C(x) \geq c s^{\alpha-\frac{1}{2}} \left\{ n(1-x^2) \left((1-x^2) \tilde{P}_{n-1}^2(x) + \frac{4n}{n-1} P_n^2(x) \right) \right\}^{\frac{1}{2}}.$$

It is proved in [1, p. 21] that

$$s^{\alpha-\frac{1}{2}} \left\{ n(1-x^2) \left((1-x^2) \tilde{P}_{n-1}^2(x) + \frac{4n}{n-1} P_n^2(x) \right) \right\}^{\frac{1}{2}} > c \quad (x \in [0, \bar{x}],$$

so for every $x \in [0, \bar{x}]$ and $n \in \mathbb{N}$, $n > N$ we have

$$\begin{aligned} L_n^{(\alpha, \beta), (\gamma, \delta)}(x) &\geq c_1 \left[\log(n\sqrt{1-x}+1) + 1 \right] - c_2 \left\{ \sqrt{n}(1-x)^\gamma(x) |P_n(x)| + \right. \\ &\quad + \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left(\frac{1}{R} \left[\log(n\sqrt{1-x}+1) + 1 \right] + \right. \\ &\quad \left. \left. + \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha+\frac{1}{2}} |P_n(x)| + 1 \right) \right\}. \end{aligned}$$

Here

$$c_1 - \frac{c_2}{R} \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \geq c_1 - \frac{c_2}{R} =: c_3 \geq c_3 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma}.$$

The number R can be chosen such that $c_3 > 0$. Then we have

$$\begin{aligned} L_n^{(\alpha, \beta), (\gamma, \delta)}(x) &\geq c_3 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left[\log(n\sqrt{1-x}+1) + 1 \right] - \\ &\quad - c_2 \sqrt{n}(1-x)^\gamma(x) |P_n(x)| - c_2 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} - \\ &\quad - c_2 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha+\frac{1}{2}} |P_n(x)| \end{aligned}$$

for all $x \in [0, \bar{x}]$ and $n \in \mathbb{N}$, $n > N$. If $x \in [\bar{x}, 1]$ then

$$1-x \leq 1-\bar{x} \sim \frac{1}{n^2}$$

(see (4.7)), and so

$$\left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left[\log(n\sqrt{1-x}+1) + 1 \right] \leq c \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \leq$$

$$\leq c \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} |P_n(x)|$$

(since $P_n(x) \sim n^\alpha$ on this interval), which means that with a suitable $c_4 > 0$ we have

$$(4.14) \quad \begin{aligned} L_n^{(\alpha, \beta), (\gamma, \delta)}(x) &\geq c_3 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} [\log(n\sqrt{1-x} + 1) + 1] - \\ &- c_2 \sqrt{n} (1-x)^\gamma (x) |P_n(x)| - c_2 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} - \\ &- c_4 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} |P_n(x)| \end{aligned}$$

for all $x \in [0, 1]$ and $n \in \mathbb{N}$, $n > N$.

4.2.4. *The final lower estimation.* From (4.8) we have

$$(4.15) \quad L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \geq c_6 (1-x)^\gamma \quad (x \in [0, 1], n \in \mathbb{N}).$$

(4.9), (4.14) and (4.15) imply

$$\begin{aligned} &c_3 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} [\log(n\sqrt{1-x} + 1) + 1] \leq L_n^{(\alpha, \beta), (\gamma, \delta)}(x) + \\ &+ c_2 \sqrt{n} (1-x)^\gamma (|P_n(x)| + |P_{n+1}(x)|) + c_2 \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} + \\ &c_4 \sqrt{n} \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} (|P_n(x)| + |P_{n+1}(x)|) \leq \\ &\leq L_n^{(\alpha, \beta), (\gamma, \delta)}(x) + \frac{c_2}{c} L_n^{(\alpha, \beta), (\gamma, \delta)}(x) + \frac{c_2}{c_6} L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \left(\sqrt{1-x} + \frac{1}{n} \right)^{-2\gamma} + \\ &+ \frac{c_4}{c} L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha - 2\gamma + \frac{1}{2}}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} c_3 (1-x)^\gamma [\log(n\sqrt{1-x} + 1) + 1] &\leq c_7 L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \\ (x \in [0, 1], n \in \mathbb{N}, n > N). \end{aligned}$$

Since (by (3.3))

$$\sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} (|P_n(x)| + |P_{n+1}(x)|) \leq c$$

$$(x \in [0, 1], n \in \mathbb{N}),$$

we have

$$L_n^{(\alpha, \beta), (\gamma, \delta)}(x) \geq c(1-x)^\gamma \left(\log(n\sqrt{1-x} + 1) + \right.$$

$$\left. + \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} (|P_n(x)| + |P_{n+1}(x)|) \right) \geq$$

$$\geq c w^{(\gamma, \delta)}(x) \phi_n^{(\alpha, \beta)}(x),$$

where

$$\phi_n^{(\alpha, \beta)}(x) = \log(n\sqrt{1-x^2} + 1) + \sqrt{n} \left(\sqrt{1-x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} \times$$

$$\times \left(\sqrt{1+x} + \frac{1}{n} \right)^{\beta + \frac{1}{2}} (|P_n^{(\alpha, \beta)}(x)| + |P_{n+1}^{(\alpha, \beta)}(x)|).$$

The above estimate holds uniformly in $x \in [0, 1]$ and $n \in \mathbb{N}$.

Theorem is proved. ■

5. Proof of Corollary

Since $L_n^{(\alpha, \beta), (\gamma, \delta)}(\pm 1) = 0$ we have

$$\max_{x \in [-1, 1]} L_n^{(\alpha, \beta), (\gamma, \delta)}(x) = L_n^{(\alpha, \beta), (\gamma, \delta)}(x_0)$$

with $x_0 \in (-1, 1)$.

From Theorem and (3.3) it follows that

$$L_n^{(\alpha, \beta), (\gamma, \delta)}(x_0) \leq c_1 \cdot 1 \cdot (\log(n+1) + c_2) \leq c_3 \log(n+1)$$

and

$$L_n^{(\alpha, \beta), (\gamma, \delta)}(x_0) \geq c_4 w^{(\gamma, \delta)}(x_0) \log \left(n\sqrt{1-x_0^2} + 1 \right) \geq$$

$$\geq c_5 \log(c_6 n + 1) \geq c_7 \log(n+1),$$

where the c_i ($i = 1 \dots 7$) constants are positive and independent of n . This proves the statement. ■

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Á. Chripkó

Department of Numerical Analysis

Eötvös Loránd University

Pázmány P. sétány 1/C.

H-1117 Budapest, Hungary

agnes.chripko@gmail.com