Some Aspects of Topological String Theory

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Thesis submitted for the degree of

Phd in Physics

2018

Abstract

In the thesis we study topological aspects of string and M-theory. We derive a large N holomorphic string expansion for the Macdonald-deformed $U(N)$ Yang-Mills theory on a closed Riemann surface. Macdonald deformation of two-dimensional Yang-Mills theory computes entropies of BPS black holes and it is also dual to refined topological string theory. In the classical limit, the expansion defines a new β-deformation for Hurwitz theory of branched covers wherein the refined partition function is a generating function for certain parameterized Euler characters. We also apply the large N expansion to observables corresponding to open surfaces and Wilson loops.

We study AKSZ constructions for the A and B sigma-models of topological string theory within a double field theory formulation that incorporates backgrounds with geometric and non-geometric fluxes. AKSZ formulations provide natural geometric methods for constructing BV quantized sigma-models. After a section condition, we relate the A- and B-model to a three-dimensional Courant sigma-model, corresponding to a generalized complex structure, which reduces to the A- or B-models on the boundary. We introduce S-duality at the level of the three-dimensional sigma-model based on the generalized complex structure, which exchanges the related AKSZ field theories, and interpret it as topological S-duality of the A- and B-models.

We also study AKSZ constructions for closed topological membranes on G_2-manifolds. These membranes were originally introduced to be the worldvolume formulation for topological M-theory, which is intended to capture a topological sector of physical M-theory. We propose two inequivalent AKSZ membrane theories, in each of which the two existing topological membranes appear as different gauge fixed versions, and their dimensional reductions give new AKSZ constructions for the topological A-model. We show that the two AKSZ membrane models originate through worldvolume dimensional reduction of a single AKSZ three-brane theory, which gives the higher Courant bracket of exceptional generalized geometry of M-theory as the underlying derived bracket.

The thesis is based on three papers [1–3].

Összefoglaló

A disszertációban a húrelmélet és az M-elmélet topologógiai vonatkozásait tanulmányozzuk. Kiszámoljuk az $U(N)$ Yang–Mills-elmélet Macdonald-deformációjának nagy N kifejtését zárt Riemann-felületeken. A Macdonald-deformált Yang–Mills-elméletben, amely a finomított topologikus húrelmélet duálisa, BPS feketelyukak entrópiája számolható. Klasszikus limeszben a kifejtés elágazó fedések Hurwitz-elméletének egy új β-deformációját definiálja, amelyben a finomított partíciós függvény parametrizált Eurel-karakterek generátor függvényét adja. Kiszámoljuk további obszervábilisek nagy N kifejtését, úgy mint a nyílt felületeknek megfelelő mennyiségekét és Wilson-hurkokét.

Tanulmányozzuk a topologikus húrelmélet A- és B-modelljének AKSZ konstrukcióját kettőzött térelméleten belül, amely geometriai és nem geometriai fluxusokat tartalmazó háttereket is leír. Az AKSZ formalizmus olyan geometriai módszer, amellyel BV-kvantált szigma-modellek konstruálhatók. Megfelelő szelési feltétel után az A- és B-modellt olyan háromdimenziós Courant szigma-modellel hozzuk kapcsolatba, amelyet egy általánosított komplex struktúra definiál, és a peremen visszaadja az A- és B-modellt. A háromdimenziós szigma-modell szintjén az általánosított komplex struktúrára alapozva S-dualitást vezetünk be, ami kicseréli a kapcsolódó AKSZ elméleteket, és ezt az A- és B-modell topologikus S-dualitásaként interpretáljuk.

Tanulmányozzuk G_2-sokaságon definiált zárt topologikus membránok AKSZ konstrukcióját is. Ezen membránokat eredetileg azért vezették be, hogy világtérfogati leírást adjanak a topologikus M-elmélet számára, melynek célja, hogy a fizikai M-elmélet topologikus szektoráról szerezzen információt. Bevezetünk két nem ekvivalens AKSZ membrán modellt, amelyekben a már ismert topologikus membránok úgy jelennek meg, mint különböző mértékrögzített esetek, és dimenziós redukcióik az A-modell új AKSZ konstrukcióihoz vezetnek. Végül megmutatjuk, hogy ezen két AKSZ membrán modell egyetlen AKSZ három-brán elmélet világfelületi dimenziós redukciójából származtatható, amelynek indukált zárójele az M-elmélet általánosított geometriájának 2-Courant zárójele.

A disszertáció három cikk eredményeire épül [1–3].

Acknowledgements

I would like to thank my supervisor, Annamária Sinkovics for introducing me to the realm of string theory and for guiding me through the first steps to be a researcher. I also thank my other supervisor, Richard Szabo for the very inspiring collaboration, and being always helpful and supportive. I thank the opportunity to visit him in Edinburgh for a few times and his hospitality during my stay.

I thank Zoltán Bajnok for offering me a position in Wigner RCP, and his support since then. I thank László Palla for the motivating discussions during the first period of my Phd. I also thank Dániel Nógradi for employing me for a short time.

I thank my family and all my friends, specially Zoltán Markó and my brother Tamás Kökényesi for their help. I also thank all my inspiring teachers and helpful secretaries in the university.

Last, but not least, I thank God for making all this possible.

Contents

Chapter 1

Introduction and outline

1 Introduction

String theory is a theory of quantum gravity, which is able to describe the known particle physics. The necessity of a theory for quantum gravity comes to the fore near to a black hole, where the effect of gravity is strong enough to take us out of the area of ordinary perturbative descriptions. Currently, the most promising and also the most popular candidate is string theory, which replaces the concept of pointlike particles with extended objects such as strings. It has a number of fascinating applications, such as gauge/geometry dualities and calculation of black hole entropies, but also leads to unsolved problems for example its background-dependence and the large number of phenomenologically realizable vacuums, i.e. the string landscape. In addition to all of these it lacks experimental confirmation and has not given any testable physical prediction so far.

Topological string theory

Calculation of quantities in string theory can usually be very complicated, but there are useful geometric methods, which work on a restriction of the original construction and help to obtain information about a subsector of string theory. In particular topological string theory has been proved to be effective in the past decades to capture topological information about physical string theory, and helped to understand some of its fundamental questions like black hole entropies and effective superpotentials.

To be more specific, the free energy of topological string theory computes F-terms in the four dimensional $\mathcal{N} = 2$ effective theory of type IIA or IIB superstring compactifications on a Calabi-Yau manifold, which is also responsible for the gravitational correction to the scattering of graviphotons. Topological string theory is defined using a so called topological sigma-model, which can be obtained after an operation, called topological twist on the supersymmetry algebra of an $\mathcal{N} = 2$ sigma-model. We will outline the definition later in more detail.

Since its birth, topological string theory has been a widely studied area by both physicists and mathematicians, and it has inspired many applications. One of the fascinating results that was not expected from the original construction is that topological string theory counts microstates of four dimensional BPS black holes, so it can give a microscopic description for a class of black hole entropies. The first part of the thesis is related to this duality.

Compactification with fluxes

Type II compactifications that preserve $\mathcal{N} = 2$ supersymmetry give the Calabi-Yau conditions in general. These compactifications have a rich mathematical structure, while they possess various phenomenological problems. A physically realistic model must have at most $\mathcal{N} = 1$ supersymmetry, because higher supersymmetry does not allow chiral interactions such as the electroweak. In addition it also suffers from the problem of moduli stabilization. After compactification, the theory is left with a number of massless fields with no potential (such as the complex structure or the Kähler deformations of the metric), which are called moduli. This would lead to an instability of the four-dimensional theory because their vacuum expectation values, which should specify the compactified theory, would not be fixed. They would result in long range forces unobserved so far in nature. Consequently a mechanism creating a potential, which stabilizes their vacuum expectation values, is necessary for realistic models.

Compactification with fluxes is not only useful to break the supersymmetry partially or completely, but also provides a mechanism that creates potential for scalar fields and stabilizes their vacuum expectation values (see [5, 6] for reviews). The basic example is the H flux for the Kalb-Ramond two-form B, but also fluxes for the RR fields have relevance in compactifications. Surprisingly these fluxes, which can be called as geometric fluxes, are not enough to stabilize all moduli in a Minkowski vacuum, but

an other family of fluxes arises if we consider their T-duals, which includes the so called non-geometric fluxes. It turns out that their T-duals are necessary in order to stabilize the moduli, but the presence of fluxes back-reacts on the compactified manifold and changes its geometrical properties making it to nicely fit into the framework of generalized geometry.

T-duality

T-duality originates from toroidal compactification of closed bosonic or type II strings, where the mass spectrum is invariant under the exchange of the winding number of strings w^i and their quantized momentum n_i in the compactified direction, with the change of the toroidal scale (i.e. $R \leftrightarrow \alpha'/R$, where R is the radius of the tori and α' is proportional to the inverse of the string tension). So the momenta and winding numbers can be combined into a 'doubled' vector $N^I = (w^i, n_i)$, which is an element of the so called generalized tangent bundle $T \oplus T^*$, whereon T-duality acts as a discrete subgroup $SO(d,d,\mathbb{Z})$ in $O(d,d)$, which leaves the matrix

$$\eta_{IJ} = \begin{pmatrix} 0 & \delta^i{}_j \\ \delta_i{}^j & 0 \end{pmatrix} \tag{1.1}$$

invariant. So far we only discussed the compactifications on torus, but this is far not the most general one. However every manifold can be treated as a torus locally, which has the stringy $SO(d,d)$ symmetry. The formalism that covariantizes T-duality symmetry is called generalized geometry.

Generalized geometry

Generalized geometry has been introduced by Hitchin together with his students [7,8], and their original motivation was to unify complex and symplectic geometry, which has led to generalized complex geometry. Generalized geometry is based on two premises. The first is to replace the tangent bundle T with the generalized tangent bundle $T \oplus T^*$, which means that a generalized vector is given by a sum of an ordinary vector and a one-form. The second premise is to replace the Lie bracket of vectors with the Courant bracket:

$$[X+\alpha, Y+\beta]_{\mathrm{C}} = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}\,\mathrm{d}(\iota_X\beta - \iota_Y\alpha)\ , \tag{1.2}$$

where $X, Y \in T$ and $\alpha, \beta \in T^*$, and $[X,Y]$ is the ordinary Lie bracket of X and Y. The Courant bracket does not satisfy the Jacobi identity. The graph of the generalized

tangent bundle with respect to the so called B-field transformation $X+\alpha \mapsto X+\alpha+\iota_X B$, where B is a closed two-form and physically related to the Kalb-Ramond B-field, is closed under the Courant bracket

$$[X+\alpha+\iota_X B, Y+\beta+\iota_Y B]_{\mathrm{C}} \;=\; [X+\alpha, Y+\beta]_{\mathrm{C}} + \iota_{[X,Y]} B \;. \tag{1.3}$$

Ordinary diffeomorphism acts on T and T^* separately, but there are more general transformations that mix vectors with one-forms, such as the B-field transformation. The ordinary diffeomorphism and the B-field transformation together with a third transformation given by a two-vector β with $X+\alpha \mapsto X+\iota_\alpha\beta+\alpha$ generate the $O(d,d)$ transformations of the generalized tangent bundle leaving the symmetric pairing

$$\langle X+\alpha, Y+\beta\rangle \;=\; \frac{1}{2}\left(\iota_X\beta+\iota_Y\alpha\right) \tag{1.4}$$

invariant.

The introduction of a generalized complex structure is very analogous to the ordinary complex structure. An endomorphism of the generalized tangent bundle $\mathbb{J}$ is called an almost complex structure, if it squares to $-\mathbb{1}_{2d}$, and obeys the hermiticity condition $\mathbb{J}^t\eta\mathbb{J}=\eta$, where η is the $O(d,d)$ invariant metric (1.1). One can define projectors $\Pi_\pm = \frac{1}{2}(\mathbb{1}_{2d} \pm \mathrm{i}\,\mathbb{J})$, which are analogous to the holomorphic and antiholomorphic projectors, and the integrability condition of the vanishing Nijenhuis tensor for the ordinary complex structures can be rewritten for generalized complex structure as

$$\Pi_\pm[\Pi_\pm(X+\alpha), \Pi_\pm(Y+\beta)]_{\mathrm{C}} = 0 \;. \tag{1.5}$$

An ordinary complex structure J as well as a symplectic structure ω can be embedded into generalized complex structures. The integrability identities of the generalized complex structure

$$\mathbb{J}_1 \;=\; \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} \tag{1.6}$$

gives the integrability conditions of the ordinary complex structure J, while the identities of

$$\mathbb{J}_2 \;=\; \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{1.7}$$

result the condition of a symplectic structure. Thus we can say that complex and symplectic geometry are two sides of generalized complex geometry.

A higher structure arises in the so called exceptional generalized geometry of M-theory, wherein the generalized tangent bundle is replaced by $T \oplus \bigwedge^2 T^*$, and the Courant bracket by its generalization

$$[X+\lambda, Y+\xi]_{2\mathrm{C}} \,=\, [X,Y] \,+\, \mathcal{L}_X\xi \,-\, \mathcal{L}_Y\lambda \,+\, \frac{1}{2}\,\mathrm{d}(\iota_X\,\lambda - \iota_Y\,\xi) \tag{1.8}$$

for two-forms λ and ξ, which we call a 2-Courant bracket. This higher generalized structure will appear in connection with our results later.

Geometric and non-geometric fluxes

We will illustrate the appearance of geometric and non-geometric fluxes through a simple example and focus on a three-torus of toroidal compactification. We start with the NSNS H-flux, which is locally determined as $H = \mathrm{d}B$ and gives a characteristic class of a gerbe. Such a flux background lies within the realm of what is called a geometric background. Starting with a constant H-flux and applying T-duality in the three different directions produces a chain of different fluxes [9, 10]

$$H_{ijk} \;\overset{T_k}{\longleftrightarrow}\; f_{ij}{}^{k} \;\overset{T_j}{\longleftrightarrow}\; Q_i{}^{jk} \;\overset{T_i}{\longleftrightarrow}\; R^{ijk}\ . \tag{1.9}$$

The second member of the T-duality chain is the metric flux f, which determines the torsion of the geometry through the Maurer-Cartan equations

$$\mathrm{d}e^k \,=\, f_{ij}{}^{k} e^i \wedge e^j \tag{1.10}$$

of one-forms e^i. The resulting geometry is called a twisted torus, and this T-duality frame is also geometric.

The other two fluxes are called non-geometric fluxes because the closed string momenta and winding modes become entangled, and the background no longer can be described within a standard manifold, as the transition functions of local charts involve T-duality transformations. The presence of Q-flux deforms the background with non-zero winding to be non-commutative

$$[x^j, x^k] \,=\, \mathrm{i}\, Q_i{}^{jk} x^i\ , \tag{1.11}$$

while the R-flux is responsible for the non-associative deformation [11]

$$[x^i, x^j, x^k] \,=\, R^{ijk}\ , \tag{1.12}$$

where the three-bracket means the Jacobiator here. The R-flux background is not even locally geometric.

Courant algebroids are central objects in generalized geometry as they are the corresponding algebraic construction to the Courant bracket introduced in (1.2). They will be defined in more detail in §3.1, but we mention that the four fluxes fit nicely into this framework [12] and appear as twists of the Courant algebroids, and their defining axioms give Bianchi identities for the fluxes.

Double field theory

Double field theory (DFT) [13–15] is a manifestly T-duality invariant low-energy formulation of string theory, where the original coordinates conjugate to closed string momentum modes are extended with dual coordinates conjugate to winding modes. Roughly saying it is a T-duality invariant 'supergravity' in doubled space, which intermediates between string theory and realistic four dimensional theories. On the other hand DFT is a natural framework to describe non-geometric fluxes (see [16–18] for reviews).

In DFT there are two natural length scale, the string length l_s and the compactification scale R. Since winding effects are suppressed if $R \gg l_s$, the dual coordinates go can to zero, and the DFT action reproduces an original supergravity action. However, if the compactification scale is so small that $R \ll l_s$, the strings can wind up enough to dominate the momentum modes, and we arrive at a T-dual description, where the ordinary coordinates can be zero while the dual ones are not. As we see, a restriction is necessary to reduce the doubling in DFT in a T-duality covariant way. So DFT is a restricted theory, which is specified after imposing the so called section condition $\eta^{IJ}\partial_I\partial_J = 0$, where I, J are doubled indices. Usually DFT is related to generalized geometry after imposing the section condition.

Since DFT contains stringy information from the beginning, its compactification includes geometric and non-geometric fluxes and provides a geometric interpretation for both of them, while compactifications of ordinary supergravities only describe geometric fluxes.

AKSZ sigma-models

Worldsheet and worldvolume sigma-models appear in the context of non-commutative and non-associative geometries. The low-energy effective field theory of open string sigma-model with constant B-field on D-brane has a non-commutative space structure [19], while in the context of closed strings non-associative geometry arises in membrane sigma-models in the presence of fluxes (e.g. in $SU(2)$ WZW models [20]). So non-commutative and non-associative structures can originate from sigma-model descriptions. Even more is true, the so called AKSZ sigma-models can capture the algebraic structure of generalized geometry: three-dimensional AKSZ sigma-models has a one-to-one correspondence to Courant algebroids [21–27], and also the fluxes appear in both sides. They are twist deformations of the Courant algebroid, and on the other side, Courant sigma-models geometrize fluxes in the sense that they are uplifts of string sigma-models to one higher dimension which can accomodate fluxes [11, 28–30].

AKSZ formulation is a natural geometric methods for constructing Batalin-Vilkovisky (BV) quantized topological sigma-models [28, 29, 31, 32]. They produce examples of topological field theories of Schwarz-type in arbitrary dimensionality such as the Poisson sigma-model, Chern-Simons theory and BF-theory; special gauge fixing action functionals also yield examples of topological field theories of Witten-type, such as the A- and B-models. We will review AKSZ construction in detail later.

In the second part of the thesis we study AKSZ constructions for topological string sigma-models and relate them to generalized complex geometry and double field theory. Then we continue with AKSZ construction for topological membranes on G_2-manifolds, which has a relevance in topological M-theory, and relate our construction to the generalized geometry of M-theory.

2 Summary and outline

The thesis is based on three papers [1–3], and contains results in two slightly different areas. The first part (Chapter 2 and 3) focuses on dualities between two-dimensional Yang-Mills theory, black hole entropy and topological string theory. Our main results centralize around the large N string expansion of a two parameter deformed $U(N)$ Yang-Mills theory on a Riemann surface, which is motivated by refined topological string theory and has an application in counting microstates of BPS black holes. We

also mention our previous paper [33], which focuses on different aspects of the two parameter deformed Yang-Mills theory. In this paper the author of this thesis mainly worked on the Douglas-Kazakov phase transition of the Yang-Mills theory in planar limit and its relation to topological string theory, but since this is an earlier result related to the author's master degree, it is not presented in this thesis.

We start with some preliminary background on two-dimensional Yang-Mills theory and topological string theory in Chapter 2. We review the specific features of two-dimensional Yang-Mills theory §1.1 and its Gross-Taylor string expansion §1.2, then introduce topological string theory very briefly in §2.1. The q-deformed Yang-Mills theory with its chiral expansion and the relation to BPS black holes and topological string theory are presented in §2.2. The introduction of (q, t)-deformed two-dimensional Yang-Mills theory can be found in §3.1, and the refinement of related dualities is summarized in §3.2.

In Chapter 3 we present our results on the large N string expansion of the (q, t)-deformed $U(N)$ Yang-Mills theory. In §1 we describe our calculation applied in the derivation of the expansion. It combines Schur-Weyl duality for quantum groups with the Etingof-Kirillov theory of generalized quantum characters, which are related to Macdonald polynomials, and Hecke algebras. In §2 we perform the chiral expansion of the partition function for closed Riemann surface and in a special limit introduce a new deformation of Hurwitz theory, which is a theory of branched covers. In §3 we continue with the expansion of other observables in the theory, such as the partition function for open surface and Wilson loops.

In the second part (Chapter 4, 5 and 6) we study AKSZ sigma-models related to generalized geometry and double field theory. In Chapter 4 we review the relevant background about AKSZ construction. In §1 we introduce AKSZ construction with two dimensional reduction method that we will use later in the thesis. Then we survey several relevant examples of two-dimensional AKSZ sigma-models in §2, which are mostly related to topological string sigma-models. In §3 we describe Courant algebroids and Courant sigma-models with the relevant examples in more detail. DFT algebroids and sigma-models are summarized in §4, and higher algebroids with the related four-dimensional AKSZ sigma-models are presented in §5.

In Chapter 5 we present our results about the double field theoretical formulation of AKSZ topological string sigma-models and its relation to generalized complex geometry. We start with their AKSZ construction within DFT in §1 and after imposing a

constraint, which has a similar role as a section condition, we reduce our DFT model to an AKSZ formulation which naturally defines a background with generalized complex structure in §2. We finish this chapter with an application in §3 and we show that topological S-duality arises from the AKSZ sigma-model of generalized complex structure.

We continue the study of AKSZ formulations in Chapter 6 with constructions for topological membranes on G_2-manifolds. In §1 we review the two different membranes on G_2-manifolds already known in the literature and present two different AKSZ constructions for them, which are unified within a single AKSZ three-brane theory. In §2 we show that it dimensionally reduces to the A-model topological string, and we further reduce it to the supersymmetric quantum mechanics in §3.

Chapter 2

Background on two-dimensional Yang-Mills theories

In the first part of the thesis we study the large N expansions of two-dimensional Yang-Mills theory with special regard on the deformations, which are motivated by topological string theory and BPS black holes. In this chapter we will go step by step introducing the original, q-deformed and (q,t)-deformed Yang-Mills theories, their relations to topological strings and black hole physics, and their large N expansions.

1 Two-dimensional Yang-Mills theory

Two-dimensional Yang-Mills theory is not just a good testing ground for gauge/string dualities, but also has physical relevance in supersymmetric black hole physics and topological string theory as well as mathematical interest in calculating branched covers of Riemann surfaces.

Gluons in two dimensions have no propagating degrees of freedom since they have no transverse directions. Such a theory does not seem to be interesting at first sight, but since there are so few degrees of freedom, it has a large group of symmetries: it is not just invariant under the original gauge transformation, but also under the area preserving diffeomorphisms of the two-dimensional surface. Consequently, it is exactly solvable and includes important information about the topology. This introductory section is based on the review papers [34] and [35].

1.1 Partition function and observables

The action of the pure bosonic Yang-Mills theory on a closed Riemann surface Σ_h classified by the genus h is given as usual

$$S_{\mathrm{YM}}[A] \;=\; -\frac{1}{2g_{\mathrm{YM}}}\int_{\Sigma_h} \mathrm{d}\mu\, \mathrm{Tr}\,(F_A^2)\;, \tag{2.1}$$

where F_A is the curvature of the gauge connection A associated to the gauge group G, g_{YM} is the coupling constant, $\mathrm{d}\mu$ is the volume form on Σ_h and the trace is computed in the fundamental representation of G. The quantum theory is defined by the path integral over the connection, which leads to the partition function

$$\mathcal{Z}_{\mathrm{YM}}(h, g_{\mathrm{YM}}) \;=\; \frac{1}{\mathrm{vol}(\mathcal{G})}\left(\frac{1}{2\pi g_{\mathrm{YM}}}\right)^{\dim \mathcal{G}/2}\int \mathcal{D}A\; \mathrm{e}^{-S_{\mathrm{YM}}[A]}\;, \tag{2.2}$$

where the group of gauge transformations is denoted by $\mathcal{G} = \Omega^0(\Sigma_h, G)$. We also mention an equivalent formulation of the partition function as it will be relevant for us later in the deformation theory of two-dimensional Yang-Mills theory. The partition function can be rewritten as

$$\mathcal{Z}_{\mathrm{YM}}(h, g_{\mathrm{YM}}) \;=\; \frac{1}{\mathrm{vol}(\mathcal{G})}\int \mathcal{D}A \int \mathcal{D}\phi\; \mathrm{e}^{-S_{\mathrm{BF}}[\phi, A]}\;, \tag{2.3}$$

where

$$S_{\mathrm{BF}}[\phi, A] \;=\; \int_{\Sigma_h} \mathrm{Tr}\left(\mathrm{i}\,\phi F_A \;-\; \frac{g_{\mathrm{YM}}}{2}\,\phi^2 \mathrm{d}\mu\right) \tag{2.4}$$

is the first order formulation of the action $S_{\mathrm{YM}}[A]$. The integration of the field $\phi \in \Omega^0(\Sigma_h, \mathfrak{g})$ leads to the equality between (2.3) and (2.2), where $\mathfrak{g}$ is the Lie-algebra of G.

The partition function $\mathcal{Z}_{\mathrm{YM}}(h, g_{\mathrm{YM}})$ has been evaluated using Migdal's combinatorial heat kernel expansion [36–39] and it can be rewritten as a sum over all irreducible representations of the gauge group G

$$\mathcal{Z}_{\mathrm{YM}}(h, g_{\mathrm{YM}}) \;=\; \sum_{\lambda\in\Lambda_+} \left(\dim(R_\lambda)\right)^{2-2h}\; \mathrm{e}^{-\frac{a}{N}C_2(R_\lambda)}\;, \tag{2.5}$$

where the dominant weights $\lambda \in \Lambda_+$ are used to label the irreducible representation R_λ, and the functions $\dim(R_\lambda)$ and $C_2(R_\lambda)$ are natural quantities in representation theory: the dimension and the quadrative Casimir of R_λ respectively. We have already dropped a normalization factor as well as we will do it in the following, and used the substitution $a = \mathrm{vol}(\Sigma_h) N g_{\mathrm{YM}}/4$, which is the only geometric parameter of the

theory besides the genus as we expected from the area preserving diffeomorphism symmetry.

For $G = U(N)$ the irreducible representation R_λ labeled by the dominant weight λ is parametrized by the N-component partition $\lambda = (\lambda_1, \ldots, \lambda_N)$; $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$, or equivalently a Young diagram $Y_\lambda \subset (\mathbb{Z}_{>0})^N$ with at most N rows. The number of rows are called the length of the partition and it is denoted by $\ell(\lambda)$, and the notation $|\lambda| = \sum_i \lambda_i$ is used for the number of boxes in λ. The dimension and the quadrative Casimir are given by the formulas

$$\dim R_\lambda \;=\; \prod_{1\leq i<j\leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i} \tag{2.6}$$

and

$$C_2(R_\lambda) \;=\; (\lambda, \lambda + 2\rho) \;=\; \sum_{i=1}^{N} \big(\lambda_i^2 + (N + 1 - 2i)\lambda_i\big) \;, \tag{2.7}$$

where the symmetric bilinear form on the weight lattice $\Lambda \cong \mathbb{Z}^N$ of $\mathfrak{g}$ is $(\lambda, \mu) = \sum_i \lambda_i \mu_i$, which is induced by the Killing form. The Weyl vector is defined as the half sum of positive roots $\mathcal{R}_+$

$$\rho \;=\; \frac{1}{2} \sum_{\alpha\in\mathcal{R}_+} \alpha \qquad \text{with} \qquad \rho_i \;=\; \frac{N - 2i + 1}{2} \;. \tag{2.8}$$

The calculation of the heat kernel expansion (2.5) is based on the triangulation of the two-dimensional Riemann surface, where the surface is cut into triangles, which are glued together along their boundaries to get the closed surface. In the partition function, this means that a holonomy is associated to each boundary of each triangle, and the triangles are glued together through the integration of their holonomies. This derivation is exact and independent of the triangulation, and also allows the definition of the partition function on open surfaces. Physically the holonomy terms correspond to insertion of defect operators along the boundaries.

Thus a partition function associated to a general surface with genus h and b boundaries is given by

$$\mathcal{Z}_{\rm YM}(h, g_{\rm YM}; U_1, \ldots, U_b) \;=\; \sum_{\lambda\in\Lambda_+} \big(\dim(R_\lambda)\big)^{2-2h-b}\; \mathrm{e}^{-\frac{a}{N} C_2(R_\lambda)} \prod_{i=1}^{b} \chi_{R_\lambda}(U_i) \;, \tag{2.9}$$

where the holonomies U_i along the boundaries are specified by their characters $\chi_{R_\lambda}(U_i)$.

Wilson loops are also natural defect observables in Yang-Mills theory, which correspond to simple closed curves on the surface Σ_h. For simplicity we only work with one single non self-intersecting Wilson loop. In this case, the closed curve divides the Riemann surface into two faces of genera h_1 and h_2 with $h = h_1 + h_2$. The expectation value of the Wilson loop operator in the representation R_λ is given by

$$W_\lambda(h_1, h_2, g_{\rm YM}) = \sum_{\mu_1,\mu_2\in\Lambda_+} \int [{\rm d}U] \left(\dim(R_{\mu_1})\right)^{1-2h_1} \left(\dim(R_{\mu_2})\right)^{1-2h_2} \times \ {\rm e}^{-\frac{a_1}{N}C_2(R_{\mu_1})}\, {\rm e}^{-\frac{a_2}{N}C_2(R_{\mu_2})}\, \chi_{R_{\mu_1}}(U)\, \chi_{R_\lambda}(U)\, \chi_{R_{\mu_2}}(U^\dagger) \ , \tag{2.10}$$

which is coming from the gluing of the two faces with area parameters a_1 and a_2 and the insertion of the Wilson loop operator.

1.2 Gross-Taylor string expansion

Large N dual of two-dimensional Yang-Mills theory with gauge group $G = SU(N)$ has a nice interpretation as a two-dimensional string theory [40]. We review the large N expansion and the related Hurwitz theory of branched covers.

The $SU(N)$ representation theory can be translated into the language of N free fermions. The idea behind the description of large N limit is that the filled Fermi sea, corresponding to the trivial representation, has two Fermi levels, which are 'far' from each other, if the excitations are small around the Fermi levels. This means that the space of all representations factorize into a tensor product of two representation, a chiral and an antichiral part:

$$R_{SU(N)} \longrightarrow R_{\rm chiral} \otimes R_{\rm antichiral} \ . \tag{2.11}$$

The chiral representation (and analogously the antichiral as well) is defined as

$$R_{\rm chiral} = \bigoplus_{n=1}^{\infty} \bigoplus_{\lambda\in\Lambda_+^n} R_\lambda \ , \tag{2.12}$$

where the set of Young diagrams with fixed n boxes is denoted by Λ_+^n. Here the irreducible representations are summed in two steps: in the first sum the constraint $\ell(\lambda) \leq N$ on Young diagrams has been dropped, then all Young diagrams with fixed boxes are summed in the second step.

If we consider excitations around both Fermi level, the most general representations that contribute in the large N expansion of $\mathcal{Z}_{\rm YM}(h, g_{\rm YM})$ (2.2) according to Gross and

Taylor are labeled by the so called 'composite' or 'coupled' Young diagrams. They are built from two diagrams λ and μ, each of them has boxes much less then N. The composite diagram is denoted by $\eta = \overline{\mu}\lambda$ and defined as

$$\eta_i \;=\; \begin{cases} \mu_1 + \lambda_i & i \leq \ell(\lambda) \;, \\ \mu_1 & \ell(\lambda) < i \leq N - \ell(\mu) \;, \\ \mu_1 - \mu_{N+1-i} & N - \ell(\mu) < i \leq N \;. \end{cases} \tag{2.13}$$

Although the two diagrams have boxes much less then N, which allows them to be described as chiral representations, the composite diagram has boxes order of N. The quadratic Casimir

$$C_2(R_{\overline{\mu}\lambda}) \;=\; C_2(R_\lambda) \;+\; C_2(R_\mu) \;+\; \frac{2|\lambda||\mu|}{N} \tag{2.14}$$

and the dimension

$$\dim(R_{\overline{\mu}\lambda}) \;=\; \dim(R_\lambda)\dim(R_\mu)\left(1 + \mathcal{O}\left(\frac{1}{N^2}\right)\right) \tag{2.15}$$

factorize in the largest order of N. As a consequence, the partition function $\mathcal{Z}_{\mathrm{YM}}(h, g_{\mathrm{YM}})$ also factorizes into a chiral and an antichiral part. Thus the chiral expansion

$$Z_h(a) \;=\; \sum_{n=0}^{\infty} \sum_{\lambda \in \Lambda_+^n} (\dim(R_\lambda))^{2-2h} \;\mathrm{e}^{-\frac{a}{N} C_2(R_\lambda)} \;, \tag{2.16}$$

is the building block of the full expansion and also the subject of our study.

The difference between $U(N)$ and $SU(N)$ representations is the appearance of the extra $U(1)$ charge in $U(N)$, which couples the chiral and the antichiral representations, but does not affect the chiral expansion significantly.

In order to study the relation of chiral expansion to string theory, the useful tool is the well known Schur-Weyl duality

$$R_{\omega_1}^{\otimes n} \;\cong\; \bigoplus_{\lambda \in \Lambda_+^n} R_\lambda \otimes r_\lambda \;, \tag{2.17}$$

which relates the n times tensor product of fundamental representation R_{ω_1} of gauge groups $U(N)$ or $SU(N)$ to the sum of the irreducible representations of the gauge groups, denoted by R_λ, and the symmetric group, denoted by r_λ.

Relations to the topology of Hurwitz space become more transparent in the limit $a \to 0$ where we have a topological theory in spacetime. Since in the thesis we study this limit, we only review this case (see e.g. [34] for the full expansion).

The chiral partition function with zero area can be expanded with delta functions on the symmetric group algebra $\mathbb{C}\mathfrak{S}_n$

$$Z_h(0) \;=\; \sum_{n=0}^{\infty} N^{n(2-2h)}\,\frac{1}{n!}\sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n}\delta\left(\Omega_n^{2-2h}\prod_{i=1}^{h}\sigma_i\tau_i\sigma_i^{-1}\tau_i^{-1}\right)\;, \tag{2.18}$$

where the delta functions are given by the character formula

$$\delta(\sigma) \;=\; \sum_{\lambda\in\Lambda_+^n}\dim(r_\lambda)\,\chi_{r_\lambda}(\sigma)\;, \tag{2.19}$$

Ω_n is a central element in the algebra given by

$$\Omega_n \;=\; \sum_{\sigma\in\mathfrak{S}_n}\left(\frac{1}{N}\right)^{n-K_\sigma}\sigma \tag{2.20}$$

and K_σ is the number of cycles in the permutation σ. The conjugacy class of σ corresponds to a Young diagram λ, which have cycles of length λ_i. In this notation $K_\sigma = \ell(\lambda)$.

The inverse of Ω_n

$$\Omega_n^{-1} \;=\; 1 + \sum_{L=1}^{\infty}\sum_{\substack{\zeta_1,\ldots,\zeta_L\in\mathfrak{S}_n\\ \zeta_1,\ldots,\zeta_L\neq 1}}(-1)^L\,N^{\sum_{s=1}^{L}(K_{\zeta_s}-n)}\prod_{s=1}^{L}\zeta_s \tag{2.21}$$

is useful to rewrite the chiral expansion as

$$\begin{aligned} Z_h(0) \;=\; \sum_{n=0}^{\infty}\sum_{B=0}^{\infty} N^{n(2-2h)-B}\,\frac{1}{n!}\sum_{L=0}^{B}\sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n}\sum_{\substack{\zeta_1,\ldots,\zeta_L\in\mathfrak{S}_n\\ \zeta_1,\ldots,\zeta_L\neq 1}}\chi(\Sigma_{h,L}) \\ \times\;\delta\left(\prod_{s=1}^{L}\zeta_s\prod_{i=1}^{h}\sigma_i\tau_i\sigma_i^{-1}\tau_i^{-1}\right)\;, \end{aligned} \tag{2.22}$$

where we have introduced the number $B = \sum_{s=1}^{L}(n-K_{\zeta_s})$ and

$$\chi(\Sigma_{h,L}) \;=\; \frac{\Gamma(3-2h)}{\Gamma(L+1)\,\Gamma(3-2h-L)}\;, \tag{2.23}$$

which is the Euler characteristic of the configuration space of L points on the Riemann surface Σ_h, i.e. the L-tuples of distinct points on Σ_h modulo the natural action of the permutation group $\mathfrak{S}_L$.

Hurwitz theory is reviewed in Appendix 1 very briefly. The sum of the delta functions reduces to sum of Hurwitz numbers defined in (A.10), the complete sum is given by

(A.9), which is the number of distinct n-sheeted branch covers of Σ_h with branching number B and L branch points. Even more can be said about the chiral expansion, the Hurwitz numbers together with the Euler character $\chi(\Sigma_{h,L})$ give the orbifold Euler character of the Hurwitz space $\mathcal{H}_{n,B,h,L}$

$$\chi_{\rm orb}\big(\mathcal{H}_{n,B,h,L}\big) \,=\, \chi(\Sigma_{h,L}) \sum_{f\in\mathcal{H}_{n,B,h,L}} \frac{1}{|{\rm Aut}(f)|} \ . \tag{2.24}$$

So the chiral partition function can be reformulated using a two-dimensional 'wordsheet theory', where the wordsheet Σ_g is the covering surface and the target Σ_h is the covered surface.

The chiral expansion assembles nicely

$$Z_h(0) \,=\, \sum_{n=0}^{\infty}\sum_{B=0}^{\infty} N^{2-2g} \sum_{L=0}^{B} \chi_{\rm orb}\big(\mathcal{H}_{n,B,h,L}\big) \ , \tag{2.25}$$

where g is determined by the Riemann-Hurwitz formula (A.4). We can define the space of holomorphic maps from the wordsheet Σ_g to the target Σ_h as the quotient space

$$\mathcal{H}(\Sigma_g \to \Sigma_h) \,=\, {}^{\mathcal{H}_{\rm conf}}\!/_{{\rm Diff}^+(\Sigma_{\rm g}) \ltimes {\rm Weyl}(\Sigma_{\rm g})} \ , \tag{2.26}$$

where $\mathcal{H}_{\rm conf}$ denotes the configuration space of the pair of metrics and holomorphic maps, ${\rm Diff}^+(\Sigma_{\rm g})$ is the orientation preserving diffeomorphisms and ${\rm Weyl}(\Sigma_{\rm g})$ is the Weyl transformations of the worldsheet Σ_g. Finally the chiral free energy has a compact form

$$F_h(0) \,=\, \sum_{g=0}^{\infty} \left(\frac{1}{N}\right)^{2g-g} \chi_{\rm orb}\big(\mathcal{H}(\Sigma_g \to \Sigma_h)\big) \ , \tag{2.27}$$

in terms of orbifold Euler character $\chi_{\rm orb}\big(\mathcal{H}(\Sigma_g \to \Sigma_h)\big)$. This formula gives a natural interpretation of the large N dual theory as a perturbative two-dimensional string theory.

2 Topological string theory, BPS black holes and q-deformation of two-dimensional Yang-Mills theory

Counting of four-dimensional BPS black hole microstates arises upon comapctification in type IIA string theory on a special class of Calabi-Yau manifolds and it is given

by a two-dimensional $U(N)$ gauge theory, where the finite gauge group parameter N corresponds to the number of D4-branes in the Calabi-Yau. The resulting gauge theory is a deformation of the original Yang-Mills theory only in two dimension, and called q-deformed Yang-Mills theory. It has further interest in string theory as its large N dual string theory is the topological string theory on the Calabi-Yau. In this section we review some pertinent background about topological strings, BPS black holes, q-deformed Yang-Mills and their relations mentioned above.

2.1 Topological string theory

The physical string theory can be described by a two-dimensional conformal field theory coupled to gravity. In the case when the conformal field theory has topological symmetry as well, the corresponding string theory is a topological string theory, which can be solved exactly and includes topological information of the geometry (see [4, 41–43] for reviews).

The topological sigma-model can be constructed from an $\mathcal{N} = 2$ supersymmetric sigma-model by the procedure called 'topological twisting' [44]. The target space of the supersymmetric sigma-model is a Calabi-Yau manifold. The topological twisting is based on the redefinition of the $U(1)$ spin current of the two-dimensional theory in order to get a scalar topological BRST charge, thus the resulting cohomological field theory has a topological invariance. The twisting procedure can be applied in two non-equivalent way leading to two non-equivalent topological sigma-models, namely the A- and B-models. The topological charge $\mathcal{Q}$ corresponding to the respective twist is given by the supersymmetric generators of the $\mathcal{N} = 2$ sigma-model as follows

$$\begin{aligned} &\text{A-twist:} \qquad \mathcal{Q} = Q_{++} + Q_{--} \ , \\ &\text{B-twist:} \qquad \mathcal{Q} = Q_{+-} + Q_{--} \ , \end{aligned} \tag{2.28}$$

where the first indices of the generators are those that belong to the two-dimensional surface, and the seconds are the R-symmetry indices.

2.1.1 Topological A-model

It is defined by maps $X^i = (X^a, X^{\bar{a}})$ from the worldsheet Σ_2 to the six-dimensional Calabi-Yau manifold M, where $a = 1, 2, 3$ are complex target space indices and we use local complex coordinates $\sigma = (z, \bar{z})$ on the Riemann surface Σ_2. We further introduce fermionic ghost fields $(\chi^a_{\bar{z}}, \chi^{\bar{a}}_z, \psi^a, \psi^{\bar{a}})$ with ghost number $(-1, -1, 1, 1)$, where ψ^i are

worldsheet scalars and χ^i are worldsheet one-forms. The action of the topological A-model is then

$$S_{\rm A} \,=\, 2t \int_{\Sigma_2} {\rm d}^2 z \, \Big(g_{a\bar{b}} \, \partial_{\bar{z}} X^a \, \partial_z X^{\bar{b}} \,+\, {\rm i} \, g_{a\bar{b}} \, \big(\chi^a_{\bar{z}} \, \nabla_z \psi^{\bar{b}} + \chi^{\bar{b}}_z \, \nabla_{\bar{z}} \psi^a \big) \,-\, R_{a\bar{b}c\bar{d}} \, \chi^a_{\bar{z}} \, \chi^{\bar{b}}_z \, \psi^c \, \psi^{\bar{d}} \Big) \\ + t \int_{\Sigma_2} {\rm d}^2 z \; X^*(k) \;, \tag{2.29}$$

where $g_{a\bar{b}}$ is the Kähler metric which obeys the Kähler identity $\partial_a g_{b\bar{c}} = \partial_b g_{a\bar{c}}$ and its complex conjugate $\partial_{\bar{a}} g_{b\bar{c}} = \partial_{\bar{c}} g_{b\bar{a}}$. The Levi-Civita connection is defined by $\nabla_z \psi^{\bar{a}} = \partial_z \psi^{\bar{a}} + \Gamma^{\bar{a}}{}_{\bar{b}\bar{c}} \, \psi^{\bar{b}} \, \partial_z X^{\bar{c}}$, and the complex Christoffel symbol is $\Gamma^a{}_{bc} = g^{a\bar{d}} \, \Gamma_{\bar{d}bc}$, where $\Gamma_{a\bar{b}\bar{c}} = \partial_{\bar{b}} g_{a\bar{c}}$. The Riemann tensor is $R_{a\bar{b}c\bar{d}} = -g_{a\bar{e}} \, \partial_c \Gamma^{\bar{e}}{}_{\bar{b}\bar{d}}$. The Kähler form $k = {\rm i} \, g_{a\bar{b}} {\rm d}X^a \wedge {\rm d}X^{\bar{b}}$, which can be complexified together with the B-field as usual, gives a topological term equal to $2\pi n$, $n \in \mathbb{Z}$, which become crucial in the quantum level.

The action is invariant under the BRST transformations

$$\delta X^a \,=\, {\rm i} \, \psi^a \;, \qquad \delta X^{\bar{a}} \,=\, {\rm i} \, \psi^{\bar{a}} \;, \qquad \delta \psi^a \,=\, 0 \qquad \text{and} \qquad \delta \psi^{\bar{a}} \,=\, 0 \;, \tag{2.30}$$

together with

$$\delta \chi^a_{\bar{z}} \,=\, -\, \partial_{\bar{z}} X^a \,-\, {\rm i} \, \Gamma^a{}_{bc} \, \psi^b \, \chi^c_{\bar{z}} \qquad \text{and} \qquad \delta \chi^{\bar{a}}_z \,=\, -\, \partial_z X^{\bar{a}} \,-\, {\rm i} \, \Gamma^{\bar{a}}{}_{\bar{b}\bar{k}} \, \psi^{\bar{b}} \, \chi^{\bar{c}}_z \;, \tag{2.31}$$

and it is BRST exact on-shell up to the topological term

$$S_{\rm A} \,=\, -t \int_{\Sigma_2} {\rm d}^2 z \; \delta \Psi_{\rm A} \,+\, t \int_{\Sigma_2} {\rm d}^2 z \; X^*(k) \;, \tag{2.32}$$

where the fermion $\Psi_{\rm A}$ is defined by

$$\Psi_{\rm A} \,=\, \int_{\Sigma_2} {\rm d}^2 z \; g_{a\bar{b}} \Big(\chi^a_{\bar{z}} \partial_z X^{\bar{b}} \,+\, \chi^{\bar{b}}_z \partial_{\bar{z}} X^a \Big) \;. \tag{2.33}$$

Such a theory has a topological invariance and called cohomological field theory. Moreover the path integral localizes over the fixed point locus of the BRST transformation, which is the space of holomorphic maps $X : \Sigma_2 \to M$. They are classified topologically by the homology class $H_2(M, \mathbb{Z})$ and called worldsheet instantons which have degree n.

The cohomology of observables are equivalent to the de Rham cohomology $H^\bullet(M)$, and a general degree p observable can be written in the form

$$\alpha_{i_1, \ldots i_p} \, \psi^{i_1} \, \ldots \, \psi^{i_p} \tag{2.34}$$

where $\alpha = \alpha_{i_1,\ldots i_p} \mathrm{d}X^{i_1} \wedge \ldots \wedge \mathrm{d}X^{i_p}$ is a closed p-form in $H^p(M)$.

It is worth to mention that the A-model action only changes in a BRST exact term if the complex structure is changed, so the observable quantities of the A-model do not depend on the complex structure, just on the Kähler structure. The opposite is true for the B-model.

2.1.2 Topological B-model

Its field contents are similar to that of the A-model, the bosonic fields X^a, $X^{\bar{a}}$ are the same maps as that of the A-model. The fermionic fields are the worldsheet scalar fields $(\eta^{\bar{a}}, \chi_a)$ with ghost number 1 and the one-form fields ρ^a with ghost number -1. The action is given by

$$\begin{aligned} S_{\mathrm{B}} \,=\, t \int_{\Sigma_2} \mathrm{d}^2 z \, \Big(& g_{a\bar{b}} \left(\partial_z X^a \, \partial_{\bar{z}} X^{\bar{b}} + \partial_{\bar{z}} X^a \, \partial_z X^{\bar{b}} \right) \,-\, g_{a\bar{b}} \left(\rho^a_z \, \nabla_{\bar{z}} \eta^{\bar{b}} + \rho^a_{\bar{z}} \, \nabla_z \eta^{\bar{b}} \right) \\ & + \rho^a_z \, \nabla_{\bar{z}} \chi_a \,-\, \rho^a_{\bar{z}} \, \nabla_z \chi_a \,-\, R^a{}_{b\bar{c}d} \, \rho^b_z \, \rho^d_{\bar{z}} \, \eta^{\bar{c}} \, \chi_a \Big) \,, \end{aligned} \tag{2.35}$$

where the definitions of the metric, Christoffel symbol, Riemann tensor and Levi-Civita connection are the same as that in the case of the A-model.

The BRST transformations are

$$\delta X^a \,=\, 0 \,, \qquad \delta X^{\bar{a}} \,=\, \eta^{\bar{a}} \,, \qquad \delta \rho^a \,=\, \mathrm{d}X^a \,, \qquad \delta \eta^{\bar{a}} \,=\, 0 \,, \qquad \text{and} \qquad \delta \chi_a \,=\, 0 \,. \tag{2.36}$$

The B-model action can also be rewritten as a sum of a BRST exact term and a topological term

$$S_{\mathrm{B}} \,=\, t \int_{\Sigma_2} \mathrm{d}^2 z \; \delta \Psi_{\mathrm{B}} \,+\, t \int_{\Sigma_2} \rho^a \, \mathrm{d}\chi_a \,, \tag{2.37}$$

where the fermion is defined by

$$\Psi_{\mathrm{B}} \,=\, \int_{\Sigma_2} \mathrm{d}^2 z \, \Big(g_{a\bar{b}} \left(\rho^a_z \, \partial_{\bar{z}} X^{\bar{b}} + \rho^a_{\bar{z}} \, \partial_z X^{\bar{b}} \right) \,-\, \Gamma^a{}_{bc} \, \rho^b_z \, \rho^c_{\bar{z}} \, \chi_a \Big) \,. \tag{2.38}$$

The B-model is a cohomological field theory as well, and the path integral localizes on constant X maps, therefore it reduces to an ordinary integral over M.

A general observable can be expressed as

$$\beta^{a_1 \ldots a_q}_{\bar{b}_1 \ldots \bar{b}_p} \, \eta^{\bar{b}_1} \ldots \eta^{\bar{b}_p} \, \chi_{a_1} \cdots \chi_{a_q} \,, \tag{2.39}$$

where $\beta = \beta^{a_1 \ldots a_q}_{\bar{b}_1 \ldots \bar{b}_p} \mathrm{d}X^{\bar{b}_1} \ldots \mathrm{d}X^{\bar{b}_p} \partial_{a_1} \wedge \ldots \partial_{a_q}$ is an element of the Dolbeault cohomology of $\wedge^q T^{(1,0)} M$-valued $(0,p)$-forms, since the BRST transformation acts on X^i and $\eta^{\bar{a}}$

as the Dolbeault differential $\overline{\partial}$. So the cohomology of observables are equivalent to the twisted Dolbeault cohomology $H^p_{\overline{\partial}}(M, \wedge^q TM)$.

Quantities in the B-model do not depend on the Kähler form, but depend on the complex structure. The action above (2.35) includes an explicit choice of the complex structure, as the holomorphic and antiholomorphic coordinates are distinguished.

2.1.3 Coupling to gravity

The coupling to gravity in the topological theory is analog to that of the bosonic string theory. The stress tensor $T(z)$ in bosonic string theory is BRST exact $T(z) = \delta b(z)$, and the free energy is calculated as the expectation value in CFT of the ghost field $b(z)$ coupled to the moduli of the metric. The stress tensor in topological string theory is also exact with respect to the topological charge: $T_{\alpha,\beta} = \delta b_{\alpha\beta}$ and $b_{\alpha\beta}$ plays the role of the ghost field. Following this analogy, the definition of the free energy of the topological string for genus $g \geq 1$ is given by

$$F_g \;=\; \int_{\overline{M}_g} \left\langle \prod_{k=1}^{6g-6} (b, \mu_k) \right\rangle \;, \tag{2.40}$$

where

$$(b, \mu_k) \;=\; \int_{\Sigma_g} \mathrm{d}^2 z \;\left(b_{zz}\, (\mu_k)_{\overline{z}}{}^{z} \;+\; b_{\overline{z}\overline{z}}\, (\overline{\mu}_k)_{z}{}^{\overline{z}} \right) \;, \tag{2.41}$$

and μ_k are refer to the infinitezimal change in the metric and they are called Beltrami differentials. The vacuum expectation value is calculated in the topological sigma-model, and then the moduli space $\overline{M}_g$ of Riemann surfaces of genus g is integrated over, which has dimension $6-6g$. The zero genus free energy is called the prepotential and it is defined purely in the topological sigma-model. They give the supergravity prepotential in type II string theories. The topological A-model strings only depend on the Kähler structure and calculate Gromow-Witten invariants of the Calabi-Yau, while topological B-model strings depend on the complex structure and has a relation to matrix models and Kodaira-Spencer theory of gravity.

2.1.4 Topological S-duality

Topological S-duality arises from the physical S-duality of type IIB superstring theory [45], and it relates the A- and B-model topological strings on the same Calabi-Yau

manifold: D-instanton contributions of one model correspond to perturbative amplitudes of the other model. The string couplings $g_{\rm A}$ and $g_{\rm B}$ of the A- and B-models are related to each other by

$$g_{\rm A} \;=\; \frac{1}{g_{\rm B}} \tag{2.42}$$

and the Kähler forms $k_{\rm A}$ and $k_{\rm B}$ of the two theories are also related by

$$k_{\rm A} \;=\; \frac{k_{\rm B}}{g_{\rm B}} \;. \tag{2.43}$$

In other words, S-duality exchanges the A- and B-models as a weak/strong coupling duality.

2.2 q-deformed Yang-Mills theory and BPS black holes

A-model topological string theory on the special Calabi-Yau

$$\mathcal{O}(p+2h-2) \oplus \mathcal{O}(-p) \to \Sigma_h \;, \tag{2.44}$$

which is a Riemann surface Σ_h fibrated by two complex line bundles with degrees $p+2h-2$ and $-p$, reduces to a q-deformed $SU(\infty)$ Yang-Mills theory on Σ_h. The geometric parameter p classifies the topologically different Calabi-Yau manifolds. In the following we introduce the q-deformation and its relation to BPS black holes.

2.2.1 q-deformed Yang-Mills theory

The topological string amplitudes as well as the partition functions of q-deformed Yang-Mills can be sewed together from building blocks in a similar way as in the original Yang-Mills theory. In topological string theory Lagrangian D3-branes are associated to each boundaries, which wrap the punctures and two dimensions in the fibers. The related Chan-Paton degrees of freedom appear as boundary holonomy terms, then gluing two surfaces through their boundary means an integration over their holonomy. The resulting two-dimensional Yang-Mills theory has gauge group $SU(\infty)$, but it can be defined for finite $SU(N)$ or $U(N)$ as well. The deformation parameter is related to the string coupling $g_s = g_{\rm A}$ by

$$q \;=\; {\rm e}^{-g_s} \;, \tag{2.45}$$

and the original $\mathfrak{sl}_N$ or $\mathfrak{gl}_N$ characters[1] in the partition functions are replaced by quantum group characters of the universal enveloping algebras $\mathcal{U}_q(\mathfrak{sl}_N)$ or $\mathcal{U}_q(\mathfrak{gl}_N)$ (see Appendix 2). In the following we summarize the definition of q-deformed Yang-Mills theory.

The partition function for a closed surface of genus h is given by

$$Z_h(q;p) \;=\; \sum_{\lambda\in\Lambda_+} \left(\dim_q R_\lambda\right)^{2-2h}\; q^{pC_2(R_\lambda)/2} \tag{2.46}$$

up to a normalization factor. The quantum dimension is defined by

$$\dim_q R_\lambda \;=\; \prod_{1\le i<j\le N} \frac{[\lambda_i-\lambda_j+j-i]_q}{[j-i]_q}\;, \tag{2.47}$$

where the symbol

$$[x]_q = \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}} \tag{2.48}$$

is called the q-number. The q-deformation only affects the dimension significantly, and the p has the role of an area parameter. An important property of the quantum dimension is that it is given by character of the quantum group element $q^{(\rho,H)}$, where H_i are the generators of the Cartan subalgebra.

The partition function for a Riemann surface of genus h with b boundaries has the form

$$Z_h(q;p;U_1,\dots,U_b) \;=\; \sum_{\lambda\in\Lambda_+} \left(\dim_q R_\lambda\right)^{2-2h-b}\; q^{\frac{p}{2}C_2(R_\lambda)}\,\chi_{R_\lambda}(U_1)\dots\chi_{R_\lambda}(U_b)\;, \tag{2.49}$$

where the holonomies are specified by characters of the quantum universal enveloping algebras $\mathcal{U}_q(\mathfrak{gl}_N)$ or $\mathcal{U}_q(\mathfrak{sl}_N)$.

The Wilson loop observable for a single non self-intersecting loop is also possible to define, and its expectation value is given by

$$\begin{aligned} W_\lambda(q;p;h_1,h_2) \;=\; \sum_{\mu_1,\mu_2\in\Lambda_+} \int [\mathrm{d}U]_q \left(\dim_q(R_{\mu_1})\right)^{1-2h_1} \left(\dim_q(R_{\mu_2})\right)^{1-2h_2} \\ \times\; q^{\frac{p}{2}C_2(R_{\mu_1})+\frac{p}{2}C_2(R_{\mu_2})}\chi_{R_{\mu_1}}(U)\,\chi_{R_\lambda}(U)\,\chi_{R_{\mu_2}}(U^\dagger)\;, \end{aligned} \tag{2.50}$$

where the surface with genus $h=h_1+h_2$ is divided into the two faces by the loop having genus h_1 and h_2.

[1] We work with the Lie-algebras $\mathfrak{sl}_N$ and $\mathfrak{gl}_N$ instead of $su(N)$ and $U(N)$ since their representation theory is equivalent.

2.2.2 Relation to BPS black holes and topological strings

A microscopic description of extremal four-dimensional black holes can be obtained by counting D-branes in superstring theories. In this study we focus on BPS black holes. Embedding into supergravity turns the extremal bound of black holes into BPS bound of supermultiplets, which keeps them to not emit any Hawking radiation. So they can carry both electric and magnetic charges, they are static and they do not emit any Hawking radiation (see [46] for a short review).

We consider BPS black hole that arises in type IIA string compactifications on the Calabi-Yau (2.44) and its partition function is given by counting D-branes wrapping holomorphic cycles in the compactified space. The D0- and D2-branes have electric charges which are summed over with given chemical potentials, while the D4- and D6-branes have magnetic charges, which are left fixed. The partition function is the sum

$$Z_{\rm BH}(P_6, P_4, \phi_2, \phi_0) \;=\; \sum_{Q_2, Q_0} \Omega(P_6, P_4, Q_2, Q_0)\; {\rm e}^{\,-\phi_2 Q_2 - \phi_0 Q_0}\;, \tag{2.51}$$

where P_6, P_4, Q_2 and Q_0 denotes the D6, D4, D2 and D0 charges respectively while ϕ_2 and ϕ_0 are the chemical potentials associated to the eletrically charged D-branes. The coefficient

$$\Omega(P_6, P_4, Q_2, Q_0) \;=\; {\rm Tr}\,_{\mathcal{H}_{P,Q}} (-1)^F \tag{2.52}$$

is the Witten index which calculates the fermionic number operator F in the fixed charged sector.

The reduction to two-dimensional $U(N)$ Yang-Mills theory is the following [47]. D2-branes wrap Σ_h, while N D4-branes wrap the total space of $\mathcal{O}(-p) \to \Sigma_h$, then the black hole partition function localizes on the surface Σ_h, and can be computed by the path integral

$$\begin{aligned} Z_{\rm BH} \;=\; \int \mathcal{D}_q A\, \mathcal{D}_q \Phi \;\exp\Bigg(& \frac{\phi_0}{4\pi^2} \int_{\Sigma_h} {\rm Tr}\;\Phi\, F_A \;+\; \frac{\phi_2}{2\pi} \int_{\Sigma_h} {\rm Tr}\;\Phi\, \omega_{\Sigma_h} \\ & - \, p\, \frac{\phi_0}{2\pi} \int_{\Sigma_h} {\rm Tr}\;\Phi^2\, \omega_{\Sigma_h} \Bigg)\;, \end{aligned} \tag{2.53}$$

where the field Φ is the holonomy of the four-dimensional gauge field A around the circle at infinity of the fiber of $\mathcal{O}(-p) \to \Sigma_h$ and ω_{Σ_h} is the volume form of Σ_h normalized to unit volume. The chemical potentials are given by

$$\phi_0 \;=\; \frac{4\pi^2}{g_s} \qquad \text{and} \qquad \phi_2 \;=\; \frac{2\pi\theta}{g_s}\;, \tag{2.54}$$

where the theta-angle θ is choosen to be zero in our case. The q-deformation appears in the measure and together with the action can be calculated using localization techniques which leads to the partition function $Z_{q\mathrm{YM}}$ of the q-deformed two-dimensional Yang-Mills in (2.46).

The large N string dual of the q-deformed Yang-Mills with gauge group $SU(N)$ is the topological A-model string theory on the Calabi-Yau (2.44) with fixed Kähler modulus

$$k = \frac{1}{2}(p + 2h - 2)Ng_s \ . \tag{2.55}$$

Its amplitudes are computed by TQFT using $SU(\infty)$ representations and they are given by the chiral expansion of the q-deformed two-dimensional $SU(N)$ Yang-Mills theory. The black hole partition function Z_{BH}, as an $U(N)$ Yang-Mills theory, in the $1/N$ expansion factorizes to a chiral and an antichiral amplitude in $SU(N)$ Yang-Mills theory, and thus in topological string theory

$$Z_{\mathrm{BH}} = \sum_{l\in\mathbb{Z}} \sum_{\lambda_1,\ldots\lambda_{|2h-2|}\in\Lambda_+} Z^{\mathrm{top}}_{R_{\lambda_1}\ldots R_{\lambda_{|2h-2|}}}(k + pg_s l)\, Z^{\mathrm{top}}_{R_{\lambda_1}\ldots R_{\lambda_{|2h-2|}}}(\overline{k} - pg_s l) \ , \tag{2.56}$$

where the chiral blocks $Z^{\text{top. str.}}_{R_{\lambda_1}\ldots R_{\lambda_{|2h-2|}}}$ are calculated by the topological string amplitudes

$$Z^{\mathrm{top}}(U_1,\ldots,U_{|2h-2|}) = \sum_{\lambda_1,\ldots\lambda_{|2h-2|}\in\Lambda_+} Z^{\mathrm{top}}_{R_{\lambda_1}\ldots R_{\lambda_{|2h-2|}}} \chi_{R_{\lambda_1}}(U_1)\ldots\chi_{R_{\lambda_{|2h-2|}}}(U_{|2h-2|}) \tag{2.57}$$

that arise from Langrangian D-branes wrapping 1-cycles in the fiber with holonomies $U_1,\ldots,U_{|2h-2|}$, which are specified by characters in $SU(\infty)$ representations $R_{\lambda_1},\ldots,$ $R_{\lambda_{|2h-2|}}$. So we can see that the large N expansion of the black hole partition function can be computed using topological string theory, and this duality originates from the large N expansion of the intermediate q-deformed Yang-Mills theory. This duality is a manifestation of the Ooguri-Strominger-Vafa (OSV) conjecture [48], which says that the microscopic entropy of the black hole can be computed in terms of topological string amplitudes.

2.2.3 Chiral expansion of q-deformed Yang-Mills theory

This part of the review is more detailed, as similar techniques introduced here is employed in our calculations later. The results and techniques were developed in [49]. Main difference between the chiral expansions of q-deformed and original two-dimensional

Yang-Mills theories is that the symmetric group algebra is replaced by its q-deformation, the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$ (see Appendix 3). In the original theory the expansion is based on the Schur-Weyl duality (2.17) of $\mathfrak{gl}_N$ characters, but now quantum Schur-Weyl duality

$$R_{\omega_1}^{\otimes n} \cong \bigoplus_{\lambda\in\Lambda_+^n} R_\lambda \otimes r_\lambda \tag{2.58}$$

arises because of the quantum group $\mathcal{U}_q(\mathfrak{gl}_N)$ characters, where R_{ω_1} and R_λ are $\mathcal{U}_q(\mathfrak{gl}_N)$ representations this time, while r_λ is the irreducible representation of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$ associated to λ. The actions of $\mathcal{U}_q(\mathfrak{gl}_N)$ and $\mathsf{H}_q(\mathfrak{S}_n)$ on $R_{\omega_1}^{\otimes n}$ are given respectively by the iterated coproduct $\Delta^{n-1} = (\Delta\otimes\mathbb{1}^{\otimes(n-1)})\circ\cdots\circ(\Delta\otimes\mathbb{1})\circ\Delta$. Thus a character in $R_{\omega_1}^{\otimes n}$ can be written in terms of irreducible representations by

$$\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}(U\,\mathsf{h}(\sigma)) \;=\; \sum_{\lambda\in\Lambda_+^n} \chi_{R_\lambda}(U)\,\chi_{r_\lambda}(\mathsf{h}(\sigma))\ , \tag{2.59}$$

where $\mathsf{h}(\sigma)\in\mathsf{H}_q(\mathfrak{S}_n)$ is the Hecke algebra element associated to $\sigma\in\mathfrak{S}_n$. Letting P_λ denote the quantum Young projector for the representation r_λ, one has

$$P_\lambda R_{\omega_1}^{\otimes n} \cong R_\lambda\otimes r_\lambda\ , \tag{2.60}$$

and P_λ is expressed with a sum of Hecke algebra elements

$$P_\lambda = \frac{d_\lambda(q)}{q^{\frac{n\,(n-1)}{4}}\,[n]_q!}\sum_{\sigma\in\mathfrak{S}_n} q^{-\ell(\sigma)}\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma^{-1})\big)\,\mathsf{h}(\sigma)\ . \tag{2.61}$$

where $d_\lambda(q)=\dim_q(r_\lambda)$ is a q-deformation of the dimension of the symmetric group representation r_λ (A.18). These equations are used to express the quantum dimension with Hecke characters, and one get

$$\begin{aligned}\dim_q(R_\lambda) \;&=\; \chi_{R_\lambda}\big(q^{(\rho,H)}\big)\\ &=\; \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\,\frac{d_\lambda(q)}{d_\lambda(1)}\sum_{\sigma\in\mathfrak{S}_n} q^{-\ell(\sigma)}\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma^{-1})\big)\ \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(q^{(\rho,H)}\,\mathsf{h}(\sigma)\big)\ .\end{aligned} \tag{2.62}$$

Calculating the trace of Hecke-elements and reformulating (2.62) in terms of central elements[2] C_T in $\mathsf{H}_q(\mathfrak{S}_n)$ labeled by conjugacy classes T in $\mathfrak{S}_n$, one arrives at the large N expansion of the quantum dimension

$$\dim_q(R_\lambda) \;=\; \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\,\frac{d_\lambda(q)}{d_\lambda(1)}\,[N]_q^n\,\chi_{r_\lambda}(\Omega_n)\ , \tag{2.63}$$

[2]We give a definition for central elements C_T later in (3.21).

where Ω_n is also central and defined by

$$\Omega_n \;=\; \sum_{T\in\mathfrak{S}_n^\vee} [N]^{K_T-n}\, q^{\frac{N-1}{2}\,\ell(T)}\, C_T \;. \tag{2.64}$$

One can rewrite the rest of the coefficients in the chiral expansion as characters of central elements with

$$\begin{aligned}\Big(\frac{[n]_q!}{q^{-\frac{n\,(n-1)}{4}}\, d_\lambda(q)}\Big)^2 \;=\; \frac{1}{d_\lambda(1)} \sum_{\sigma,\tau\in\mathfrak{S}_n} q^{-\ell(\sigma)-\ell(\tau)} \\ \times\;\; \chi_{r_\lambda}\big(\mathsf{h}(\sigma)\,\mathsf{h}(\tau)\,\mathsf{h}(\sigma^{-1})\,\mathsf{h}(\tau^{-1})\big)\end{aligned} \tag{2.65}$$

and

$$d_\lambda(q) \;=\; \chi_{r_\lambda}(D_n)\;, \tag{2.66}$$

where D_n is defined in the completed Hecke algebra $\widehat{\mathsf{H}}_q(\mathfrak{S}_n)$ by

$$D_n \;=\; \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!} \sum_{k=0}^{\infty} (-1)^k \sum_{\substack{\sigma_1,\ldots,\sigma_k\in\mathfrak{S}_n\\ \sigma_i\neq 1}} q^{-\sum_i \ell(\sigma_i)} \prod_{j=1}^{k} \mathsf{h}(\sigma_j^{-1})\,\mathsf{h}(\sigma_j)\;. \tag{2.67}$$

Central elements can be collected together under one single character using the property

$$\chi_{r_\lambda}(C)\,\chi_{r_\lambda}(\mathsf{h}(\sigma)) \;=\; d_\lambda(1)\,\chi_{r_\lambda}(C\,\mathsf{h}(\sigma)) \tag{2.68}$$

for a general central element C. A delta-function on Hecke algebras can be defined by

$$\delta\big(\mathsf{h}(\sigma)\big) \;=\; \begin{cases} 1 & \text{if} \quad \sigma \;=\; 1\;. \\ 0 & \text{otherwise}\;, \end{cases} \tag{2.69}$$

and extended over $\mathsf{H}_q(\mathfrak{S}_n)$ by $\mathbb{C}$-linearity. It can be expressed as the sum of characters of $\mathsf{H}_q(\mathfrak{S}_n)$ given by

$$\delta\big(\mathsf{h}(\sigma)\big) = \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!} \sum_{\lambda\in\Lambda_+^n} d_\lambda(q)\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big)\;. \tag{2.70}$$

Then one gets the chiral expansion of the partition function (2.46) in the topological limit ($p=0$)

$$\begin{aligned} Z_h(q;0) \;=\; \sum_{n=0}^{\infty} \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!} \sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n} [N]_q^{n\,(2-2h)} \\ \times\;\; \delta\left(D_n\,\Omega_n^{2-2h} \prod_{i=1}^{h} q^{-\ell(\sigma_i)-\ell(\tau_i)}\,\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\right)\;, \end{aligned} \tag{2.71}$$

furthermore the Ω_n factors can be expanded as follows

$$Z_h(q;0) \;=\; \sum_{n=0}^{\infty}\sum_{L=0}^{\infty} \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\;\chi(\Sigma_{h,L}) \sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n} \;\sum_{\substack{T_1,\ldots,T_L\in\mathfrak{S}_n^\vee\\ T_1,\ldots,T_L\neq 1}} [N]_q^{n\,(2-2h)-\sum_{i=1}^L(K_{T_i}-n)}$$
$$\times\; q^{\frac{N-1}{2}\,\sum_{i=1}^L(i-1)K_{T_i}}\;\delta\left(D_n\,C_{T_1}\ldots C_{T_L}\prod_{i=1}^{h} q^{-\ell(\sigma_i)-\ell(\tau_i)}\,\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\right)\;. \tag{2.72}$$

To perform the chiral expansion of the partition function (2.49) for open surface one have to implement the transformation first

$$\begin{aligned} &Z_h(q;0;C_1,\ldots,C_b)\\ &\quad:=\;\int_{T^b}\prod_{i=1}^{b}[\mathrm{d}U_i]_q\;\sum_{n_i=0}^{\infty}\frac{q^{-\frac{n_i\,(n_i-1)}{4}}}{[n_i]_q!}\;\mathrm{Tr}_{R_{\omega_1}^{\otimes n_i}}\big(\,C_i\,U_i^{\dagger}\,\big)\;Z_{q\mathrm{YM}}(U_1,\ldots,U_b)\;, \end{aligned} \tag{2.73}$$

which changes the basis from holonomies to central elements. The integral measure $[\mathrm{d}U_i]_q$ on the maximal torus T is the Haar measure on the gauge group. We clarified this Fourier-like transformation in [1] (see Appendix 4 together with its refinement). The chiral expansion of the transformed partition function is given by

$$\begin{aligned} Z_h(q;0;C_1,\ldots,C_b) \;&=\; \sum_{n=0}^{\infty}\Big(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\Big)^b\sum_{\lambda\in\Lambda_+^n}\big(\dim_q(R_\lambda)\big)^{2-2h-b}\prod_{j=1}^{b}\chi_{r_\lambda}(C_i)\\ &=\;\sum_{n=0}^{\infty}[N]_q^{n\,(2-2h-b)}\;\Big(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\Big)^b\sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n}\\ &\quad\times\;\delta\Big((E_n)^{b-1}\,\Omega_n^{2-2h-b}\prod_{i=1}^{h}q^{-\ell(\sigma_i)-\ell(\tau_i)}\,\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\;\prod_{j=1}^{b}C_j\Big)\;, \end{aligned} \tag{2.74}$$

where the central element E_n is defined by

$$E_n\;:=\;\sum_{\sigma\in\mathfrak{S}_n}q^{-\ell(\sigma)}\,\mathsf{h}(\sigma^{-1})\,\mathsf{h}(\sigma)\;. \tag{2.75}$$

The Wilson loop observable defined in (2.50) is also expanded in transformed ver-

sion

$$
\begin{aligned}
W(q;0;h_1,h_2;C) &= \frac{q^{-\frac{n_\lambda\,(n_\lambda-1)}{4}}}{[n_\lambda]_q!} \sum_{\lambda\in\Lambda_+^{n_\lambda}} \chi_{r_\lambda}(C)\; W_\lambda \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \big([N]_q\big)^{n_1\,(1-2h_1)+n_2\,(1-2h_2)}\; \delta_{n_1+n_\lambda,n_2}\, \frac{q^{-\frac{n_1\,(n_1-1)}{4}}}{[n_1]_q!}\, \frac{q^{-\frac{n_\lambda\,(n_\lambda-1)}{4}}}{[n_\lambda]_q!} \\
&\quad\times \sum_{\sigma_1\in\mathfrak{S}_{n_1}}\sum_{\sigma_2\in\mathfrak{S}_{n_\lambda}} q^{-\ell(\sigma_1)-\ell(\sigma_2)}\,\delta\Big(C\,\mathsf{h}\big(\sigma_2^{-1}\big)\Big) \\
&\quad\times\, \delta\Big(D_{n_1}\,\Omega_{n_1}^{1-2h_1}\,\Pi_{n_1}^{(h_1)}\,\mathsf{h}\big(\sigma_1^{-1}\big)\Big)\,\delta\Big(\Omega_{n_2}^{1-2h_2}\,\Pi_{n_2}^{(h_2)}\,\big(\mathsf{h}(\sigma_1)\cdot\mathsf{h}(\sigma_2)\big)\Big)\,,
\end{aligned}
\tag{2.76}
$$

where n_λ is the number of boxes corresponding to the representation R_λ with $n_1+n_\lambda = n_2$, and $\mathsf{h}(\sigma_1)\cdot\mathsf{h}(\sigma_2)$ acts on $\mathsf{H}_q(\mathfrak{S}_{n_2})$ via $\mathsf{g}_1,\ldots,\mathsf{g}_{n_1-1}\in\mathsf{H}_q(\mathfrak{S}_{n_1})\subset\mathsf{H}_q(\mathfrak{S}_{n_2})$ and $\mathsf{g}_{n_1+1},\ldots,\mathsf{g}_{n_1+n_\lambda-1}\in\mathsf{H}_q(\mathfrak{S}_{n_\lambda})\subset\mathsf{H}_q(\mathfrak{S}_{n_2})$. The notation

$$
\Pi_n^{(h)} = \sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n} \prod_{i=1}^{h} q^{-\ell(\sigma_i)-\ell(\tau_i)}\,\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1}) \tag{2.77}
$$

is also defined for simplicity.

3 Refinement

The two-dimensional (q,t)-deformed or Macdonald-deformed Yang-Mills theory is a two-parameter deformation of the usual two-dimensional Yang-Mills theory. It can be thought of as a refinement of the q-deformation in a sense that Macdonald polynomials are refined versions of original Schur polynomials, or as a quantum deformation of the classical β-deformation which can be characterised in certain cases by β-ensembles of random matrix models. In this section we consider the partition function defining the gauge theory together with more general amplitudes, and its geometrical interpretations in the context of refined topological string amplitudes. We also review shortly the refinement of the dualities related to BPS black holes.

3.1 Macdonald deformation of two-dimensional Yang-Mills theory

We start with a combinatorial definition of the partition function of (q,t)-deformed Yang-Mills theory, then review the theory of generalized characters, and end with

the introduction of more general amplitudes such as partition functions for arbitrary open surfaces and Wilson loop observabels.

3.1.1 The partition function

The partition function for the Macdonald deformation of Yang-Mills theory with gauge group G on a closed oriented Riemann surface Σ_h of genus h can be written as a generalization of the q-deformed Yang-Mills theory (2.46) given by [50–52]

$$Z_h(q,t;p) = \sum_{\lambda\in\Lambda_+} \frac{\dim_{q,t}(R_\lambda)^{2-2h}}{(g_\lambda)^{1-h}}\, q^{\frac{p}{2}\,(\lambda,\lambda)}\, t^{p\,(\rho,\lambda)} \ . \tag{2.78}$$

Here the degree $p\in\mathbb{Z}$ and the deformation parameters $q,t\in\mathbb{C}^*$ satisfy $|q|<1$ and $|t|<1$ in order to ensure that the series (2.78) has a non-zero radius of convergence; we shall sometimes assume $q,t\in(0,1)$ for convenience. For simplicity of presentation, below we shall write some formulas for the case when the refinement parameter

$$\beta = \frac{\log t}{\log q} \tag{2.79}$$

is a positive integer, and then extend our final results to arbitrary $\beta\in\mathbb{C}$ by analytic continuation. The refined quantum dimension of the representation R_λ is

$$\dim_{q,t}(R_\lambda) = \prod_{m=0}^{\beta-1}\prod_{\alpha\in\mathcal{R}_+} \frac{\big[(\lambda+\beta\,\rho,\alpha)+m\big]_q}{\big[(\beta\,\rho,\alpha)+m\big]_q} \ . \tag{2.80}$$

The Macdonald metric is given by

$$g_\lambda = \frac{1}{N!}\prod_{m=0}^{\beta-1}\prod_{\alpha\in\mathcal{R}_+} \frac{\big[(\lambda+\beta\,\rho,\alpha)+m\big]_q}{\big[(\lambda+\beta\,\rho,\alpha)-m\big]_q} \ . \tag{2.81}$$

We shall often assume that the rank N is such that $\rho\in\mathbb{Z}^N$, which in particular can be supposed in the large N expansion that we consider in the following.

In this thesis we shall specialize to the unitary gauge group $G=U(N)$. In this case there are convenient combinatorial expressions available for the dimension and metric factors. The Weyl vector is given in (2.8) and the dominant weights $\lambda\in\Lambda_+$ are parametrized by Young diagrams $Y_\lambda\subset(\mathbb{Z}_{>0})^2$ with at most N rows, Then the refined quantum dimension and Macdonald metric have the equivalent forms

$$\begin{aligned}\dim_{q,t}(R_\lambda) &= t^{\frac12\,(\|\lambda^{\rm t}\|-N\,|\lambda|)}\prod_{(i,j)\in Y_\lambda}\frac{1-t^{N-i+1}\,q^{j-1}}{1-t^{\lambda^{\rm t}_j-i+1}\,q^{\lambda_i-j}} \ ,\\ g_\lambda &= g_\emptyset\prod_{(i,j)\in Y_\lambda}\frac{1-t^{\lambda^{\rm t}_j-i}\,q^{\lambda_i-j+1}}{1-t^{\lambda^{\rm t}_j-i+1}\,q^{\lambda_i-j}}\,\frac{1-t^{N-i+1}\,q^{j-1}}{1-t^{N-i}\,q^{j}} \ ,\end{aligned} \tag{2.82}$$

where $|\lambda| := \sum_{i=1}^N \lambda_i$ and $\|\lambda\| := \sum_{i=1}^N \lambda_i^2$, while

$$g_\emptyset = \frac{1}{N!} \prod_{m=0}^{\beta-1} \prod_{1\leq i<j\leq N} \frac{\big[\beta\,(j-i)+m\big]_q}{\big[\beta\,(j-i)-m\big]_q} \ . \tag{2.83}$$

The products in (2.82) run over all boxes (i,j) of the Young diagram Y_λ with $1 \leq i \leq N$, $1 \leq j \leq \lambda_i$, and λ^{t} corresponds to the transposed Young diagram, i.e. λ_i^{t} is the number of entries $\leq i$ in Y_λ.

3.1.2 Generalized characters and Macdonald polynomials

Before we discuss boundary partition functions and Wilson loop observables, we need the theory of generalized characters in order to introduce the refinement of quantum group characters (see e.g. [53, §2] and [35, §6.1]). Then Macdonald deformation corresponds to the deformation wherein original quantum group characters are replaced with generalized characters.

If V, W are finite-dimensional representations of $\mathcal{U}_q(\mathfrak{gl}_N)$, and $\Phi : V \to V \otimes W$ is a non-zero intertwining operator for $\mathcal{U}_q(\mathfrak{gl}_N)$, then the vector-valued function

$$\chi_\Phi(U) = \mathrm{Tr}_{\,V}\big(\Phi\, U\big) \tag{2.84}$$

on the maximal torus $T \subset G$ is called a *generalized character*. Contrary to the classical case $q = 1$, if the representation W is non-trivial then $\chi_\Phi(U)$ is not invariant under the action of the Weyl group $\mathfrak{S}_N$ on T. Since the operator Φ preserves weight, the vector $\chi_\Phi(U)$ actually takes values in the weight zero subspace $W_0 \subset W$.

To compute the generalized character explicitly, let V^* denote the dual $\mathcal{U}_q(\mathfrak{gl}_N)$-module, and let v_i, v^i be dual bases for V, V^*. We can then identify Φ with an intertwiner $\Phi : V^* \otimes V \to W$ and

$$\chi_\Phi(U) = \Phi\big(v^i \otimes U v_i\big) \ , \tag{2.85}$$

where throughout we use the Einstein summation convention for repeated upper and lower indices. Since $v^i \otimes v_i = (\mathbb{1}_{V^*} \otimes q^{-(\rho,H)})\,\mathbf{1}_{\mathbb{C}}$, where $H = (H_1, \ldots, H_N)$ are the Cartan generators of $\mathfrak{gl}_N$ and $\mathbf{1}_{\mathbb{C}} = \imath(1)$ with $\imath : \mathbb{C} \to V^* \otimes V$ an embedding of $\mathcal{U}_q(\mathfrak{gl}_N)$-modules, we can also write the generalized character as

$$\chi_\Phi(U) = \Phi\big((\mathbb{1}_{V^*} \otimes q^{-(\rho,H)}\, U)\mathbf{1}_{\mathbb{C}}\big) \ . \tag{2.86}$$

In the special instance where $W = \mathbb{C}$ is the trivial representation of $\mathcal{U}_q(\mathfrak{gl}_N)$ and $\Phi : V \to V$ is the identity operator, so that $\Phi : V^* \otimes V \to \mathbb{C}$ is the canonical dual pairing, then $\chi_{\mathbb{1}_V}(U) = \chi_V(U) = \mathrm{Tr}_V(U)$ is the usual character of U in the representation V.

Now let $V = R_\lambda$ for fixed $\lambda \in \Lambda_+$ and $W = W_{\beta-1}$ for fixed $\beta \in \mathbb{Z}_{>0}$ where

$$W_{\beta-1} := R_{\omega_1}^{\odot(\beta-1)\,N} \otimes (\det)^{-(\beta-1)} \tag{2.87}$$

is the q-deformation of the traceless $(\beta-1)\,N$-th symmetric power of the first fundamental representation $R_{\omega_1} = \mathbb{C}^N$ of G, which is a finite-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{gl}_N)$ with highest weight $(\beta-1)\,N\,\omega_1 - (\beta-1)\,(1,\dots,1) = (\beta-1)\,(N-1,-1,\dots,-1)$. By [54, Lemma 1], the space of intertwining operators $\mathrm{Hom}_{\mathcal{U}_q(\mathfrak{gl}_N)}\big(R_{\lambda'}, R_{\lambda'} \otimes W_{\beta-1}\big)$ for $\mathcal{U}_q(\mathfrak{gl}_N)$ is one-dimensional if $\lambda' = \lambda_\beta := \lambda + (\beta-1)\,\rho$ for a highest weight λ and zero otherwise; recall that λ is a dominant weight of $\mathfrak{gl}_N$ if and only if it is of the form

$$\lambda = a\,(1,\dots,1) + \sum_{i=1}^{N} n_i\,\omega_i \tag{2.88}$$

for some $n_i \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{C}$, where $\omega_i = (1^i\ 0^{N-i})$, $i = 1,\dots,N$ are the fundamental weights of $\mathfrak{gl}_N$. It follows that a non-zero $\mathcal{U}_q(\mathfrak{gl}_N)$-homomorphism $\Phi_\lambda : R_{\lambda_\beta} \to R_{\lambda_\beta} \otimes W_{\beta-1}$ is unique up to normalization. As the weight zero subspace $\big(W_{\beta-1}\big)_0$ is one-dimensional, the corresponding generalized character

$$\chi_{\Phi_\lambda}(U) := \mathrm{Tr}_{R_{\lambda_\beta}}\big(\Phi_\lambda\,U\big) \tag{2.89}$$

can be regarded as taking values in $\mathbb{C}$. By [54, Theorem 1], if λ is a partition these generalized characters are given in terms of the monic form $M_\lambda(x;q,t)$ of the Macdonald polynomials at $t = q^\beta$, where $U = \mathrm{e}^{(z,H)}$ and $x = \mathrm{e}^{z}$. We choose the normalization of Φ_λ and the identification $\big(W_{\beta-1}\big)_0 \cong \mathbb{C}$ in such a way so that

$$\chi_{\Phi_\lambda}(U) = \frac{M_\lambda(x;q,t)}{\sqrt{g_\lambda}}\ . \tag{2.90}$$

In the unrefined limit $\beta = 1$, we have $g_\lambda = 1$ and the Macdonald polynomials reduce to the Schur polynomials $M_\lambda(x;q,q) = s_\lambda(x)$ (independently of q), which coincide with the ordinary characters $\chi_{R_\lambda}(U) = \mathrm{Tr}_{R_\lambda}(U)$ of the irreducible representation R_λ.

Then the refined partition function on open surface of genus h with b boundaries is given by

$$Z_{h,b}(q,t;p;U_1,\ldots,U_b) = \sum_{\lambda\in\Lambda_+} \frac{\dim_{q,t}(R_\lambda)^{2-2h-b}}{(g_\lambda)^{1-h-b/2}}\, q^{\frac{p}{2}\,(\lambda,\lambda)}\, t^{p\,(\rho,\lambda)} \prod_{i=1}^{b} \chi_{\Phi_\lambda}(U_i)\ , \tag{2.91}$$

which is a one-parameter deformed version of (2.49). The refinement of the Wilson loop observable (2.50) is defined by [35]

$$\begin{aligned} W_\lambda(q,t;p;h_1,h_2) \;=\; \sum_{\mu,\nu\in\Lambda_+} \int_T [\mathrm{d}U]_{q,t}\, \frac{\dim_{q,t}(R_\mu)^{1-2h_1}}{(g_\mu)^{\frac{1}{2}-h_1}}\, \frac{\dim_{q,t}(R_\nu)^{1-2h_2}}{(g_\nu)^{\frac{1}{2}-h_2}}\; q^{\frac{p}{2}\,(\lambda,\lambda)}\, t^{p\,(\rho,\lambda)} \\ \times\, \chi_{\Phi_\mu}(U)\, \chi_{\Phi_\lambda}(U)\, \chi_{\Phi_\nu}(U^\dagger)\ . \end{aligned} \tag{2.92}$$

3.2 Refined dualities

In this section we introduce refined topological string theory based on the motivation coming from M-theory and survey the refinement of dualities between BPS black holes, refined topological string theory and two dimensional Yang-Mills theory.

3.2.1 M-theory motivation

The refined topological string theory only exists for the A-model and originally it is based on the refinement of the topological vertex on toric manifolds [55, 56]. The complete geometrical picture of refined topological string theory involves M-theory on a particular eleven-dimensional manifold [52], which we now describe (see [1] for a more detailed description).

The refined topological string partition function is given by the index of M-theory on the geometry

$$(X \times \mathsf{TN} \times S^1)_{\epsilon_1,\epsilon_2}\ , \tag{2.93}$$

where X denotes the Calabi-Yau ((2.44) in our case), TN the Taub-NUT spacetime, and the circle product is twisted: going around the S^1 circle, the two complex coordinates of TN are rotated by

$$(z,w,x) \;\longmapsto\; (q^n\, z, t^{-n}\, w, x + 2\pi\, r\, n)\ , \tag{2.94}$$

where $(z,w) \in \mathbb{C}^2$ denotes the coordinates on TN, $x \in \mathbb{R}$ the coordinate on the circle, and $n \in \mathbb{Z}$. The deformation parameters

$$q = \mathrm{e}^{-\epsilon_1} \qquad \text{and} \qquad t = \mathrm{e}^{\epsilon_2}\ , \tag{2.95}$$

are given by the equivariant parameters

$$\epsilon_1 = \frac{1}{\sqrt{\beta}}\, g_s \qquad \text{and} \qquad \epsilon_2 = -\sqrt{\beta}\, g_s \tag{2.96}$$

of the Ω-background [57], where g_s is the topological string coupling constant. The partition function of M-theory on this geometry defines the refined topological string theory, and it is computed by an five-dimensional index of the resulting theory on $\mathsf{TN} \times S^1$ (see [52] for the details). In the case when $\epsilon_1 = \epsilon_2 = g_s$ it gives the partition function of the ordinary topological string theory.

The Ω-background symmetry $(\epsilon_1, \epsilon_2) \mapsto (-\epsilon_2, -\epsilon_1)$ corresponds to the inversion symmetry $\beta \mapsto \frac{1}{\beta}$ of the refinement parameter together with the rank change $N \mapsto \beta\,(N-1)+1$. It acts on the Macdonald deformation parameters as $(q,t) \mapsto (t,q)$ which corresponds to the symmetry $p \mapsto 2-2h-p$ that exchanges the two line bundle summands of the Calabi-Yau fibration over Σ_h.

3.2.2 Refined black hole partition function

The BPS black hole partition function defined in (2.51) and (2.52) is not the most general partition function which counts microstates of BPS black holes in four dimensions. One can include information about the spin [52] by replacing the Witten index with the spin character

$$\mathrm{Tr}_{\,\mathcal{H}_{\mathrm{BPS}}} (-1)^F \; \mathrm{e}^{\,-2\gamma J_3} \;, \tag{2.97}$$

where J_3 is the three-dimensional generator of rotations and γ is the conjugate chemical potential. This is not an index, but one can take the gravity decoupling limit, where an $SU(2)$ R-symmetry appears, and this can be used to form a genuine index [58]

$$\mathrm{Tr}_{\,\mathcal{H}_{P,Q}} (-1)^{2J_3}\, \mathrm{e}^{\,-2\gamma(J_3-R)} \;=\; \sum_{J_3,R} \Omega(P,Q;J_3,R)\; \mathrm{e}^{\,-2\gamma(J_3-R)} \;. \tag{2.98}$$

This is used to define the refined partition function of BPS black hole with the summation over electric charges as in the unrefined case

$$Z_{\mathrm{ref\,BH}}(P_6,P_4,\phi_2,\phi_0;\gamma) \;=\; \sum_{Q_2,Q_0,J_3,R} \Omega(P_6,P_4,Q_2,Q_0;J_3,R)\; \mathrm{e}^{\,-2\gamma(J_3-R)-\phi_2 Q_2-\phi_0 Q_0} \;. \tag{2.99}$$

The refined partition function $Z_{\mathrm{ref\,BH}}$ corresponding to the compactified Calabi-Yau (2.44) reduces to a two-dimensional gauge theory on the base surface as before, but now the gauge theory is the Macdonald deformation of the two-dimensional $U(N)$

Yang-Mills theory introduced in §3.1.1. In the Lagrangian formalism, the action is the same as of the unrefined in (2.53), the refinement appears in the measure of the path integral. The parameters ϵ_1 and ϵ_2 are given by the chemical potentials ϕ_2, ϕ_2 and γ with

$$\epsilon_1 = \frac{4\pi^2}{\phi_0} \qquad \text{and} \qquad \epsilon_2 = \frac{4\pi^2}{\phi_0}\left(1 - \frac{\gamma}{2\pi\,\mathrm{i}}\right) . \tag{2.100}$$

The large N dual of the (q,t)-deformed $SU(N)$ gauge theory is the refined string theory, and the dualities are very analogous to the original q-deformed case described in §2.2.2, only the string coupling g_s is replaced in the Kähler modulus with parameter ϵ_2

$$k = \frac{1}{2}(p + 2h - 2)N\epsilon_2 \ , \tag{2.101}$$

and so the refined black hole partition function (2.56) factorizes into a chiral and an antichiral part as well as the unrefined partition function (2.56), where the ϵ_1 has the role of the string coupling g_s this time. The duality is developed to the refined OSV conjecture, which says that the microscopic description of BPS black holes with spin is computed by refined topological string amplitudes.

Chapter 3

Chiral expansion of Macdonald-deformed two-dimensional Yang-Mills theory

In this chapter we present our results on the chiral expansion of Macdonald deformed two-dimensional Yang-Mills theory, and introduce a deformed Hurwitz theory of branched covers. We finish the chapter with the chiral expansion of observables in the theory. The results have been published in [1].

1 Generalized quantum characters as Hecke characters

In this section we develop a combinatorial description of the dimension factors defined in (2.80) for the quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_N)$ appearing in (2.78) in terms of characters of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$ of type A_{n-1}. Our final result is summarised in Proposition 3.39. For this, we shall use quantum Schur-Weyl duality between $\mathcal{U}_q(\mathfrak{gl}_N)$ and $\mathsf{H}_q(\mathfrak{S}_n)$, which has been discussed in §2.2.3 in Chapter 2. See Appendix 2 for relevant definitions and properties of quantum groups which are used throughout, and Appendix 3 for those pertaining to Hecke algebras.

1.1 Quantum Schur-Weyl duality

Generalized characters have been introduced in §3.1.2 in Chapter 2, and they were also used to express the refined quantum dimension. Using quantum Schur-Weyl duality introduced in (2.58) we can write the generalized characters $\chi_{\Phi_\lambda}(U)$ for $\lambda_\beta \in \Lambda_+^n$ as combinatorial expansions over the symmetric group $\mathfrak{S}_n$ involving characters of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$. For this, we introduce a $\mathcal{U}_q(\mathfrak{gl}_N)$-intertwiner

$$\Phi_n \,:\, R_{\omega_1}^{\otimes n} \;\longrightarrow\; R_{\omega_1}^{\otimes n} \otimes W_{\beta-1} \tag{3.1}$$

for each $n \geq 0$, where $W_{\beta-1}$ is given by (2.87). This intertwiner can be defined in the following (non-canonical) way: As a $\mathcal{U}_q(\mathfrak{gl}_N)$-module the vector space $R_{\omega_1}^{\otimes n}$ decomposes into irreducible unitary representations as

$$R_{\omega_1}^{\otimes n} = \bigoplus_{\lambda \in \Lambda_+^n} R_\lambda^{\oplus d_\lambda(1)} \tag{3.2}$$

where $d_\lambda(1) = \dim(r_\lambda)$. We can use the projector property $\sum_{\lambda \in \Lambda_+^n} P_\lambda = \mathbb{1}_{R_{\omega_1}^{\otimes n}}$ to write

$$\Phi_n = \sum_{\lambda,\mu \in \Lambda_+^n} \big(P_\mu \otimes \mathbb{1}_{W_{\beta-1}}\big)\, \Phi_n\, P_\lambda \ , \tag{3.3}$$

with

$$\big(P_\mu \otimes \mathbb{1}_{W_{\beta-1}}\big)\, \Phi_n\, P_\lambda := \delta_{\lambda,\mu} \sum_{\lambda \in \Lambda_+^n} \Phi_{\lambda_{\beta-2}} \otimes \mathbb{1}_{r_\lambda} \ , \tag{3.4}$$

where we used (3.2) and $\Phi_{\lambda_{\beta-2}} \in \mathrm{Hom}_{\mathcal{U}_q(\mathfrak{gl}_N)}\big(R_\lambda, R_\lambda \otimes W_{\beta-1}\big)$. In the large N limit, if λ is a dominant weight then so are λ_β and $\lambda_{\beta-2}$, and thus $\mathrm{Hom}_{\mathcal{U}_q(\mathfrak{gl}_N)}\big(R_\lambda, R_\lambda \otimes W_{\beta-1}\big)$ is non-zero and one-dimensional if $\lambda \in \Lambda_+$. This gives an identification of underlying linear transformations

$$\Phi_n = \bigoplus_{\lambda \in \Lambda_+^n} \Phi_{\lambda_{\beta-2}} \otimes \mathbb{1}_{r_\lambda} \ . \tag{3.5}$$

We evaluate the trace $\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, U\, P_\lambda\big)$ in two different ways. Firstly, using (2.60) and (3.2) along with (2.85) we easily get

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, U\, P_\lambda\big) \;=\; \mathrm{Tr}_{R_\lambda}(\Phi_{\lambda_{\beta-2}}\, U)\, d_\lambda(1) \;=\; \chi_{\Phi_{\lambda_{\beta-2}}}(U)\, d_\lambda(1) \ . \tag{3.6}$$

Secondly, we substitute the explicit expansion (2.61), and hence for any weight $\lambda \in \Lambda_+^n$ we can express the vector-valued trace as

$$\mathrm{Tr}_{R_\lambda}(\Phi_{\lambda_{\beta-2}}\, U) \;=\; \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\,\frac{d_\lambda(q)}{d_\lambda(1)}\sum_{\sigma\in\mathfrak{S}_n} q^{-\ell(\sigma)}\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma^{-1})\big)\;\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, U\,\mathsf{h}(\sigma)\big)\;. \tag{3.7}$$

It will prove useful later on to derive directly a formula for the inverse of this transformation of characters, generalizing (2.59) by

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, U\,\mathsf{h}(\sigma)\big) = \sum_{\lambda\in\Lambda_+^n}\chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big)\;\chi_{\Phi_{\lambda_{\beta-2}}}(U)\;. \tag{3.8}$$

A short proof is presented in Appendix 5 as Lemma (A.32).

By multiplying the left-hand side and the right-hand side of the character formula (3.8) with $q^{-\ell(\sigma)}\,\chi_{r_{\lambda'}}\big(\mathsf{h}(\sigma^{-1})\big)$, summing over all permutations $\sigma\in\mathfrak{S}_n$ and using the orthogonality relations for Hecke characters [49]

$$\sum_{\sigma\in\mathfrak{S}_n} q^{-\ell(\sigma)}\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big)\,\chi_{r_{\lambda'}}\big(\mathsf{h}(\sigma^{-1})\big) = \delta_{\lambda,\lambda'}\;q^{\frac{n\,(n-1)}{4}}\;[n]_q!\;\frac{d_\lambda(1)}{d_\lambda(q)}\;, \tag{3.9}$$

we arrive at the expression (3.7).

1.2 Refined quantum dimensions

We are finally ready to derive our Hecke character expansion for the refined quantum dimensions. Firstly we note that the refined quantum dimension (2.80) and the Macdonald metric (2.81) are both invariant under any shift of the dominant weight λ by the maximal partition $(1^N) := (1,\ldots,1)$ of length N, i.e.

$$\dim_{q,t}(R_{\lambda+a\,(1^N)}) = \dim_{q,t}(R_\lambda) \qquad \text{and} \qquad g_{\lambda+a\,(1^N)} = g_\lambda\;. \tag{3.10}$$

We shall assume that a is an integer. The refined quantum dimension is obtained by the specialization $U = t^{(\rho,H)}$ in the generalized characters (2.90), i.e.

$$\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} \;=\; \chi_{\Phi_\lambda}\big(q^{\beta\,(\rho,H)}\big) \;=\; \mathrm{Tr}_{R_{\lambda_\beta}}\big(\Phi_\lambda\, q^{\beta\,(\rho,H)}\big)\;. \tag{3.11}$$

We wish to substitute in the expansion (3.7), but the Hecke characters and dimensions are only defined for partitions, whereas λ_β is not necessarily a partition. Hence we

use the shift symmetry (3.10) to get a partition $\lambda_\beta + a\,(1^N)$, which is true as long as $a \geq \frac{N-1}{2}\,(\beta-1)$. For definiteness we use the lowest value

$$a = \tfrac{N-1}{2}\,(\beta-1) \tag{3.12}$$

which for large N can be regarded as integral. In the large N expansion that we consider later on, we will typically also consider the limit $\beta \to 1$ such that the quantity aN is finite. Then we get

$$\begin{aligned}\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} \;=\; \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\,\frac{d_{\lambda_\beta+a\,(1^N)}(q)}{d_{\lambda_\beta+a\,(1^N)}(1)}\sum_{\sigma\in\mathfrak{S}_n} q^{-\ell(\sigma)}\,\chi_{r_{\lambda_\beta+a\,(1^N)}}\big(\mathsf{h}(\sigma^{-1})\big) \\ \times\;\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\,q^{\beta\,(\rho,H)}\,\mathsf{h}(\sigma)\big)\end{aligned} \tag{3.13}$$

for $\lambda_\beta + a\,(1^N) \in \Lambda_+^n$, i.e. $\lambda \in \Lambda_+^{n-a\,N}$. We can easily check that this expression agrees with the anticipated formula for the quantum dimension $\dim_q(R_\lambda) = \mathrm{Tr}_{\,R_\lambda}\big(q^{(\rho,H)}\big)$ in the unrefined limit: In the limit $\beta = 1$, we have $\lambda_\beta = \lambda$, $a = 0$, and the intertwiners Φ_λ and Φ_n become identity operators on finite-dimensional modules, so that (3.13) coincides with the q-dimension formula (2.62).

To manipulate the sum in (3.13), let us first explicitly specify how the Hecke operators act. From [59] we can express $\mathsf{h}(\sigma) \in \mathsf{H}_q(\mathfrak{S}_n)$ as a product of generators g_i, $i = 1,\ldots,n-1$ in the same way that we express $\sigma \in \mathfrak{S}_n$ in the form of a reduced word; we say that $\mathsf{h}(\sigma) \in \mathsf{H}_q(\mathfrak{S}_n)$ is a reduced word if $\sigma \in \mathfrak{S}_n$ is a reduced word. From [49], g_i acts on $R_{\omega_1}^{\otimes n}$ as the braiding operator

$$\mathsf{g} = q^{1/2}\;\check{\mathsf{R}} \tag{3.14}$$

on the tensor product $R_{\omega_1} \otimes R_{\omega_1}$ in the $(i, i+1)$ slot of $R_{\omega_1}^{\otimes n}$, where $\check{\mathsf{R}} = \mathsf{P}\,\mathsf{R}$ with P the flip operator $\mathsf{P}(v \otimes w) = w \otimes v$; here R is the FRT quantum R-matrix [60]

$$\mathsf{R} = q^{1/2}\sum_{i=1}^{N} H_i \otimes H_i + \sum_{i\neq j} H_i \otimes H_j + \big(q^{1/2} - q^{-1/2}\big)\sum_{i>j} E_{ij} \otimes E_{ji}\ , \tag{3.15}$$

where $H_i = E_{ii}$ and E_{ij} for $i, j = 1,\ldots,N$ act on the standard basis $\{e_k\} \subset R_{\omega_1} = \mathbb{C}^N$ of the fundamental representation as

$$E_{ij}e_k = \delta_{jk}\;e_i\ . \tag{3.16}$$

Let us define the (q,t)-*trace* of an element $x \in \mathsf{H}_q(\mathfrak{S}_n)$ by

$$\mathrm{Tr}_{\,q,t}(x) := \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\,t^{(\rho,H)}\,x\big)\ , \tag{3.17}$$

where Φ_n is the intertwiner (3.1). This terminology is justified by the cyclicity property

$$\mathrm{Tr}_{\,q,t}(x\,y) = \mathrm{Tr}_{\,q,t}(y\,x) \tag{3.18}$$

for all $x, y \in \mathsf{H}_q(\mathfrak{S}_n)$. The proof of this statement can be found in Appendix 5 as Lemma (A.34).

We can use the cyclicity to truncate the reduced words $\mathsf{h}(\sigma)$ to minimal words in the sum (3.13). A word is said to be *minimal* if it is both reduced and contains no generators g_i more than once. Using the Hecke relations on g_i and cyclicity of the (q,t)-trace, we can then truncate the sum in (3.13) to conjugacy classes T in $\mathfrak{S}_n$ and express the (q,t)-trace $\mathrm{Tr}_{\,q,t}\big(\mathsf{h}(\sigma)\big)$ for any $\sigma \in T$ as the (q,t)-trace $\mathrm{Tr}_{\,q,t}\big(\mathsf{h}(m_T)\big)$ of the minimal word $m_T \in T$ in the conjugacy class [59]. Hence following the derivation of [49, eq. (2.38)], we can write the expansion (3.13) as

$$\begin{aligned}\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} \;=\; \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\,\frac{d_{\lambda_\beta+a\,(1^N)}(q)}{d_{\lambda_\beta+a\,(1^N)}(1)}\sum_{T\in\mathfrak{S}_n^\vee} q^{-\ell(T)}\,\chi_{r_{\lambda_\beta+a\,(1^N)}}(C_T) \\ \times\;\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,q^{\beta\,(\rho,H)}\,\mathsf{h}(m_T)\big)\end{aligned} \tag{3.19}$$

where $\ell(T)$ is the length of the permutation $m_T \in \mathfrak{S}_n$ and C_T are the same central elements of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$ as in [49]. To write C_T explicitly, we need to express an arbitrary character of the Hecke algebra in terms of characters of minimal words [59] using cyclicity property of the (q,t)-trace and the Hecke relations (A.15) from Appendix 3 as

$$\chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big) = \sum_{T\in\mathfrak{S}_n^\vee}\alpha_T(\sigma)\,\chi_{r_\lambda}\big(\mathsf{h}(m_T)\big)\ , \tag{3.20}$$

where the expansion coefficients $\alpha_T(\sigma)$ do not depend on the representation r_λ. Then the central element C_T is defined by

$$C_T = q^{\ell(T)}\sum_{\sigma\in\mathfrak{S}_n} q^{-\ell(\sigma)}\,\alpha_T(\sigma^{-1})\,\mathsf{h}(\sigma)\ . \tag{3.21}$$

This central element is the same as that was used for the expansion of the unrefined Ω_n element in (2.64). The quantum Young projector (2.61) can be rewritten as

$$P_\lambda = \frac{d_\lambda(q)}{q^{\frac{n\,(n-1)}{4}}\,[n]_q!}\sum_{T\in\mathfrak{S}_n^\vee} q^{-\ell(T)}\,\chi_{r_\lambda}\big(\mathsf{h}(m_T)\big)\,C_T\ . \tag{3.22}$$

This transformation can be inverted to express C_T in terms of the central elements P_λ, because the determinant of the transformation matrix is non-zero. To see that

the determinant of $\chi_{r_\lambda}\big(\mathsf{h}(m_T)\big)$ is non-zero, we use the orthogonality relation (3.9) and expand it into minimal words using (3.20). Then we take the determinant of the obtained expression and use multiplicativity of the determinant to find that it is non-vanishing. Hence the centrality of the elements C_T follows from the centrality of the projectors.

1.3 (q,t)-traces of minimal words

We are left with the problem of evaluating the (q,t)-traces $\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, q^{\beta\,(\rho,H)}\,\mathsf{h}(m_T)\big)$ of minimal words; we shall follow the strategy of [49, Appendix B]. For $n=1$, the R-matrix $\check{\mathsf{R}}$ acts trivially while $q^{\beta\,(\rho,H)}e_i = t^{\rho_i}\,e_i$, where $\rho_i = \frac{N+1}{2} - i$. Using

$$\dim_{q,t}(R_{\omega_1}) = [N]_t = \mathrm{Tr}_{R_{\omega_1}}(t^{(\rho,H)}) \qquad \text{and} \qquad g_{\omega_1} = g_\emptyset\,\frac{\big[N\big]_t}{\big[\beta\,(N-1)+1\big]_q} \tag{3.23}$$

where generally

$$[N]_{t^k} := \frac{[k\,\beta\,N]_q}{[k\,\beta]_q}\ , \tag{3.24}$$

we then easily find for the (q,t)-trace

$$\mathrm{Tr}_{R_{\omega_1}}\big(\Phi_1\,t^{(\rho,H)}\big) = \frac{\dim_{q,t}(R_{\omega_1})}{\sqrt{g_{\omega_1}}} = \bigg(\frac{\big[N\big]_t\,\big[\beta\,(N-1)+1\big]_q}{g_\emptyset}\bigg)^{1/2}\ . \tag{3.25}$$

Note that here the intertwining operator $\Phi_1 = \Phi_{\omega_1} : R_{\omega_1} \to R_{\omega_1}\otimes W_{\beta-1}$ simply acts in the (q,t)-trace proportionally to 1 to rescale the normalization of the trace of $t^{(\rho,H)}$ by the Macdonald measure factor $(g_{\omega_1})^{-1/2}$. In the unrefined limit $\beta=1$, this expression reduces as expected to the quantum dimension of the fundamental representation $\dim_q(R_{\omega_1}) = [N]_q$. Below we shall also need the generalization of the trace formula in (3.23) to arbitrary powers $t^{k\,(\rho,H)}$, $k\in\mathbb{Z}_{>0}$, which is given by

$$\mathrm{Tr}_{R_{\omega_1}}\big(t^{k\,(\rho,H)}\big) = [N]_{t^k}\ . \tag{3.26}$$

Next we set $n=2$. We can use the FRT formula (3.15) for the R-matrix to compute

$$\begin{aligned}\big(t^{k\,(\rho,H)}\otimes\mathbb{1}_{R_{\omega_1}}\big)\,\check{\mathsf{R}}(e_i\otimes e_j) = t^{k\,\rho_j}\,e_j\otimes e_i\,+\,t^{k\,\rho_j}\,\big(q^{1/2}-1\big)\,\delta_{ij}\,e_i\otimes e_j \\ +\,t^{k\,\rho_i}\,\big(q^{1/2}-q^{-1/2}\big)\,\theta_{ij}\,e_i\otimes e_j\end{aligned} \tag{3.27}$$

for any $k\in\mathbb{Z}_{>0}$, where $\theta_{ij}:=1$ if $i<j$ and $\theta_{ij}:=0$ otherwise. From this expression one can easily derive the partial traces

$$\big(\mathrm{Tr}_{R_{\omega_1}}\otimes\mathbb{1}_{R_{\omega_1}}\big)\Big(\big(t^{k\,(\rho,H)}\otimes\mathbb{1}_{R_{\omega_1}}\big)\,\mathsf{g}_1\Big) = t^{k\,\frac{N+1}{2}}\,\frac{q-1}{t^k-1}\,\mathbb{1}_{R_{\omega_1}} + \frac{t^k-q}{t^k-1}\,t^{k\,(\rho,H)} \tag{3.28}$$

as operators in $\mathcal{U}_q(\mathfrak{gl}_N)$ acting on the fundamental representation R_{ω_1}. In the unrefined limit $t = q$ at $k = 1$, this operator reduces to $q^{\frac{N+1}{2}}\, \mathbb{1}_{R_{\omega_1}}$ as in [49, eq. (B.5)]; in the general case it is also diagonal but no longer proportional to the identity operator.

Let us decompose the representation $W_{\beta-1}$ into its one-dimensional weight subspaces $\big(W_{\beta-1}\big)_\alpha \cong \mathbb{C}w^\alpha$; in particular, the isomorphism $\big(W_{\beta-1}\big)_0 \cong \mathbb{C}$ is given by mapping $w^0 \mapsto 1$. Then one can find explicit formulas for the matrix elements of $\Phi_2 : R_{\omega_1}^{\otimes 2} \to R_{\omega_1}^{\otimes 2} \otimes W_{\beta-1}$ in the following way: we write

$$\Phi_2(e_i \otimes e_j) = P_{ij}{}^{kl}{}_\alpha \; e_k \otimes e_l \otimes w^\alpha \;, \tag{3.29}$$

then the condition that Φ_2 is an intertwiner can be rewritten as a system of linear equations for the expansion coefficients $P_{ij}{}^{kl}{}_\alpha$. Since Φ_2 is uniquely determined up to the normalization in (2.90), this linear system has a unique solution which determines $P_{ij}{}^{kl}{}_\alpha$ as a rational function in $q^{1/2}$ and $t^{1/2} = q^{\beta/2}$; with (2.90) the intertwining operators $\Phi_\lambda : R_{\lambda_\beta} \to R_{\lambda_\beta} \otimes W_{\beta-1}$ are normalized in such a way that if v_{λ_β} is a highest weight vector of R_{λ_β}, then the component of $\Phi_\lambda(v_{\lambda_\beta})$ in $R_{\lambda_\beta} \otimes \big(W_{\beta-1}\big)_0$ is $(g_{\lambda_\beta})^{-1/2}\, v_{\lambda_\beta} \otimes w^0$. Setting $P_{ij}{}^{kl} := P_{ij}{}^{kl}{}_0$, the (q,t)-trace of the generator g_1 can then be written as

$$\begin{aligned}
\mathrm{Tr}_{R_{\omega_1}^{\otimes 2}}\big(\Phi_2\, t^{(\rho,H)}\, \mathsf{g}_1\big) \;&:=\; \big(\mathrm{Tr}_{R_{\omega_1}} \otimes \mathrm{Tr}_{R_{\omega_1}}\big)\big(\Phi_2\, (t^{(\rho,H)} \otimes t^{(\rho,H)})\, \mathsf{g}_1\big) \\
&=\; q^{1/2}\, t^{N+1} \Big(\sum_{i,j=1}^{N} t^{-i-j}\, P_{ji}{}^{ij} + \big(q^{1/2}-1\big) \sum_{i=1}^{N} t^{-2i}\, P_{ii}{}^{ii} \\
&\qquad + \big(q^{1/2} - q^{-1/2}\big) \sum_{i<j} t^{-i-j}\, P_{ij}{}^{ij} \Big) \;.
\end{aligned} \tag{3.30}$$

In the unrefined limit $\beta = 1$, we have $P_{ij}{}^{kl} = \delta_i{}^k\, \delta_j{}^l$ and it is easy to check that this expression reduces to $q^{\frac{N+1}{2}}\, [N]_q$ as in [49, eq. (B.5)]. In the general case we have

$$P_{ij}{}^{kl} = (g_{\omega_1})^{-1}\; \delta_i{}^k\, \delta_j{}^l \;, \tag{3.31}$$

which has been proved in Lemma (A.38) in Appendix 5.

Using this we can straightforwardly express the (q,t)-trace (3.30) in terms of q-numbers as

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes 2}}\big(\Phi_2\, t^{(\rho,H)}\, \mathsf{g}_1\big) = \frac{\big[\beta\,(N-1)+1\big]_q}{g_\emptyset} \Big(t^{\frac{N+1}{2}}\, \frac{q-1}{t-1} + \frac{t-q}{t-1}\, \frac{[N]_{t^2}}{[N]_t} \Big) \;. \tag{3.32}$$

We can use the general results of Lemma (A.38) together with the partial trace formula (3.28) and the traces (3.26) to calculate the (q,t)-trace

$$\begin{aligned}
\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, t^{(\rho,H)}\, \mathsf{g}_1\, \mathsf{g}_2 \cdots \mathsf{g}_{n-1}\big) \;&:=\; \big(\mathrm{Tr}_{R_{\omega_1}}\big)^{\otimes n}\big(\Phi_{\omega_1}^{\otimes n}\, (t^{(\rho,H)})^{\otimes n}\, \mathsf{g}_1\, \mathsf{g}_2 \cdots \mathsf{g}_{n-1}\big) \\
&=\; (g_{\omega_1})^{-n/2}\, \mathrm{Tr}_{R_{\omega_1}}\Big(t^{(\rho,H)}\, \big(\mathrm{Tr}_{R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}\big)\big((t^{(\rho,H)} \otimes \mathbb{1}_{R_{\omega_1}})\, \mathsf{g}_1\big) \\
&\qquad \cdots\; \big(\mathrm{Tr}_{R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes(n-2)}\big)\big((t^{(\rho,H)} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes(n-2)})\, \mathsf{g}_1\big) \\
&\qquad\qquad \times\; \big(\mathrm{Tr}_{R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes(n-1)}\big)\big((t^{(\rho,H)} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes(n-1)})\, \mathsf{g}_1\big)\Big)
\end{aligned} \tag{3.33}$$

recursively in n. The derivation is presented as Lemma (A.44) in Appendix 5.

It is straightforward to show that Lemma (A.44) reduces to (3.32) for $n = 2$, while for $n = 3$ it reads as

$$\begin{aligned}
\mathrm{Tr}_{R_{\omega_1}^{\otimes 3}}\big(\Phi_3\, t^{(\rho,H)}\, \mathsf{g}_1\, \mathsf{g}_2\big) \;=\; (g_{\omega_1})^{-3/2} \Bigg(& t^{N+1}\, \frac{q-1}{t-1} \Big(\frac{q-1}{t-1} + \frac{t-q}{t^2-1} \Big)\, [N]_t \\
& + t^{\frac{N+1}{2}}\, \frac{(q-1)(t-q)}{(t-1)^2}\, [N]_{t^2} \;+\; \frac{(t-q)\,\big(t^2-q\big)}{(t-1)\,\big(t^2-1\big)}\, [N]_{t^3} \Bigg)\;.
\end{aligned} \tag{3.34}$$

Let us look at the unrefined limit $q = t$ of Lemma (A.44). In this case $(q^{-1};q)_{k+1} = 0$ for $k > 0$ from the definition (A.54), so only the $k = 0$ term in (A.44) is non-zero and the sum over partitions in ζ_m receives a non-vanishing contribution from only the maximal partition $\lambda = (1^m)$ with $\ell(\lambda) = m$ parts, and $L_{(1,\dots,1)} = 1$, so that

$$\zeta_m(q,q) = \prod_{i=1}^{m} \frac{1}{q} = q^{-m} \qquad \text{for} \quad m \geq 0\;. \tag{3.35}$$

Then the $q = t$ limit of the (q,t)-trace formula (A.44) becomes

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, t^{(\rho,H)}\, \mathsf{g}_1\, \mathsf{g}_2 \cdots \mathsf{g}_{n-1}\big)\Big|_{q=t} \;=\; q^{(n-1)\,\frac{N+1}{2}}\; [N]_q \tag{3.36}$$

which coincides with the unrefined quantum trace formula of [49, eq. (B.6)]. The ensuing simplicity of the unrefined limit as compared to the general case in Lemma (A.44) is explained in terms of the combinatorics of symmetric functions in Appendix 6.

We can now use Lemma (A.44) to evaluate $\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, t^{(\rho,H)}\, \mathsf{h}(m_T)\big)$. If the conjugacy class T is composed of permutations which have $\mu_i(T)$ cycles of length i, then $n = \sum_i i\, \mu_i(T)$ and we get

$$\begin{aligned}
&\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, t^{(\rho,H)}\, \mathsf{h}(m_T)\big) \;= \\
&(g_{\omega_1})^{-n/2}\, \frac{q^n\, t^{n\,\frac{N+1}{2}}}{(q-1)^{\sum_i \mu_i(T)}} \prod_{i=1}^{n} \Bigg(\sum_{k=0}^{i-1} t^{-(k+1)\,\frac{N+1}{2}}\, \frac{\big(q^{-1};t\big)_{k+1}}{(t;t)_k}\, \zeta_{i-1-k}(q,t)\; [N]_{t^{k+1}} \Bigg)^{\mu_i(T)}\;.
\end{aligned} \tag{3.37}$$

Let us rewrite this formula in terms of the partitions $\mu = \mu(T)$ which parameterize the conjugacy classes $T = T_\mu$ as

$$\begin{aligned} &\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, t^{(\rho,H)}\, \mathsf{h}(m_{T_\mu})\big) = \\ &\qquad (g_{\omega_1})^{-n/2}\, \frac{q^n\, t^{n\,\frac{N+1}{2}}}{(q-1)^{\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} \Big(\sum_{k=1}^{\mu_i} t^{-k\,\frac{N+1}{2}}\, \frac{\big(q^{-1};t\big)_k}{(t;t)_{k-1}}\, \zeta_{\mu_i-k}(q,t)\; [N]_{t^k} \Big) \end{aligned} \tag{3.38}$$

where $\ell(\mu) = \sum_i \mu_i(T)$ is the length of the partition μ.

We can finally substitute the formula (3.38) into (3.19) to get the main result of this section.

Proposition 3.39 *The refined quantum dimensions can be expressed as*

$$\begin{aligned} \frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} \;=\; & \frac{q^{-\frac{n\,(n-5)}{4}}\, t^{n\,\frac{N+1}{2}}}{(g_{\omega_1})^{n/2}\,[n]_q!}\, \frac{d_{\lambda_\beta+a\,(1^N)}(q)}{d_{\lambda_\beta+a\,(1^N)}(1)} \sum_{\mu\in\Lambda_+^n} \frac{q^{-\ell^*(\mu)}}{(q-1)^{\ell(\mu)}}\, \chi_{r_{\lambda_\beta+a\,(1^N)}}(C_\mu) \\ & \times \prod_{i=1}^{\ell(\mu)} \Big(\sum_{k=1}^{\mu_i} t^{-k\,\frac{N+1}{2}}\, \frac{\big(q^{-1};t\big)_k}{(t;t)_{k-1}}\, \zeta_{\mu_i-k}(q,t)\; [N]_{t^k} \Big) \end{aligned}$$

for $\lambda \in \Lambda_+^{n-a\,N}$, where $\ell^*(\mu) = \sum_i (i-1)\,\mu_i = n - \ell(\mu)$ is the colength of the partition μ (the complement to its length) which coincides with the length of the permutation (the minimal number of generators) that belongs to the conjugacy class labelled by μ, and the central Hecke algebra element $C_\mu := C_{T_\mu}$ is defined by (3.21)*. The coefficients $\zeta_m(q,t)$ are defined in* (A.44)*.*

It easy to see that this refined quantum dimension formula reduces at $\beta = 1$ to the quantum dimension formula (2.63) with (2.64).

2 Chiral expansion of the partition function

To explore the relations between the (q,t)-deformed Yang-Mills theory on Σ_h and a refined topological string theory, we consider the topological limit of the gauge theory which is the limit of degree $p = 0$. In this section we will study the partition function which from (2.78) is given by

$$Z_h(q,t;0) = \sum_{n=0}^{\infty} \sum_{\lambda\in\Lambda_+^n} \Big(\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} \Big)^{2-2h}\,, \tag{3.40}$$

analogously to [49]. The chiral expansion is the asymptotic large N expansion defined by dropping the constraint $\ell(\lambda) \leq N$ on the lengths of the partitions λ, as we described

in §1.2. Our main results of this part are summarised in Propositions (3.60) and (3.78).

2.1 Generalised quantum Ω-factors

Let us begin by rewriting the refined quantum dimension from Proposition 3.39 in a simpler condensed form. Following (2.20) and (2.64), we define the element

$$\Omega_n(q,t) \;=\; \frac{t^{n\,\frac{N+1}{2}}}{\big([N]_t\big)^n} \sum_{\mu\in\Lambda_+^n} \Big(\frac{q}{q-1}\Big)^{\ell(\mu)} \times \prod_{i=1}^{\ell(\mu)} \Big(\sum_{k=1}^{\mu_i} t^{-k\,\frac{N+1}{2}} \frac{\big(q^{-1};t\big)_k}{(t;t)_{k-1}}\, \zeta_{\mu_i-k}(q,t)\; [N]_{t^k} \Big)\; C_\mu \;. \tag{3.41}$$

This is a sum of central elements C_μ of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$, so $\Omega_n(q,t)$ is also central. The normalization is chosen so that the identity is the leading term at large N. For this, we note that, under the assumptions $q,t\in(0,1)$, the largest terms of $\Omega_n(q,t)$ in the large N expansion come from rectangular partitions of the form $\mu=(m,\dots,m)$ which give the leading term

$$\Big(\frac{q\,(q^{-1};t)_m}{(q-1)\,(t;t)_{m-1}}\Big)^{\ell(\mu)} \;. \tag{3.42}$$

For $\beta\geq 1$ and $k\in\mathbb{Z}_{>0}$ we have

$$\Big|\,\frac{1-q^{-1}\,t^k}{1-t^k}\,\Big| = \Big|\,\frac{1-t^{k-\frac{1}{\beta}}}{1-t^k}\,\Big| < 1 \;. \tag{3.43}$$

This implies that the absolute value of (3.42) is less than 1, unless $\mu=(1,\dots,1)$ in which case it is equal to 1. Hence the maximal partition $(1,\dots,1)$ is the leading term which corresponds to the identity permutation, and we can write

$$\Omega_n(q,t) \;=\; 1 \,+\, \Omega'_n(q,t) \;, \tag{3.44}$$

where $\Omega'_n(q,t)$ has the same form as $\Omega_n(q,t)$ except that the sum runs over all non-maximal partitions μ of n.

The $q=t$ limit of (3.41)

$$\Omega_n(q,q) = \sum_{\mu\in\Lambda_+^n} q^{\frac{N-1}{2}\,\ell(\mu)}\; \big([N]_q\big)^{-\ell^*(\mu)}\; C_\mu \tag{3.45}$$

coincides with the unrefined element defined by (2.64). As we discuss further below, the power $\big([N]_q\big)^{-\ell^*(\mu)}$ appearing here suggests an interpretation of $\Omega_n(q,q)$ in terms

of branch points on Σ_h in a topological string theory of worldsheet branched covers of the target Riemann surface Σ_h, with string coupling $g_{\rm str} = \frac{1}{[N]_q}$.

With this new notation we can write the result of Proposition (3.39) as

$$\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} = \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\,\frac{d_{\lambda_\beta+a\,(1^N)}(q)}{d_{\lambda_\beta+a\,(1^N)}(1)}\,\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^n\,\chi_{r_{\lambda_\beta+a\,(1^N)}}\big(\Omega_n(q,t)\big)\ . \tag{3.46}$$

This formula is very similar to the unrefined one from (2.63), except that in our case the expansion parameter is

$$\frac{[N]_t}{\sqrt{g_{\omega_1}}} = \Bigg(\frac{\big[N\big]_t\,\big[\beta\,(N-1)+1\big]_q}{g_\emptyset}\Bigg)^{1/2} \tag{3.47}$$

and $g_\emptyset$ is of order 1 in the large N limit. This expansion parameter respects the Ω-background symmetry $(\epsilon_1,\epsilon_2)\mapsto(-\epsilon_2,-\epsilon_1)$ described in §3.2.1; however, fixing $p=0$ breaks this symmetry of the topological partition function in the ensuing large N expansion.

2.2 Chiral series

We next collect every central element under a single character using the formula (2.68). This implies

$$\begin{aligned} Z_h(q,t;0) = \sum_{n=a\,N}^{\infty}\ \sum_{\lambda\in\Lambda_+^{n-a\,N}}\ &\Big(\frac{q^{-\frac{n\,(n-1)}{4}}\,d_{\lambda_\beta+a\,(1^N)}(q)}{[n]_q!}\Big)^{2-2h}\,\frac{1}{d_{\lambda_\beta+a\,(1^N)}(1)} \\ &\times\ \Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)}\,\chi_{r_{\lambda_\beta+a\,(1^N)}}\big(\Omega_n(q,t)^{2-2h}\big)\ . \end{aligned} \tag{3.48}$$

Because of our normalization (3.44), the element $\Omega_n(q,t)$ is always formally invertible in the large N expansion.

By (2.65) we have

$$\begin{aligned} \Big(\frac{[n]_q!}{q^{-\frac{n\,(n-1)}{4}}\,d_{\lambda_\beta+a\,(1^N)}(q)}\Big)^2 \ &= \ \frac{1}{d_{\lambda_\beta+a\,(1^N)}(1)}\sum_{\sigma,\tau\in\mathfrak{S}_n} q^{-\ell(\sigma)-\ell(\tau)} \\ &\qquad\times\ \chi_{r_{\lambda_\beta+a\,(1^N)}}\big(\mathsf{h}(\sigma)\,\mathsf{h}(\tau)\,\mathsf{h}(\sigma^{-1})\,\mathsf{h}(\tau^{-1})\big) \end{aligned} \tag{3.49}$$

which yields

$$
\begin{aligned}
Z_h(q,t;0) \;=\; & \sum_{n=a\,N}^{\infty} \sum_{\lambda\in\Lambda_+^{n-a\,N}} \frac{1}{d_{\lambda_\beta+a\,(1^N)}(1)} \Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)} \\
& \times \Big(\frac{q^{-\frac{n\,(n-1)}{4}}\, d_{\lambda_\beta+a\,(1^N)}(q)}{[n]_q!}\Big)^2 \sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n} q^{-\sum_i\,(\ell(\sigma_i)+\ell(\tau_i))} \\
& \times \chi_{r_{\lambda_\beta+a\,(1^N)}}\Big(\Omega_n(q,t)^{2-2h} \prod_{i=1}^{h} \mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\Big) \;,
\end{aligned}
\tag{3.50}
$$

where we used (2.68) and the centrality property of (3.49) from [49]. We use the element defined in (2.67) to express the q-dimension $d_{\lambda_\beta+a\,(1^N)}(q)$ in terms of the character of D_n as

$$
d_{\lambda_\beta+a\,(1^N)}(q) \;=\; \chi_{r_{\lambda_\beta+a\,(1^N)}}(D_n) \;. \tag{3.51}
$$

Since D_n is central in $\widehat{\mathsf{H}}_q(\mathfrak{S}_n)$, using (2.68) we get

$$
\begin{aligned}
Z_h(q,t;0) \;=\; & \sum_{n=a\,N}^{\infty} \frac{q^{-\frac{n\,(n-1)}{2}}}{\big([n]_q!\big)^2} \Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)} \\
& \times \sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n} q^{-\sum_i\,(\ell(\sigma_i)+\ell(\tau_i))} \sum_{\lambda\in\Lambda_+^{n-a\,N}} d_{\lambda_\beta+a\,(1^N)}(q) \\
& \times \chi_{r_{\lambda_\beta+a\,(1^N)}}\Big(D_n\,\Omega_n(q,t)^{2-2h} \prod_{i=1}^{h} \mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\Big) \;.
\end{aligned}
\tag{3.52}
$$

The delta-function on Hecke algebras are defined by (2.69) and (2.70). To write the partition function in terms of delta-functions as in the unrefined case, we have to take the sum over all partitions of n. There is a bijection between partitions $\alpha\in\Lambda_+^n$ such that $\alpha_i \geq (\beta-1)\,\rho_i + a = (\beta-1)\,(N-i)$ for $i=1,\ldots,N$ and partitions in $\Lambda_+^{n-a\,N}$. Thus we need to construct a step function $\Theta_n(\beta)$ on partitions that cuts off the contributions involving smaller partitions and allows us to sum over all $\alpha\in\Lambda_+^n$; it is defined by the property

$$
\chi_\alpha\big(\Theta_n(\beta)\big) \;=\; \begin{cases} d_\alpha(1) & \text{if} \quad \alpha_i \geq (\beta-1)\,(N-i) \;, \\ 0 & \text{otherwise} \;, \end{cases} \tag{3.53}
$$

for $\alpha\in\Lambda_+^n$. The sum of quantum Young projectors (2.61) given by

$$
\Theta_n(\beta) \;=\; \sum_{\substack{\mu\in\Lambda_+^n \\ \mu_i\geq(\beta-1)\,(N-i)}} P_\mu \tag{3.54}
$$

fulfills this criterion, and it is a central element of $\mathsf{H}_q(\mathfrak{S}_n)$ because the projectors are central.

We now insert $\big(d_{\lambda_\beta+a\,(1^N)}(1)\big)^{-1}\,\chi_{\lambda_\beta+a\,(1^N)}\big(\Theta_n(\beta)\big)=1$ in (3.52) and using (2.68) we get

$$\begin{aligned} Z_h(q,t;0) \;=\; & \sum_{n=a\,N}^{\infty} \frac{q^{-\frac{n\,(n-1)}{2}}}{\big([n]_q!\big)^2}\,\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)} \\ & \times \sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n} q^{-\sum_i\,(\ell(\sigma_i)+\ell(\tau_i))} \sum_{\lambda\in\Lambda_+^{n-a\,N}} d_{\lambda_\beta+a\,(1^N)}(q) \\ & \times\; \chi_{r_{\lambda_\beta+a\,(1^N)}}\Big(\Theta_n(\beta)\,D_n\,\Omega_n(q,t)^{2-2h}\,\prod_{i=1}^{h}\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\Big)\;. \end{aligned} \tag{3.55}$$

A partition $\alpha\in\Lambda_+^n$ satisfies $\alpha_i\geq(\beta-1)\,(N-i)$ for $i=1,\ldots,N$ if and only if it can be written as $\alpha=\lambda_\beta+a\,(1^N)$ for some $\lambda\in\Lambda_+^{n-a\,N}$. The contributions involving $\alpha_i<(\beta-1)\,(N-i)$ for some i in the partition function vanish because of the step function $\Theta_n(\beta)$. Hence we can shift the summation range and sum over all $\alpha\in\Lambda_+^n$ to get

$$\begin{aligned} Z_h(q,t;0) \;=\; & \sum_{n=a\,N}^{\infty} \frac{q^{-\frac{n\,(n-1)}{2}}}{\big([n]_q!\big)^2}\,\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)} \sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n} q^{-\sum_i\,(\ell(\sigma_i)+\ell(\tau_i))} \\ & \times \sum_{\alpha\in\Lambda_+^n} d_\alpha(q)\,\chi_{r_\alpha}\Big(\Theta_n(\beta)\,D_n\,\Omega_n(q,t)^{2-2h}\,\prod_{i=1}^{h}\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\Big)\;, \end{aligned} \tag{3.56}$$

and then using the expression for the delta-function from (2.70) we arrive at

$$\begin{aligned} Z_h(q,t;0) \;=\; & \sum_{n=\lceil a\,N\rceil}^{\infty} \Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)}\,\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!} \\ \times & \sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n} \delta\Big(\Theta_n(\beta)\,D_n\,\Omega_n(q,t)^{2-2h}\,\prod_{i=1}^{h} q^{-\ell(\sigma_i)-\ell(\tau_i)}\,\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\Big)\;. \end{aligned} \tag{3.57}$$

Now we can analytically continue β away from integer values, because this expansion only depends on β in the q-numbers and in a, and we can choose a larger value for a; the smallest choice is a_0 such that $a_0\,N=\lceil a\,N\rceil$, and $\lceil a\,N\rceil$ also vanishes in the unrefined limit. The restriction on the sum in the definition of the step function $\Theta_n(\beta)$ from (3.54) can also be continued in a straightforward way. This expansion is the refined version of (2.71); it is a refined quantum deformation of the large N chiral

Gross-Taylor expansion given by (2.18). The Hecke element $\Theta_n(\beta)$ does not have any unrefined analog, as it reduces to $\Theta_n(\beta = 1) = \sum_{\mu \in \Lambda_+^n} P_\mu = 1$ by (2.61) together with the expression for the delta-function from (2.70) and we recover the unrefined expansion.

Finally, using (3.44) we can expand the Ω-factors $\Omega_n(q,t)^{2-2h}$ in the completion $\widehat{\mathsf{H}}_q(\mathfrak{S}_n)$ via the power series

$$\Omega_n(q,t)^{2-2h} \;=\; \sum_{L=0}^{\infty} d(2-2h,L)\; \Omega'_n(q,t)^L \;, \tag{3.58}$$

where

$$d(m,L) \;=\; \frac{\Gamma(m+1)}{\Gamma(L+1)\,\Gamma(m-L+1)} \;. \tag{3.59}$$

As explained in [34], the binomial coefficient $d(2-2h,L)$ is the Euler characteristic $\chi(\Sigma_{h,L})$ of the configuration space of L points on the Riemann surface Σ_h, also given by (2.23). In this way we arrive at

Proposition 3.60 *The chiral series for the partition function of topological (q,t)-deformed Yang-Mills theory on Σ_h is given by*

$$\begin{aligned} Z_h(q,t;0) \;=\; & \sum_{n=\lceil a\,N\rceil}^{\infty} \sum_{L=0}^{\infty} \big(g_{\omega_1}\big)^{n\,(h-1)}\; \big([N]_t\big)^{n\,(2-2h-L)}\; \frac{\chi(\Sigma_{h,L})}{[n]_q!}\; q^{-\frac{n\,(n-1)}{4}}\; t^{n\,L\,\frac{N+1}{2}} \\ & \times \sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h \in \mathfrak{S}_n} \prod_{l=1}^{L} \sum_{\substack{\mu^l \in \Lambda_+^n \\ \mu^l \neq (1^n)}} \Big(\frac{q}{q-1}\Big)^{\ell(\mu^l)} \\ & \times \prod_{i=1}^{\ell(\mu^l)} \Big(\sum_{k=1}^{\mu_i^l} t^{-k\,\frac{N+1}{2}}\; \frac{\big(q^{-1};t\big)_k}{(t;t)_{k-1}}\, \zeta_{\mu_i^l-k}(q,t)\; [N]_{t^k} \Big) \\ & \times \delta\Big(\Theta_n(\beta)\, D_n\, C_{\mu^1} \cdots C_{\mu^L} \prod_{j=1}^{h} q^{-\ell(\sigma_j)-\ell(\tau_j)}\, \mathsf{h}(\sigma_j)\, \mathsf{h}(\tau_j)\, \mathsf{h}(\sigma_j^{-1})\, \mathsf{h}(\tau_j^{-1})\Big) \;. \end{aligned} \tag{3.61}$$

Here the central Hecke algebra elements $\Theta_n(\beta)$ and D_n are defined in (3.54) *and* (2.67)*.*

This is a refined quantum deformation of the symmetric group enumeration of covering maps of the Riemann surface Σ_h, analogously to the description in terms of quantum spectral curves discussed in Appendix 7. In particular, following [49] it is tempting to suppose that this expansion is captured by a balanced topological string theory [61] with target space the M-theory compactification described in §3.2.1, which

would naturally compute Euler characters of certain moduli spaces of curves in this background. We elaborate further on these points below.

In the unrefined limit $q = t$, the asymptotic expansion of Proposition 3.60 becomes (2.71) independently of the parts of the partitions $\mu^1, \ldots, \mu^L$.

2.3 β-deformed Hurwitz theory

To understand better the geometrical effect of refinement as it occurs in the expansion of Proposition 3.60, let us consider the classical limit $q \to 1$ with β fixed. In this limit the Macdonald polynomials reduce to the Jack polynomials which are ordinary generalized characters of irreducible $U(N)$ representations [54], and the Hecke algebra reduces to the ordinary group algebra of the symmetric group $\mathbb{C}[\mathfrak{S}_n]$. It is straightforward to show that the Ω-factors reduce to

$$\lim_{q\to 1} \Omega_n(q,t)\big|_{t=q^\beta} \;=\; \sum_{\mu\in\Lambda_+^n} \frac{\Delta_\mu(\beta)}{N^{\ell^*(\mu)}}\, c_\mu \;, \tag{3.62}$$

where $c_\mu = \sum_{\sigma\in T_\mu} \sigma$ is the undeformed version of (3.21), and we have defined

$$\Delta_\mu(\beta) \;=\; \prod_{i=1}^{\ell(\mu)} \Big(\sum_{k=1}^{\mu_i} \gamma_k(\beta) \sum_{\lambda\in\Lambda_+^{\mu_i-k}} \beta^{-\ell(\lambda)}\, \frac{\ell(\lambda)!}{z_\lambda} \prod_{j=1}^{\ell(\lambda)} \gamma_{\lambda_j}(\beta) \Big) \;, \tag{3.63}$$

with

$$\begin{aligned} \gamma_1(\beta) \;&=\; 1 \;, \\ \gamma_k(\beta) \;&=\; \prod_{l=1}^{k-1} \frac{\beta\, l - 1}{\beta\, l} \;=\; \frac{\Gamma(k-\frac{1}{\beta})}{\Gamma(1-\frac{1}{\beta})\,\Gamma(k)} \qquad \text{for} \quad k>1 \;. \end{aligned} \tag{3.64}$$

The integer

$$z_\lambda = \prod_{i=1}^{|\lambda|} i^{m_i(\lambda)}\, m_i(\lambda)! \tag{3.65}$$

is the order of the stabilizer, under conjugation, of any element of the conjugacy class T_λ. In the unrefined limit, $\Delta_\mu(\beta) \to 1$ as $\beta \to 1$ and (3.62) coincides with the unrefined Ω-factor from (2.20), in which case the weights of the sum depend only on the colengths $\ell^*(\mu) = n - \ell(\mu)$ of the partitions $\mu \in \Lambda_+^n$ (the same is true of the unrefined q-deformed Ω-factors (3.45)). In marked contrast, for $\beta \neq 1$ the weights depend explicitly on the parts of the partition μ through the combinatorial coefficients $\Delta_\mu(\beta)$.

The expansion into Euler characters given in Proposition 3.60 reduces to

$$\begin{aligned}\widetilde{Z}_h(\beta) &:= \lim_{q\to 1} Z_h(q,t;0)\big|_{t=q^\beta} \\ &= \sum_{n=\lceil a\,N\rceil}^{\infty} \tilde{g}(\beta)^{n\,(h-1)} \sum_{L=0}^{\infty} \frac{\chi(\Sigma_{h,L})}{n!} \prod_{l=1}^{L} \sum_{\substack{\mu^l\in\Lambda_+^n \\ \mu^l\neq(1^n)}} \varDelta_{\mu^l}(\beta)\; N^{n\,(2-2h)-\sum_{l=1}^{L}\,\ell^*(\mu^l)} \\ &\qquad \times \sum_{\lambda\in\Lambda_+^n} \omega_\lambda(\beta) \sum_{\sigma_1,\tau_1,\ldots,\sigma_h,\tau_h\in\mathfrak{S}_n} \delta\Big(c_\lambda\, c_{\mu^1}\cdots c_{\mu^L} \prod_{i=1}^{h} [\sigma_i,\tau_i]\Big)\ ,\end{aligned} \tag{3.66}$$

where $[\sigma,\tau]=\sigma\,\tau\,\sigma^{-1}\,\tau^{-1}$ denotes the *group* commutator and

$$\tilde{g}(\beta) := \frac{N}{\beta\,(N-1)+1} \prod_{m=0}^{\beta-1} \prod_{1\le i<j\le N} \frac{\beta\,(j-i)+m}{\beta\,(j-i)-m}\ . \tag{3.67}$$

We used (3.22) and (3.23), and defined a new deformation weight

$$\omega_\lambda(\beta) \;=\; \frac{1}{n!} \sum_{\substack{\mu\in\Lambda_+^n \\ \mu_i\ge(\beta-1)\,(N-i)}} d_\mu\, \chi_{r_\mu}(m_{T_\lambda})\ , \tag{3.68}$$

which reduces to $\delta(c_\lambda)=\delta_{\lambda,(1^n)}$ in the $\beta=1$ limit. Rewriting this expansion entirely as sums over elements of the symmetric group $\mathfrak{S}_n$ reveals that it is a β-deformation of the ordinary chiral Gross-Taylor series given in (2.22), containing an extra class sum, and with extra deformation weights $\varDelta_\mu(\beta)$ and $\omega_\lambda(\beta)$. In this expansion the weights depend explicitly on the parts of the partitions and not only on their colengths, although they are decoupled according to distinct partitions.

To make contact with Hurwitz theory, we shall collect terms with a fixed value of the integer

$$B=\sum_{l=1}^{L} \ell^*(\mu^l) \tag{3.69}$$

and set

$$2g-2=n\,(2h-2)+B\ , \tag{3.70}$$

which has the form as a Riemann-Hurwitz formula defined in (A.4), where h is the genus of the covered surface Σ_h, g is the genus of the covering surface Σ_g, n is the number of covering sheets, L is the number of fixed branched point and B is the branching number of the cover. The Hurwitz number introduced in (A.10) counts the number of branched covers like this, whose branch points have ramification profiles

specified by partitions. We incorporate all weights corresponding to the partitions, and express (3.66) with Hurwitz numbers. Thus we have

$$\begin{aligned}\widetilde{Z}_h(\beta) \;=\; & \sum_{n=\lceil a\,N\rceil}^{\infty} \tilde{g}(\beta)^{n\,(h-1)} \sum_{\lambda\in\Lambda_+^n} \omega_\lambda(\beta) \sum_{B=0}^{\infty} \Big(\frac{1}{N}\Big)^{2g-2} \sum_{L=0}^{B} \chi(\Sigma_{h,L}) \\ & \times \sum_{\substack{\mu^1,\ldots,\mu^L\in\Lambda_+^n \\ \mu^l\neq(1^n)\,,\,\sum_{l=1}^L \ell^*(\mu^l)=d}} \Delta_{\mu^1}(\beta)\cdots\Delta_{\mu^L}(\beta)\; H_{h,n}\big(\lambda,\mu^1,\ldots,\mu^L\big)\;. \end{aligned} \tag{3.71}$$

From this expression we can infer at least four novel aspects of the closed string expansion of the β-deformation of two-dimensional Yang-Mills theory, interpreted from the geometric point of view of Hurwitz theory:

1. Branched covers of index $n < \lceil a\,N\rceil$ do not contribute to the string expansion. This feature has important ramifications for the planar limit of the gauge theory which we discuss below.

2. The refinement introduces an additional weighting by the quantity (3.67) such that the expansion parameter does not simply distinguish the genera of the covering worldsheets Σ_g. Below we shall replace this weight with its leading term $\tilde{g}(\beta) = \frac{1}{\beta}$ in the large N limit.

3. The string expansion (3.71) generically involves deformations of the enumeration of branched covers $f : \Sigma_g \to \Sigma_h$ in terms of Hurwitz numbers $H_{h,n}(\mu^1,\ldots,\mu^L)$ which include an additional marked point with holonomy in the representation (2.87) of $U(N)$; the inclusion of such marked points is the earmark of refinement and is captured by the intertwining operators defining the generalised characters [35,53]. Accordingly, the Hurwitz numbers $H_{h,n}\big(\lambda,\mu^1,\ldots,\mu^L\big)$ account for additional branching over this marked point with ramification profile $\lambda\in\Lambda_+^n$. Due to the deformation weights $\omega_\lambda(\beta)$, for $\beta\neq 1$ their contributions are strongly suppressed in the large N limit.

4. The expansion (3.71) involves *weighted* sums of Hurwitz numbers, with deformation weights $\Delta_{\mu^l}(\beta)$ and $\omega_\lambda(\beta)$ depending explicitly on the parts of the partitions μ^l and λ which label the winding numbers of closed strings around the branch points in the target space Σ_h. This deformation obstructs a rewriting of the partition function as a generating function of characters of Hurwitz spaces of holomorphic maps $f : \Sigma_g \to \Sigma_h$, as occurs in the unrefined case [34], and as such an interpretation as a balanced topological string theory [61] with string coupling $g_{\rm str} = \frac{1}{N}$ and two-dimensional target space Σ_h is not immediately evident. In fact, this weighting suggests that the string expansion involves contributions from *marked*

covers $f^{\rm m} : \Sigma_g \to \Sigma_h$; a marking of a branched cover $f : \Sigma_g \to \Sigma_h$ consists of a marking of each of its branch points z^l for $l = 1, \ldots, L$, i.e. a choice of labelling $\big\{w_1^l, \ldots, w_{\ell(\mu^l)}^l\big\} = f^{-1}(z^l)$ such that μ_i^l is the ramification index at w_i^l. An automorphism $\alpha : \Sigma_g \to \Sigma_g$ of a marked cover preserves the labels w_i^l, and we denote the corresponding marked cover counts by $H_{h,n}^{\rm m}(\mu^1, \ldots, \mu^L)$. The action of the automorphism group $\mathrm{Aut}(f)$ on the labels of $f^{\rm m}$ gives a group homomorphism

$$\mathrm{Aut}(f) \;\longrightarrow\; \prod_{l=1}^{L} \mathrm{Aut}\big(\mu^l\big) \tag{3.72}$$

whose kernel is $\mathrm{Aut}(f^{\rm m})$ and whose image has index given by the number m of markings of f (up to isomorphism). It follows that $|\mathrm{Aut}(f)|\, m = |\mathrm{Aut}(f^{\rm m})| \prod_{l=1}^{L} |\mathrm{Aut}(\mu^l)|$, and hence the combinatorial expansion (3.71) can be written in terms of marked Hurwitz numbers as

$$\begin{aligned} \widetilde{Z}_h(\beta) \;&=\; \sum_{n=\lceil a\,N\rceil}^{\infty} \beta^{-n\,(h-1)} \sum_{\lambda\in\Lambda_+^n} \frac{\omega_\lambda(\beta)}{\big|\mathrm{Aut}(\lambda)\big|} \sum_{B=0}^{\infty} \Big(\frac{1}{N}\Big)^{2g-2} \sum_{L=0}^{B} \chi(\Sigma_{h,L}) \\ &\quad\times \sum_{\substack{\mu^1,\ldots,\mu^L\in\Lambda_+^n \\ \mu^l\neq(1^n)\,,\,\sum_{l=1}^L \ell^*(\mu^l)=d}} \frac{\Delta_{\mu^1}(\beta)}{\big|\mathrm{Aut}\big(\mu^1\big)\big|} \cdots \frac{\Delta_{\mu^L}(\beta)}{\big|\mathrm{Aut}\big(\mu^L\big)\big|}\; H_{h,n}^{\rm m}\big(\lambda, \mu^1, \ldots, \mu^L\big)\;. \end{aligned} \tag{3.73}$$

While the refined weights (3.68) have a natural meaning as deformations of the identity, it would be interesting to understand better the geometrical significance of the combinatorial weights (3.63) in terms of orbifold Euler characteristics of moduli spaces of Riemann surfaces, as suggested by the appearence of the binomial-type coefficients (3.64). We can give further insight into this perspective following the geometric interpretation of refinement from Appendix 7. Let $\mathcal{H}_{n,B,h,L}$ denote the Hurwitz space of isomorphism classes of n-sheeted branched covers $f : \Sigma_g \to \Sigma_h$ with branching number B and L branch points. It has the structure of a discrete fibration

$$\pi_{n,B,h,L} \,:\, \mathcal{H}_{n,B,h,L} \;\longrightarrow\; \Sigma_{h,L} \tag{3.74}$$

over the configuration space of L indistinguishable points on Σ_h, which sends the class of a holomorphic map $f : \Sigma_g \to \Sigma_h$ to the branch locus of f. There is also a natural map

$$\mathcal{H}_{n,B,h,L} \;\longrightarrow\; \mathcal{M}_g \tag{3.75}$$

which sends the class of the cover $f : \Sigma_g \to \Sigma_h$ to the class of the curve Σ_g; the image of $\mathcal{H}_{n,B,h,L}$ under this map is a subvariety of the moduli space $\mathcal{M}_g$ of genus g curves.

Recall from Appendix 7 that, in the planar limit of the gauge theory on the sphere, refinement can be interpreted geometrically as replacing the orbifold Euler characters $\chi_{\rm orb}(\mathcal{M}_g)$ with the parameterized Euler characters (A.105). It is natural to think of pulling back the corresponding characteristic classes under the map (3.75), and for fixed N we define the *parameterized Euler character*

$$\chi_{n,B,h,L}(\beta) \;:=\; \sum_{\lambda\in\Lambda_+^n} \frac{\omega_\lambda(\beta)}{\beta^{n\,(h-1)}} \sum_{\substack{\mu^1,\ldots,\mu^L\in\Lambda_+^n\\ \mu^l\neq(1^n)\,,\,\sum_{l=1}^L \ell^*(\mu^l)=B}} \Delta_{\mu^1}(\beta)\cdots\Delta_{\mu^L}(\beta) \times \chi(\Sigma_{h,L})\,H_{h,n}\big(\lambda,\mu^1,\ldots,\mu^L\big) \tag{3.76}$$

for $n\geq\lceil a\,N\rceil$. Via the fibration (3.74), in the unrefined limit it reduces to the orbifold Euler character

$$\chi_{n,B,h,L}(1)=\chi_{\rm orb}(\mathcal{H}_{n,B,h,L}) \tag{3.77}$$

given by (2.24). Then we can rewrite (3.71) in the more suggestive form

Proposition 3.78 *The chiral series for the partition function of topological β-deformed Yang-Mills theory on Σ_h is the generating function*

$$\widetilde{Z}_h(\beta)=\sum_{n=\lceil a\,N\rceil}^{\infty}\sum_{B=0}^{\infty}\Big(\frac{1}{N}\Big)^{2g-2}\sum_{L=0}^{B}\chi_{n,B,h,L}(\beta)$$

for the parameterized Euler characters (3.76)*, where g is determined from n, h and B by the Riemann-Hurwitz formula* (3.70)*.*

This generalizes the string theory interpretation of the unrefined case §1.2, wherein the orbifold Euler characters of Hurwitz spaces $\chi_{\rm orb}(\mathcal{H}_{n,B,h,L})$ are replaced under refinement by the parameterized Euler characters $\chi_{n,B,h,L}(\beta)$. As in Appendix 7, it is natural to expect that these β-deformed characters are themselves associated to characteristic classes of some related moduli spaces; in particular, for $\beta=2$ the deformation weights are given by

$$\Delta_\mu(2)=2^{\ell^*(\mu)}\prod_{i=1}^{\ell(\mu)}\Big(\sum_{k=1}^{\mu_i}\frac{(2k-3)!!}{(k-1)!}\sum_{\lambda\in\Lambda_+^{\mu_i-k}}\frac{\ell(\lambda)!}{z_\lambda}\prod_{j=1}^{\ell(\lambda)}\frac{(2\lambda_j-3)!!}{(\lambda_j-1)!}\Big)\,. \tag{3.79}$$

However, in the present case the characters are non-polynomial functions of β; below we will compare their forms explicitly with the parameterized Euler characters (A.105).

For the case of a spherical target space $\Sigma_0 = \mathbb{P}^1$, certain classes of Hurwitz numbers can be expressed as integrals of psi-classes and Hodge classes over the Deligne-Mumford moduli spaces of punctured curves $\overline{\mathcal{M}}_{g,n}$ [62]. The corresponding partition functions are annihilated by the differential operator of a quantum curve, see e.g. [63] for a review; it would be interesting to see if there is a similar quantum spectral curve underlying the partition function $\widetilde{Z}_0(\beta)$. On the other hand, orbifold Hurwitz numbers lead to partition functions which are annihilated by the difference operator of a quantum curve [63], and it would be interesting to understand the general (q,t)-deformed partition function $Z_0(q,t;0)$ also in this context.

2.4 Planar limit

In the planar limit of Appendix 7, the pertinent generalised Selberg integrals can also be expressed in terms of Jack symmetric functions [64], which gives a geometrical meaning to the refinement parameter β as a combinatorial invariant of non-orientability for maps of graphs into surfaces. Let us now compare the leading term of the partition function (3.57) for $h \geq 2$ in the classical limit $q = 1$ with the parameterized Euler characteristics (A.105). This amounts to setting the Ω-factors $\Omega_n(q,t)$ to 1 and keeping only the $n = a\,N$ term of the sum in (3.57), which yields

$$Z_h^{\rm pl}(q,t;0) \;=\; \frac{q^{-\frac{a\,N\,(a\,N-1)}{4}}}{[a\,N]_q!}\,\big(d_{(\beta-1)\,\rho+a\,(1^N)}(q)\big)^{2-2h}\,\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{a\,N\,(2-2h)} \;. \tag{3.80}$$

In the classical limit this becomes

$$\widetilde{Z}_h^{\rm pl}(\beta) = \frac{1}{(a\,N)!}\,\big(d_{(\beta-1)\,\rho+a\,(1^N)}\big)^{2-2h}\,\Big(\frac{N^4}{\tilde{g}(\beta)}\Big)^{a\,N\,(1-h)} \;, \tag{3.81}$$

where we used (3.23). We can rewrite (3.67) in the form

$$\tilde{g}(\beta) = \frac{N}{\beta\,(N-1)+1}\,\frac{\prod\limits_{i=1}^{N}\frac{\Gamma(\beta\,i)}{\Gamma(\beta)}}{\prod\limits_{m=0}^{\beta-1}\prod\limits_{i=1}^{N-1}\frac{\beta^i\,\Gamma\big(i+1-\frac{m}{\beta}\big)}{\Gamma\big(1-\frac{m}{\beta}\big)}} \;, \tag{3.82}$$

and using the dimension formula (A.19) in Appendix 3 we can write the dimension of the symmetric group representation corresponding to the partition $(\beta-1)\,\rho + a\,(1^N)$ as

$$d_{(\beta-1)\,\rho+a\,(1^N)} = \frac{\beta^{\frac{N\,(N-1)}{2}}\,\Gamma(a\,N+1)\,G(N+1)}{\prod\limits_{i=1}^{N-1}\Gamma(1+\beta\,i)} \;, \tag{3.83}$$

where $G(z)$ is the Barnes G-function with the property that $G(N+1) = \prod_{i=1}^{N} \Gamma(i)$. The appearence of this Barnes function suggests, following [65], that our asymptotic expansion could be related to refined topological closed string theory on the resolved conifold geometry.

The corresponding free energy $\widetilde{F}_h^{\rm pl}(\beta) := -\log \widetilde{Z}_h^{\rm pl}(\beta)$ can be expanded as a power series in β, whose coefficients are combinations of Bernoulli numbers, in much the same way that we dealt with the partition function (A.97). The resulting expansion is somewhat complicated, so we content ourselves with an integral representation from which the expansion is straightforwardly extracted. For this, we use the integral formula for the gamma-function [66]

$$\log \Gamma(z) = \int_0^\infty \frac{\mathrm{d}x}{x} \, \frac{1}{\mathrm{e}^{\,x} - 1} \, \big((z-1)\,(1 - \mathrm{e}^{\,-x}) + \mathrm{e}^{\,-x\,(z-1)} - 1 \big) \ , \tag{3.84}$$

which holds for $\mathfrak{Re}(z) > 0$. After some calculation, one infers the free energy

$$\begin{aligned} \widetilde{F}_h^{\rm pl}(\beta) \ = \ & (h-1) \big(N\,(N-1)\,\log\beta \ - \ \tfrac{1}{2}\,a\,N^2\,(N-1)\,\beta\,\log\beta \ - \ 3\,a\,N\,\log N \\ & - \ a\,N\,\log(\beta\,(N-1)+1) \big) \ + \ \int_0^\infty \frac{\mathrm{d}x}{x} \, \frac{1}{\mathrm{e}^{\,x} - 1} \, \mathcal{F}_h^{\beta,N}(x) \ , \end{aligned} \tag{3.85}$$

where we have defined

$$\begin{aligned} \mathcal{F}_h^{\beta,N}(x) = \ & a\,N \big(1 + \tfrac{\beta}{2}\,N\,(N-1)\,(1-h) \big) \big(1 - \mathrm{e}^{\,-x} \big) \\ & + \ (2h-2) \Big(\frac{1 - \mathrm{e}^{\,-\beta\,(N-1)\,x}}{1 - \mathrm{e}^{\,\beta\,x}} \ - \ \frac{1 - \mathrm{e}^{\,-(N-1)\,x}}{1 - \mathrm{e}^{\,x}} \Big) + (2h-1) \ \big(\mathrm{e}^{\,-a\,N\,x} - 1 \big) \\ & + \ a\,N\,(h-1) \Big(\frac{\mathrm{e}^{\,-\beta\,(N-1)\,x} + \mathrm{e}^{\,x} - \mathrm{e}^{\,-(\beta\,N-1)\,x} - 1}{1 - \mathrm{e}^{\,\beta\,x}} - N + 1 \\ & \qquad + \ N\,\mathrm{e}^{\,-(\beta-1)\,x} + \tfrac{\beta}{2}\,N\,(N-1)\,\mathrm{e}^{\,-\beta\,x} \big(\mathrm{e}^{\,x} - 1 \big) \Big) \ . \end{aligned} \tag{3.86}$$

In the $\beta = 1$ limit (which also induces $a = 0$) the planar free energy vanishes, as we expect from (3.80).

The power series expansion in β can now be obtained by using the generating function (A.102) to expand the denominators $(1 - \mathrm{e}^{\,\beta\,x})^{-1}$ and the integral identities of [66, Appendix A]. For example, we can readily compute the contribution

$$-\int_0^\infty \frac{\mathrm{d}x}{x} \, \frac{1}{\mathrm{e}^{\,x} - 1} \, \frac{1 - \mathrm{e}^{\,-\beta\,(N-1)\,x}}{1 - \mathrm{e}^{\,\beta\,x}} \ = \ \sum_{n=0}^{\infty} \mathcal{F}_n^N \ \beta^n \tag{3.87}$$

where

$$\begin{aligned} \mathcal{F}_0^N \ &= \ (N-1) \big(\tfrac{1}{\varepsilon} + \tfrac{1}{2}\,\log\varepsilon + \tfrac{1}{2}\,(\gamma - \log 2\pi) \big) \ , \\ \mathcal{F}_n^N \ &= \ \zeta(n)\,(n-1)! \sum_{k=0}^{n} (N-1)^{k+1} \, \frac{B_{n-k}}{(k+1)!\,(n-k)!} \qquad \text{for} \quad n \geq 1 \ . \end{aligned} \tag{3.88}$$

Here $\varepsilon \to 0^+$ gives the leading one-loop linear and logarithmic divergences, γ is the Euler-Mascheroni constant, and $\zeta(z)$ is the Riemann zeta-function. These formulas explicitly illustrate the analytic dependence of the parameterized Euler characteristics (3.76) on the refinement parameter β, as compared to the polynomial characters (A.105).

3 Chiral expansions of defect observables

As we discussed in §1.1 in Chapter 2 two-dimensional Yang-Mills theory also involves observables corresponding to insertions in the partition function of operators supported on real codimension one defects in Σ_h. In this section we extend the chiral expansion of §2 to these observables.

3.1 Boundaries

We first describe the large N expansion of the refinement of q-deformed Yang-Mills theory on open Riemann surfaces of genus h with b boundaries. The corresponding partition function has been introduced in (2.91). Again we consider only the topological gauge theory and study

$$Z_{h,b}(q,t;0;U_1,\ldots,U_b) = \sum_{n=0}^{\infty} \sum_{\lambda\in\Lambda_+^n} \Big(\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}}\Big)^{2-2h-b} \prod_{i=1}^{b} \chi_{\Phi_\lambda}(U_i) \ . \tag{3.89}$$

This partition function was also considered in [52] but with a different normalization for the boundary characters.

We begin by using the transformation from Appendix 4 to change to a basis of central elements C_i of $\mathsf{H}_q(\mathfrak{S}_\infty)$ and set

$$\begin{aligned} Z_{h,b}(q,t;0;C_1,\ldots,C_b) \ := \int_{T^b} \prod_{i=1}^{b} [\mathrm{d}U_i]_{q,t} \sum_{n_i=a\,N}^{\infty} \frac{q^{-\frac{n_i\,(n_i-1)}{4}}}{[n_i]_q!} \, \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n_i}}\big(\,\Phi_{n_i}\,\Theta_{n_i}(\beta)\,C_i\,U_i^\dagger\,\big) \\ \times\ Z_{h,b}(q,t;0;U_1,\ldots,U_b) \ , \end{aligned} \tag{3.90}$$

where the intertwining operator Φ_{n_i} is defined in §1.1 in Chapter 2, the step function $\Theta_{n_i}(\beta)$ is defined in (3.54), and the integration measure $[\mathrm{d}U]_{q,t}$ on the maximal torus $T \subset G$ given by (A.28) defines the Macdonald inner product of generalized characters as an integral over holonomies [54]. Since the commutant of the representation of

$\mathcal{U}_q(\mathfrak{gl}_N)$ on $R_{\omega_1}^{\otimes n}$ is the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$, the map (3.90) may be regarded as the refined version of the quantum Fourier transformation of the boundary holonomy amplitudes. The generalized characters are orthonormal with respect to the Macdonald inner product, i.e.

$$\int_T [\mathrm{d}U]_{q,t}\ \chi_{\Phi_\lambda}(U)\,\chi_{\Phi_{\lambda'}}(U^\dagger)\ =\ \delta_{\lambda,\lambda'} \tag{3.91}$$

for $\lambda,\lambda'\in\Lambda_+$. Using Lemma A.32 we can write

$$\begin{aligned}\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\,\Phi_n\,\Theta_n(\beta)\,C_i\,U_i^\dagger\,\big)\ &=\ \sum_{\lambda\in\Lambda_+^n}\chi_{r_\lambda}\big(\Theta_n(\beta)\,C_i\big)\,\chi_{\Phi_{\lambda_{\beta-2}}}(U_i^\dagger)\\ &=\ \sum_{\mu\in\Lambda_+^{n-a\,N}}\chi_{r_{\mu_\beta+a\,(1^N)}}(C_i)\,\chi_{\Phi_{\mu+a\,(1^N)}}(U_i^\dagger)\ ,\end{aligned} \tag{3.92}$$

which yields

$$\begin{aligned}Z_{h,b}(q,t;0;C_1,\dots,C_b)\ =\ &\sum_{n=a\,N}^{\infty}\Big(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\Big)^b\sum_{n'=0}^{\infty}\sum_{\lambda\in\Lambda_+^{n'}}\Big(\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}}\Big)^{2-2h-b}\\ &\times\sum_{\mu_1,\dots,\mu_b\in\Lambda_+^{n-a\,N}}\prod_{i=1}^b\chi_{r_{\mu_{i\beta}+a\,(1^N)}}(C_i)\ \delta_{\lambda,\mu_i+a\,(1^N)}\\ =\ &\sum_{n=a\,N}^{\infty}\Big(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\Big)^b\sum_{\mu\in\Lambda_+^{n-a\,N}}\Big(\frac{\dim_{q,t}(R_\mu)}{\sqrt{g_\mu}}\Big)^{2-2h-b}\\ &\times\prod_{i=1}^b\chi_{r_{\mu_\beta+a\,(1^N)}}(C_i)\end{aligned} \tag{3.93}$$

where we used (3.91) and (3.10).

We can now use (3.46) to expand the refined quantum dimensions and from (2.68) we get

$$\begin{aligned}Z_{h,b}(q,t;0;C_1,\dots,C_b)\ =\ &\sum_{n=\lceil a\,N\rceil}^{\infty}\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h-b)}\Big(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\Big)^b\\ &\times\sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n}\delta\Big(\Theta_n(\beta)\,(E_n)^{b-1}\,\Omega_n(q,t)^{2-2h-b}\\ &\times\prod_{i=1}^h q^{-\ell(\sigma_i)-\ell(\tau_i)}\,\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\ \prod_{j=1}^b C_j\Big)\ ,\end{aligned} \tag{3.94}$$

where the central element E_n is defined by (2.75) with the properties

$$E_n^{-1} \;=\; \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\;D_n \qquad \text{in} \quad \widehat{\mathsf{H}}_q(\mathfrak{S}_n) \tag{3.95}$$

and

$$\chi_{r_\lambda}(E_n) \;=\; q^{\frac{n\,(n-1)}{4}}\;[n]_q!\;\frac{d_\lambda(1)^2}{d_\lambda(q)} \qquad \text{for} \quad \lambda \in \Lambda_+^n\;. \tag{3.96}$$

For fixed n this expression reduces to (2.74) in the unrefined limit, whereas our derivation gives the full partition function summed over all indices n. In particular, this partition function is a refined quantum deformation of the counting of holomorphic maps with specified monodromies C_j at the boundaries [67]; by expanding the Ω-factors, in the classical limit $q = 1$ it can be expressed in terms of parameterized Euler characters as in §2.3.

Let us look at some of the basic amplitudes which are the building blocks for the entire (q,t)-deformed gauge theory. The topological disk amplitude (with puncture of holonomy in the representation (2.87)) is the case $h = 0$, $b = 1$ in (3.89) which evaluates to

$$Z_{0,1}(q,t;0;U) \;=\sum_{n=0}^{\infty}\sum_{\lambda\in\Lambda_+^n}\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}}\;\chi_{\Phi_\lambda}(U) = \delta_{q,t}\big(U,q^{\beta\,(\rho,H)}\big)\;, \tag{3.97}$$

where $\delta_{q,t}$ is the delta-function in the measure $[\mathrm{d}U]_{q,t}$. This shows that the wavefunction $\Psi(U)$ for a disk in the topological (q,t)-deformed gauge theory is supported on generalized quantum group holonomies of flat connections on a disk, generalising the unrefined case of [49, eq. (3.7)] wherein $\delta_{q,q}$ is the delta-function in the Haar measure for $U(N)$. Dually, we can represent the disk partition function in a form that depends solely on Hecke algebra quantities by using (3.94) to write

$$Z_{0,1}(q,t;0;C) \;=\sum_{n=\lceil a\,N\rceil}^{\infty}\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^n\;\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\;\delta\big(\Theta_n(\beta)\,\Omega_n(q,t)\,C\big)\;, \tag{3.98}$$

independently of the central elements (2.75). Similarly, the punctured topological cylinder amplitude is obtained from (3.89) with $h = 0$, $b = 2$, giving

$$Z_{0,2}(q,t;0;U_1,U_2) \;=\sum_{n=0}^{\infty}\sum_{\lambda\in\Lambda_+^n}\chi_{\Phi_\lambda}(U_1)\,\chi_{\Phi_\lambda}(U_2) = \delta_{q,t}(U_1,U_2) \tag{3.99}$$

with the dual formulation

$$Z_{0,2}(q,t;0;C_1,C_2) \;=\sum_{n=\lceil a\,N\rceil}^{\infty}\Big(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\Big)^2\;\delta\big(\Theta_n(\beta)\,E_n\,C_1\,C_2\big) \tag{3.100}$$

independently of the Ω-factors (3.41).

3.2 Wilson loops

The natural closed defect observables of the gauge theory are of course the Wilson loops which correspond to simple closed curves on the surface Σ_h. As before, we consider the large N expansion of the expectation value of the single Wilson loop observable defined by (2.92). In the topological limit $p = 0$ we can use the orthonormality relation (3.91) to obtain

$$W_\lambda(q,t;0;h_1,h_2) \;=\; \sum_{\mu,\nu\in\Lambda_+} \Big(\frac{\dim_{q,t}(R_\mu)}{\sqrt{g_\mu}}\Big)^{1-2h_1} \Big(\frac{\dim_{q,t}(R_\nu)}{\sqrt{g_\nu}}\Big)^{1-2h_2} \widetilde{N}^{\nu}_{\mu\,\lambda}\;, \tag{3.101}$$

where $\widetilde{N}^{\nu}_{\mu\,\lambda}$ are refined fusion coefficients defined by the relation

$$\chi_{\Phi_\mu}(U)\,\chi_{\Phi_\lambda}(U) \;=\; \sum_{\nu\in\Lambda_+} \widetilde{N}^{\nu}_{\mu\,\lambda}\;\chi_{\Phi_\nu}(U) \tag{3.102}$$

expressing the completeness of the Macdonald polynomials $M_\lambda(x;q,t)$ in the ring of symmetric functions. We compare them to the Littlewood-Richardson coefficients $N^{\nu}_{\mu\,\lambda} \in \mathbb{Z}_{\geq 0}$ which give the multiplicities in the decomposition of tensor products of irreducible $U(N)$-modules as

$$R_\mu \otimes R_\lambda = \bigoplus_{\nu\in\Lambda_+} R_\nu^{\oplus N^{\nu}_{\mu\,\lambda}}\;, \tag{3.103}$$

and the same decomposition is true as $\mathcal{U}_q(\mathfrak{gl}_N)$-modules. To suitably express $\widetilde{N}^{\nu}_{\mu\,\lambda}$ and expand the Wilson loops we need the preliminary lemmatas A.62 and A.68, which are proved in Appendix 5.

To work out the large N expansion of the Wilson loop (3.101), we use the expansion of the Littlewood-Richardson coefficients in terms of Hecke characters given by [49]

$$\begin{aligned} N^{\nu}_{\mu\,\lambda} \;=\; & \frac{q^{-\frac{n_1\,(n_1-1)}{4}}}{[n_1]_q!}\,\frac{q^{-\frac{n_2\,(n_2-1)}{4}}}{[n_2]_q!}\,\frac{d_\mu(q)}{d_\mu(1)}\,\frac{d_\lambda(q)}{d_\lambda(1)} \sum_{\sigma_1\in\mathfrak{S}_{n_1}} \sum_{\sigma_2\in\mathfrak{S}_{n_2}} q^{-\ell(\sigma_1)-\ell(\sigma_2)} \\ & \times\; \chi_{r_\mu}\big(\mathsf{h}(\sigma_1^{-1})\big)\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma_2^{-1})\big)\,\chi_{r_\nu}\big(\mathsf{h}(\sigma_1)\cdot\mathsf{h}(\sigma_2)\big)\;, \end{aligned} \tag{3.104}$$

where $|\mu| = n_1$, $|\lambda| = n_2$, $|\nu| = n_1 + n_2 =: n$, and $\mathsf{h}(\sigma_1)\cdot\mathsf{h}(\sigma_2)$ acts on $\mathsf{H}_q(\mathfrak{S}_n)$ via $\mathsf{g}_1,\ldots,\mathsf{g}_{n_1-1} \in \mathsf{H}_q(\mathfrak{S}_{n_1}) \subset \mathsf{H}_q(\mathfrak{S}_n)$ and $\mathsf{g}_{n_1+1},\ldots,\mathsf{g}_{n_1+n_2-1} \in \mathsf{H}_q(\mathfrak{S}_{n_2}) \subset \mathsf{H}_q(\mathfrak{S}_n)$. We rewrite the expectation value of the Wilson loop (3.101) using (3.10), Lemma A.68 and Lemma A.62 to get

$$\begin{aligned} W_\lambda(q,t;0;h_1,h_2) \;=\; & \sum_{n_1=a\,N}^{\infty} \sum_{n=2a\,N}^{\infty} \sum_{\mu\in\Lambda_+^{n_1-a\,N}} \sum_{\nu\in\Lambda_+^{n-2a\,N}} N^{\nu_\beta+2a\,(1^N)}_{\mu_\beta+a\,(1^N)\;\lambda_\beta+a\,(1^N)} \\ & \times \Big(\frac{\dim_{q,t}(R_{\mu+a\,(1^N)})}{\sqrt{g_{\mu+a\,(1^N)}}}\Big)^{1-2h_1} \Big(\frac{\dim_{q,t}(R_{\nu+2a\,(1^N)})}{\sqrt{g_{\nu+2a\,(1^N)}}}\Big)^{1-2h_2}\;, \end{aligned} \tag{3.105}$$

for $|\lambda| = n_2 - a\,N$. Again we expand a transformed version of the Wilson loop given by

$$W(q,t;0;h_1,h_2;C) \;=\; \frac{q^{-\frac{n_2\,(n_2-1)}{4}}}{[n_2]_q!} \sum_{\lambda\in\Lambda_+^{n_2-a\,N}} \chi_{r_{\lambda_\beta+a\,(1^N)}}(C)\; W_\lambda(q,t;0;h_1,h_2)\;, \tag{3.106}$$

where C is an arbitrary central element of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_{n_2})$. Using now the expansion of the Littlewood-Richardson coefficients $N_{\mu\,\lambda}^{\nu}$ from (3.104), the expansion of the refined quantum dimensions from (3.46), the character of the central element D_n from (3.51), the definition of the step function $\Theta_n(\beta)$ from (3.54), the properties (3.49) and (2.68), and the definition of the delta-functions on Hecke algebras from (2.70) we finally arrive at the chiral series for Wilson loop observables given by

$$\begin{aligned} W(q,t;0;h_1,h_2;C) \;=\; & \sum_{n_1=\lceil a\,N\rceil}^{\infty} \sum_{n=\lceil 2a\,N\rceil}^{\infty} \Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n_1\,(1-2h_1)+n\,(1-2h_2)}\, \delta_{n_1+n_2,n}\, \frac{q^{-\frac{n_1\,(n_1-1)}{4}}}{[n_1]_q!} \\ & \times\; \frac{q^{-\frac{n_2\,(n_2-1)}{4}}}{[n_2]_q!} \sum_{\sigma_1\in\mathfrak{S}_{n_1}} \sum_{\sigma_2\in\mathfrak{S}_{n_2}} q^{-\ell(\sigma_1)-\ell(\sigma_2)}\, \delta\Big(\Theta_{n_2}(\beta)\, C\, \mathsf{h}\big(\sigma_2^{-1}\big)\Big) \\ & \times\; \delta\Big(\Theta_{n_1}(\beta)\, D_{n_1}\, \Omega_{n_1}(q,t)^{1-2h_1}\, \Pi_{n_1}^{(h_1)}\, \mathsf{h}\big(\sigma_1^{-1}\big)\Big) \\ & \times\; \delta\Big(\Theta_n^2(\beta)\, \Omega_n(q,t)^{1-2h_2}\, \Pi_n^{(h_2)}\, \big(\mathsf{h}(\sigma_1)\cdot\mathsf{h}(\sigma_2)\big)\Big)\;, \end{aligned} \tag{3.107}$$

where we have defined

$$\Theta_n^2(\beta) \;=\; \sum_{\substack{\mu\in\Lambda_+^n \\ \mu_i\geq(\beta-1)\,\rho_i+2a}} P_\mu \tag{3.108}$$

and $\Pi_n^{(h)}$ is defined previously in (2.77). This expression is the refined version of (2.76). It is a refined quantum deformation of the counting of covering worldsheets with boundary that maps to the corresponding Wilson graph on Σ_h according to the specified monodromy C [67,68]; the expansion into parameterized orbifold Euler characters in the classical limit $q=1$ proceeds as in §2.3.

Chapter 4

Sigma-models from AKSZ constructions

1 Aspects of the AKSZ construction

In this section we survey some pertinent background about AKSZ construction and BV quantization, and describe two dimensional reduction method that we will apply in the thesis.

1.1 AKSZ sigma-models

We begin by briefly introducing the ingredients of AKSZ theory. A more complete review can be found in [29]. The AKSZ construction is a BV quantized sigma-model formulation, and it gives a geometric solution to the classical master equation

$$(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}})_{\mathrm{BV}} = 0 \tag{4.1}$$

given by the BV bracket, which imposes BRST symmetry. The solution $\boldsymbol{\mathcal{S}}$ is called the *AKSZ action*, which is just a BV action.

The basic ingredients of AKSZ theory consists of two classes of supermanifolds. The *super worldvolume* or source $(\mathcal{W}, Q_{\mathcal{W}}, \mu)$ consists of a differential graded (dg-)manifold, which is a graded manifold $\mathcal{W}$ equiped with a cohomological vector field $Q_{\mathcal{W}}$, i.e. $Q_{\mathcal{W}}$ is of degree 1 and its Lie derivative $\mathcal{L}_{Q_{\mathcal{W}}}$ squares to zero, and a measure μ which is invariant under $Q_{\mathcal{W}}$. In the thesis we take $\mathcal{W} = T[1]\Sigma_d$, the tangent bundle of a d-dimensional oriented worldvolume manifold Σ_d with the degree of its fibers shifted by 1, which is isomorphic to the exterior algebra of differential forms $(\Omega(\Sigma_d), \wedge)$. We

choose the cohomological vector field $Q_{\mathcal{W}}$ corresponding to the de Rham differential, which in local affine coordinates $\hat{z}^{\hat{\mu}} = (\sigma^\mu, \theta^\mu) \in T[1]\Sigma_d$, with degree 0 coordinates σ^μ on Σ_d and degree 1 fiber coordinates θ^μ, has the form $Q_{\mathcal{W}} = \theta^\mu \frac{\partial}{\partial \sigma^\mu} =: \boldsymbol{D}$, where repeated upper and lower indices are always implicitly understood to be summed over. The measure in local coordinates can be written in the form $\mu = \mathrm{d}^d\hat{z} := \mathrm{d}^d\sigma\, \mathrm{d}^d\theta$.

The *target* $(\mathcal{M}, Q_\gamma, \omega)$ is a symplectic dg-manifold, which is a graded manifold $\mathcal{M}$ with a cohomological vector field Q_γ, and a graded symplectic form ω for which Q_γ is a Hamiltonian vector field: $\iota_{Q_\gamma}\omega = \mathrm{d}\gamma$ for some Hamiltonian function γ on $\mathcal{M}$, where ι_Q denotes contraction of a differential form along the vector field Q. In order to reproduce the BV formalism, the symplectic structure ω is taken to be of degree $d+1$, so that the Hamiltonian function γ is of degree d.

The AKSZ space of fields is the mapping space

$$\boldsymbol{\mathcal{M}} = \mathsf{Map}\big(T[1]\Sigma_d\,,\, \mathcal{M}\big) \tag{4.2}$$

consisting of smooth maps from $(T[1]\Sigma_d, \boldsymbol{D}, \mu)$ to $(\mathcal{M}, Q_\gamma, \omega)$, which we refer to as *superfields* in the following. We can introduce local coordinates on $\boldsymbol{\mathcal{M}}$ via the superfields

$$\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z}^{\hat{\mu}}) \;=\; \boldsymbol{\phi}^*(\hat{X}^{\hat{\imath}})(\hat{z}^{\hat{\mu}})\;, \tag{4.3}$$

for local coordinates $\hat{z}^{\hat{\mu}} \in \mathcal{W}$, $\hat{X}^{\hat{\imath}} \in \mathcal{M}$ and $\boldsymbol{\phi} \in \boldsymbol{\mathcal{M}}$. The cohomological vector fields $Q_{\mathcal{W}} = \boldsymbol{D}$ and Q_γ induce a cohomological vector field $\boldsymbol{Q}$ on $\boldsymbol{\mathcal{M}}$ in the following way. For $\boldsymbol{\phi} \in \boldsymbol{\mathcal{M}}$ and $\hat{z} \in \mathcal{W}$, use local coordinates to define

$$\begin{aligned} \boldsymbol{Q}_0 \;&=\; \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\; \boldsymbol{D}\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})\, \frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})}\;, \\ \boldsymbol{Q}_\gamma \;&=\; \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\; Q_\gamma^{\hat{\imath}}\big(\hat{\boldsymbol{X}}(\hat{z})\big)\, \frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})}\;, \end{aligned} \tag{4.4}$$

where $Q_\gamma^{\hat{\imath}}(\hat{X})\,\partial/\partial\hat{X}^{\hat{\imath}}$ is the local form of the vector field Q_γ on $\mathcal{M}$, while the de Rham differential on $\boldsymbol{\mathcal{M}}$ is given by the vector field

$$\boldsymbol{\delta} \;=\; \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\; \boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})\, \frac{\overrightarrow{\delta}}{\delta\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})} \tag{4.5}$$

with ghost number 1. Relevant definitions and formulas in differential calculus on mapping superspaces are summarized in Appendix 8. Then $\boldsymbol{\mathcal{M}}$ is a dg-manifold with the cohomological vector field

$$\boldsymbol{Q} \;=\; \boldsymbol{Q}_0 + \boldsymbol{Q}_\gamma\;. \tag{4.6}$$

We note that $\boldsymbol{Q}_0$ has ghost number[1] $|\boldsymbol{D}| - d = 1 - d$ and $\boldsymbol{Q}_\gamma$ has ghost number

[1]We use the terminology 'ghost number' for the degree of a superfield $\boldsymbol{\phi}$ in $\boldsymbol{\mathcal{M}}$.

$|Q_\gamma| - d = 1 - d$ as well, where $|Q_\gamma|$ denotes the degree of Q_γ. If a vector field on $\boldsymbol{\mathcal{M}}$ acts as a derivative, its ghost number is shifted by $d-1$, because a vector field based with coordinate $\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})$ has ghost number $|\hat{X}^{\hat{\imath}}| - 1$, but a functional derivative with respect to $\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})$ has ghost number $|\hat{X}^{\hat{\imath}}| + d$.

Given an n-form $\alpha \in \Omega^n(\mathcal{M})$, we can lift it to an n-form $\boldsymbol{\alpha} \in \Omega^n(\boldsymbol{\mathcal{M}})$ by transgression to the mapping space as

$$\boldsymbol{\alpha} = \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*(\alpha)\ , \tag{4.7}$$

where $\mathrm{ev} : T[1]\Sigma_d \times \boldsymbol{\mathcal{M}} \to \mathcal{M}$ is the evaluation map. As we see, $\boldsymbol{\alpha}$ is an n-form functional of the fields in $\boldsymbol{\mathcal{M}}$, and due to the integration $\boldsymbol{\alpha}$ has ghost number $|\alpha| - d$, where $|\alpha|$ denotes the total degree of α (i.e. the form degree coming from the grading of $\boldsymbol{\delta}$ plus the degree of the graded coordinates). In particular, since transgression is a chain map, from the degree $d+1$ symplectic form ω on $\mathcal{M}$ and a Liouville potential ϑ, such that $\omega = \mathrm{d}\vartheta$, we get the symplectic form $\boldsymbol{\omega}$ of ghost number 1 and Liouville potential $\boldsymbol{\vartheta}$ on $\boldsymbol{\mathcal{M}}$, such that $\boldsymbol{\omega} = \boldsymbol{\delta}\boldsymbol{\vartheta}$. Furthermore, the cohomological vector field $\boldsymbol{Q}$ on $\boldsymbol{\mathcal{M}}$ is also Hamiltonian with Hamiltonian function $-\iota_{\boldsymbol{Q}_0}\boldsymbol{\vartheta} + \boldsymbol{\gamma}$ of degree 0: $\iota_{\boldsymbol{Q}_\gamma}\boldsymbol{\omega} = \boldsymbol{\delta}\boldsymbol{\gamma}$, where $\iota_{\boldsymbol{Q}_0}$ and $\iota_{\boldsymbol{Q}_\gamma}$ have ghost number 0, while $\boldsymbol{\gamma}$ has ghost number $|\gamma| - d = 0$. In other words, the mapping space of superfields $\boldsymbol{\mathcal{M}}$ is itself a symplectic dg-manifold.

The BV bracket $(\,\cdot\,,\,\cdot\,)_{\mathrm{BV}}$ is the graded Poisson bracket of ghost number 1 on $\boldsymbol{\mathcal{M}}$ defined from $\boldsymbol{\omega}$, and it corresponds to the graded Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ of degree $-d+1$ on $\mathcal{M}$ defined from ω, since the transgression map $\int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*$ is a Lie algebra homomorphism from $(\mathcal{M}, \{\,\cdot\,,\,\cdot\,\})$ to $(\boldsymbol{\mathcal{M}}, (\,\cdot\,,\,\cdot\,)_{\mathrm{BV}})$:

$$\int_{\mathcal{W}} \mu\ \mathrm{ev}^*\big(\{F,G\}\big) = \Big(\int_{\mathcal{W}} \mu\ \mathrm{ev}^*(F)\,,\ \int_{\mathcal{W}} \mu\ \mathrm{ev}^*(G)\Big)_{\mathrm{BV}}\ , \tag{4.8}$$

where F and G are any local functions on $\mathcal{M}$. In particular, the cohomological vector fields as derivatives can be represented through derived brackets as

$$Q_\gamma = \{\gamma,\,\cdot\,\} \qquad \text{and} \qquad \vec{\boldsymbol{Q}} = (\boldsymbol{\mathcal{S}},\,\cdot\,)_{\mathrm{BV}}\ , \tag{4.9}$$

where the Hamiltonian $\boldsymbol{\mathcal{S}}$ on $\boldsymbol{\mathcal{M}}$ is defined to be the AKSZ action, which is the desired BV action. To explicitly specify it, we choose a Liouville potential ϑ on $\mathcal{M}$, and consider its zero locus $\mathcal{L}$ which is a Lagrangian submanifold of $\mathcal{M}$. We pick a submanifold $\mathcal{L}' \subseteq \mathcal{L}$ and restrict the space of fields $\boldsymbol{\mathcal{M}}$ to the subspace $\boldsymbol{\mathcal{M}}_{\mathcal{L}'} \subset \boldsymbol{\mathcal{M}}$ consisting of maps that send the boundary $\partial\mathcal{W} = T[1]\partial\Sigma_d$ into $\mathcal{L}'$. This assigns

boundary conditions on our fields, and now we can write the degree 0 AKSZ action $\boldsymbol{\mathcal{S}}$ on $\boldsymbol{\mathcal{M}}_{\mathcal{L}'}$ in the form

$$\boldsymbol{\mathcal{S}} \,=\, \boldsymbol{\mathcal{S}}_0 \,+\, \boldsymbol{\gamma} \ , \tag{4.10}$$

where

$$\boldsymbol{\mathcal{S}}_0 = -\iota_{\boldsymbol{Q}_0}\boldsymbol{\vartheta} \tag{4.11}$$

is the kinetic term and the Hamiltonian function

$$\boldsymbol{\gamma} = \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*(\gamma) \tag{4.12}$$

is the interaction term. The cocycle conditions $Q_\gamma^2 = 0$ and $\vec{\boldsymbol{Q}}^2 = 0$ are equivalent to $\{\gamma,\gamma\} = 0$ and $(\boldsymbol{\gamma},\boldsymbol{\gamma})_{\mathrm{BV}} = 0$, hence the AKSZ action is a solution of the classical master equation $(\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{S}})_{\mathrm{BV}} = 0$. In the BV formalism, the cohomological vector field $\vec{\boldsymbol{Q}}$ corresponds to the BRST charge.

A *canonical transformation* is associated to a degree $d-1$ function α on $\mathcal{M}$. We use the notation δ_α for the corresponding Hamiltonian vector field, and $\mathrm{e}^{\,\delta_\alpha}$ and $\mathrm{e}^{\,\boldsymbol{\delta_\alpha}}$ for the respective canonical transformations. The action of the canonical transformation on $\boldsymbol{\gamma}$ is given by $\mathrm{e}^{\,\boldsymbol{\delta_\alpha}}\boldsymbol{\gamma} = \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*(\,\mathrm{e}^{\,\delta_\alpha}\gamma)$, which preserves the classical master equation as

$$\{\,\mathrm{e}^{\,\delta_\alpha}\gamma,\,\mathrm{e}^{\,\delta_\alpha}\gamma\} = \,\mathrm{e}^{\,\delta_\alpha}\{\gamma,\gamma\} = 0\ , \tag{4.13}$$

due to $\{\gamma,\gamma\} = 0$. If $\alpha|_{\mathcal{L}'} = 0$, then the AKSZ action $\boldsymbol{\mathcal{S}}_0 + \boldsymbol{\gamma}$ is equivalent to $\boldsymbol{\mathcal{S}}_0 + \mathrm{e}^{\,\boldsymbol{\delta_\alpha}}\boldsymbol{\gamma}$ up to a canonical transformation. The canonical transformation $\mathrm{e}^{\,\boldsymbol{\delta_\alpha}}$ is an example of a *duality transformation*, which in the AKSZ formalism is defined to be a symplectomorphism, i.e. a diffeomorphism between underlying symplectic manifolds

$$\boldsymbol{f}\ :\ (\boldsymbol{\mathcal{M}},\boldsymbol{\omega})\ \longrightarrow\ (\boldsymbol{\mathcal{M}}',\boldsymbol{\omega}')\ , \tag{4.14}$$

satisfying

$$\boldsymbol{f}^*\boldsymbol{\omega}' = \boldsymbol{\omega}\ . \tag{4.15}$$

In other words, $\boldsymbol{f}$ is a coordinate transformation on symplectic manifolds which leaves the symplectic structure invariant.

We can then introduce a boundary term in the AKSZ action using the ingredients of a canonical transformation. Let β be a degree $d-1$ function on $\mathcal{M}$ as before, and further assume that $\{\beta,\beta\} = 0$ and $\mathrm{e}^{\,\delta_\beta}\gamma\big|_{\mathcal{L}'} = 0$. Then the AKSZ action $\boldsymbol{\mathcal{S}}_0 + \boldsymbol{\gamma}$ on $\boldsymbol{\mathcal{M}}_{\mathcal{L}'}$ is equivalent to the AKSZ action

$$\boldsymbol{\mathcal{S}} \,=\, \boldsymbol{\mathcal{S}}_0 \,+\, \boldsymbol{\gamma} - \oint_{T[1]\partial\Sigma_d} \mathrm{d}^{d-1}\hat{z}\ \mathrm{ev}^*(\beta) \tag{4.16}$$

on $\boldsymbol{\mathcal{M}}_{\mathcal{L}'_\beta}$, which is given by shifting the Liouville potential ϑ to $\vartheta - \mathrm{d}\beta$ with $\mathcal{L}'_\beta = \mathrm{e}^{\,\delta_\beta}\mathcal{L}'$ the new zero locus of the shifted Liouville potential.

A common choice of target for the AKSZ construction is to take $\mathcal{M}$ to be an N-manifold, which is a graded manifold with no coordinates of negative degree. In this case the triple $(\mathcal{M}, Q_\gamma, \omega)$ is called a *QP-manifold of degree* $n = d - 1$; if the N-manifold $\mathcal{M}$ is concentrated in degrees $0, 1, \ldots, n$, then $(\mathcal{M}, Q_\gamma, \omega)$ is called a symplectic Lie n-algebroid, and it arises from an n-graded vector bundle over the degree 0 body $M = \mathcal{M}_0$ of $\mathcal{M}$ [29]; in particular, functions of degree $n - 1$ can be identified with sections of a vector bundle $E \to M$ equiped with the structure of a Leibniz algebroid. For example, in the simplest dimension $d = 1$ with target a degree 0 QP-manifold, one necessarily has $Q_\gamma = 0$ and thus a symplectic Lie 0-algebroid is just an ordinary symplectic manifold $(\mathcal{M}, \omega)$; in this case the degree 1 Hamiltonian function γ is locally constant on $\mathcal{M}$ and the AKSZ construction produces a topological quantum mechanics given as a one-dimensional Chern-Simons theory whose Chern-Simons form is a Liouville potential ϑ [69, 70].

In the following we will describe gauge fixing and dimensional reduction methods of AKSZ theories which we will apply in different contexts through the thesis. Some of these techniques and approaches seems to be novel, as we have not found them in the literature. Furthermore we shall survey the AKSZ topological field theories associated with the first few non-trivial members in the hierarchy of QP-structures on the target manifold for dimensions $d = 2, 3$ and 4, in the context of the string and membrane models of interest in the thesis. We shall also deal with targets that have negative degree coordinates and hence unravel new constructions even in low dimension.

1.2 Gauge fixing in the superfield formalism

The entire field content of a system with degenerate symmetries is usually specified by separating it into 'fields', which includes the original physical and ghost fields from the BRST picture, and dual 'antifields', which correspond to the equations of motion and define canonically conjugate variables with respect to the symplectic phase space structure on the space $\boldsymbol{\mathcal{M}}$ of all fields. In AKSZ constructions the fields and antifields are not distinguished from the onset. The theory is specified once the antifields are assigned, and different choices yield different field theories.

In the usual BV quantized theories, the fields and antifields are distinguished from the start. One chooses a gauge fixing fermion $\Psi[\phi]$, which is a functional of the fields ϕ^a (but not the antifields) of ghost number $U = -1$, and then the antifields ϕ^+_a are fixed to the variations $\phi^+_a = \frac{\delta\Psi}{\delta\phi^a}$. This can be reformulated in terms of the BV symplectic structure on the space of superfields $\boldsymbol{\mathcal{M}}$. For this, we consider the case where the source dg-manifold is the superworldvolume $\mathcal{W} = T[1]\Sigma_d$ with local coordinates $\hat{z} = (\sigma, \theta)$ and write a generic BV symplectic structure on superfields in its canonical form as

$$\boldsymbol{\omega} \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;\; \boldsymbol{\delta\phi}^+_a(\hat{z})\,\boldsymbol{\delta\phi}^a(\hat{z})\;, \tag{4.17}$$

where we chose a convenient ordering of antifields $\boldsymbol{\phi}^+_a$ and fields $\boldsymbol{\phi}^a$ in this way. We write $|a|$ for the degree of the superfield $\boldsymbol{\phi}^a$; then its antifield $\boldsymbol{\phi}^+_a$ has degree $d-1-|a|$. If the Liouville potential is chosen as

$$\boldsymbol{\vartheta} \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;\boldsymbol{\phi}^+_a(\hat{z})\,\boldsymbol{\delta\phi}^a(\hat{z})\;, \tag{4.18}$$

then the kinetic part of the AKSZ action is

$$\boldsymbol{\mathcal{S}}_0 \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;(-1)^{|a|}\,\boldsymbol{\phi}^+_a(\hat{z})\,\boldsymbol{D\phi}^a(\hat{z})\;. \tag{4.19}$$

We choose a gauge fixing fermion $\boldsymbol{\Psi}[\boldsymbol{\phi}]$, which is a functional on superfields $\boldsymbol{\phi}(\hat{z}) \in \boldsymbol{\mathcal{M}}$, and fix the antifields to

$$\boldsymbol{\phi}^+_a(\hat{z}) \,=\, (-1)^{|a|\,(d+1)}\,\frac{\overrightarrow{\delta}\boldsymbol{\Psi}}{\delta\boldsymbol{\phi}^a(\hat{z})}\;, \tag{4.20}$$

where an extra sign factor has been introduced, which depends on the dimension of the worldvolume. The left-acting functional derivative is defined in the usual way by

$$\lim_{\epsilon\to 0}\,\frac{\boldsymbol{\Psi}[\boldsymbol{\phi}+\epsilon\,\boldsymbol{\xi}\,] \,-\, \boldsymbol{\Psi}[\boldsymbol{\phi}]}{\epsilon} \,=:\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;\boldsymbol{\xi}(\hat{z})\,\frac{\overrightarrow{\delta}\boldsymbol{\Psi}}{\delta\boldsymbol{\phi}(\hat{z})}\;. \tag{4.21}$$

The BV symplectic form (4.29) in the gauge that is fixed by $\boldsymbol{\Psi}[\boldsymbol{\phi}]$ according to (4.20) is

$$\begin{aligned}\boldsymbol{\omega}_{\boldsymbol{\Psi}} \,&=\, \sum_a (-1)^{|a|\,(d+1)}\,\int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}_1\;\boldsymbol{\delta}\frac{\overrightarrow{\delta}\boldsymbol{\Psi}}{\delta\boldsymbol{\phi}^a(\hat{z}_1)}\,\boldsymbol{\delta\phi}^a(\hat{z}_1)\\ &=\, (-1)^{d+1}\,\sum_{a,b}(-1)^{|b|\,(|a|+1)+|a|\,d}\,\int_{T[1]\Sigma_d}\mathrm{d}^d\hat{z}_1\,\int_{T[1]\Sigma_d}\mathrm{d}^d\hat{z}_2\\ &\qquad\times\;\frac{\overrightarrow{\delta}^2\boldsymbol{\Psi}}{\delta\boldsymbol{\phi}^b(\hat{z}_2)\,\delta\boldsymbol{\phi}^a(\hat{z}_1)}\,\boldsymbol{\delta\phi}^b(\hat{z}_2)\,\boldsymbol{\delta\phi}^a(\hat{z}_1)\;.\end{aligned} \tag{4.22}$$

Interchanging variables and indices yields sign changes which are given by

$$\begin{aligned}
\frac{\vec{\delta}^2\boldsymbol{\Psi}}{\delta\boldsymbol{\phi}^b(\hat{z}_2)\,\delta\boldsymbol{\phi}^a(\hat{z}_1)} &= (-1)^{(|a|+d)\,(|b|+d)}\,\frac{\vec{\delta}^2\boldsymbol{\Psi}}{\delta\boldsymbol{\phi}^a(\hat{z}_1)\,\delta\boldsymbol{\phi}^b(\hat{z}_2)}\ , \\
\boldsymbol{\delta}\boldsymbol{\phi}^b(\hat{z}_2)\,\boldsymbol{\delta}\boldsymbol{\phi}^a(\hat{z}_1) &= (-1)^{(|a|+1)\,(|b|+1)}\,\boldsymbol{\delta}\boldsymbol{\phi}^a(\hat{z}_1)\,\boldsymbol{\delta}\boldsymbol{\phi}^b(\hat{z}_2)\ , \\
\int_{T[1]\Sigma_d}\mathrm{d}^d\hat{z}_1\,\int_{T[1]\Sigma_d}\mathrm{d}^d\hat{z}_2 &= (-1)^d\int_{T[1]\Sigma_d}\mathrm{d}^d\hat{z}_2\,\int_{T[1]\Sigma_d}\mathrm{d}^d\hat{z}_1\ .
\end{aligned} \tag{4.23}$$

This shows that the gauge fixed BV symplectic form is a product of a symmetric and an antisymmetric expression, and hence $\boldsymbol{\omega}_{\boldsymbol{\Psi}} = 0$. Thus gauge fixing with a fermion in the sense of (4.20) means that one chooses a Lagrangian submanifold $\boldsymbol{\mathcal{L}}$ of the space of all fields $\boldsymbol{\mathcal{M}}$, i.e. a subspace $\boldsymbol{\mathcal{L}} \subset \boldsymbol{\mathcal{M}}$ on which the symplectic form $\boldsymbol{\omega}$ vanishes and which has half the dimension of $\boldsymbol{\mathcal{M}}$. In the following we use this prescription generally: A choice of gauge in BV quantization is equivalent to a choice of a Lagrangian submanifold $\boldsymbol{\mathcal{L}}$ in $\boldsymbol{\mathcal{M}}$. The Batalin-Vilkovisky theorem [71] ensures that the path integral over $\boldsymbol{\mathcal{L}}$ is independent of the choice of representative for the homology class of the Lagrangian submanifold $\boldsymbol{\mathcal{L}}$. By the localization theorem, the path integral localizes over the fixed point locus of the BV–BRST charge $\boldsymbol{Q}$ in the Lagrangian subspace $\boldsymbol{\mathcal{L}}$. From a physical point of view, the Lagrangian submanifold intersects the gauge orbits orthogonally, i.e. the action of the BV–BRST charge $(\boldsymbol{\mathcal{S}},\,\cdot\,)_{\mathrm{BV}}$ vanishes on Lagrangian submanifolds, as the BV bracket acts as zero there. Thus the BV gauge symmetry is completely fixed on Lagrangian submanifolds.[2]

Let us now reformulate these observations in terms of the expansion coefficients of superfields. An arbitrary superfield $\boldsymbol{\phi}^a$ can be expanded in terms of the degree 1 fiber coordinates θ^μ of $\mathcal{W} = T[1]\Sigma_d$ in the form

$$\boldsymbol{\phi}^a(\hat{z}) = \phi^{(0)\,a}(\sigma) + \phi^{(1)\,a}_{\mu_1}(\sigma)\,\theta^{\mu_1} + \frac{1}{2}\,\phi^{(2)\,a}_{\mu_1\mu_2}(\sigma)\,\theta^{\mu_1}\,\theta^{\mu_2} + \cdots + \frac{1}{d!}\,\phi^{(d)\,a}_{\mu_1\cdots\mu_d}(\sigma)\,\theta^{\mu_1}\cdots\theta^{\mu_d}\ , \tag{4.24}$$

where $\phi^{(p)\,a}$ are the degree $|a|-p$ coefficients of $\boldsymbol{\phi}^a$ which can be identified with p-forms on Σ_d. The BV symplectic form can be written as an integral over the original worldvolume Σ_d as

$$\begin{aligned}
\boldsymbol{\omega} &= \sum_{p=0}^{d}\int_{\Sigma_d}\boldsymbol{\delta}\phi^{(p)\,+}_a\wedge\boldsymbol{\delta}\phi^{(p)\,a} \\
&= \sum_{p=0}^{d}\frac{1}{p!}\int_{\Sigma_d}\mathrm{d}^d\sigma\,\sum_a(-1)^{|a|+p}\,\boldsymbol{\delta}\widetilde{\phi}^{(p)\,+;\,\mu_1\cdots\mu_p}_a(\sigma)\,\boldsymbol{\delta}\phi^{(p)\,a}_{\mu_1\cdots\mu_p}(\sigma)\ ,
\end{aligned} \tag{4.25}$$

[2]We have not studied the Gribov problem in this context.

where $\widetilde{\phi}_a^{(p)\,+}$ is the dual antifield of $\phi^{(p)\,a}$ defined by

$$\widetilde{\phi}_a^{(p)\,+;\,\mu_{d-p+1}\cdots\mu_d} \;=\; (-1)^{d\,(d+1+|a|+p)+|a|\,(p+1)+p}\,\frac{1}{(d-p)!}\,\epsilon^{\mu_1\cdots\mu_d}\,\left(\phi_a^+\right)^{(d-p)}_{\mu_1\cdots\mu_{d-p}}\;. \tag{4.26}$$

Here $\epsilon^{\mu_1\cdots\mu_d}$ is the Levi-Civita tensor density on Σ_d, and $\left(\phi_a^+\right)^{(d-p)}$ are the expansion coefficients of the superfield $\boldsymbol{\phi}_a^+$. The BV symplectic form with this sign convention gives the canonical Poisson bracket relations

$$\left\{\phi^{(p)\,a}_{\mu_1\cdots\mu_p}\,,\,\widetilde{\phi}_b^{(p')\,+;\,\nu_1\cdots\nu_{p'}}\right\} \;=\; \delta^{pp'}\;\delta^a{}_b\;\delta_{\mu_1\cdots\mu_p}{}^{\nu_1\ldots\nu_{p'}}\;. \tag{4.27}$$

Gauge fixing with a fermion $\Psi[\phi]$ then takes the more familiar form

$$\widetilde{\phi}_a^{(p)\,+} \;=\; \frac{\overrightarrow{\delta}\Psi}{\delta\phi^{(p)\,a}}\;, \tag{4.28}$$

and it gives a vanishing symplectic structure $\boldsymbol{\omega}_{\boldsymbol{\Psi}} = 0$, whence the fermion Ψ generates a Lagrangian submanifold in the terms of the expansion coefficients as well. In order to apply our reduction techniques later, we have introduced the gauge fixing procedure of BV quantized superfields in this detailed context, since we have not found it in the literature.

1.3 Dimensional reduction by gauge fixing

In the following we introduce a particular gauge fixing as a novel dimensional reduction technique which reduces a given AKSZ theory on the superworldvolume $\mathcal{W} = T[1]\Sigma_d$ to an AKSZ theory on its boundary $\partial\mathcal{W} = T[1]\partial\Sigma_d$. We consider the case when the fields (but not the antifields) occur in even number and can be paired: a superfield $\boldsymbol{\phi}^a(\hat{z})$ with ghost degree $|a|$ is paired with another superfield $\boldsymbol{\chi}_a(\hat{z})$ with ghost degree $d-2-|a|$, and vice versa. The BV symplectic form is written in the canonical form

$$\boldsymbol{\omega} \;=\; \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;\left((-1)^{d+|a|}\,\boldsymbol{\delta}\boldsymbol{\phi}_a^+(\hat{z})\,\boldsymbol{\delta}\boldsymbol{\phi}^a(\hat{z}) \;+\; (-1)^{|a|}\,\boldsymbol{\delta}\boldsymbol{\chi}^{a\,+}(\hat{z})\,\boldsymbol{\delta}\boldsymbol{\chi}_a(\hat{z})\right)\;, \tag{4.29}$$

where we chose a convenient ordering of antifields $\boldsymbol{\phi}_a^+, \boldsymbol{\chi}^{a\,+}$ and fields $\boldsymbol{\phi}^a, \boldsymbol{\chi}_a$ in this way. The ghost degrees of the antifields $\boldsymbol{\phi}_a^+$ and $\boldsymbol{\chi}^{a\,+}$ are $d-1-|a|$ and $|a|+1$ respectively.

We choose a submanifold as gauge fixing on the space of superfields $\boldsymbol{\mathcal{M}}$. It is given by the constraints[3]

$$\boldsymbol{\phi}_a^+ \,=\, \boldsymbol{D}\boldsymbol{\chi}_a \qquad \text{and} \qquad \boldsymbol{\chi}^{a\,+} \,=\, (-1)^{|a|\,(d+1)+1}\,\boldsymbol{D}\boldsymbol{\phi}^a \;, \tag{4.30}$$

which reduces the BV symplectic form to

$$\boldsymbol{\omega}_{\text{gf}} \,=\, \oint_{T[1]\partial\Sigma_d} \mathrm{d}^{d-1}\hat{z}\;(-1)^{d+|a|+1}\,\boldsymbol{\delta}\boldsymbol{\chi}_a(\hat{z})\,\boldsymbol{\delta}\boldsymbol{\phi}^a(\hat{z})\;. \tag{4.31}$$

In the following we refer to this gauge as the *exact gauge*. If the worldvolume has no boundaries or the boundary conditions give $\boldsymbol{\omega}_{\text{gf}} = 0$, the submanifold is a Lagrangian submanifold as well, and hence it gives a full gauge fixing. Otherwise the submanifold is not Lagrangian, and therefore it only gives a full gauge fixing in the bulk $\mathcal{W} \setminus \partial\mathcal{W}$ but not on the boundary $\partial\mathcal{W}$.

If the Liouville potential is chosen as

$$\boldsymbol{\vartheta} \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;\big((-1)^{d+|a|}\,\boldsymbol{\phi}_a^+(\hat{z})\,\boldsymbol{\delta}\boldsymbol{\phi}^a(\hat{z}) \,+\, (-1)^{(d+1)\,|a|}\,\boldsymbol{\chi}_a(\hat{z})\,\boldsymbol{\delta}\boldsymbol{\chi}^{a\,+}(\hat{z})\big)\;, \tag{4.32}$$

then the kinetic part of the AKSZ action is given by

$$\begin{aligned}\boldsymbol{\mathcal{S}}_0 \,=\, -\iota_{\boldsymbol{Q}_0}\boldsymbol{\vartheta} \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;\big((-1)^{d+|a|+1}\,\boldsymbol{\phi}_a^+(\hat{z})\,\boldsymbol{D}\boldsymbol{\phi}^a(\hat{z}) \\ +\,(-1)^{(d+1)\,|a|+1}\,\boldsymbol{\chi}_a(\hat{z})\,\boldsymbol{D}\boldsymbol{\chi}^{a\,+}(\hat{z})\big)\;.\end{aligned} \tag{4.33}$$

We have not specified any boundary conditions yet. They are needed in order to derive consistent equations of motion. The variation of the action $\boldsymbol{\mathcal{S}}_0$ gives

$$\begin{aligned}\delta\boldsymbol{\mathcal{S}}_0 \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;\big((-1)^{d+|a|+1}\,\delta\boldsymbol{\phi}_a^+(\hat{z})\,\boldsymbol{D}\boldsymbol{\phi}^a(\hat{z}) \,+\, (-1)^{d+|a|+1}\,\boldsymbol{\phi}_a^+(\hat{z})\,\boldsymbol{D}\delta\boldsymbol{\phi}^a(\hat{z}) \\ +\,(-1)^{(d+1)\,|a|+1}\,\boldsymbol{\chi}_a(\hat{z})\,\boldsymbol{D}\delta\boldsymbol{\chi}^{a\,+}(\hat{z}) \,+\, (-1)^{(d+1)\,|a|+1}\,\boldsymbol{\chi}_a(\hat{z})\,\boldsymbol{D}\delta\boldsymbol{\chi}^{a\,+}(\hat{z})\big)\;.\end{aligned} \tag{4.34}$$

The equations of motion for $\boldsymbol{\phi}^a$ and $\boldsymbol{\chi}_a$ are obtained via integration by parts. The boundary terms of the variation

$$\delta\boldsymbol{\mathcal{S}}_0\big|_{T[1]\partial\Sigma_d} \,=\, \oint_{T[1]\partial\Sigma_d} \mathrm{d}^{d-1}\hat{z}\;\big(\boldsymbol{\phi}_a^+(\hat{z})\,\delta\boldsymbol{\phi}^a(\hat{z}) \,-\, (-1)^{d\,(|a|+1)}\,\boldsymbol{\chi}_a(\hat{z})\,\delta\boldsymbol{\chi}^{a\,+}(\hat{z})\big) \tag{4.35}$$

must vanish on their own. The straightforward boundary conditions $\boldsymbol{\phi}_a^+|_{T[1]\partial\Sigma_d} = 0$, $\boldsymbol{\chi}^{a\,+}|_{T[1]\partial\Sigma_d} = 0$ and $\delta\boldsymbol{\phi}^a|_{T[1]\partial\Sigma_d} = 0$, $\delta\boldsymbol{\chi}_a|_{T[1]\partial\Sigma_d} = 0$ result in a vanishing reduced

[3]It is important to note that the fields and antifields in this gauge are assigned in the bulk $\mathcal{W} \setminus \partial\mathcal{W}$.

kinetic action on the boundary, so they are not suitable for us. On the other hand, the boundary variation term $\delta\boldsymbol{\mathcal{S}}_0\big|_{T[1]\partial\Sigma_d}$ in the partial exact gauge fixing reduces to

$$\begin{aligned}\delta\boldsymbol{\mathcal{S}}_{0,\mathrm{gf}}\big|_{T[1]\partial\Sigma_d} &= \oint_{T[1]\partial\Sigma_d} \mathrm{d}^{d-1}\hat{z}\,\big(\boldsymbol{D}\boldsymbol{\chi}_a(\hat{z})\,\delta\boldsymbol{\phi}^a(\hat{z}) \,+\, (-1)^{d+|a|}\,\boldsymbol{\phi}^a(\hat{z})\,\boldsymbol{D}\delta\boldsymbol{\chi}_a(\hat{z})\big)\\ &= \oint_{T[1]\partial\Sigma_d} \mathrm{d}^{d-1}\hat{z}\;\boldsymbol{D}\big(\boldsymbol{\chi}_a(\hat{z})\,\delta\boldsymbol{\phi}^a(\hat{z})\big)\\ &= 0\ .\end{aligned} \tag{4.36}$$

As we see, the exact gauge is consistent with the necessary boundary conditions, which means the equations of motion are well-defined in this gauge, and the master equation also holds because the interaction term reduces to the boundary as well. This is not true without the exact gauge or suitable boundary conditions. Hence the exact gauge fixing appears here as a boundary condition.

The gauge fixed kinetic action

$$\boldsymbol{\mathcal{S}}_{0,\mathrm{gf}} \;=\; \oint_{T[1]\partial\Sigma_d} \mathrm{d}^{d-1}\hat{z}\;(-1)^{d+|a|+1}\,\boldsymbol{\chi}_a(\hat{z})\,\boldsymbol{D}\boldsymbol{\phi}^a(\hat{z}) \tag{4.37}$$

can be derived from the Liouville potential

$$\boldsymbol{\vartheta}_{\mathrm{b}} \;=\; \oint_{T[1]\partial\Sigma_d} \mathrm{d}^{d-1}\hat{z}\;(-1)^{d+|a|+1}\,\boldsymbol{\chi}_a(\hat{z})\,\boldsymbol{\delta}\boldsymbol{\phi}^a(\hat{z})\ , \tag{4.38}$$

with $\boldsymbol{\omega}_{\mathrm{gf}} = \boldsymbol{\delta}\boldsymbol{\vartheta}_{\mathrm{b}}$, but with the opposite sign:

$$\boldsymbol{\mathcal{S}}_{0,\mathrm{gf}} \;=\; \iota_{\boldsymbol{Q}_{0,\mathrm{b}}}\boldsymbol{\vartheta}_{\mathrm{b}}\ , \tag{4.39}$$

where $\boldsymbol{Q}_{0,\mathrm{b}}$ is the cohomological vector field on $T[1]\partial\Sigma_d$. The interaction term enters into the picture in a simpler way. Let us assume that the Hamiltonian functional $\boldsymbol{\gamma} = \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*(\gamma)$, which satisfies the equation $\iota_{\boldsymbol{Q}_\gamma}\boldsymbol{\omega} = \boldsymbol{\delta}\boldsymbol{\gamma}$, reduces to the boundary in the exact gauge as

$$\boldsymbol{\gamma}_{\mathrm{gf}} \;=\; -\int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;\boldsymbol{D}\,\mathrm{ev}^*(\beta) \;=\; -\oint_{\mathrm{T}[1]\partial\Sigma_{\mathrm{d}}} \mathrm{d}^{\mathrm{d}-1}\hat{\mathrm{z}}\ \mathrm{ev}^*(\beta) \;=:\; -\boldsymbol{\beta} \tag{4.40}$$

for a function β on the target graded manifold $\mathcal{M}$ with degree $d-1$. Then the full action (4.10) reduces in the exact gauge to

$$\boldsymbol{\mathcal{S}}_{\mathrm{gf}} \;=\; \iota_{\boldsymbol{Q}_{0,\mathrm{b}}}\boldsymbol{\vartheta}_{\mathrm{b}} \,-\, \boldsymbol{\beta}\ , \tag{4.41}$$

which satisfies the BV master equation, and thus gives an AKSZ action on the boundary.

1.4 Dimensional reduction by effective actions

In the thesis we shall also apply another dimensional reduction method, called 'Losev's trick' [72], which is not specific to boundary reductions. We briefly recall the technique following [73], see also [74] where a similar technique is employed.

The symplectic structure ω on the target supermanifold $\mathcal{M}$ induces a natural second order differential operator Δ, which in local coordinates is given by

$$\Delta = \frac{1}{2}\,\omega^{\hat{\imath}\hat{\jmath}}\,\frac{\vec{\partial}}{\partial \hat{X}^{\hat{\imath}}}\,\frac{\vec{\partial}}{\partial \hat{X}^{\hat{\jmath}}}\ , \tag{4.42}$$

where $\omega^{\hat{\imath}\hat{\jmath}}$ is the inverse of $\omega_{\hat{\imath}\hat{\jmath}}$. This pulls back to give the BV Laplacian $\boldsymbol{\Delta}$ for the BV bracket $(\,\cdot\,,\,\cdot\,)_{\mathrm{BV}}$ on the space of AKSZ fields $\boldsymbol{\mathcal{M}}$. The AKSZ action $\boldsymbol{\mathcal{S}}$ satisfies the BV quantum master equation $\boldsymbol{\Delta}\,\mathrm{e}^{-\boldsymbol{\mathcal{S}}/\hbar} = 0$ on $\boldsymbol{\mathcal{M}}$, which is equivalent to $\frac{1}{2}\,(\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{S}})_{\mathrm{BV}} = \hbar\,\boldsymbol{\Delta}\boldsymbol{\mathcal{S}}$. This ensures independence of the BRST-invariant quantum field theory on the choice of gauge fixing, provided we define the path integral by equiping $\boldsymbol{\mathcal{M}}$ with a measure $\boldsymbol{\mu}$ which is compatible with $\boldsymbol{\omega}$ [71].

Borrowing standard terminology from renormalization of quantum field theory, let us now assume that the space of AKSZ fields can be decomposed into a direct product $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_{\mathrm{UV}} \times \boldsymbol{\mathcal{M}}_{\mathrm{IR}}$ of ultraviolet (UV) and infrared (IR) degrees of freedom, with a compatible decomposition of the canonical symplectic form $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathrm{UV}} + \boldsymbol{\omega}_{\mathrm{IR}}$. Then the BV Laplacian also decomposes as $\boldsymbol{\Delta} = \boldsymbol{\Delta}_{\mathrm{UV}} + \boldsymbol{\Delta}_{\mathrm{IR}}$. One now 'integrates out' the ultraviolet degrees of freedom to get an effective action. The integration requires a gauge fixing on the ultraviolet sector $\boldsymbol{\mathcal{M}}_{\mathrm{UV}}$ of the space of superfields, which means a choice of a Lagrangian submanifold $\boldsymbol{\mathcal{L}} \subset \boldsymbol{\mathcal{M}}_{\mathrm{UV}}$. Then the effective BV action $\boldsymbol{\mathcal{S}}_{\mathrm{eff}}$ in the infrared sector is defined as

$$\mathrm{e}^{-\boldsymbol{\mathcal{S}}_{\mathrm{eff}}/\hbar} := \int_{\boldsymbol{\mathcal{L}}} \sqrt{\boldsymbol{\mu}_{\boldsymbol{\mathcal{L}}}}\ \mathrm{e}^{-\boldsymbol{\mathcal{S}}/\hbar}\ , \tag{4.43}$$

where $\sqrt{\boldsymbol{\mu}_{\boldsymbol{\mathcal{L}}}}$ is the measure on $\boldsymbol{\mathcal{L}}$ induced by $\boldsymbol{\mu}$. Therefore the effective action satisfies the quantum master equation $\boldsymbol{\Delta}_{\mathrm{IR}}\,\mathrm{e}^{-\boldsymbol{\mathcal{S}}_{\mathrm{eff}}/\hbar} = 0$. A change of gauge fixing in the ultraviolet sector only changes $\mathrm{e}^{-\boldsymbol{\mathcal{S}}_{\mathrm{eff}}/\hbar}$ by a $\boldsymbol{\Delta}_{\mathrm{IR}}$-exact term. Similarly, the value of the partition function is independent of the particular choice of splitting $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_{\mathrm{UV}} \times \boldsymbol{\mathcal{M}}_{\mathrm{IR}}$ by the Batalin-Vilkovisky theorem [71].

2 String sigma-models

In this section we describe several relevant examples of two-dimensional AKSZ sigma-models which are related to string sigma-models and topological A- and B-models.

2.1 B-fields and the Poisson sigma-model

In dimension $d = 2$, the AKSZ theory with target space a degree 1 QP-manifold describes the topological sigma-model for closed strings in an NS–NS B-field background. In the worldsheet sigma-model approach, the fundamental field is a map $X : \Sigma_2 \to M$ from a closed and oriented Riemann surface Σ_2 to a target space M. Denoting the local coordinates by $(X^i) \in M$ and $(\sigma^\mu) \in \Sigma_2$, the string field X is described by a set of functions $\big(X^i(\sigma^\mu)\big)$ on Σ_2. The topological part of the bosonic string action is

$$I_{\Sigma_2,\,B} \;=\; \int_{\Sigma_2} X^*(B) \;=\; \frac{1}{2}\int_{\Sigma_2} B_{ij}\,\mathrm{d}X^i\wedge\mathrm{d}X^j \;, \tag{4.44}$$

where $B = \frac{1}{2}\,B_{ij}\,\mathrm{d}X^i\wedge\mathrm{d}X^j$ is the Kalb-Ramond two-form field on M. If B is non-degenerate, it corresponds to an almost symplectic structure on M and we can write the classically equivalent first order string sigma-model

$$I_{\Sigma_2,\,\pi} \;=\; \int_{\Sigma_2}\Big(\chi_i\wedge\mathrm{d}X^i \,+\, \frac{1}{2}\,\pi^{ij}\,\chi_i\wedge\chi_j\Big) \;, \tag{4.45}$$

where B_{ij} is the inverse of $-\pi^{ij}$ and $\chi = (\chi_i) \in \Omega^1(\Sigma_2, X^*T^*M)$ is an auxiliary one-form. The bivector $\pi = \frac{1}{2}\,\pi^{ij}\,\frac{\partial}{\partial X^i}\wedge\frac{\partial}{\partial X^j}$ on M is a Poisson bivector on-shell, which is equivalent to a flat B-field $\mathrm{d}B = 0$, so that the Kalb-Ramond field corresponds to a symplectic structure on M. This is the action functional of the Poisson sigma-model [75, 76].

The AKSZ formulation of the Poisson sigma-model is studied in [32]. We take $\mathcal{W} = T[1]\Sigma_2$, and $\mathcal{M} = T^*[1]M$ with degree 0 base coordinates X^i on M and degree 1 fiber coordinates χ_i. The canonical symplectic form on $\mathcal{M}$ is

$$\omega_2 = \mathrm{d}\chi_i\wedge\mathrm{d}X^i \;, \tag{4.46}$$

which leads to the canonical graded Poisson bracket $\{\chi_i, X^j\} = \delta_i{}^j$ on the local coordinates of $\mathcal{M}$. We choose the Liouville potential to be $\vartheta = \chi_i\,\mathrm{d}X^i$. The most

general form of a degree 2 Hamiltonian function γ on $\mathcal{M}$ is given by a $(0,2)$-tensor $\pi = \pi^{ij}\,\frac{\partial}{\partial X^i}\otimes\frac{\partial}{\partial X^j}$ on M as

$$\gamma_\pi \;=\; \frac{1}{2}\,\pi^{ij}(X)\,\chi_i\,\chi_j\;. \tag{4.47}$$

The corresponding cohomological vector field Q_{γ_π} on $\mathcal{M}$ is

$$Q_{\gamma_\pi} \;=\; \pi^{ij}\,\chi_j\,\frac{\partial}{\partial X^i} \;+\; \frac{1}{2}\,\frac{\partial\pi^{ij}}{\partial X^k}\,\chi_i\,\chi_j\,\frac{\overrightarrow{\partial}}{\partial\chi_k}\;. \tag{4.48}$$

Compatibility of Q_{γ_π} with ω_π implies $\pi\in\Gamma(\bigwedge^2 TM)$ and the classical master equation $\{\gamma_\pi,\gamma_\pi\}=0$ implies that $\pi=\frac{1}{2}\,\pi^{ij}\,\frac{\partial}{\partial X^i}\wedge\frac{\partial}{\partial X^j}$ must be a Poisson bivector on M, i.e. $\pi^{l[i}\,\frac{\partial\pi^{jk]}}{\partial X^l}=0$ or equivalently $[\pi,\pi]_{\rm S}=0$, where $[\,\cdot\,,\,\cdot\,]_{\rm S}$ denotes the Schouten bracket on multivectors. In other words, a QP1-manifold or symplectic Lie 1-algebroid is the same thing as a Poisson manifold (M,π), which by construction is also a Lie algebroid on the cotangent bundle T^*M. The Hamiltonian function determines a derived bracket which defines a Poisson bracket on $C^\infty(M)$ through

$$\{f,g\}_\pi \;=\; \pi(\mathrm{d}f\wedge\mathrm{d}g) \;=\; -\,\{\{f,\gamma\},g\}\;. \tag{4.49}$$

The kinetic part of the AKSZ action is inherited from the cohomological vector field $Q_{\mathcal{W}}$ on $\mathcal{W}=T[1]\Sigma_2$, and is given by[4]

$$\boldsymbol{\mathcal{S}}_0^{(2)} \;=\; \int_{T[1]\Sigma_2}\mathrm{d}^2\hat{z}\;\boldsymbol{\chi}_i\,\boldsymbol{D}\boldsymbol{X}^i\;, \tag{4.50}$$

where as before the superworldsheet differential is $\boldsymbol{D}=\theta^\mu\,\frac{\partial}{\partial\sigma^\mu}=Q_{\mathcal{W}}$. The BV bracket has the form

$$(\,\cdot\,,\,\cdot\,)_{\rm BV} \;=\; \int_{T[1]\Sigma_2}\mathrm{d}^2\hat{z}\;\frac{\delta}{\delta\boldsymbol{X}^i}\wedge\frac{\delta}{\delta\boldsymbol{\chi}_i}\;, \tag{4.51}$$

where

$$\frac{\delta}{\delta\boldsymbol{X}^i}\wedge\frac{\delta}{\delta\boldsymbol{\chi}_i} := \frac{\overleftarrow{\delta}}{\delta\boldsymbol{X}^i}\,\frac{\overrightarrow{\delta}}{\delta\boldsymbol{\chi}_i} - \frac{\overleftarrow{\delta}}{\delta\boldsymbol{\chi}_i}\,\frac{\overrightarrow{\delta}}{\delta\boldsymbol{X}^i}\;. \tag{4.52}$$

Together these ingredients give the AKSZ action for the Poisson sigma-model as

$$\boldsymbol{\mathcal{S}}_\pi^{(2)} \;=\; \int_{T[1]\Sigma_2}\mathrm{d}^2\hat{z}\;\Big(\boldsymbol{\chi}_i\,\boldsymbol{D}\boldsymbol{X}^i \;+\; \frac{1}{2}\,\boldsymbol{\pi}^{ij}\,\boldsymbol{\chi}_i\,\boldsymbol{\chi}_j\Big)\;, \tag{4.53}$$

where $\boldsymbol{f}=\boldsymbol{\phi}^*(f)=f(\boldsymbol{\phi})$ for a function f on $\mathcal{M}$ and $\boldsymbol{\phi}\in\boldsymbol{\mathcal{M}}$. Integrating over the odd coordinates θ^μ and restricting to the degree 0 fields in (4.53) recovers the

[4]The upper index of a particular AKSZ action, which is locked in brackets, is used to indicate the dimension of the worldvolume, where the AKSZ action is defined on.

classical action (4.45), and in this sense the action (4.53) provides a BV quantization of the original Poisson sigma-model, which yields the Cattaneo-Felder path integral approach [77] of the Kontsevich star-product. By the fixed point theorem, the path integral localizes onto critical points of the action $\boldsymbol{\mathcal{S}}_{\pi}^{(2)}$, which are the fixed points of the cohomological vector field Q_γ that defines the Poisson cohomology $H_\pi^\bullet(M)$ of M.

2.2 AKSZ formulations of the A-model

The topological A- and B-models coupled to gravity are the topological A- and B-model string theories, which have been widely studied for more than 20 years. They were also one of the first examples of the AKSZ construction in [31]. In the following we review their relevant AKSZ constructions, which reduce to the A- or B-model in a particular gauge. The reader can find details about their field-antifield choices and gauge fixing in the indicated references, and therefore we only define their AKSZ sigma-models.

We begin with the A-model, whose AKSZ constructions were mostly related to the Poisson sigma-model or the B-field coupling. We found rather different constructions related to the AKSZ membranes on G_2-manifolds, which we will discuss later.

<u>*A1.*</u> The original AKSZ construction [31] is formulated in the same way as the Poisson sigma-model in §2.1 but with zero kinetic term. Thus it has the same target QP1-manifold with the same symplectic structure and Hamiltonian as those of the Poisson sigma-model, where the Poisson bivector π is given by the inverse of the Kähler form on the target Calabi-Yau manifold. The AKSZ action thus constructed is

$$\boldsymbol{\mathcal{S}}_{\mathrm{A1}}^{(2)} = \frac{1}{2}\int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\ \boldsymbol{\pi}^{ij}\,\boldsymbol{\chi}_i\,\boldsymbol{\chi}_j\ . \tag{4.54}$$

<u>*A2.*</u> A complete Poisson sigma-model formulation for the A-model with kinetic term (4.53) appeared in [78] as

$$\boldsymbol{\mathcal{S}}_{\mathrm{A2}}^{(2)} = \boldsymbol{\mathcal{S}}_{\pi}^{(2)} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\ \Big(\boldsymbol{\chi}_i\,\boldsymbol{D}\boldsymbol{X}^i + \frac{1}{2}\boldsymbol{\pi}^{ij}\,\boldsymbol{\chi}_i\,\boldsymbol{\chi}_j\Big)\ . \tag{4.55}$$

The equation of motion for $\boldsymbol{\chi}_i$ reduces it to the AKSZ action (4.54) up to a sign, so they are classically equivalent.

A3. The BV quantized topological NS–NS B-field coupling is not strictly speaking constructed by the AKSZ formalism, but it is nevertheless worth mentioning as a BV action which gives the A-model [79] with the same field definitions as those of the Poisson sigma-model:

$$\boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{A3}} \,=\, \frac{1}{2}\int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\;\boldsymbol{B}_{ij}\,\boldsymbol{DX}^i\,\boldsymbol{DX}^j\;, \tag{4.56}$$

where the two-form B is the Kähler form, and the flat condition $\mathrm{d}B = 0$ is equivalent to the Poisson condition of its inverse π. It is of course not surprising that the B-field coupling is classically equivalent to the Poisson sigma-model as well.

A4. In [79, 80] an AKSZ Poisson sigma-model together with the topological B-field coupling is used as an AKSZ formulation of the A-model with action

$$\boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{A4}} \,=\, \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\;\Big(\boldsymbol{\chi}_i\,\boldsymbol{DX}^i \,+\, \frac{1}{2}\,\boldsymbol{\pi}^{ij}\,\boldsymbol{\chi}_i\,\boldsymbol{\chi}_j \,+\, \frac{1}{4}\,\boldsymbol{B}_{ij}\,\boldsymbol{DX}^i\,\boldsymbol{DX}^j\Big)\;, \tag{4.57}$$

where B_{ij} is the inverse of π^{ij}. The last term has no effect in the BV bracket since $\mathrm{d}B = 0$.

Zucchini model. The BV sigma-model of [79] is not strictly speaking given by an AKSZ construction, since it involves BV quantized kinetic terms which do not arise from a Louville potential of the BV symplectic form. It has the same field content and BV symplectic form as those of the Poisson sigma-model, and the BV action is given by

$$\boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{Z}} \,=\, \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\;\Big(\boldsymbol{\chi}_i\,\boldsymbol{DX}^i \,+\, \frac{1}{2}\,\boldsymbol{\pi}^{ij}\,\boldsymbol{\chi}_i\,\boldsymbol{\chi}_j \,+\, \frac{1}{2}\,\boldsymbol{\omega}_{ij}\,\boldsymbol{DX}^i\,\boldsymbol{DX}^j \,+\, \boldsymbol{J}^i{}_j\,\boldsymbol{\chi}_i\,\boldsymbol{DX}^j\Big)\;, \tag{4.58}$$

where π is a bivector and ω is a two-form, and together with the tensor $(1,1)$-tensor J they satisfy the identities

$$\begin{aligned} J^i{}_k\,J^k{}_j \,+\, \pi^{ik}\,\omega_{kj} \,+\, \delta^i{}_j \,&=\, 0\;,\\ J^i{}_k\,\pi^{kj} \,+\, J^j{}_k\,\pi^{ki} \,&=\, 0\;,\\ \omega_{ik}\,J^k{}_j \,+\, \omega_{jk}\,J^k{}_i \,&=\, 0\;. \end{aligned} \tag{4.59}$$

The master equation imposes further constraints

$$\begin{aligned} \pi^{[i|l}\,\partial_l\pi^{jk]} \,&=\, 0\;,\\ J^l{}_i\,\partial_l\pi^{jk} \,+\, 2\,\pi^{jl}\,\partial_{[i}J^k{}_{l]} \,+\, \pi^{kl}\,\partial_l J^j{}_i \,-\, J^j{}_l\,\partial_i\pi^{lk} \,&=\, 0\;,\\ 2\,J^l{}_{[i|}\,\partial_l J^k{}_{|j]} \,-\, 2\,J^k{}_l\,\partial_{[i}J^l{}_{j]} \,+\, 3\,\pi^{kl}\,\partial_{[l}\omega_{ij]} \,&=\, 0\;,\\ J^l{}_i\,\partial_{[l}\omega_{jk]} \,+\, J^l{}_j\,\partial_{[l}\omega_{ki]} \,+\, J^l{}_k\,\partial_{[l}\omega_{ij]} \,-\, \partial_{[i}\big(\omega_{j|l}\,J^l{}_{|k]}\big) \,&=\, 0\;, \end{aligned} \tag{4.60}$$

where $\partial_i = \partial/\partial X^i$, which are the same identities as the integrability condition of a generalized complex structure $\mathbb{J}$ in the form

$$\mathbb{J}^I{}_J = \begin{pmatrix} J^i{}_j & \pi^{ij} \\ \omega_{ij} & -J^j{}_i \end{pmatrix} , \tag{4.61}$$

where the doubled indices I, J have been introduced. The Zucchini model reduces to the Poisson sigma-model upon setting $J = 0$ and $\omega = 0$, which is the A-model. If in addition ω is non-zero it adds a B-field coupling, which is just another copy of the A-model. These two cases are those that appear in (1.6) and (1.7) as the ordinary complex and symplectic structures embedded in generalized complex structure.

2.3 AKSZ formulations of the B-model

AKSZ constructions for the topological B-model are more diverse and have different superfield contents. We do not enumerate all of them here, nor the original construction from [31], since they are similar to the ones described below.

<u>*B1.*</u> The base degree 0 manifold of the target QP-manifold, which is a Calabi-Yau threefold M, is equiped with a complex structure which splits the local coordinate indices to $i = (a, \overline{a})$, where $a = 1, 2, 3$. The target QP-manifold $\mathcal{M}$ is defined by its coordinates: X^a, $X^{\overline{a}}$, $\widetilde{X}_{\overline{a}}$ have degree 0, and χ_a, $\chi_{\overline{a}}$, $\widetilde{\chi}^{\overline{a}}$ have degree 1. The symplectic form on $\mathcal{M}$ is

$$\omega_{\mathrm{B1}} = \mathrm{d}X^a \wedge \mathrm{d}\chi_a + \mathrm{d}X^{\overline{a}} \wedge \mathrm{d}\chi_{\overline{a}} + \mathrm{d}\widetilde{X}_{\overline{a}} \wedge \mathrm{d}\widetilde{\chi}^{\overline{a}} . \tag{4.62}$$

The B-model is constructed in [81] by the AKSZ action

$$\boldsymbol{\mathcal{S}}'^{(2)}_{\mathrm{B1}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z} \, \big(\boldsymbol{\chi}_a \boldsymbol{D}\boldsymbol{X}^a + \boldsymbol{\chi}_{\overline{a}} \boldsymbol{D}\boldsymbol{X}^{\overline{a}} + \widetilde{\boldsymbol{X}}_{\overline{a}} \boldsymbol{D}\widetilde{\boldsymbol{\chi}}^{\overline{a}} + \boldsymbol{\chi}_{\overline{a}} \widetilde{\boldsymbol{\chi}}^{\overline{a}} \big) . \tag{4.63}$$

We can enlarge its field content with the addition of new coordinates $\widetilde{\chi}^a$ and $\widetilde{X}_a$ whose contribution to the symplectic structure is defined by the term

$$\mathrm{d}\widetilde{X}_a \wedge \mathrm{d}\widetilde{\chi}^a , \tag{4.64}$$

and furthermore we also add the term $\widetilde{\boldsymbol{X}}_a \boldsymbol{D}\widetilde{\boldsymbol{\chi}}^a + \boldsymbol{\chi}_a \widetilde{\boldsymbol{\chi}}^a$ to the AKSZ action (4.63) which can be set to zero with gauge fixing $\widetilde{\boldsymbol{\chi}}^a = 0$. Introducing the new fields leads to an extended AKSZ action for the B-model given by

$$\boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{B1}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z} \, \big(\boldsymbol{\chi}_i \boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{X}}_i \boldsymbol{D}\widetilde{\boldsymbol{\chi}}^i + \boldsymbol{\chi}_i \widetilde{\boldsymbol{\chi}}^i \big) . \tag{4.65}$$

B2. The AKSZ construction of the B-model with an explicit complex structure J was studied in [74], see also [29]. It has the same field content as the first construction of the B-model: X^i, $\widetilde{X}_i$ are degree 0 coordinates and χ_i, ζ^i are degree 1 coordinates. The symplectic structure only differs in a sign from the first construction:

$$\omega_{\mathrm{B2}} \,=\, \mathrm{d}X^i \wedge \mathrm{d}\chi_i \,-\, \mathrm{d}\widetilde{X}_i \wedge \mathrm{d}\widetilde{\chi}^i \ . \tag{4.66}$$

The AKSZ action is given by

$$\boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{B2}} \,=\, \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\, \big(\boldsymbol{\chi}_i\, \boldsymbol{D}\boldsymbol{X}^i \,-\, \widetilde{\boldsymbol{X}}_i\, \boldsymbol{D}\widetilde{\boldsymbol{\chi}}^i \,+\, J^i{}_j\, \boldsymbol{\chi}_i\, \widetilde{\boldsymbol{\chi}}^j\big) \ , \tag{4.67}$$

where $J^i{}_j$ is a constant complex structure on the target manifold. The first construction is just a special case of this: If we take $J^i{}_j = \mathrm{i}\,\delta^i{}_j$, and rescale the fields $\widetilde{\boldsymbol{X}}_i$ and $\widetilde{\boldsymbol{\chi}}^i$ by i, then the action (4.67) reduces to $\boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{B1}}$.

The case of non-constant complex structure J was also studied in [74], and an AKSZ sigma-model was proposed, of which the master equation gives the integrability condition

$$J^l{}_{[i|}\, \partial_l J^k{}_{|j]} \,-\, J^k{}_l\, \partial_{[i} J^l{}_{j]} \,=\, 0 \ , \tag{4.68}$$

and the condition

$$J^i{}_k\, J^k{}_j \,=\, -\,\delta^i{}_j \tag{4.69}$$

is added by hand. The field content is the same as that of the constant case, and the action constructed by the AKSZ formalism is given by

$$\boldsymbol{\mathcal{S}}^{(2)}_{J} \,=\, \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\, \big(\boldsymbol{\chi}_i\, \boldsymbol{D}\boldsymbol{X}^i \,-\, \widetilde{\boldsymbol{X}}_i\, \boldsymbol{D}\widetilde{\boldsymbol{\chi}}^i \,+\, \boldsymbol{J}^i{}_j\, \boldsymbol{\chi}_i\, \widetilde{\boldsymbol{\chi}}^j \,+\, \boldsymbol{\partial}_j \boldsymbol{J}^i{}_k\, \widetilde{\boldsymbol{X}}_i\, \widetilde{\boldsymbol{\chi}}^j\, \widetilde{\boldsymbol{\chi}}^k\big) \ . \tag{4.70}$$

3 Courant sigma-models

The next dimension $d = 3$ is particularly relevant to extending the Poisson sigma-model to closed string backgrounds with non-zero NS–NS three-form flux $H = \mathrm{d}B$, or to M-theory backgrounds with three-form C-field. In this setting the closed strings are replaced with membranes described by maps $X = (X^i)$ from a closed three-dimensional worldvolume Σ_3 to the target space M. The topological part of the bosonic membrane action is the Wess-Zumino coupling

$$I_{\Sigma_3,H} \,=\, \int_{\Sigma_3} X^*(H) \,=\, \frac{1}{3!} \int_{\Sigma_3} H_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \ . \tag{4.71}$$

This action is classically equivalent to the first order membrane sigma-model action

$$I'_{\Sigma_3,H} \,=\, \int_{\Sigma_3} \Big(F_i \wedge \big(\mathrm{d}X^i - \psi^i\big) \,-\, \chi_i \wedge \mathrm{d}\psi^i \,+\, \frac{1}{3!}\, H_{ijk}\, \psi^i \wedge \psi^j \wedge \psi^k \Big) \;, \tag{4.72}$$

where $\psi = (\psi^i) \in \Omega^1(\Sigma_3, X^*TM)$ and $\chi = (\chi_i) \in \Omega^1(\Sigma_3, X^*T^*M)$ are one-forms, while $F = (F_i) \in \Omega^2(\Sigma_3, X^*T^*M)$ is an auxiliary two-form. The corresponding AKSZ sigma-model is defined on worldvolume superfields with target space a QP-manifold of degree 2, which corresponds to the standard Courant algebroid. This is true for general, an AKSZ sigma-model with source dg-manifold $\mathcal{W} = T[1]\Sigma_3$ and target space a QP-manifold of degree 2, corresponds to a Courant algebroid [24], and vice versa.

In this section we review the Courant algebroid and the corresponding three-dimensional Courant sigma-model. We describe its specific examples which are relevant for us: the standard and the contravariant Courant sigma-models. We also study their boundary reductions in the exact gauge.

3.1 Courant algebroids

A *Courant algebroid* on a manifold M is a vector bundle E over M equiped with a symmetric non-degenerate bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$ on its fibers, an anchor map $\rho : E \to TM$, and a binary bracket of sections $[\,\cdot\,,\,\cdot\,]_{\mathrm{D}}$, called the Dorfman bracket, which together satisfy

$$\begin{aligned} [e_1, [e_2, e_3]_{\mathrm{D}}]_{\mathrm{D}} \,&=\, [[e_1, e_2]_{\mathrm{D}}, e_3]_{\mathrm{D}} \,+\, [e_2, [e_1, e_3]_{\mathrm{D}}]_{\mathrm{D}} \;, \\ \rho(e_1)\langle e_2, e_3\rangle \,&=\, \langle [e_1, e_2]_{\mathrm{D}}, e_3\rangle \,+\, \langle e_2, [e_1, e_3]_{\mathrm{D}}\rangle \;, \\ \rho(e_1)\langle e_2, e_3\rangle \,&=\, \langle e_1, [e_2, e_3]_{\mathrm{D}} \,+\, [e_3, e_2]_{\mathrm{D}}\rangle \;, \end{aligned} \tag{4.73}$$

where e_1, e_2, e_3 are sections of E.

Let us now review the correspondence between Courant algebroids and QP-manifolds of degree 2, following [29, 82] for the most part. We choose local Darboux coordinates on the QP-manifold $\mathcal{M}$ (X^i, ζ^a, F_i) with degrees $(0, 1, 2)$ in which the graded symplectic structure is given as

$$\omega \,=\, \mathrm{d}F_i \wedge \mathrm{d}X^i \,+\, \frac{1}{2}\, k_{ab}\, \mathrm{d}\zeta^a \wedge \mathrm{d}\zeta^b \;. \tag{4.74}$$

Here we have introduced a constant metric k_{ab} on the degree 1 subspace, which is a local coordinate expression of the symmetric pairing in the corresponding Courant

algebroid. The graded Poisson brackets of the coordinates are canonical in the sense that

$$\{X^i, F_j\} \,=\, \delta^i{}_j \qquad \text{and} \qquad \{\zeta^a, \zeta^b\} \,=\, k^{ab} \ , \tag{4.75}$$

where k^{ab} is the inverse of k_{ab}.

The most general form of the degree 3 Hamiltonian function γ is given by

$$\gamma_{\rho,T} \,=\, \rho^i{}_a(X)\, F_i\, \zeta^a \,+\, \frac{1}{3!}\, T_{abc}(X)\, \zeta^a\, \zeta^b\, \zeta^c \ , \tag{4.76}$$

where the functions $\rho^i{}_a$ and T_{abc} on M give the local forms of the anchor map and fluxes, respectively. The three operations on the Courant algebroid are given by taking derived brackets defined by $\gamma_{\rho,T}$ and the graded Poisson bracket through

$$[e_1, e_2]_{\mathrm{D}} = \{\{e_1, \gamma_{\rho,T}\}, e_2\} \ , \qquad \langle e_1, e_2\rangle = \{e_1, e_2\} \qquad \text{and} \qquad \rho(e) = \{e, \{\gamma_{\rho,T}, \cdot\}\} \ . \tag{4.77}$$

These operations are defined on degree 1 functions e with local expression $e = f_a(X)\, \zeta^a$, where f_a is a degree 0 function on the body $M = \mathcal{M}_0$ of $\mathcal{M}$, which are identified as local sections of a vector bundle E over M. They satisfy the Courant algebroid axioms in (4.73) as a consequence of the classical master equation $\{\gamma_{\rho,T}, \gamma_{\rho,T}\} = 0$.

Conversely, given a Courant algebroid on a vector bundle E over M, we define the target QP-manifold $\mathcal{M}$ of degree 2 to be the symplectic submanifold of $T^*[2]E[1]$ corresponding to the isometric embedding $E \hookrightarrow E \oplus E^*$ with respect to the Courant algebroid pairing and the canonical dual pairing. Then X^i are local coordinates on M, F_i are local fiber coordinates of the shifted cotangent bundle $T^*[2]M$, and ζ^a are local fiber coordinates of the shifted vector bundle $E[1]$. In other words, a QP-manifold of degree 2 or a symplectic Lie 2-algebroid is the same thing as a Courant algebroid.

In the thesis we shall only work with Courant algebroids on the generalized tangent bundle

$$E = TM \oplus T^*M \ . \tag{4.78}$$

The corresponding QP-manifold of degree 2 is then simply $\mathcal{M} = T^*[2]T[1]M$. The local degree 1 coordinates are dual pairs[5] $\zeta^I = (\psi^i, \chi_i)$ and the symplectic form is

$$\omega_3 \,=\, \mathrm{d}F_i \wedge \mathrm{d}X^i \,+\, \mathrm{d}\chi_i \wedge \mathrm{d}\psi^i \ . \tag{4.79}$$

[5]We use capital indices $I, J, \ldots = 1, \ldots 2d$ for the generalized tangent bundle, where d is the dimension of M.

For the Liouville potential we choose $\vartheta = F_i\,\mathrm{d}X^i - \chi_i\,\mathrm{d}\psi^i$. Its zero locus is $\mathcal{L}_3 = \{F_i = 0,\, \psi^i = 0\}$. In the Hamiltonian function given by (4.76), the three-form T_{IJK} encodes the allowed geometric and non-geometric supergravity fluxes for given $\rho^i{}_I$. The local sections of the generalized tangent bundle are identified symbolically as the degree 1 functions with

$$A^i\,\chi_i \,+\, \alpha_i\,\psi^i \;\longleftrightarrow\; A^i\,\frac{\partial}{\partial X^i} \,+\, \alpha_i\,\mathrm{d}X^i \;. \tag{4.80}$$

Standard Courant algebroid

The simplest Hamiltonian function with $\rho^i{}_I = \delta^i{}_I$ and $T_{IJK} = 0$ is given by

$$\gamma_0 \,=\, F_i\,\psi^i \;. \tag{4.81}$$

Its derived brackets on degree 1 functions (4.80) gives the *standard Courant algebroid* on the generalized tangent bundle $E = TM \oplus T^*M$, which features in generalized geometry [7,8]. It is an extension of the Lie algebroid of tangent vectors by cotangent vectors with the three operations

$$\begin{aligned} \langle A+\alpha, B+\beta\rangle \,&=\, \iota_A\beta \,+\, \iota_B\alpha \;, \\ \rho(A+\alpha) \,&=\, A \;, \\ [A+\alpha, B+\beta]_{\mathrm{D},0} \,&=\, [A,B] \,+\, \mathcal{L}_A\beta \,-\, \iota_B\,\mathrm{d}\alpha \;, \end{aligned} \tag{4.82}$$

where the sections of $E = TM \oplus T^*M$ are composed of vector fields A, B and one-forms α, β. The antisymmetrization of the standard Dorfman bracket in (4.82) given by

$$[A+\alpha, B+\beta]_{\mathrm{C}} \,=\, [A,B] + \mathcal{L}_A\beta - \mathcal{L}_B\alpha - \frac{1}{2}\,\mathrm{d}(\iota_A\beta - \iota_B\alpha) \;, \tag{4.83}$$

is called the Courant bracket, and is has already been introduced previously in (1.2). It is the natural bracket in generalized geometry which is compatible with the commutator algebra of generalized Lie derivatives [7,8].

Only the simplest case of pure NS–NS flux $T_{IJK} = H_{ijk}$ is consistent with the choice of anchor map $\rho^i{}_I = \delta^i{}_I$ of the standard Courant algebroid, which is necessarily closed by the classical master equation. Given a Kalb-Ramond two-form field B on M, with $H = \mathrm{d}B$, canonical transformation of the Hamiltonian function (4.81) by the degree 2 function $B = \frac{1}{2}\,B_{ij}(X)\,\psi^i\,\psi^j$ on $\mathcal{M}$ yields the twisted Hamiltonian function

$$\gamma_H \,:=\, \mathrm{e}^{\,\delta_B}\gamma_0 \,=\, F_i\,\psi^i \,+\, \frac{1}{3!}\,H_{ijk}\,\psi^i\,\psi^j\,\psi^k \;. \tag{4.84}$$

The NS–NS H-flux thus appears as a twisting of the standard Courant algebroid, which gives rise to a deformation of the Dorfman bracket through an extra term as

$$[A+\alpha,B+\beta]_{\mathrm{D},H} \;=\; [A,B] \,+\, \mathcal{L}_A\beta \,-\, \iota_B\,\mathrm{d}\alpha \,+\, \iota_A\iota_B H \;. \tag{4.85}$$

Poisson Courant algebroid

Consider the Hamiltonian defined through a bivector π and a three-vector R on M by setting $\rho^i{}_I = \pi^{ij}$, $T_{IJK} = (\partial_i\pi^{jk}, R^{ijk})$ to give

$$\gamma_{\pi,R} \;=\; \pi^{ij}\,F_i\,\chi_j \,-\, \frac{1}{2}\,\partial_i\pi^{jk}\,\psi^i\,\chi_j\,\chi_k \,+\, \frac{1}{3!}\,R^{ijk}(X)\,\chi_i\,\chi_j\,\chi_k \;. \tag{4.86}$$

The master equation $\{\gamma_{\pi,R},\gamma_{\pi,R}\} = 0$ gives the constraints

$$[\pi,\pi]_{\mathrm{S}} \;=\; 0 \qquad \text{and} \qquad [\pi,R]_{\mathrm{S}} \;=\; 0 \;. \tag{4.87}$$

Note that the R-flux also enters here as a twist: The Hamiltonian (4.86) can be regarded as a canonical transformation $\gamma_{\pi,R} = \mathrm{e}^{\,\delta_\beta}\gamma_{\pi,0}$ by a degree 2 function $\beta = \frac{1}{2}\,\beta^{ij}(X)\,\chi_i\,\chi_j$ with $R = \mathrm{d}_\beta\beta$ where $\mathrm{d}_\beta = [\beta,\,\cdot\,]_{\mathrm{S}}$, regarded as a bivector β on M which is T-dual to the B-field of the H-flux frame. The corresponding Courant algebroid is the Poisson Courant algebroid [83], for which the identities are equivalent to the Poisson condition for π if $R = 0$: The *Poisson Courant algebroid* is the Courant algebroid on the generalized tangent bundle $E = TM \oplus T^*M$ over a Poisson manifold (M,π) with the operations

$$\begin{aligned}
\langle A+\alpha,B+\beta\rangle \;&=\; \iota_A\beta \,+\, \iota_B\alpha \;,\\
\rho(A+\alpha) \;&=\; \iota_\alpha\pi \;,\\
[A+\alpha,B+\beta]_{\mathrm{D};\pi,R} \;&=\; [\alpha,\beta]_\pi \,+\, \mathcal{L}^\pi_\alpha Y \,-\, \iota_\beta\,\mathrm{d}_\pi X \,-\, \iota_\alpha\iota_\beta R \;,
\end{aligned} \tag{4.88}$$

where $\mathcal{L}^\pi_\alpha = \iota_\alpha\,\mathrm{d}_\pi + \mathrm{d}_\pi\,\iota_\alpha$ and $[\,\cdot\,,\,\cdot\,]_\pi$ is the Koszul bracket on one-forms given by

$$[\alpha,\beta]_\pi \;=\; \mathcal{L}_{\iota_\alpha\pi}\beta \,-\, \mathcal{L}_{\iota_\beta\pi}\alpha \,-\, \mathrm{d}\big(\pi(\alpha\wedge\beta)\big) \;. \tag{4.89}$$

3.2 Standard Courant sigma-model

It is evident from the general construction that Courant algebroids are uniquely encoded (up to isomorphism) in the corresponding AKSZ topological membrane theories, which are called Courant sigma-models [27]. In the particular example of the

standard Courant algebroid on $E = TM \oplus T^*M$ twisted by a closed NS–NS three-form flux H, the mapping space $\boldsymbol{\mathcal{M}}$ of superfields supports the canonical BV symplectic structure

$$\boldsymbol{\omega}_3 = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \big(\boldsymbol{\delta X}^i\, \boldsymbol{\delta F}_i + \boldsymbol{\delta\psi}^i\, \boldsymbol{\delta\chi}_i\big)\ , \tag{4.90}$$

where the ghost number 2 superfields $\boldsymbol{F}_i$ and ghost number 0 superfields $\boldsymbol{X}^i$, as well as the conjugate pairs of ghost number 1 superfields $\boldsymbol{\chi}_i$ and $\boldsymbol{\psi}^i$, contain each other's antifields respectively. The AKSZ construction leads to the action

$$\boldsymbol{\mathcal{S}}_H^{(3)} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \Big(\boldsymbol{F}_i\, \boldsymbol{DX}^i - \boldsymbol{\chi}_i\, \boldsymbol{D\psi}^i + \boldsymbol{F}_i\, \boldsymbol{\psi}^i + \frac{1}{3!}\, \boldsymbol{H}_{ijk}\, \boldsymbol{\psi}^i\, \boldsymbol{\psi}^j\, \boldsymbol{\psi}^k\Big)\ , \tag{4.91}$$

which solves the classical master equation $\big(\boldsymbol{\mathcal{S}}_H^{(3)}, \boldsymbol{\mathcal{S}}_H^{(3)}\big)_{\mathrm{BV}} = 0$. Integrating over θ^μ and restricting to degree 0 fields in (4.91) yields the first order membrane sigma-model action (4.72).

The standard Courant sigma-model on an open worldvolume is well-defined if, as usual, one specifies its boundary conditions. Instead we consider it in exact gauge as an illustration. The exact gauge defined in §1.3 reads here as

$$\boldsymbol{F}_i \ = \ \boldsymbol{D\chi}_i \qquad \text{and} \qquad \boldsymbol{\psi}^i \ = \ -\,\boldsymbol{DX}^i\ . \tag{4.92}$$

It gives the gauge fixed BV symplectic structure

$$\boldsymbol{\omega}_{3,\mathrm{gf}} \ = \ \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\ \boldsymbol{\delta X}^i\, \boldsymbol{\delta\chi}_i\ , \tag{4.93}$$

and reduces the AKSZ action (4.91) without H-flux to zero. With H-flux, the AKSZ action leads to a pure Wess-Zumino coupling

$$\boldsymbol{\mathcal{S}}_{H,\mathrm{gf}}^{(3)} \ = \ -\,\frac{1}{3!} \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\ \boldsymbol{H}_{ijk}\, \boldsymbol{DX}^i\, \boldsymbol{DX}^j\, \boldsymbol{DX}^k \tag{4.94}$$

which is no longer an AKSZ action, as there are no BV gauge degrees of freedom in the bulk. This is reminescent of the fact that the equation of motion for $\boldsymbol{F}_i$ also gives the same action (4.94) up to a sign. If $H_{ijk} = \partial_{[i}B_{jk]}$ is exact, then we obtain the boundary AKSZ action

$$-\,\frac{1}{2} \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\ \boldsymbol{B}_{ij}\, \boldsymbol{DX}^i\, \boldsymbol{DX}^j\ , \tag{4.95}$$

which is the quantization of the NS–NS B-field coupling. Hence the exact gauge is nicely applicable for boundary reductions of topological membranes describing flux deformations of string sigma-models.

Our observation is that the standard Courant sigma-model is related to the B-model on its boundary via the exact gauge. We will unfold this later in the thesis, but now we only present the observation as a motivation to study Courant sigma-models in the context of A- and B-models.

Although we have found that the standard Courant sigma-model has a trivial boundary reduction in the exact gauge, we can obtain a non-trivial boundary theory if we extend its field content and then set the extra fields to zero with gauge fixing.

The standard Courant sigma-model without H-flux is given by the Hamiltonian γ_0 in (4.81) and the symplectic form ω_3 in (4.79). The AKSZ action is

$$\boldsymbol{\mathcal{S}}_0^{(3)} \,=\, \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \big(\boldsymbol{F}_i\, \boldsymbol{D}\boldsymbol{X}^i \,-\, \boldsymbol{\chi}_i\, \boldsymbol{D}\boldsymbol{\psi}^i \,+\, \boldsymbol{F}_i\, \boldsymbol{\psi}^i\big)\ . \tag{4.96}$$

We double its fields with the introduction of degree 0 coordinates $\widetilde{X}_i$, degree 1 coordinates $\widetilde{\chi}^i$, $\widetilde{\psi}_i$ and degree 2 coordinates $\widetilde{F}^i$ on the target QP2-manifold. The extra term in the symplectic structure is

$$\mathrm{d}\widetilde{X}_i \wedge \mathrm{d}\widetilde{F}^i \,+\, \mathrm{d}\widetilde{\chi}^i \wedge \mathrm{d}\widetilde{\psi}_i\ , \tag{4.97}$$

which together with the extended Hamiltonian

$$\gamma_0 \,+\, \widetilde{\gamma}_0 \,=\, F_i\, \psi^i \,-\, \widetilde{\psi}_i\, \widetilde{F}^i \tag{4.98}$$

defines the AKSZ action

$$\widetilde{\boldsymbol{\mathcal{S}}}_0^{(3)} \,=\, \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \big(\boldsymbol{F}_i\, \boldsymbol{D}\boldsymbol{X}^i \,-\, \boldsymbol{\chi}_i\, \boldsymbol{D}\boldsymbol{\psi}^i \,+\, \boldsymbol{F}_i\, \boldsymbol{\psi}^i \,-\, \widetilde{\boldsymbol{\psi}}_i\, \widetilde{\boldsymbol{F}}^i\big)\ , \tag{4.99}$$

where we did not introduce all the possible kinetic terms. This extended standard Courant sigma-model is comparable to the membrane sigma-model in [74], which was introduced in order to uplift the AKSZ construction of the B-model in (4.67) to an AKSZ membrane theory with generalized complex structure. Our construction arrives at a different B-model construction and uses less fields, but does not include the generalized complex structure.

The last term in (4.99) decouples from the original standard Courant sigma-model. To see this we can choose a different gauge than that we will choose for the boundary reduction, but we use the same field-antifield decomposition. For example, if we set $\widetilde{\boldsymbol{F}}^i = 0$ and $\widetilde{\boldsymbol{\chi}}^i = 0$ as a partial gauge fixing, we can trivially integrate out the fields $\widetilde{\boldsymbol{X}}_i$ and $\widetilde{\boldsymbol{\psi}}_i$, which gives the action of the standard Courant sigma-model in (4.96).

Alternatively, we can arrive at the same conclusion if we rescale the fields by a real parameter λ in a way which leaves the symplectic structure invariant:

$$\widetilde{\boldsymbol{\chi}}^i \longrightarrow \lambda\,\widetilde{\boldsymbol{\chi}}^i \qquad \text{and} \qquad \widetilde{\boldsymbol{\psi}}_i \longrightarrow \frac{1}{\lambda}\,\widetilde{\boldsymbol{\psi}}_i \ , \tag{4.100}$$

which is a duality transformation given by a symplectomorphism at the BV level. Then we take the $\lambda \to \infty$ limit: the term $\widetilde{\boldsymbol{\psi}}_i\,\widetilde{\boldsymbol{F}}^i$ in the action tends to zero and we get the standard Courant sigma-model in this way as well. Later on we will employ a similar rescaling technique to propose topological S-duality of A- and B-models in the level of their AKSZ formulations.

Now we reduce the extended standard Courant sigma-model to its boundary with the previously defined exact gauge from §1.3. In this case it means the specific gauge choice

$$\widetilde{\boldsymbol{\chi}}^i = -\,\boldsymbol{D}\boldsymbol{X}^i \ , \qquad \boldsymbol{F}_i = \boldsymbol{D}\widetilde{\boldsymbol{\psi}}_i \ , \qquad \boldsymbol{\chi}_i = -\,\boldsymbol{D}\widetilde{\boldsymbol{X}}_i \qquad \text{and} \qquad \widetilde{\boldsymbol{F}}^i = \boldsymbol{D}\boldsymbol{\psi}^i \ . \tag{4.101}$$

The BV symplectic form becomes

$$\oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\,\big(\boldsymbol{\delta}\boldsymbol{X}^i\,\boldsymbol{\delta}\widetilde{\boldsymbol{\psi}}_i + \boldsymbol{\delta}\widetilde{\boldsymbol{X}}_i\,\boldsymbol{\delta}\widetilde{\boldsymbol{\chi}}^i\big) \ , \tag{4.102}$$

which is the same BV symplectic form induced by (4.62) and (4.64) with the relabelling $\widetilde{\boldsymbol{\psi}}_i \to \boldsymbol{\chi}_i$. Our gauge fixing also reduces the AKSZ action in (4.99) to the boundary action

$$\widetilde{\boldsymbol{\mathcal{S}}}^{(3)}_{0,\mathrm{gf}} = \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\,\big(\widetilde{\boldsymbol{\psi}}_i\,\boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{X}}_i\,\boldsymbol{D}\widetilde{\boldsymbol{\chi}}^i + \widetilde{\boldsymbol{\psi}}_i\,\widetilde{\boldsymbol{\chi}}^i\big) = \boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{B1}} \ , \tag{4.103}$$

which is the same action as that of the B-model in (4.65) with the same relabelling $\widetilde{\boldsymbol{\psi}}_i \to \boldsymbol{\chi}_i$ as before.

3.3 Contravariant Courant sigma-model

The contravariant Courant sigma-model was introduced in [84] as the Courant sigma-model corresponding to a Poisson Courant algebroid. It is defined by the AKSZ action

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(3)}_{\pi,R} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\,\Big(&\boldsymbol{F}_i\,\boldsymbol{D}\boldsymbol{X}^i - \boldsymbol{\chi}_i\,\boldsymbol{D}\boldsymbol{\psi}^i + \boldsymbol{\pi}^{ij}\,\boldsymbol{F}_i\,\boldsymbol{\chi}_j \\ &- \frac{1}{2}\,\boldsymbol{\partial}_i\boldsymbol{\pi}^{jk}\,\boldsymbol{\psi}^i\,\boldsymbol{\chi}_j\,\boldsymbol{\chi}_k + \frac{1}{3!}\,\boldsymbol{R}^{ijk}\,\boldsymbol{\chi}_i\,\boldsymbol{\chi}_j\,\boldsymbol{\chi}_k\Big) \ .\end{aligned} \tag{4.104}$$

In the absence of R-flux the master equation gives the Poisson condition for the bivector π, so we can expect that the contravariant Courant sigma-model is closely related to the Poisson sigma-model. This relation turns out to be the exact gauge boundary reduction. We use the same gauge fixing as we used for the standard Courant sigma-model in (4.92) which gives the boundary BV symplectic form (4.93). The resulting boundary AKSZ action is that of the Poisson sigma-model in (4.53):

$$\boldsymbol{\mathcal{S}}^{(3)}_{\pi,0;\mathrm{gf}} = \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\,\Big(\boldsymbol{\chi}_i\,\boldsymbol{DX}^i + \frac{1}{2}\,\boldsymbol{\pi}^{ij}\,\boldsymbol{\chi}_i\,\boldsymbol{\chi}_j\Big) = \boldsymbol{\mathcal{S}}^{(2)}_{\pi}\ . \tag{4.105}$$

We also found that, in the degenerate limit, where the anchor of the contravariant Courant sigma-model is set to zero, and in the exact gauge, it coincides precisely with the membrane sigma-model of [11] which quantizes the nonassociative phase space and geometry of the R-flux background [20, 85, 86]. This clarifies more precisely the geometrical meaning of the model of [11] in terms of a Courant algebroid structure. An alternative geometric description as a certain reduction of the standard Courant sigma-model for the target space of double field theory is discussed in [93], which we study in §4. A vanishing anchor map ρ with non-zero R-flux means that the bivector field π is identically zero, and the Dorfman bracket is given solely by the three-vector R in the simple form

$$[X+\alpha, Y+\beta]_{\mathrm{D};0,R} = -\,\iota_\alpha\iota_\beta R\ , \tag{4.106}$$

so that the tangent bundle TM decouples completely from this structure.

We choose the exact gauge (4.92). In this case our gauge choice is not compatible with boundary conditions, because the BV master equation forces the flux term to be zero on the boundary, which means $\boldsymbol{\chi}_i = 0$ on $T[1]\partial\Sigma_3$ if $R \neq 0$. This can be circumvented by adding a non-topological boundary term to the action as in [11]. We introduce this as a strictly classical term after the full gauge fixing, and for brevity avoid here issues concerning its quantization. Hence the AKSZ action (4.104) with $\pi = 0$ reduces to

$$\boldsymbol{\mathcal{S}}^{(3)}_{0,R;\mathrm{gf}} = \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\,\boldsymbol{\chi}_i\,\boldsymbol{DX}^i + \frac{1}{3!}\int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\,\boldsymbol{R}^{ijk}\,\boldsymbol{\chi}_i\,\boldsymbol{\chi}_j\,\boldsymbol{\chi}_k\ . \tag{4.107}$$

There is still a gauge degree of freedom on the boundary fields remaining, therefore we choose $\chi_i^{(0)} = \chi_i^{(2)} = X^{i\,(1)} = 0$. The bulk fields are not antifields in the BV sense, so we cannot set any of them to zero. Instead we eliminate the non-zero parts in

the bulk by their equations of motion. The gauge fixed action for constant R in the superfield expansion (4.24) is

$$\boldsymbol{\mathcal{S}}^{(3)}_{0,R;\mathrm{gf}} = \oint_{\partial\Sigma_3} \mathrm{d}^2\sigma\ \chi_i \wedge \mathrm{d}X^i + \frac{1}{3!}\int_{\Sigma_3} \mathrm{d}^3\sigma\ \big(R^{ijk}\,\chi_i\wedge\chi_j\wedge\chi_k - R^{ijk}\,f_i\,\chi_j\wedge\phi_k + R^{ijk}\,f_i\,f_j\,C_k\big) \tag{4.108}$$

where $\chi_i^{(1)} = \chi_i$ is a degree 0 one-form, $\chi_i^{(0)} = f_i$ is a degree 1 function, $\chi_i^{(2)} = \phi_i$ is a degree -1 two-form and $\chi_i^{(3)} = C_i$ is a degree -2 three-form. Both f_i and ϕ_i vanish on the boundary, due to the boundary gauge fixing. The equations of motion of the three non-zero degree fields sets the last two bulk terms to zero, and they are consistent with each other. Now we introduce a boundary term given by the inverse of a target space metric g^{ij} since we need χ_i to be non-zero on the boundary. Finally we arrive at the action containing only degree 0 fields:

$$\boldsymbol{\mathcal{S}}^{(3)}_{0,R;\mathrm{gf}} = \oint_{\partial\Sigma_3} \mathrm{d}^2\sigma\ \chi_i \wedge \mathrm{d}X^i + \frac{1}{3!}\int_{\Sigma_3} \mathrm{d}^3\sigma\ R^{ijk}\,\chi_i\wedge\chi_j\wedge\chi_k + \oint_{\partial\Sigma_3} \mathrm{d}^2\sigma\ g^{ij}\,\chi_i\wedge *\chi_j\ , \tag{4.109}$$

where $*$ is the Hodge duality operator corresponding to a chosen metric on the membrane worldvolume Σ_3. This is precisely the string sigma-model derived in [11] which quantizes the non-geometric R-flux background.

4 DFT membrane sigma-models

The algebroid structure of double field theory was studied in [82, 88–94]; in particular, the notion of DFT algebroid was introduced in [93] whose derived bracket is the C-bracket of double field theory, and whose corresponding membrane sigma-model naturally captures the T-duality orbit of geometric and non-geometric flux backgrounds in a single unified description. DFT algebroids correspond to topological membrane sigma-models, which can be obtained by reducing (or projecting) larger AKSZ sigma-models. In this section we describe the construction of DFT membrane sigma-model in AKSZ theory, which we shall apply later.

4.1 DFT algebroids

The definition of DFT algebroid starts with the large Courant algebroid which is a straightforward doubled version of a general Courant algebroid. We formulate the definition from the graded symplectic geometry viewpoint, since this is explicitly relevant

in AKSZ constructions. For simplicity we consider the case when the large Courant algebroid is a Courant algebroid corresponding to the QP2-manifold $T^*[2]T[1]T^*M$, where the doubling of the original base manifold M appears as the total space of the cotangent bundle T^*M. We use the doubled index $I = 1, \ldots, 2d$ to label coordinates on the base space T^*M, which can be split into the first d indices $I = 1, \ldots, d$, which are the original indices labelling coordinates on M, and the second d dual indices $I = d+1, \ldots, 2d$ labelling the covectors of T^*M; both sets of indices are labeled by $i = 1, \ldots, d$.

The symplectic form coming from (4.79) is

$$\begin{aligned} \omega_{\rm DFT} &= \mathrm{d}X^I \wedge \mathrm{d}F_I \,+\, \mathrm{d}\chi_I \wedge \mathrm{d}\psi^I \\ &= \mathrm{d}X^i \wedge \mathrm{d}F_i \,+\, \mathrm{d}\widetilde{X}_i \wedge \mathrm{d}\widetilde{F}^i \,+\, \mathrm{d}\chi_i \wedge \mathrm{d}\psi^i \,+\, \mathrm{d}\widetilde{\chi}^i \wedge \mathrm{d}\widetilde{\psi}_i \ , \end{aligned} \tag{4.110}$$

where the splitting of a general doubled coordinate ϕ^I has been used: $\phi^I = (\phi^i, \widetilde{\phi}_i)$. As we know from (4.80) a degree one function $A^I\,\chi_I + \alpha_I\,\psi^I$ corresponds to a section of $T(T^*M) \oplus T^*(T^*M)$ symbolically as

$$A^i\,\chi_i + \widetilde{A}_i\,\widetilde{\chi}^i + \alpha_i\,\psi^i + \widetilde{\alpha}^i\,\widetilde{\psi}_i \quad \longleftrightarrow \quad A^i\,\frac{\partial}{\partial X^i} + \widetilde{A}_i\,\frac{\partial}{\partial \widetilde{X}_i} + \alpha_i\,\mathrm{d}X^i + \widetilde{\alpha}^i\,\mathrm{d}\widetilde{X}_i \ , \tag{4.111}$$

and the derived brackets (4.77) with a given general Hamiltonian (4.76):

$$\begin{aligned} \gamma_{\rm DFT} \,=& \rho^I{}_J\,F_I\,\psi^J \,+\, \beta^{IJ}\,F_I\,\chi_J \,+\, \frac{1}{3!}\,T^{(0)}{}_{IJK}\,\psi^I\,\psi^J\,\psi_K \,+\, \frac{1}{2}\,T^{(1)}{}_{IJ}{}^K\,\psi^I\,\psi^J\,\chi_K \\ &+\, \frac{1}{2}\,T^{(2)}{}_I{}^{JK}\,\psi^I\,\chi_J\,\chi_K \,+\, \frac{1}{3!}\,T^{(3)\,IJK}\,\chi_I\,\chi_J\,\chi_K \ , \end{aligned} \tag{4.112}$$

define a Courant algebroid on $T(T^*M) \oplus T^*(T^*M)$.

The DFT algebroid is based on the projection to DFT vectors, which halves the number of degree 1 coordinates. We introduce a new basis for the subspace of degree 1 fields spanned by χ_I and ψ^I given as

$$\tau^I_\pm \,=\, \frac{1}{2}\,\big(\psi^I \,\pm\, \eta^{IJ}\,\chi_J\big) \ , \tag{4.113}$$

where η_{IJ} is the $O(d,d)$-invariant constant metric defined in (1.1). The projection p_+ to the subspace L_+ spanned by τ^I_+ yields the projection to DFT vectors, which are vectors under $O(d,d)$. The corresponding sub-bundle of $T(T^*M) \oplus T^*(T^*M)$ is also denoted L_+.

For the symplectic structure $\omega_{\rm DFT}$ the projection means

$$\mathrm{d}\chi_I \wedge \mathrm{d}\psi^I \,=\, \eta_{IJ}\,\mathrm{d}\tau^I_+ \wedge \mathrm{d}\tau^J_+ \,-\, \eta_{IJ}\,\mathrm{d}\tau^I_- \wedge \mathrm{d}\tau^J_- \ \xrightarrow{\mathsf{p}_+}\ \eta_{IJ}\,\mathrm{d}\tau^I_+ \wedge \mathrm{d}\tau^J_+ \ . \tag{4.114}$$

The coordinates τ_+^I are counted twice in the symplectic structure compared to the original (4.110), so we halve their contribution in the symplectic structure solely:

$$\omega_{\mathrm{DFT},+} \;=\; \mathrm{d}X^I \wedge \mathrm{d}F_I \;+\; \frac{1}{2}\,\eta_{IJ}\,\mathrm{d}\tau_+^I \wedge \mathrm{d}\tau_+^J \;. \tag{4.115}$$

For the Liouville potential we take $\vartheta_{\mathrm{DFT},+} = F_I\,\mathrm{d}X^I - \frac{1}{2}\,\eta_{IJ}\,\tau_+^I\,\mathrm{d}\tau_+^J$. To specify its zero locus $\mathcal{L}_{\mathrm{DFT},+}$ as a Lagrangian submanifold, we choose a polarisation which is defined by a projector $\mathcal{P}$ on L_+ of rank d that is maximally isotropic with respect to the $O(d,d)$ metric (1.1):

$$\mathcal{P}^i{}_K\,\eta^{KL}\,\mathcal{P}^j{}_L = 0 \;. \tag{4.116}$$

It acts on the basis τ_+^I to give degree 1 coordinates

$$\tau^i = \mathcal{P}^i{}_J\,\tau_+^J \tag{4.117}$$

which span a d-dimensional subspace of L_+; then $\mathcal{L}_{\mathrm{DFT},+} := \{F_I = 0,\, \tau^i = 0\}$. Different polarizations define different Lagrangian submanifolds which are all related by $O(d,d)$ transformations: Acting with $\mathcal{O} \in O(d,d)$ changes the polarization as

$$\left(\begin{smallmatrix}\mathcal{P}\\ \widetilde{\mathcal{P}}\end{smallmatrix}\right) \longmapsto \left(\begin{smallmatrix}\mathcal{P}'\\ \widetilde{\mathcal{P}}'\end{smallmatrix}\right) = \left(\begin{smallmatrix}\mathcal{P}\\ \widetilde{\mathcal{P}}\end{smallmatrix}\right)\mathcal{O} \;, \tag{4.118}$$

where $\widetilde{\mathcal{P}} = 1 - \mathcal{P}$ is the complementary projector.

The Hamiltonian is projected to the subspace L_+ as

$$\gamma_{\mathrm{DFT},+} \;=\; (\rho_+)^I{}_J\,F_I\,\tau_+^J \;+\; \frac{1}{3!}\,(T_+)_{IJK}\,\tau_+^I\,\tau_+^J\,\tau_+^K \;, \tag{4.119}$$

where the new structure functions are defined by

$$(\rho_+)^I{}_J \;=\; \rho^I{}_J \;+\; \beta^{IK}\,\eta_{KJ} \;, \tag{4.120}$$

and

$$\begin{aligned}(T_+)_{IJK} \;=\; T^{(0)}{}_{IJK} \;+\; 3\,T^{(1)}{}_{[IJ}{}^{K'}\,\eta_{K]K'} \;+\; 3\,T^{(2)}{}_{[I|}{}^{J'K'}\,\eta_{J'|J|}\,\eta_{K'|K]} \\ +\; T^{(3)\,I'J'K'}\,\eta_{I'[I|}\,\eta_{J'|J|}\,\eta_{K'|K]} \;.\end{aligned} \tag{4.121}$$

The Hamiltonian $\gamma_{\mathrm{DFT},+}$ defined by these functions does not necessarily satisfy the master equation, despite the fact that the original Hamiltonian γ_{DFT} of the large Courant algebroid defined by $\rho^I{}_J$, β^{IJ}, $T^{(0)}{}_{IJK}$, $T^{(1)}{}_{IJ}{}^K$, $T^{(2)}{}_I{}^{JK}$ and $T^{(3)\,IJK}$ does.

The C-bracket is defined on DFT vectors of L_+, which correspond to the degree 1 functions A in the subspace spanned by $e_I = \eta_{IJ}\,\tau_+^J$:

$$A \;=\; A^I\,e_I \;=\; \frac{1}{2}\,A^I\,\big(\chi_I + \eta_{IJ}\,\psi^J\big) \;. \tag{4.122}$$

It can be obtained from derived brackets of the QP2-manifold together with the symmetric pairing and anchor map as

$$
\begin{aligned}
[\![A_1, A_2]\!]_{L_+} &= -\frac{1}{2}\big(\{\{A_1, \gamma_{\mathrm{DFT},+}\}, A_2\} \,-\, \{\{A_2, \gamma_{\mathrm{DFT},+}\}, A_1\}\big) \ , \\
\langle A_1, A_2\rangle_{L_+} &= \{A_1, A_2\} \ , \\
\rho_+(A) &= \{A, \{\gamma_{\mathrm{DFT},+}, \cdot\}\} \ ,
\end{aligned}
\tag{4.123}
$$

for DFT vectors A, A_1 and A_2. Since the master equation does not hold for $\gamma_{\mathrm{DFT},+}$, the algebraic structure is not that of a Courant algebroid, but called a DFT algebroid in [93]:

A *DFT algebroid* on T^*M is a vector bundle L_+ of rank $2d$ over T^*M equiped with a non-degenerate symmetric form $\langle\,\cdot\,,\,\cdot\,\rangle_{L_+}$ on its fibres, an anchor map $\rho_+ : L_+ \to T(T^*M)$, and a skew-symmetric bracket of sections $[\![\,\cdot\,,\,\cdot\,]\!]_{L_+}$, called the C-bracket, which together satisfy

$$
\begin{aligned}
\langle \mathcal{D}_+ f, \mathcal{D}_+ g\rangle_{L_+} &= \frac{1}{4}\,\langle \mathrm{d}f, \mathrm{d}g\rangle_{L_+} \ , \\
[\![A, f\,B]\!]_{L_+} &= f\,[\![A, B]\!]_{L_+} \,+\, \big(\rho_+(A)f\big)\,B \,-\, \langle A, B\rangle_{L_+}\,\mathcal{D}_+ f \ , \\
\rho_+(C)\langle A, B\rangle_{L_+} &= \big\langle [\![C, A]\!]_{L_+} \,+\, \mathcal{D}_+\langle C, A\rangle_{L_+}\,,\, B\big\rangle_{L_+} \\
&\quad + \big\langle A\,,\, [\![C, B]\!]_{L_+} \,+\, \mathcal{D}_+\langle C, B\rangle_{L_+}\big\rangle_{L_+} \ ,
\end{aligned}
\tag{4.124}
$$

for all sections $A, B, C \in C^\infty(T^*M, L_+)$ and functions $f, g \in C^\infty(T^*M)$, where $\mathcal{D}_+ : C^\infty(T^*M) \to C^\infty(T^*M, L_+)$ is the derivative defined through $\langle \mathcal{D}_+ f, A\rangle_{L_+} = \frac{1}{2}\,\rho_+(A)f$.

4.2 AKSZ construction of DFT membrane sigma-models

The large Courant sigma-model constructed by AKSZ theory corresponds to the large Courant algebroid in the spirit of §3. Then we execute the projection by p_+ on the level of AKSZ fields, which means selecting a special submanifold (by projection to DFT vectors) containing half of the ghost number 1 fields. This method is quite natural, because fields with identical properties appear twice, and we keep only one field of each identical pair. Note that there are infinitely many possibilities to perform the reduction on ghost number 1 fields, but only this projection to DFT vectors gives the right C-bracket structure of double field theory. One can think of the other reductions as a class of duality transformations, which leads out of the realm of the original physical double field theory.

The AKSZ action of the large Courant sigma-model corresponding to (4.112) is defined by

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(3)}_{\mathrm{DFT}} \,=\, \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \Big(&\boldsymbol{F}_I\, \boldsymbol{D}\boldsymbol{X}^I \,-\, \boldsymbol{\chi}_I\, \boldsymbol{D}\boldsymbol{\psi}^I \,+\, \boldsymbol{\rho}^I{}_J\, \boldsymbol{F}_I\, \boldsymbol{\psi}^J \,+\, \boldsymbol{\beta}^{IJ}\, \boldsymbol{F}_I\, \boldsymbol{\chi}_J \\ &+\, \frac{1}{3!}\, \boldsymbol{T}^{(0)}{}_{IJK}\, \boldsymbol{\psi}^I\, \boldsymbol{\psi}^J\, \boldsymbol{\psi}_K \,+\, \frac{1}{2}\, \boldsymbol{T}^{(1)}{}_{IJ}{}^K\, \boldsymbol{\psi}^I\, \boldsymbol{\psi}^J\, \boldsymbol{\chi}_K \\ &+\, \frac{1}{2}\, \boldsymbol{T}^{(2)}{}_I{}^{JK}\, \boldsymbol{\psi}^I\, \boldsymbol{\chi}_J\, \boldsymbol{\chi}_K \,+\, \frac{1}{3!}\, \boldsymbol{T}^{(3)\,IJK}\, \boldsymbol{\chi}_I\, \boldsymbol{\chi}_J\, \boldsymbol{\chi}_K\Big)\ .\end{aligned} \tag{4.125}$$

The BV symplectic structure coming from (4.110) is given by

$$\begin{aligned}\boldsymbol{\omega}_{\mathrm{DFT}} &= \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \big(\boldsymbol{\delta}\boldsymbol{X}^I\, \boldsymbol{\delta}\boldsymbol{F}_I \,+\, \boldsymbol{\delta}\boldsymbol{\chi}_I\, \boldsymbol{\delta}\boldsymbol{\psi}^I\big) \\ &= \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \big(\boldsymbol{\delta}\boldsymbol{X}^I\, \boldsymbol{\delta}\boldsymbol{F}_I \,+\, \eta_{IJ}\, \boldsymbol{\delta}\boldsymbol{\tau}^I_+\, \boldsymbol{\delta}\boldsymbol{\tau}^J_+ \,-\, \eta_{IJ}\, \boldsymbol{\delta}\boldsymbol{\tau}^I_-\, \boldsymbol{\delta}\boldsymbol{\tau}^J_-\big)\ ,\end{aligned} \tag{4.126}$$

where we performed a duality transformation originating from (4.113):

$$\boldsymbol{\tau}^I_\pm \,=\, \frac{1}{2}\big(\boldsymbol{\psi}_I \,\pm\, \eta_{IJ}\, \boldsymbol{\chi}^J\big)\ . \tag{4.127}$$

Now we restrict the superfields to the submanifold $\boldsymbol{\tau}^I_- = 0$. This is not a partial gauge fixing, since both fields and antifields are set to zero. The corresponding action is given by

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(3)}_{\mathrm{DFT},+} \,=\, \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \Big(&\boldsymbol{F}_I\, \boldsymbol{D}\boldsymbol{X}^I \,-\, \eta_{IJ}\, \boldsymbol{\tau}^I_+\, \boldsymbol{D}\boldsymbol{\tau}^J_+ \,+\, (\boldsymbol{\rho}_+)^I{}_J\, \boldsymbol{F}_I\, \boldsymbol{\tau}^J_+ \\ &+\, \frac{1}{3!}\, (\boldsymbol{T}_+)_{IJK}\, \boldsymbol{\tau}^I_+\, \boldsymbol{\tau}^J_+\, \boldsymbol{\tau}^K_+\Big)\ .\end{aligned} \tag{4.128}$$

The reason we do not define this action directly from a DFT algebroid is that the action $\boldsymbol{\mathcal{S}}^{(3)}_{\mathrm{DFT},+}$ does not satisfy the BV master equation, so it cannot be constructed by AKSZ theory, nor does it define a BV quantized sigma-model. In order for it to define a BV quantized theory or an AKSZ theory we have to impose additional conditions on the structure functions $(\rho_+)^I{}_J$ and $(T_+)_{IJK}$ coming from the BV master equation for the reduced action. This can be done with the section condition. We will use this method later in the thesis to study the topological A- and B-models within the framework of double field theory.

5 AKSZ threebranes and Lie algebroids up to homotopy

Just as it proves useful to view closed strings as modes of membranes when deforming their target spaces by fluxes, it is likewise useful to view membranes as modes of higher degrees of freedom, threebranes, particularly when the membranes are regarded as the fundamental objects in M-theory with background four-form fluxes $G = \mathrm{d}C$. With notation as previously, the threebrane theory is defined on a four-dimensional worldvolume Σ_4, and the topological part of the bosonic action is

$$I_{\Sigma_4,G} \,=\, \int_{\Sigma_4} X^*(G) \,=\, \frac{1}{4!}\int_{\Sigma_4} G_{ijkl}\,\mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\mathrm{d}X^k\wedge\mathrm{d}X^l\ . \tag{4.129}$$

This action is classically equivalent to the first order threebrane sigma-model action

$$I'_{\Sigma_4,G} \,=\, \int_{\Sigma_4}\Big(F_i\wedge\big(\psi^i-\mathrm{d}X^i\big)+\chi_i\wedge\mathrm{d}\psi^i+\frac{1}{4!}\,G_{ijkl}\,\psi^i\wedge\psi^j\wedge\psi^k\wedge\psi^l\Big)\ , \tag{4.130}$$

where $\psi\in\Omega^1(\Sigma_4,X^*TM)$ and $\chi\in\Omega^2(\Sigma_4,X^*T^*M)$, while $F\in\Omega^3(\Sigma_4,X^*T^*M)$ is an auxiliary three-form. In dimension $d=4$, the target superspace of the AKSZ construction is a QP-manifold of degree 3, which is equivalent to a higher algebroid structure introduced in [95] that arises from a homotopy deformation of a Lie algebroid. It is called a *Lie algebroid up to homotopy.*

5.1 Lie algebroids up to homotopy

Let E_0 be a vector bundle over a manifold M. We consider a general QP-manifold of degree 3 on $\mathcal{M}=T^*[3]E_0[1]$, regarded as a symplectic Lie 3-algebroid on E_0 with underlying N-manifold

$$\mathcal{M} \,=\, M\longleftarrow E_0[1]\longleftarrow E_0[1]\oplus E_0^*[2]\longleftarrow T^*[3]E_0[1]\ . \tag{4.131}$$

The local coordinates on $\mathcal{M}$ are denoted (X^i,ψ^a,χ_a,F_i) with degrees $(0,1,2,3)$, where X^i are local coordinates on M, ψ^a are local fiber coordinates of the shifted vector bundle $E_0[1]$, χ_a are dual fiber coordinates of $T^*[3]E_0[1]\to E_0[1]$, and F_i are local fiber coordinates of the shifted cotangent bundle $T^*[3]M$. The canonical symplectic structure is given by

$$\omega \,=\, \mathrm{d}F_i\wedge\mathrm{d}X^i\,+\,\mathrm{d}\psi^a\wedge\mathrm{d}\chi_a\ . \tag{4.132}$$

The most general form of a degree 4 Hamiltonian function $\gamma_{k,\rho,T}$ on $\mathcal{M}$ is given by a sum

$$\gamma_{k,\rho,T} \;=\; \gamma_k \,+\, \gamma_\rho \,+\, \gamma_T \;, \tag{4.133}$$

where

$$\begin{aligned} \gamma_k \;&=\; \frac{1}{2}\, k^{ab}(X)\, \chi_a\, \chi_b \;, \\ \gamma_\rho \;&=\; \rho^i{}_a(X)\, F_i\, \psi^a \,+\, \frac{1}{2}\, f^a{}_{bc}(X)\, \chi_a\, \psi^b\, \psi^b \;, \\ \gamma_T \;&=\; \frac{1}{4!}\, T_{abcd}(X)\, \psi^a\, \psi^b\, \psi^c\, \psi^d \;, \end{aligned} \tag{4.134}$$

are given by functions k^{ab}, $\rho^i{}_a$, $f^a{}_{bc}$ and T_{abcd} on M. A Lie algebroid up to homotopy is defined with respect to this decomposition of the Hamiltonian function as the vector bundle E_0 over M with a symmetric pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on sections of E_0^*, an anchor map $\rho:\, E_0 \,\to\, TM$, an antisymmetric bracket $[\,\cdot\,,\,\cdot\,]_{\rm uth}$ on sections of E_0, a de Rham-type differential d on sections of $\bigwedge^\bullet E_0$, and a four-form Ω on E_0. We can identify sections e of E_0 with degree 2 functions $e \,=\, f^a(X)\,\chi_a$ and sections α of E_0^* with degree 1 functions $\alpha = g_a(X)\,\psi^a$, where f^a and g_a are degree 0 functions on M. Then the five operations are defined via derived brackets as

$$\begin{aligned} \langle \alpha_1, \alpha_2 \rangle \;&=\; \{\{\gamma_k, \alpha_1\}, \alpha_2\} \;, \\ \rho(e) \;&=\; \{\{\gamma_\rho, e\},\, \cdot\,\} \;, \\ [e_1, e_2]_{\rm uth} \;&=\; \{\{\gamma_\rho, e_1\}, e_2\} \;, \\ \mathsf{d} \;&=\; \{\gamma_\rho,\, \cdot\,\} \;, \\ \Omega(e_1, e_2, e_3, e_4) \;&=\; \{\{\{\{\gamma_T, e_1\}, e_2\}, e_3\}, e_4\} \;. \end{aligned} \tag{4.135}$$

The pairing additionally defines a symmetric bundle map $\mathfrak{d} : E^* \to E$ by

$$\mathfrak{d}\alpha \;=\; -\,\{\gamma_k, \alpha\} \;. \tag{4.136}$$

The classical master equation $\{\gamma_{k,\rho,T}, \gamma_{k,\rho,T}\} = 0$ implies that these operations obey the identities

$$\begin{aligned} [e_1, f\, e_2]_{\rm uth} \;&=\; f\, [e_1, e_2]_{\rm uth} \,+\, \big(\rho(e_1) f\big)\, e_2 \qquad \text{for} \quad f \in C^\infty(M) \;, \\ [[e_1, e_2]_{\rm uth}, e_3]_{\rm uth} \,+\, \text{cyclic} \;&=\; \mathfrak{d}\, \Omega(e_1, e_2, e_3,\, \cdot\,) \;, \\ \rho \circ \mathfrak{d} \;&=\; 0 \;, \\ \rho(e)\langle \alpha_1, \alpha_2 \rangle \;&=\; \langle \mathcal{L}_e \alpha_1, \alpha_2 \rangle \,+\, \langle \alpha_1, \mathcal{L}_e \alpha_2 \rangle \qquad \text{with} \quad \mathcal{L}_e \,:=\, \{\{\gamma_\rho, e\},\, \cdot\,\} \;, \\ \mathsf{d} \circ \Omega \;&=\; 0 \;, \end{aligned} \tag{4.137}$$

and we also note that $\mathsf{d}^2 \neq 0$ in general. In other words, a symplectic Lie 3-algebroid is a vector bundle with operations $([\,\cdot\,,\,\cdot\,]_{\rm uth}, \rho, \mathfrak{d}, \Omega)$ characterized by the algebraic identities (4.137). A particularly interesting feature behind the algebraic structure of a Lie algebroid up to homotopy is that its bracket can be extended to all degree 2 functions on $\mathcal{M} = T^*[3]E_0[1]$, which are identified as the sections of $E := E_0 \oplus \bigwedge^2 E_0^*$. This leads to a higher analogue of the Courant bracket

$$[\,\cdot\,,\,\cdot\,]_{\rm 2C} \,=\, \{\{\gamma,\,\cdot\,\},\,\cdot\,\}\ , \tag{4.138}$$

where now the full Hamiltonian function is used. We shall call it a *2-Courant bracket* in the following.

The simplest relevant example for us is what we shall call the *standard Lie algebroid up to homotopy*, which is the case $E_0 = TM$. The symplectic structure is

$$\omega_4 \,=\, {\rm d}X^i \wedge {\rm d}F_i \,+\, {\rm d}\psi^i \wedge {\rm d}\chi_i\ . \tag{4.139}$$

We choose the Liouville potential given by $\vartheta = F_i\,{\rm d}X^i + \chi_i\,{\rm d}\psi^i$. The simplest Hamiltonian function from (4.133) and (4.134) has identity anchor map $\rho^i{}_j = \delta^i{}_j$ with all other structure functions equal to zero, and is given by

$$\gamma_0 \,=\, F_i\,\psi^i\ . \tag{4.140}$$

The cohomological vector field is again the de Rham vector field $Q_{\gamma_0} = \psi^i\,\frac{\partial}{\partial X^i}$ on $\mathcal{M} = T^*[3]T[1]M$. In this instance, the derived bracket on degree 2 functions

$$A^i\,\chi_i + \frac{1}{2}\,\lambda_{ij}\,\psi^i\,\psi^j \;\longleftrightarrow\; A^i\,\frac{\partial}{\partial X^i} + \frac{1}{2}\,\lambda_{ij}\,{\rm d}X^i \wedge {\rm d}X^j \tag{4.141}$$

gives the standard 2-Courant bracket introduced previously in (1.8) on the vector bundle

$$E = TM \oplus \bigwedge^2 T^*M\ , \tag{4.142}$$

which reads explicitly as

$$[A+\lambda, B+\xi]_{\rm 2C} \,=\, [A,B] \,+\, \mathcal{L}_A\xi \,-\, \mathcal{L}_B\lambda \,+\, \frac{1}{2}\,{\rm d}(\iota_B\,\lambda - \iota_A\,\xi) \tag{4.143}$$

for vector fields A, B and two-forms λ, ξ on M.[6] The standard 2-Courant bracket (4.143) appears in exceptional generalized geometry as the natural bracket which is compatible with the commutator algebra of generalized Lie derivatives [99,100].

[6]This is called a Vinogradov algebroid in [96–98].

One can also introduce a flux deformation by an additional term γ_T in the Hamiltonian function, which twists the standard 2-Courant bracket by a four-form which is necessarily closed by the classical master equation. Given an M-theory three-form C-field on M, with four-form flux $G = \mathrm{d}C$, canonical transformation of the Hamiltonian function γ_0 by the degree 3 function $C = \frac{1}{3!}\, C_{ijk}(X)\, \psi^i\, \psi^j\, \psi^k$ on $\mathcal{M}$ yields the twisted Hamiltonian function

$$\gamma_G \;:=\; \mathrm{e}^{\,\delta_C}\gamma_0 \;=\; F_i\, \psi^i \;+\; \frac{1}{4!}\, G_{ijkl}\, \psi^i\, \psi^j\, \psi^k\, \psi^l \;, \tag{4.144}$$

and it gives the twisted standard 2-Courant bracket as

$$[A+\lambda, B+\xi]_{2\mathrm{C},G} \;=\; [A,B] \;+\; \mathcal{L}_A\xi \;-\; \mathcal{L}_B\lambda \;+\; \frac{1}{2}\,\mathrm{d}(\iota_B\,\lambda - \iota_A\,\xi) \;+\; \iota_A\iota_B G \;. \tag{4.145}$$

5.2 Twisted standard 2-Courant sigma-model and double dimensional reduction

One can now use the AKSZ construction to build BV quantized sigma-models in four dimensions based on degree 3 QP-manifolds, which we shall call *2-Courant sigma-models*. For the standard Lie algebroid up to homotopy on $E_0 = TM$ twisted by a closed four-form flux G, the BV bracket is

$$(\,\cdot\,,\,\cdot\,)_{\mathrm{BV}} = \int_{T[1]\Sigma_4} \mathrm{d}^4\hat{z}\, \left(\frac{\delta}{\delta \boldsymbol{X}^i} \wedge \frac{\delta}{\delta \boldsymbol{F}_i} + \frac{\delta}{\delta \boldsymbol{\chi}_i} \wedge \frac{\delta}{\delta \boldsymbol{\psi}^i} \right) , \tag{4.146}$$

and the classical master equation is solved by the topological threebrane action

$$\boldsymbol{\mathcal{S}}_G^{(4)} \;=\; \int_{T[1]\Sigma_4} \mathrm{d}^4\hat{z}\, \Big(\boldsymbol{F}_i\, \boldsymbol{D}\boldsymbol{X}^i \;+\; \boldsymbol{\psi}^i\, \boldsymbol{D}\boldsymbol{\chi}_i \;+\; \boldsymbol{F}_i\, \boldsymbol{\psi}^i \;+\; \frac{1}{4!}\, \boldsymbol{G}_{ijkl}\, \boldsymbol{\psi}^i\, \boldsymbol{\psi}^j\, \boldsymbol{\psi}^k\, \boldsymbol{\psi}^l \Big) \;. \tag{4.147}$$

Integrating over θ^μ and restricting to degree 0 fields in (4.147) recovers the classical action (4.130).

As a simple example of the dimensional reduction technique described in §1.4, we present our observation on the reduction of the standard 2-Courant sigma-model to the standard Courant sigma-model with flux deformations. To motivate the reduction of threebrane flux to membrane flux, consider the simple topological threebrane action (4.129) given by the pullback of a closed four-form flux G on a d-dimensional manifold M by the worldvolume map $X : \Sigma_4 \to M$. For simplicity we take Σ_4 to be a closed manifold. We perform a double dimensional reduction on a circle taking both the worldvolume and the target to be product manifolds $\Sigma_4 = \Sigma_3 \times S^1$ and $M = \widehat{M} \times S^1$,

with $\widehat{M}$ a manifold of dimension $d-1$, and wrap the S^1 of the worldvolume around the S^1 of the target space; in other words, we regard the membranes as modes of threebranes wrapping S^1. We write the local coordinates on the worldvolume Σ_4 as $\sigma = (\widehat{\sigma}, t)$, where $\widehat{\sigma} \in \Sigma_3$ and t is the coordinate on S^1. The target space coordinate indices are $I = (i, d)$, where $i = 1, \ldots, d-1$ label directions along $\widehat{M}$.

Wrapping the target circle means that the map X has the local expression

$$X \,=\, \big(X^I(\sigma)\big) \,=\, \big(\widehat{X}^i(\widehat{\sigma}), w\, t\big) \tag{4.148}$$

with the reduced map $\widehat{X} : \Sigma_3 \to \widehat{M}$ and $X^d = w\, t$, where w is a winding number. The dimensional reduction of the action $I_{\Sigma_4,G}$ from (4.129) is then given by $I_{\Sigma_3,H}$ from (4.71), where the closed three-form flux H on $\widehat{M}$ is given by

$$H_{ijk}(\widehat{X}) \,=\, w \int_{S^1} \mathrm{d}t\; G_{ijkd}(\widehat{X}, t)\ . \tag{4.149}$$

Hence the threebrane flux G reduces to a membrane flux H under double dimensional reduction on a circle. We shall now show that this reduction also works at the level of the full AKSZ sigma-models.

We start with the G-twisted standard 2-Courant sigma-model given by (4.147), and use the dimensional reduction method of §1.4. We write the expansion of an arbitrary superfield $\boldsymbol{\phi} \in \boldsymbol{\mathcal{M}}$ with respect to the coordinate direction t as

$$\boldsymbol{\phi} \,=\, \widehat{\boldsymbol{\phi}} \,+\, \boldsymbol{\phi}_t\, \theta^t\ , \tag{4.150}$$

where neither $\widehat{\boldsymbol{\phi}}$ nor $\boldsymbol{\phi}_t$ contain the odd coordinate θ^t. If $\boldsymbol{\phi}$ has ghost number n, then $\widehat{\boldsymbol{\phi}}$ has ghost number n and $\boldsymbol{\phi}_t$ has ghost number $n-1$. We choose the infrared fields to be $(\boldsymbol{F}_t)_i$, $\widehat{\boldsymbol{X}}^i$, $\widehat{\boldsymbol{\psi}}^i$ and $(\boldsymbol{\chi}_t)_i$. On the ultraviolet fields we fix the gauge by choosing the Lagrangian submanifold $\boldsymbol{\mathcal{L}}$ defined by

$$\boldsymbol{X}^I_t \,=\, 0\ , \qquad \widehat{\boldsymbol{X}}^d \,=\, -\,w\, t\ , \qquad \boldsymbol{\psi}^i_t \,=\, 0\ , \qquad \boldsymbol{\psi}^d_t \,=\, w \qquad \text{and} \qquad \widehat{\boldsymbol{\psi}}^d \,=\, 0\ . \tag{4.151}$$

The equations of motion for $\widehat{\boldsymbol{F}}_i$ and $\widehat{\boldsymbol{\chi}}_i$ give $\partial_t \widehat{\boldsymbol{X}}^i = 0$ and $\partial_t \widehat{\boldsymbol{\psi}}^i = 0$, and in this way we get the AKSZ action of the H-twisted standard Courant sigma-model (4.91) and its BV symplectic form (4.90) with the definitions of the fields

$$\boldsymbol{X}^i \,=\, \widehat{\boldsymbol{X}}^i\ , \qquad \boldsymbol{F}_i \,=\, \int_{S^1} \mathrm{d}t\; (\boldsymbol{F}_t)_i\ , \qquad \boldsymbol{\psi}^i \,=\, \widehat{\boldsymbol{\psi}}^i \qquad \text{and} \qquad \boldsymbol{\chi}_i \,=\, \int_{S^1} \mathrm{d}t\; (\boldsymbol{\chi}_t)_i\ , \tag{4.152}$$

and H-flux as in (4.149). We refer to this type of gauge fixing as a double dimensional reduction on a circle.

It is worth stressing that the kinetic terms are necessary in this construction because without the term $\widehat{\boldsymbol{F}}_d\,\partial_t\widehat{\boldsymbol{X}}^d$, the term $\widehat{\boldsymbol{F}}_d\,\boldsymbol{\psi}_t^d$ gives $w\,\widehat{\boldsymbol{F}}_d$, which would yield $w=0$ and vanishing H-flux on-shell. An interesting feature here is that the term coming from the Liouville potential $\psi^i\,\mathrm{d}\chi_i$ of the threebrane has been reversed via the reduction to the Liouville potential $-\chi_i\,\mathrm{d}\psi^i$ of the membrane. Note also that this dimensional reduction can be done at the purely bosonic level without the ghost fields: Starting from (4.130), we use the expression (4.148) for the wrapping of X, and then the equations of motion for the three-form field F_I and reduced two-form fields gives the bosonic part of the standard Courant sigma-model with H-flux in (4.72).

By a direct computation in local coordinates, it is further possible to show that the standard 2-Courant bracket (4.143) on $M=\widehat{M}\times S^1$ suitably reduces to the standard Courant bracket (4.83) on $\widehat{M}$. The dimensional reduction of the 2-Courant sigma-model to the Courant sigma-model is analogous to the reduction discussed by [100] in the context of $SL(5)$ exceptional field theory, wherein the $SL(5)$ generalized Courant bracket reduces to the $O(3,3)$ generalized Courant bracket (C-bracket) of double field theory.

Chapter 5

Double field theory and generalized complex geometry for the A/B-models

In this chapter we reformulate the AKSZ constructions of the A- and B-models in the framework of double field theory as a single membrane sigma-model, and study its relation to generalized complex geometry and S-duality. In the following we rely on our paper [2].

The A- and B-model topological string theories in backgrounds with H-flux are captured by generalized complex geometry [101–103]. They have been extensively studied by introducing AKSZ string and open membrane sigma-models with generalized complex structures, which reduce upon gauge fixing to the A- and B-models [74, 78–80, 104]. These were based on AKSZ constructions for generalized complex geometry, which describes the Kähler structure of A-model and the complex structure of B-model within one generalized complex structure. The novelty of our approach is that their AKSZ sigma-models are reformulated within double field theory on both the string and membrane levels, which gives a more natural explanation of how their AKSZ constructions are related to generalized complex structures. It also highlights some new aspects, such as how topological S-duality appears on the level of AKSZ sigma-models and can be traced back to generalized complex geometry.

Based on our observation that the Poisson sigma-model on doubled spaces captures both the A- and B-models with different choices of the doubled Poisson structure, we propose an open AKSZ membrane sigma-model, inspired by the approach of [84] to T-duality between geometric and non-geometric fluxes, which gives back the doubled

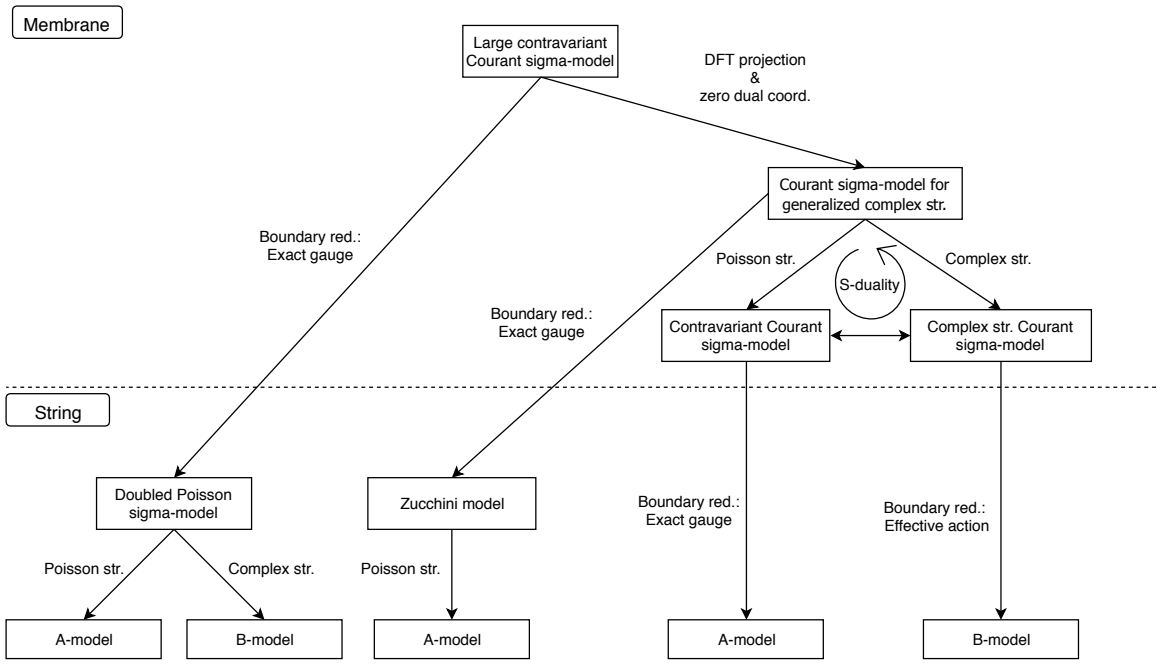

Figure 5.1. Schematic presentation of the different reductions and connections between the AKSZ string and membrane sigma-models related to the topological A- and B-models.

Poisson sigma-model on the boundary in a specific gauge. Then we reduce the fields in a way which can be interpreted as the same reduction performed in [93], where it was called a DFT projection. The resulting AKSZ membrane sigma-model captures a particular class of generalized complex structures given by an initial Poisson and complex structure. It therefore corresponds to a Courant algebroid for the generalized complex structure with the identities of its integrability condition; to the best of our knowledge this is a new example of Courant algebroid. We also show that the AKSZ membrane theory can be reduced through gauge fixing to the A- or B-models on the boundary if the generalized complex structure is given by a purely Poisson or complex structure respectively. Furthermore, we find a realization of topological S-duality, which exchanges the weakly and strongly coupled sectors of the topological A- and B-model string theories [45], on the level of the AKSZ construction. Our result is based on an S-duality which maps Poisson and complex structure Courant algebroids into each other, and lies within the Courant algebroid for the generalized complex structure. This duality is promoted to the AKSZ membrane sigma-model and interpreted as S-duality which relates the couplings g_{A} and g_{B} of the A- and B-models in the usual way: $g_{\mathrm{A}} = 1/g_{\mathrm{B}}$. In Figure 5.1 we summarize the relations between the different AKSZ string and membrane sigma-models appearing in the

thesis.

1 Double field theory for the A- and B-models

Our starting point is a double field theoretic formulation of the AKSZ constructons of A- and B-models, which, on the one hand, leads to the AKSZ formulation with generalized complex geometry, and on the other hand, shows a possible new direction in the study of A- and B-models.

1.1 Doubled Poisson sigma-model

We shall start by proposing an AKSZ Poisson sigma-model with doubled target space coordinates, which gives the A- and B-models separately. Let us consider an AKSZ Poisson sigma-model with target QP1-manifold $T^*[1]T^*M$ with degree 0 and degree 1 coordinates

$$X^I = \begin{pmatrix} X^i \\ \widetilde{X}_i \end{pmatrix} \qquad \text{and} \qquad \chi_I = \begin{pmatrix} \chi_i \\ \widetilde{\chi}^i \end{pmatrix} \tag{5.1}$$

respectively. The doubled Poisson structure is denoted by Ω^{IJ}, and it depends on both degree 0 coordinates X^i and $\widetilde{X}_i$. The symplectic form and AKSZ action are as introduced in §2.1 in Chapter 4:

$$\omega_{\mathrm{D2}} = \mathrm{d}\chi_I \wedge \mathrm{d}X^I \ , \tag{5.2}$$

and

$$\boldsymbol{\mathcal{S}}_{\Omega}^{(2)} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z} \left(\boldsymbol{\chi}_I \, \boldsymbol{D}\boldsymbol{X}^I + \frac{1}{2} \boldsymbol{\Omega}^{IJ} \boldsymbol{\chi}_I \boldsymbol{\chi}_J \right) . \tag{5.3}$$

The master equation imposes the Poisson condition for Ω^{IJ}:

$$\Omega^{[I|L} \partial_L \Omega^{JK]} = 0 \ , \tag{5.4}$$

where the doubled derivative ∂_I is defined by

$$\partial_I = \begin{pmatrix} \partial/\partial X^i \\ \partial/\partial \widetilde{X}_i \end{pmatrix} . \tag{5.5}$$

We will now show that particular choices of Ω^{IJ} give the A- or B-models.

A-model

The A-model is obtained by using the doubled Poisson structure

$$\Omega_{\mathrm{A}}^{IJ} = \begin{pmatrix} \pi^{ij} & 0 \\ 0 & 0 \end{pmatrix} , \tag{5.6}$$

where the bivector π^{ij} only depends on X^i. The AKSZ action $\boldsymbol{\mathcal{S}}_{\Omega_{\mathrm{A}}}^{(2)}$ defined by Ω_{A}^{IJ} is given by

$$\boldsymbol{\mathcal{S}}_{\Omega_{\mathrm{A}}}^{(2)} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z} \left(\boldsymbol{\chi}_i \, \boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{\chi}}^i \, \boldsymbol{D}\widetilde{\boldsymbol{X}}_i + \frac{1}{2} \boldsymbol{\pi}^{ij} \, \boldsymbol{\chi}_i \, \boldsymbol{\chi}_j \right) , \tag{5.7}$$

where the additional term $\widetilde{\boldsymbol{\chi}}^i \, \boldsymbol{D}\widetilde{\boldsymbol{X}}_i$ can be removed with a partial gauge fixing. Thus it yields the original Poisson sigma-model, which is the AKSZ construction of the A-model (4.55). Then the constraint (5.4) reduces to the original constraint that π defines a Poisson structure on M.

B-model

The B-model cannot be obtained simultaneously with the A-model, as it arises from a different doubled Poisson structure. The AKSZ construction given by a constant complex structure in (4.67) can be obtained using

$$\Omega_{\mathrm{B}}^{IJ} = \begin{pmatrix} 0 & J^i{}_j \\ -J^j{}_i & 0 \end{pmatrix} . \tag{5.8}$$

It gives the AKSZ construction of the B-model after the sign flip $\widetilde{\boldsymbol{X}}_i \to -\widetilde{\boldsymbol{X}}_i$. The AKSZ construction of the B-model in (4.65) can be derived directly with the choice $J^i{}_j = \mathrm{i}\,\delta^i_j$ from the doubled Poisson sigma-model using

$$\Omega_{\mathrm{B}}'^{IJ} = \begin{pmatrix} 0 & \delta^i{}_j \\ -\delta^j{}_i & 0 \end{pmatrix} . \tag{5.9}$$

The AKSZ formulation of the B-model for an arbitrary complex structure $J^i{}_j$, which only depends on X^i, is given in (4.70). Our doubled Poisson sigma-model also includes this construction and it is given by choosing

$$\Omega_{J}^{IJ} = \begin{pmatrix} 0 & J^i{}_j \\ -J^j{}_i & -2\,\partial_{[i} J^k{}_{j]} \, \widetilde{X}_k \end{pmatrix} \tag{5.10}$$

after the sign flip $\widetilde{\boldsymbol{X}}_i \to -\widetilde{\boldsymbol{X}}_i$. The constraint (5.4) gives the same constraint as in the original construction, which is the integrability condition (4.68).[1]

[1]Since the complex structure appears as a Poisson structure on doubled space, it can be quantized using the Cattaneo-Felder approach [77]. In particular, the case of a constant complex structure can be quantized in closed form analogously to the Moyal star-product, but on doubled space.

General doubled Poisson structure

We write a general doubled Poisson structure in the block matrix form

$$\Omega_{\mathrm{G}}^{IJ} = \begin{pmatrix} \mathsf{P}^{ij} & \mathsf{J}^{i}{}_{j} \\ -\mathsf{J}^{j}{}_{i} & \mathsf{Q}_{ij} \end{pmatrix} , \tag{5.11}$$

where the blocks P^{ij}, $\mathsf{J}^{i}{}_{j}$ and Q_{ij} are constrained by (5.4), and they depend on both X^i and $\widetilde{X}_i$. The corresponding AKSZ action is

$$\boldsymbol{\mathcal{S}}_{\Omega_{\mathrm{G}}}^{(2)} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z} \, \Big(\boldsymbol{\chi}_i \, \boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{\chi}}^i \, \boldsymbol{D}\widetilde{\boldsymbol{X}}_i + \frac{1}{2} \, \mathbf{P}^{ij} \, \boldsymbol{\chi}_i \, \boldsymbol{\chi}_j + \mathbf{J}^{i}{}_{j} \, \boldsymbol{\chi}_i \, \widetilde{\boldsymbol{\chi}}^j + \frac{1}{2} \, \mathbf{Q}_{ij} \, \widetilde{\boldsymbol{\chi}}^i \, \widetilde{\boldsymbol{\chi}}^j \Big) . \tag{5.12}$$

It is tempting to try to relate Ω_{G}^{IJ} to the general complex structure $\mathbb{J}^{I}{}_{J}$ in (4.61), but the identities (4.60) are not equivalent to (5.4). We will return to this problem later. It is also interesting to note that the action $\boldsymbol{\mathcal{S}}_{\Omega_{\mathrm{G}}}^{(2)}$ gives the Zucchini model (4.58) if we replace $\widetilde{\boldsymbol{\chi}}^i$ with $\boldsymbol{D}\boldsymbol{X}^i$, and nothing depends on $\widetilde{X}_i$. But they are different BV theories with different constraints on their block structures.

1.2 Large contravariant Courant sigma-model

We have seen that the A- and B-models on the AKSZ level appear to be two different particular cases of the same two-dimensional AKSZ theory on a doubled target space. These Poisson sigma-models can be uplifted to the membrane level as a contravariant Courant sigma-model from §3.3 in Chapter 4, which gives them back on the boundary in the exact gauge. The novelty of the membrane description is that one can introduce flux terms in the bulk. We shall now study the doubled contravariant Courant sigma-model with doubled Poisson structures introduced in §1.1. These AKSZ constructions will play a similar role later in the study of the A- and B-models within double field theory as the large Courant sigma-model in §4 in Chapter 4.

The BV symplectic form of the contravariant Courant sigma-model in doubled space with QP2-manifold $T^*[2]T[1]T^*M$ is given by (4.126). The AKSZ action comes from (4.104) and is given by

$$\begin{aligned} \boldsymbol{\mathcal{S}}_{\Omega,\mathcal{R}}^{(3)} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z} \, \Big(& \boldsymbol{F}_I \, \boldsymbol{D}\boldsymbol{X}^I - \boldsymbol{\chi}_I \, \boldsymbol{D}\boldsymbol{\psi}^I + \boldsymbol{\Omega}^{IJ} \, \boldsymbol{F}_I \, \boldsymbol{\chi}_J \\ & - \frac{1}{2} \, \boldsymbol{\partial}_I \boldsymbol{\Omega}^{JK} \, \boldsymbol{\psi}^I \, \boldsymbol{\chi}_J \, \boldsymbol{\chi}_K + \frac{1}{3!} \, \boldsymbol{\mathcal{R}}^{IJK} \, \boldsymbol{\chi}_I \, \boldsymbol{\chi}_J \, \boldsymbol{\chi}_K \Big) , \end{aligned} \tag{5.13}$$

with the definition of a general three-vector flux $\mathcal{R}^{IJK}$ on T^*M, which is allowed in the contravariant Courant sigma-model.

A-model

The doubled Poisson structure Ω_{A}^{IJ} in (5.6) gives the original contravariant Courant sigma-model action (4.104) on M after the gauge fixing $\widetilde{\boldsymbol{F}}^i = 0$ and $\widetilde{\boldsymbol{\chi}}^i = 0$, which leaves only the R-flux $\mathcal{R}^{ijk} = R^{ijk}$. One needs to assume that R^{ijk} only depends on X^i in order to reduce the action purely to M. We have already seen that the contravariant Courant sigma-model in the exact gauge further reduces to the Poisson sigma-model formulation of the A-model in §3.3 in Chapter 4.

B-model

The doubled Poisson structure $\Omega_{\mathrm{B}}^{\prime IJ}$ defined in (5.9) with vanishing flux $\mathcal{R} = 0$ gives the standard Courant sigma-model and a 'dual' standard Courant sigma-model with action

$$\boldsymbol{\mathcal{S}}^{(3)}_{\Omega'_{\mathrm{B}},0} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\,\big(\boldsymbol{F}_i\,\boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{F}}^i\,\boldsymbol{D}\widetilde{\boldsymbol{X}}_i - \boldsymbol{\chi}_i\,\boldsymbol{D}\boldsymbol{\psi}^i - \widetilde{\boldsymbol{\chi}}^i\,\boldsymbol{D}\widetilde{\boldsymbol{\psi}}_i + \boldsymbol{F}_i\,\widetilde{\boldsymbol{\chi}}^i - \widetilde{\boldsymbol{F}}^i\,\boldsymbol{\chi}_i\big)\ . \tag{5.14}$$

It can be seen that the standard and the dual standard Courant sigma-models are decoupled from each other (in both the action and the symplectic form), so they can be separately gauge fixed. For example, the gauge $\widetilde{\boldsymbol{F}}^i = 0$ and $\boldsymbol{\psi}^i = 0$ yields the standard Courant sigma-model, which is related to the B-model as described in §3.2 in Chapter 4. The exact gauge defined in §1.3 in Chapter 4 reads here as

$$\boldsymbol{F}_i = \boldsymbol{D}\boldsymbol{\chi}_i\ , \qquad \boldsymbol{\psi}^i = -\,\boldsymbol{D}\boldsymbol{X}^i\ , \qquad \widetilde{\boldsymbol{F}}^i = \boldsymbol{D}\widetilde{\boldsymbol{\chi}}^i \qquad \text{and} \qquad \widetilde{\boldsymbol{\psi}}_i = -\,\boldsymbol{D}\widetilde{\boldsymbol{X}}_i\ , \tag{5.15}$$

and it gives the action of the B-model (4.65) on the boundary:

$$\boldsymbol{\mathcal{S}}^{(3)}_{\Omega'_{\mathrm{B}},0;\mathrm{gf}} = \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\,\big(\boldsymbol{\chi}_i\,\boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{X}}_i\,\boldsymbol{D}\widetilde{\boldsymbol{\chi}}^i + \boldsymbol{\chi}_i\,\widetilde{\boldsymbol{\chi}}^i\big) = \boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{B1}}\ . \tag{5.16}$$

The B-model construction with general constant complex structure is similar: the doubled Poisson structure Ω_{B}^{IJ} in (5.8) leads to the AKSZ action

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(3)}_{\Omega_{\mathrm{B}},0} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\,\big(&\boldsymbol{F}_i\,\boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{F}}^i\,\boldsymbol{D}\widetilde{\boldsymbol{X}}_i - \boldsymbol{\chi}_i\,\boldsymbol{D}\boldsymbol{\psi}^i \\ &- \widetilde{\boldsymbol{\chi}}^i\,\boldsymbol{D}\widetilde{\boldsymbol{\psi}}_i + J^i{}_j\,\boldsymbol{F}_i\,\widetilde{\boldsymbol{\chi}}^j - J^j{}_i\,\widetilde{\boldsymbol{F}}^i\,\boldsymbol{\chi}_j\big)\ .\end{aligned} \tag{5.17}$$

The exact gauge (5.15) gives the B-model construction (4.67) on the boundary:

$$\boldsymbol{\mathcal{S}}^{(3)}_{\Omega_{\mathrm{B}},0;\mathrm{gf}} = \oint_{T[1]\partial\Sigma_2} \mathrm{d}^2\hat{z}\,\big(\boldsymbol{\chi}_i\,\boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{X}}_i\,\boldsymbol{D}\widetilde{\boldsymbol{\chi}}^i + J^i{}_j\,\boldsymbol{\chi}_i\,\widetilde{\boldsymbol{\chi}}^j\big) = \boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{B2}}\ . \tag{5.18}$$

Finally, we consider the choice of doubled Poisson structure Ω_J from (5.10) which is associated to a non-constant complex structure. The corresponding AKSZ action is given by

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(3)}_{\Omega_J,0} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \big(&\boldsymbol{F}_i\, \boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{F}}^i\, \boldsymbol{D}\widetilde{\boldsymbol{X}}_i - \boldsymbol{\chi}_i\, \boldsymbol{D}\boldsymbol{\psi}^i - \widetilde{\boldsymbol{\chi}}^i\, \boldsymbol{D}\widetilde{\boldsymbol{\psi}}_i + \boldsymbol{J}^i{}_j\, \boldsymbol{F}_i\, \widetilde{\boldsymbol{\chi}}^j \\ &- \boldsymbol{J}^j{}_i\, \widetilde{\boldsymbol{F}}^i\, \boldsymbol{\chi}_j - 2\, \boldsymbol{\partial}_{[i}\boldsymbol{J}^k{}_{j]}\, \widetilde{\boldsymbol{X}}_k\, \widetilde{\boldsymbol{F}}^i\, \widetilde{\boldsymbol{\chi}}^j - \boldsymbol{\partial}_i \boldsymbol{J}^j{}_k\, \boldsymbol{\psi}^i\, \boldsymbol{\chi}_j\, \widetilde{\boldsymbol{\chi}}^k \\ &+ \boldsymbol{\partial}_{[i}\boldsymbol{J}^k{}_{j]}\, \widetilde{\boldsymbol{\psi}}_k\, \widetilde{\boldsymbol{\chi}}^i\, \widetilde{\boldsymbol{\chi}}^j + \boldsymbol{\partial}_i \boldsymbol{\partial}_j \boldsymbol{J}^l{}_k\, \widetilde{\boldsymbol{X}}_l\, \boldsymbol{\psi}^i\, \widetilde{\boldsymbol{\chi}}^j\, \widetilde{\boldsymbol{\chi}}^k\big)\ .\end{aligned} \tag{5.19}$$

The master equation does not give the integrability condition (4.68) for $J^i{}_j$ this time. We will use the DFT projection later to obtain the right Courant algebroid whose relations give the integrability condition. In the exact gauge (5.15), the action $\boldsymbol{\mathcal{S}}^{(3)}_{\Omega_J,0}$ reduces to the boundary action for the non-constant complex structure defined in (4.70) after a sign flip:

$$\boldsymbol{\mathcal{S}}^{(3)}_{\Omega_J,0;\mathrm{gf}} = \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\, \big(\boldsymbol{\chi}_i\, \boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{X}}_i\, \boldsymbol{D}\widetilde{\boldsymbol{\chi}}^i + \boldsymbol{J}^i{}_j\, \boldsymbol{\chi}_i\, \widetilde{\boldsymbol{\chi}}^j - \boldsymbol{\partial}_j \boldsymbol{J}^i{}_k\, \widetilde{\boldsymbol{X}}_i\, \widetilde{\boldsymbol{\chi}}^j\, \widetilde{\boldsymbol{\chi}}^k\big) = \boldsymbol{\mathcal{S}}^{(2)}_J\ . \tag{5.20}$$

General doubled Poisson structure

The general AKSZ action (5.13) can be expanded in block form using the general doubled Poisson structure Ω^{IJ}_{G} from (5.11) as

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(3)}_{\Omega_{\mathrm{G}},0} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \Big(&\boldsymbol{F}_i\, \boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{F}}^i\, \boldsymbol{D}\widetilde{\boldsymbol{X}}_i - \boldsymbol{\chi}_i\, \boldsymbol{D}\boldsymbol{\psi}^i - \widetilde{\boldsymbol{\chi}}^i\, \boldsymbol{D}\widetilde{\boldsymbol{\psi}}_i + \mathbf{P}^{ij}\, \boldsymbol{F}_i\, \boldsymbol{\chi}_j \\ &+ \mathbf{Q}_{ij}\, \widetilde{\boldsymbol{F}}^i\, \widetilde{\boldsymbol{\chi}}^j + \mathbf{J}^i{}_j\, \boldsymbol{F}_i\, \widetilde{\boldsymbol{\chi}}^j - \mathbf{J}^j{}_i\, \widetilde{\boldsymbol{F}}^i\, \boldsymbol{\chi}_j - \frac{1}{2}\, \boldsymbol{\partial}_i \mathbf{P}^{jk}\, \boldsymbol{\psi}^i\, \boldsymbol{\chi}_j\, \boldsymbol{\chi}_k \\ &- \frac{1}{2}\, \boldsymbol{\partial}_i \mathbf{Q}_{jk}\, \boldsymbol{\psi}^i\, \widetilde{\boldsymbol{\chi}}^j\, \widetilde{\boldsymbol{\chi}}^k - \boldsymbol{\partial}_i \mathbf{J}^j{}_k\, \boldsymbol{\psi}^i\, \boldsymbol{\chi}_j\, \widetilde{\boldsymbol{\chi}}^k - \frac{1}{2}\, \widetilde{\boldsymbol{\partial}}^i \mathbf{P}^{jk}\, \widetilde{\boldsymbol{\psi}}_i\, \boldsymbol{\chi}_j\, \boldsymbol{\chi}_k \\ &- \frac{1}{2}\, \widetilde{\boldsymbol{\partial}}^i \mathbf{Q}_{jk}\, \widetilde{\boldsymbol{\psi}}_i\, \widetilde{\boldsymbol{\chi}}^j\, \widetilde{\boldsymbol{\chi}}^k - \widetilde{\boldsymbol{\partial}}^i \mathbf{J}^j{}_k\, \widetilde{\boldsymbol{\psi}}_i\, \boldsymbol{\chi}_j\, \widetilde{\boldsymbol{\chi}}^k\Big)\ ,\end{aligned} \tag{5.21}$$

where the dual derivative is defined by $\widetilde{\partial}^i = \partial/\partial\widetilde{X}_i$. As expected it reduces in the exact gauge (5.15) on the boundary to the action $\boldsymbol{\mathcal{S}}^{(2)}_{\Omega_{\mathrm{G}}}$ given by (5.12).

Fluxes in the A- and B-models

Introducing $\mathcal{R}$-flux in (5.13) gives four different terms

$$\begin{aligned}\frac{1}{3!}\,\boldsymbol{\mathcal{R}}^{IJK}\,\boldsymbol{\chi}_I\,\boldsymbol{\chi}_J\,\boldsymbol{\chi}_K \;=\; &\frac{1}{3!}\,\boldsymbol{R}^{ijk}\,\boldsymbol{\chi}_i\,\boldsymbol{\chi}_j\,\boldsymbol{\chi}_k \;+\; \frac{1}{2}\,\boldsymbol{Q}_i{}^{jk}\,\widetilde{\boldsymbol{\chi}}^i\,\boldsymbol{\chi}_j\,\boldsymbol{\chi}_k \\ &+\; \frac{1}{2}\,\boldsymbol{F}_{ij}{}^k\,\widetilde{\boldsymbol{\chi}}^i\,\widetilde{\boldsymbol{\chi}}^j\,\boldsymbol{\chi}_k \;+\; \frac{1}{3!}\,\boldsymbol{H}_{ijk}\,\widetilde{\boldsymbol{\chi}}^i\,\widetilde{\boldsymbol{\chi}}^j\,\widetilde{\boldsymbol{\chi}}^k\;.\end{aligned} \tag{5.22}$$

Both geometric and non-geometric fluxes can appear in the membrane formulations of the A- and B-models, but the gauge fixings leave only $\boldsymbol{R}^{ijk}$ in the case of the original contravariant Courant sigma-model and $\boldsymbol{H}_{ijk}$ in the case of the standard Courant sigma-model. One of the main features of our new construction for the A- and B-models is that it allows for the introduction of four different fluxes. The compatibility condition $[\Omega,\mathcal{R}]_{\mathrm{S}}=0$ between the fluxes and the doubled Poisson bivector Ω can be derived from the master equation (4.87). The same fluxes (5.22) can be defined in the AKSZ theories (5.21) as well.

2 Generalized complex geometry and A/B-models

In this section we continue our study of A/B-models with the reduction of doubled degrees of freedom, which leads to their reformulation within generalized complex geometry.

2.1 Courant sigma-model for generalized complex geometry

So far we have introduced a contravariant Courant sigma-model on doubled space, which reduces to the topological A- and B-models in the exact gauge. We shall now treat it as a large Courant sigma-model and use the projection to DFT vectors from §4 in Chapter 4, which halves the number of degree 1 coordinates. Explicitly, the degree 1 coordinates χ_I and ψ^I are transformed to $\tau^I_\pm$ in (4.113), which in components can be written as

$$\tau^I_\pm \;=\; \frac{1}{2}\begin{pmatrix}\psi^i \pm \widetilde{\chi}^i\\ \widetilde{\psi}_i \pm \chi_i\end{pmatrix}\;. \tag{5.23}$$

The coordinates τ^I_- are projected out by $\frac{1}{2}\,\mathsf{p}_+$ in the same way they were in (4.115), hence in Darboux coordinates

$$\tau^I_+ \;=\; \begin{pmatrix}q^i\\ p_i\end{pmatrix} \tag{5.24}$$

the symplectic structure ω_{DFT} from (4.110) becomes

$$\omega_{\mathrm{DFT}} \xrightarrow{\frac{1}{2}\,\mathsf{p}_+} \omega_{\mathrm{DFT},+} = \mathrm{d}X^i \wedge \mathrm{d}F_i + \mathrm{d}\widetilde{X}_i \wedge \mathrm{d}\widetilde{F}^i + \mathrm{d}q^i \wedge \mathrm{d}p_i \ . \tag{5.25}$$

In the Hamiltonian (4.112) we substitute

$$\chi_i \longrightarrow p_i \ , \qquad \psi^i \longrightarrow q^i \ , \qquad \widetilde{\chi}^i \longrightarrow q^i \qquad \text{and} \qquad \widetilde{\psi}_i \longrightarrow p_i \ , \tag{5.26}$$

which together with the symplectic form $\omega_{\mathrm{DFT},+}$ reduces the AKSZ action $\boldsymbol{\mathcal{S}}^{(3)}_{\Omega_{\mathrm{G}},0}$ from (5.21) to the action

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(3)}_{\Omega_{\mathrm{G}},+} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \Big(& \boldsymbol{F}_i\, \boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{F}}^i\, \boldsymbol{D}\widetilde{\boldsymbol{X}}_i - \boldsymbol{p}_i\, \boldsymbol{D}\boldsymbol{q}^i + \mathsf{P}^{ij}\, \boldsymbol{F}_i\, \boldsymbol{p}_j + \mathsf{Q}_{ij}\, \widetilde{\boldsymbol{F}}^i\, \boldsymbol{q}^j \\ & + \mathsf{J}^i{}_j\, \boldsymbol{F}_i\, \boldsymbol{q}^j - \mathsf{J}^j{}_i\, \widetilde{\boldsymbol{F}}^i\, \boldsymbol{p}_j - \frac{1}{2}\, \boldsymbol{\partial}_i \mathsf{P}^{jk}\, \boldsymbol{q}^i\, \boldsymbol{p}_j\, \boldsymbol{p}_k - \frac{1}{2}\, \boldsymbol{\partial}_i \mathsf{Q}_{jk}\, \boldsymbol{q}^i\, \boldsymbol{q}^j\, \boldsymbol{q}^k \\ & + \boldsymbol{\partial}_i \mathsf{J}^k{}_j\, \boldsymbol{q}^i\, \boldsymbol{q}^j\, \boldsymbol{p}_k - \frac{1}{2}\, \widetilde{\boldsymbol{\partial}}^i \mathsf{P}^{jk}\, \boldsymbol{p}_i\, \boldsymbol{p}_j\, \boldsymbol{p}_k - \frac{1}{2}\, \widetilde{\boldsymbol{\partial}}^k \mathsf{Q}_{ij}\, \boldsymbol{q}^i\, \boldsymbol{q}^j\, \boldsymbol{p}_k \\ & - \widetilde{\boldsymbol{\partial}}^j \mathsf{J}^k{}_i\, \boldsymbol{q}^i\, \boldsymbol{p}_j\, \boldsymbol{p}_k \Big) \ ,\end{aligned} \tag{5.27}$$

where for simplicity we imposed the $O(d,d)$-invariant boundary condition[2]

$$(\boldsymbol{p}_i\, \boldsymbol{q}^i)\big|_{T[1]\partial\Sigma_3} = 0 \ . \tag{5.28}$$

We reduce the dual coordinates in (5.27) with a gauge fixing $\widetilde{\boldsymbol{F}}^i = 0$ and assume that none of the blocks P^{ij}, $\mathsf{J}^i{}_j$ or Q_{ij} depend on $\widetilde{X}_i$. The resulting action is not necessarily an AKSZ action as it does not satisfy the master equation. Instead we impose the master equation as a further constraint on the blocks in order to satisfy the quantization condition, and we define the reduced action with the constrained blocks, which we write symbolically as

$$\mathsf{P}^{ij} \xrightarrow{\text{master}} \pi^{ij} \ , \qquad \mathsf{J}^i{}_j \xrightarrow{\text{master}} J^i{}_j \qquad \text{and} \qquad \mathsf{Q}_{ij} \xrightarrow{\text{master}} \omega_{ij} \ . \tag{5.29}$$

The reduced AKSZ action is given by

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(3)}_{\mathrm{Z}} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \Big(& \boldsymbol{F}_i\, \boldsymbol{D}\boldsymbol{X}^i - \boldsymbol{p}_i\, \boldsymbol{D}\boldsymbol{q}^i + \boldsymbol{\pi}^{ij}\, \boldsymbol{F}_i\, \boldsymbol{p}_j + \boldsymbol{J}^i{}_j\, \boldsymbol{F}_i\, \boldsymbol{q}^j \\ & - \frac{1}{2}\, \boldsymbol{\partial}_i \boldsymbol{\pi}^{jk}\, \boldsymbol{q}^i\, \boldsymbol{p}_j\, \boldsymbol{p}_k - \frac{1}{2}\, \boldsymbol{\partial}_i \boldsymbol{\omega}_{jk}\, \boldsymbol{q}^i\, \boldsymbol{q}^j\, \boldsymbol{q}^k + \boldsymbol{\partial}_i \boldsymbol{J}^k{}_j\, \boldsymbol{q}^i\, \boldsymbol{q}^j\, \boldsymbol{p}_k \Big) \ .\end{aligned} \tag{5.30}$$

[2]This boundary condition will be compatible with our further reduction to the B-model, but not to the A-model. To be compatible with the latter reduction we need to start with a different kinetic term for the large Courant algebroid, in order to obtain the right kinetic term of the Courant sigma-model for the generalized complex structure.

The special property of this AKSZ action is that in the exact gauge

$$\boldsymbol{F}_i \,=\, \boldsymbol{D}\boldsymbol{p}_i \qquad \text{and} \qquad \boldsymbol{q}^i \,=\, -\boldsymbol{D}\boldsymbol{X}^i \tag{5.31}$$

it gives the Zucchini action (4.58):

$$\boldsymbol{\mathcal{S}}^{(3)}_{\mathrm{Z,gf}} \,=\, \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\,\Big(\boldsymbol{p}_i\,\boldsymbol{D}\boldsymbol{X}^i + \frac{1}{2}\,\boldsymbol{\pi}^{ij}\,\boldsymbol{p}_i\,\boldsymbol{p}_j + \frac{1}{2}\,\boldsymbol{\omega}_{ij}\,\boldsymbol{D}\boldsymbol{X}^i\,\boldsymbol{D}\boldsymbol{X}^j - \boldsymbol{J}^i{}_j\,\boldsymbol{p}_i\,\boldsymbol{D}\boldsymbol{X}^j\Big) \,=\, \boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{Z}}\ , \tag{5.32}$$

on the boundary after the sign flip $J^i{}_j \to -J^i{}_j$.

One may naturally expect that the master equation for $\boldsymbol{\mathcal{S}}^{(3)}_{\mathrm{Z}}$ will give the constraints of a generalized complex structure (4.60) as the Zucchini model does, but this is not precisely true. If ω_{ij} vanishes then we get the same identities as those of a generalized complex structure with $\omega = 0$, or if we set $\omega = \pi^{-1}$ then $\mathrm{d}\omega = 0$ and the term involving ω vanishes, thus we arrive at the same AKSZ action. Otherwise the ω term generally prevents the constraints from being the identities of a generalized complex structure.

Thus we propose a *Courant sigma-model*

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(3)}_{\pi,J} \,=\, \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\,\Big(&\boldsymbol{F}_i\,\boldsymbol{D}\boldsymbol{X}^i \,-\, \boldsymbol{p}_i\,\boldsymbol{D}\boldsymbol{q}^i \,+\, \boldsymbol{\pi}^{ij}\,\boldsymbol{F}_i\,\boldsymbol{p}_j \,+\, \boldsymbol{J}^i{}_j\,\boldsymbol{F}_i\,\boldsymbol{q}^j \\ &-\, \frac{1}{2}\,\boldsymbol{\partial}_i\boldsymbol{\pi}^{jk}\,\boldsymbol{q}^i\,\boldsymbol{p}_j\,\boldsymbol{p}_k \,+\, \boldsymbol{\partial}_i\boldsymbol{J}^k{}_j\,\boldsymbol{q}^i\,\boldsymbol{q}^j\,\boldsymbol{p}_k\Big)\end{aligned} \tag{5.33}$$

for the generalized complex structure

$$\mathbb{J}^I{}_J \,=\, \begin{pmatrix} J^i{}_j & \pi^{ij} \\ 0 & -J^j{}_i \end{pmatrix}\ . \tag{5.34}$$

In the language of symplectic dg-geometry this means that the master equation for the Hamiltonian

$$\gamma_{\pi,J} \,=\, \pi^{ij}\,F_i\,p_j \,+\, J^i{}_j\,F_i\,q^j \,-\, \frac{1}{2}\,\partial_i\pi^{jk}\,q^i\,p_j\,p_k \,+\, \partial_i J^k{}_j\,q^i\,q^j\,p_k \tag{5.35}$$

with the symplectic form

$$\omega_3 \,=\, \mathrm{d}X^i \wedge \mathrm{d}F_i \,+\, \mathrm{d}q^i \wedge \mathrm{d}p_i \tag{5.36}$$

gives the conditions

$$\begin{aligned} \pi^{[i|l}\,\partial_l\pi^{jk]} \,&=\, 0\ , \\ J^l{}_i\,\partial_l\pi^{jk} \,+\, 2\,\pi^{jl}\,\partial_{[i}J^k{}_{l]} \,+\, \pi^{kl}\,\partial_l J^j{}_i \,-\, J^j{}_l\,\partial_i\pi^{lk} \,&=\, 0\ , \\ J^l{}_{[i|}\,\partial_l J^k{}_{|j]} \,-\, J^k{}_l\,\partial_{[i}J^l{}_{j]} \,&=\, 0\ . \end{aligned} \tag{5.37}$$

The first identity says that π^{ij} satisfies the Poisson condition, the third says $J^i{}_j$ satisfies the integrability condition of the ordinary complex structure, and the second identity is an additional compatibility condition needed to combine them into a generalized complex structure.

The Hamiltonian $\gamma_{\pi,J}$ defines a *Courant algebroid for the generalized complex structures* (5.34) over the target space M, with the Dorfman bracket and anchor

$$[e_1, e_2]_{\mathrm{D};\pi,\mathrm{J}} \,=\, \{\{e_1, \gamma_{\pi,J}\}, e_2\} \qquad \text{and} \qquad \rho(e) \,=\, \{e, \{\gamma_{\pi,J}, \cdot\}\} \tag{5.38}$$

where the functions e, e_1 and e_2 have degree 1. It would be interesting in its own right to study further this new Courant algebroid structure.

2.2 Dimensional reductions to the A- and B-models

The relation of the Courant sigma-model (5.33) to the A-model is quite straightforward. If we set J to zero, and only keep π non-zero, the remaining identity from (5.37) is the Poisson condition. The resulting AKSZ action is just that of the contravariant Courant sigma-model, which reduces to the Poisson sigma-model on its boundary in the exact gauge, and thus to the A-model as well.

The relation to the B-model is not immediately apparent. Let π be zero, and J non-zero. The remaining identity from (5.37) is the integrability condition for the ordinary complex structure J. The Hamiltonian associated to the resulting AKSZ action is

$$\gamma_{0,J} \,=\, J^i{}_j\, F_i\, q^j \,+\, \partial_i J^k{}_j\, q^i\, q^j\, p_k \;, \tag{5.39}$$

from which a *Courant algebroid for a generic complex structure* can be derived with the Dorfman bracket and anchor

$$[e_1, e_2]_{\mathrm{D};0,\mathrm{J}} \,=\, \{\{e_1, \gamma_{0,J}\}, e_2\} \qquad \text{and} \qquad \rho(e) \,=\, \{e, \{\gamma_{0,J}, \cdot\}\} \tag{5.40}$$

respectively, where again e, e_1 and e_2 are degree 1 functions. This structure is similar to that of the Poisson Courant algebroid, which is the derived Courant algebroid for a generic Poisson structure.

We apply the dimensional reduction method introduced in §1.4 in Chapter 4 on the AKSZ action with the ordinary complex structure solely:

$$\boldsymbol{\mathcal{S}}^{(3)}_{0,J} \,=\, \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \big(\boldsymbol{F}_i\, \boldsymbol{D}\boldsymbol{X}^i \,-\, \boldsymbol{p}_i\, \boldsymbol{D}\boldsymbol{q}^i \,+\, \boldsymbol{J}^i{}_j\, \boldsymbol{F}_i\, \boldsymbol{q}^j \,+\, \boldsymbol{\partial}_i \boldsymbol{J}^k{}_j\, \boldsymbol{q}^i\, \boldsymbol{q}^j\, \boldsymbol{p}_k\big) \;. \tag{5.41}$$

The reduction method requires that the membrane worldvolume Σ_3 be a product manifold, hence we apply it in a neighbourhood of the boundary. For this, choose an open subset U of Σ_3 which includes $\partial\Sigma_3$:

$$\Sigma_3\big|_U \;=\; \partial\Sigma_3 \times \mathbb{R}^+ \;, \tag{5.42}$$

where $\mathbb{R}^+$ is the half-line parameterized with coordinate t, for which the $t = 0$ points belong to the boundary. Then the worldvolume Σ_3 is covered by open sets as

$$\Sigma_3 = U \cup U' \;, \tag{5.43}$$

where the open set U' does not include the boundary, i.e. it is contained in the bulk interior $\Sigma_3 \setminus \partial\Sigma_3$. Then the BV symplectic form is given by the sum of two integrals over the covering sets U and U' as[3]

$$\begin{aligned}\boldsymbol{\omega}_3 \;&=\; \boldsymbol{\omega}_{3|U} \,+\, \boldsymbol{\omega}_{3|U'} \\ &:=\; \int_{T[1]U} \mathrm{d}^3\hat{z}\,\big(\boldsymbol{\delta X}^i\,\boldsymbol{\delta F}_i \,+\, \boldsymbol{\delta q}^i\,\boldsymbol{\delta p}_i\big) \,+\, \int_{T[1]U'} \mathrm{d}^3\hat{z}\,\big(\boldsymbol{\delta X}'^i\,\boldsymbol{\delta F}'_i \,+\, \boldsymbol{\delta q}'^i\,\boldsymbol{\delta p}'_i\big) \;,\end{aligned} \tag{5.44}$$

where we have rescaled the fields $\boldsymbol{F}_i$, $\boldsymbol{p}_i$, $\boldsymbol{F}'_i$ and $\boldsymbol{p}'_i$ with a suitable partition of unity subordinate to the covering (5.43). These fields are chosen so that the decomposition of the AKSZ action

$$\begin{aligned}\boldsymbol{\mathcal{S}}^{(3)}_{0,J} \;&=\; \boldsymbol{\mathcal{S}}^{(3)}_{0,J|U} \,+\, \boldsymbol{\mathcal{S}}^{(3)}_{0,J|U'} \\ &:=\; \int_{T[1]U} \mathrm{d}^3\hat{z}\,\big(\boldsymbol{F}_i\,\boldsymbol{DX}^i \,-\, \boldsymbol{p}_i\,\boldsymbol{Dq}^i \,+\, \boldsymbol{J}^i{}_j\,\boldsymbol{F}_i\,\boldsymbol{q}^j \,+\, \boldsymbol{\partial}_i\boldsymbol{J}^k{}_j\,\boldsymbol{q}^i\,\boldsymbol{q}^j\,\boldsymbol{p}_k\big) \\ &\quad+\, \int_{T[1]U'} \mathrm{d}^3\hat{z}\,\big(\boldsymbol{F}'_i\,\boldsymbol{DX}'^i \,-\, \boldsymbol{p}'_i\,\boldsymbol{Dq}'^i \,+\, \boldsymbol{J}^i{}_j\,\boldsymbol{F}'_i\,\boldsymbol{q}'^j \,+\, \boldsymbol{\partial}_i\boldsymbol{J}^k{}_j\,\boldsymbol{q}'^i\,\boldsymbol{q}'^j\,\boldsymbol{p}'_k\big)\end{aligned} \tag{5.45}$$

is independent of the choice of partition of unity.

First we deal with the boundary contributions. They are defined on a product manifold $\partial\Sigma_3 \times \mathbb{R}^+$, so we can apply the method of §1.4 in Chapter 4. We use a uniform notation for an arbitrary superfield $\boldsymbol{\phi}$:

$$\boldsymbol{\phi} \;=\; \widehat{\boldsymbol{\phi}} \,+\, \boldsymbol{\phi}_t\,\theta^t \;, \tag{5.46}$$

where the component superfields $\widehat{\boldsymbol{\phi}}$ and $\boldsymbol{\phi}_t$ do not depend on the odd coordinate θ^t of $T[1]\mathbb{R}^+$. The integrals over U factorize and we get the BV symplectic form

$$\boldsymbol{\omega}_{3|U;\mathrm{gf}} \;=\; -\oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\,\int_{\mathbb{R}^+} \mathrm{d}t\,\big(\boldsymbol{\delta}\widehat{\boldsymbol{X}}^i\,\boldsymbol{\delta}(\boldsymbol{F}_t)_i \,+\, \boldsymbol{\delta}\widehat{\boldsymbol{q}}^i\,\boldsymbol{\delta}(\boldsymbol{p}_t)_i\big) \;, \tag{5.47}$$

[3]We use the same notation for the boundary fields as well for brevity, but they are not the identical to those used earlier.

where we have used a different gauge fixing and also different antifields than those which were used for the reduction to the A-model: we have set $\boldsymbol{X}^i_t$ and $\boldsymbol{q}^i_t$ to zero. The gauge fixed boundary action is

$$\boldsymbol{\mathcal{S}}^{(3)}_{0,J|U;\mathrm{gf}} = \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z} \int_{\mathbb{R}^+} \mathrm{d}t \, \big(\widehat{\boldsymbol{F}}_i \, \partial_t \widehat{\boldsymbol{X}}^i + \widehat{\boldsymbol{p}}_i \, \partial_t \widehat{\boldsymbol{q}}^i - (\boldsymbol{F}_t)_i \, \widehat{\boldsymbol{D}} \widehat{\boldsymbol{X}}^i - (\boldsymbol{p}_t)_i \, \widehat{\boldsymbol{D}} \widehat{\boldsymbol{q}}^i \\ - \boldsymbol{J}^i{}_j \, (\boldsymbol{F}_t)_i \, \widehat{\boldsymbol{q}}^j + \boldsymbol{\partial}_i \boldsymbol{J}^k{}_j \, \widehat{\boldsymbol{q}}^i \, \widehat{\boldsymbol{q}}^j \, (\boldsymbol{p}_t)_k \big) \ . \tag{5.48}$$

The first two terms determine the boundary conditions. Integrating out the fields $\widehat{\boldsymbol{F}}_i$ and $\widehat{\boldsymbol{p}}_i$ imposes the condition that the fields $\widehat{\boldsymbol{X}}^i$ and $\widehat{\boldsymbol{q}}^i$ are independent of t, while the zero modes of $\widehat{\boldsymbol{F}}_i$ and $\widehat{\boldsymbol{p}}_i$ on $\mathbb{R}^+$ lead to the condition that $\widehat{\boldsymbol{X}}^i$ and $\widehat{\boldsymbol{q}}^i$ vanish at $t = 0$ which means they vanish on the boundary.

We introduce the new notations

$$\begin{aligned} \boldsymbol{\chi}_i &= -\int_{\mathbb{R}^+} \mathrm{d}t \, (\boldsymbol{F}_t)_i \ , & \boldsymbol{X}^i &= \widehat{\boldsymbol{X}}^i \ , \\ \widetilde{\boldsymbol{X}}_i &= -\int_{\mathbb{R}^+} \mathrm{d}t \, (\boldsymbol{p}_t)_i \ , & \widetilde{\boldsymbol{\chi}}^i &= \widehat{\boldsymbol{q}}^i \ , \end{aligned} \tag{5.49}$$

and rewrite the BV symplectic form and the boundary AKSZ action with them as

$$\boldsymbol{\omega}_{3|U;\mathrm{gf}} = \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z} \, \big(\boldsymbol{\delta} \boldsymbol{X}^i \, \boldsymbol{\delta} \boldsymbol{\chi}_i + \boldsymbol{\delta} \widetilde{\boldsymbol{X}}_i \, \boldsymbol{\delta} \widetilde{\boldsymbol{\chi}}^i \big) \ , \tag{5.50}$$

and

$$\begin{aligned} \boldsymbol{\mathcal{S}}^{(3)}_{0,J|U;\mathrm{gf}} &= \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z} \, \big(\boldsymbol{\chi}_i \, \boldsymbol{D} \boldsymbol{X}^i + \widetilde{\boldsymbol{X}}_i \, \boldsymbol{D} \widetilde{\boldsymbol{\chi}}^i + \boldsymbol{J}^i{}_j \, \boldsymbol{\chi}_i \, \widetilde{\boldsymbol{\chi}}^j - \boldsymbol{\partial}_j \boldsymbol{J}^i{}_k \, \widetilde{\boldsymbol{X}}_i \, \widetilde{\boldsymbol{\chi}}^j \, \widetilde{\boldsymbol{\chi}}^k \big) \\ &= \boldsymbol{\mathcal{S}}^{(2)}_J \ , \end{aligned} \tag{5.51}$$

which give the BV symplectic form corresponding to (4.66) and the AKSZ action (4.70) for the B-model after the sign flip $\widetilde{\boldsymbol{X}}_i \to -\widetilde{\boldsymbol{X}}_i$. Hence the action $\boldsymbol{\mathcal{S}}^{(3)}_{0,J}$ defined in (5.41) reduces to the B-model action in a neighbourhood of the boundary $\partial\Sigma_3$.

For the bulk contributions, one can gauge fix the bulk fields independently from the boundary fields using the same fields and antifields that were used for the reduction to the A-model. The exact gauge was a gauge fixing on the bulk as well, and not only on the boundary. Hence if we set $\boldsymbol{F}'_i$ and $\boldsymbol{q}'^i$ to zero, we get a vanishing bulk action $\boldsymbol{\mathcal{S}}^{(3)}_{0,J|U';\mathrm{gf}}$, and thus the action $\boldsymbol{\mathcal{S}}^{(3)}_{0,J}$ in (5.41) can be reduced entirely to the B-model action on the boundary.

We recall that, in all schemes presented in the thesis, the reductions to the A- and B-models differ significantly: not only are the gauge choices different, but the antifields

are also assigned differently, and the boundary conditions differ as well. But they both appear as boundary AKSZ sigma-models while the bulk fields are gauge fixed completely.

3 Topological S-duality in generalized complex geometry

As an application of the formalism developed in the thesis, in this section we shall demonstrate how the topological S-duality described in §2.1.4 in Chapter 2 is realised geometrically in generalized complex geometry using our Courant algebroids and AKSZ sigma-models.

3.1 Duality between Poisson and complex structure Courant algebroids

A duality transformation in the language of QP-manifolds is a transformation of supercoordinates which leaves the symplectic structure invariant. One of the simplest non-trivial cases is the renormalization of the fields by a scale transformation: a coordinate is scaled inversely with respect to its dual coordinate. In the following we study the Courant algebroid for generalized complex structures in this context.

The symplectic form ω_3 given by (5.36) is left invariant under the scale transformation

$$p_i \longmapsto \lambda\, p_i \qquad \text{and} \qquad q^i \longmapsto \frac{1}{\lambda}\, q^i \ , \tag{5.52}$$

with a constant parameter $\lambda \in \mathbb{R}$, which transforms the Hamiltonian $\gamma_{\pi,J}$ from (5.35) to

$$\gamma^{\lambda}_{\pi,J} \,=\, \lambda\, \pi^{ij}\, F_i\, p_j \,-\, \frac{\lambda}{2}\, \partial_i \pi^{jk}\, q^i\, p_j\, p_k \,+\, \frac{1}{\lambda}\, J^i{}_j\, F_i\, q^j \,+\, \frac{1}{\lambda}\, \partial_i J^k{}_j\, q^i\, q^j\, p_k \ . \tag{5.53}$$

The scale transformation has no effect on the identities (5.37), and $\gamma^{\lambda}_{\pi,J}$ satisfies the master equation as well.

Now we can take both the large or small λ limit. They give different Courant algebroids, namely the Poisson and the complex structure Courant algebroid respectively:

$$\frac{1}{\lambda}\, \gamma_{0,J} \xleftarrow{\lambda \ll 1} \gamma^{\lambda}_{\pi,J} \xrightarrow{\lambda \gg 1} \lambda\, \gamma_{\pi,0} \ , \tag{5.54}$$

where the Hamiltonian $\gamma_{0,J}$ is defined in (5.39) while $\gamma_{\pi,0}$ is defined in (4.86). After the limits are taken the parameter λ can be scaled back to obtain the original Hamiltonians which are independent of λ.

Thus scaling with λ introduces a type of weak/strong duality, which interpolates continuously between Poisson and complex structure Courant algebroids within the Courant algebroid for generalized complex geometry, and it exchanges them between the two limits. In the following we relate this duality to the topological S-duality between the A- and B-models based on our AKSZ constructions and boundary reductions from §1.

3.2 Topological S-duality

In the following we promote our duality to the level of AKSZ constructions. We start with the AKSZ action given by the Hamiltonian $\lambda\,\gamma_{\pi,J}$ defined in (5.53):

$$\boldsymbol{\mathcal{S}}^{(3)}_{\mathrm{A/B}} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\,\Big(\frac{1}{\lambda}\,\boldsymbol{F}_i\,\boldsymbol{DX}^i - \frac{1}{\lambda}\,\boldsymbol{p}_i\,\boldsymbol{Dq}^i + \boldsymbol{\pi}^{ij}\,\boldsymbol{F}_i\,\boldsymbol{p}_j - \frac{1}{2}\,\boldsymbol{\partial}_i\boldsymbol{\pi}^{jk}\,\boldsymbol{q}^i\,\boldsymbol{p}_j\,\boldsymbol{p}_k + \boldsymbol{J}^i{}_j\,\boldsymbol{F}_i\,\boldsymbol{q}^j + \boldsymbol{\partial}_i\boldsymbol{J}^k{}_j\,\boldsymbol{q}^i\,\boldsymbol{q}^j\,\boldsymbol{p}_k\Big)\,, \tag{5.55}$$

where we explicitly introduced an overall constant $1/\lambda$ as a membrane tension in the definition of the action, which does not affect the BV quantization of the sigma-model. Now we perform the scaling duality (5.52). Since it leaves the BV symplectic form invariant, the kinetic terms do not change, only the interaction terms. The scale transformed AKSZ action is given by

$$\boldsymbol{\mathcal{S}}^{\lambda\,(3)}_{\mathrm{A/B}} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\,\Big(\frac{1}{\lambda}\,\boldsymbol{F}_i\,\boldsymbol{DX}^i - \frac{1}{\lambda}\,\boldsymbol{p}_i\,\boldsymbol{Dq}^i + \lambda\,\boldsymbol{\pi}^{ij}\,\boldsymbol{F}_i\,\boldsymbol{p}_j - \frac{\lambda}{2}\,\boldsymbol{\partial}_i\boldsymbol{\pi}^{jk}\,\boldsymbol{q}^i\,\boldsymbol{p}_j\,\boldsymbol{p}_k + \frac{1}{\lambda}\,\boldsymbol{J}^i{}_j\,\boldsymbol{F}_i\,\boldsymbol{q}^j + \frac{1}{\lambda}\,\boldsymbol{\partial}_i\boldsymbol{J}^k{}_j\,\boldsymbol{q}^i\,\boldsymbol{q}^j\,\boldsymbol{p}_k\Big)\,. \tag{5.56}$$

The large λ limit gives the contravariant Courant sigma-model without kinetic terms, which reduces to the A-model action given by (4.54) in the exact gauge in the same way that it reduced in §3.3 in Chapter 4:

$$\boldsymbol{\mathcal{S}}^{\lambda\,(3)}_{\mathrm{A/B}} \xrightarrow{\lambda\gg 1} \frac{\lambda}{2}\oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\;\boldsymbol{\pi}^{ij}\,\boldsymbol{p}_i\,\boldsymbol{p}_j\,. \tag{5.57}$$

Here λ appears as the inverse of the A-model string coupling:

$$\lambda = \frac{1}{g_{\mathrm{A}}}\,. \tag{5.58}$$

On the other hand, if we take λ to be small, we get the AKSZ action of the complex structure Courant algebroid with an overall membrane tension $1/\lambda$, which can be reduced to the B-model on its boundary as in §2.2:

$$\boldsymbol{\mathcal{S}}^{\lambda\,(3)}_{\mathrm{A/B}} \xrightarrow{\lambda \ll 1} \frac{1}{\lambda} \oint_{T[1]\partial\Sigma_3} \mathrm{d}^2\hat{z}\, \Big(\boldsymbol{\chi}_i\, \boldsymbol{D}\boldsymbol{X}^i + \widetilde{\boldsymbol{X}}_i\, \boldsymbol{D}\widetilde{\boldsymbol{\chi}}^i + \boldsymbol{J}^i{}_j\, \boldsymbol{\chi}_i\, \widetilde{\boldsymbol{\chi}}^j - \boldsymbol{\partial}_j \boldsymbol{J}^i{}_k\, \widetilde{\boldsymbol{X}}_i\, \widetilde{\boldsymbol{\chi}}^j\, \widetilde{\boldsymbol{\chi}}^k \Big) \,. \tag{5.59}$$

Here λ appears as the B-model string coupling this time:

$$\lambda \,=\, g_{\mathrm{B}} \;, \tag{5.60}$$

which together with (5.58) gives the relation (2.42). However, it says nothing about the scaling relation of Kähler forms in (2.43). This is due to the fact that the scalings of π and J are not fixed to each other by the constraints (5.37).

Chapter 6

AKSZ constructions for topological membranes on G_2-manifolds

In this chapter we describe our results in topological membranes related to topological M-theory, which was published in the paper [3].

Topological M-theory was originally proposed as a unification of the topological A- and B-models [105,106], and is intended to capture a topological sector of physical M-theory. It can be constructed on seven-dimensional manifolds of G_2-holonomy where it has reduced $\mathcal{N} = 1$ supersymmetry. The theory of [105] is based on a Hitchin-type form theory of G_2-manifolds, and its dimensional reduction on a circle gives Hitchin's form theories of the topological A- and B-models.

The A- and B-models have worldsheet formulations as string theories where they are given by two-dimensional topological sigma-models. Hence it is natural to expect that topological M-theory has a worldvolume formulation and its fundamental objects are topological membranes. Two different membrane theories have been proposed for this purpose. One is constructed using the Mathai-Quillen formalism in [107], which reduces on a circle to the Mathai-Quillen construction of the A-model [108, 109] and its path integral localizes on associative three-cycles. The other one is introduced in [110] as a BRST gauge fixed version of the simple topological action constructed by pullback to the membrane worldvolume of the harmonic three-form associated to the G_2-structure, which also reduces to the A-model and localizes on associative three-cycles.

Both types of topological membranes are intended to be the fundamental objects of the same theory, which inevitably raises the question of whether they can be described

within a single membrane model. We give a unified treatment of these objects within AKSZ formulation. We propose two different BV quantized sigma-models for topological membranes on G_2-manifolds given by the AKSZ formulation, which each give back the membrane theories discussed above in particular gauges. Our distinct AKSZ membrane theories have the special feature that they can be unified within a single AKSZ three-brane sigma-model, which reproduces them through a worldvolume dimensional reduction. The AKSZ three-brane theory yields the standard 2-Courant bracket as its derived bracket, which fits it into the context of exceptional generalized geometry in M-theory. The derived bracket is also the same as the anomaly-free current algebra of topological membranes induced on the generalized tangent bundle $T \oplus \bigwedge^2 T^*$ of G_2-manifolds [96]. Furthermore we dimensionally reduce our AKSZ membranes to give new AKSZ constructions for the A-model after gauge fixing and canonical transformation. Performing a further dimensional reduction of one of these string models then gives a novel AKSZ construction for supersymmetric quantum mechanics.

1 AKSZ theories of topological membranes on G_2-manifolds

In this section we study the two topological membrane theories on G_2-manifolds. The first is the topological membrane model of [107] which is based on the Mathai-Quillen formalism,[1] the second one is the BRST model of [110]. We supplement the Mathai-Quillen construction with an auxiliary field, analogously to the construction of [110], and we give AKSZ formulations which reproduce both membrane models after gauge fixing.

1.1 Topological membrane theories

Mathai-Quillen membrane sigma-model

Let us begin by reviewing the topological membrane theory of [107], which we call the Mathai-Quillen membrane sigma-model. Let M_7 be an oriented seven-dimensional Riemannian manifold with G_2-structure, which is equivalent to equiping M_7 with a global three-form Φ that is closed, $\mathrm{d}\Phi = 0$, and coclosed, $\mathrm{d} * \Phi = 0$, where $*$ is the

[1] For further details about the Mathai-Quillen formalism in general, see e.g. [108, 109].

Hodge duality operator with respect to the metric g of M_7. Given an embedding map $X : \Sigma_3 \to M_7$, let us introduce a local section of the cotangent bundle T^*M_7 by

$$\Xi_I \;=\; \frac{1}{3!}\,(*\Phi)_{IJKL}\,\partial_\mu X^J\,\partial_\nu X^K\,\partial_\rho X^L\,\epsilon^{\mu\nu\rho}\;, \tag{6.1}$$

where Greek indices label local coordinates σ^μ on the worldvolume Σ_3, with $\partial_\mu := \frac{\partial}{\partial\sigma^\mu}$, and capital Latin indices label coordinates X^I on M_7 this time, with $\partial_I := \frac{\partial}{\partial X^I}$. The symbol $\epsilon^{\mu\nu\rho}$ is the Levi-Civita tensor density on Σ_3. If Ξ_I vanishes, then $X(\Sigma_3) \subset M_7$ is called an associative three-cycle.

We further introduce a ghost field ψ^I on Σ_3 with ghost number 1 and an antighost field χ^I on Σ_3 with ghost number -1. Then the action of the Mathai-Quillen membrane sigma-model is

$$I_{\rm MQ} \;=\; \int_{\Sigma_3} {\rm d}^3\sigma\,\Big(\frac{1}{2}\,g^{IJ}\,\Xi_I\,\Xi_J \;+\; {\rm i}\,\chi^I\,\big(\delta\Xi_I - \Gamma^K{}_{IJ}\,\psi^J\,\Xi_K\big) \;-\; \frac{1}{4}\,R_{IJKL}\,\psi^I\,\psi^J\,\chi^K\,\chi^L\Big)\;, \tag{6.2}$$

where

$$\delta\Xi_I - \Gamma^K{}_{IJ}\,\psi^J\,\Xi_K \;=\; \frac{1}{2}\,(*\Phi)_{IJKL}\,\nabla_\mu\psi^J\,\partial_\nu X^K\,\partial_\rho X^L\,\epsilon^{\mu\nu\rho}\;, \tag{6.3}$$

with $\nabla_\mu\psi^I = \partial_\mu\psi^I + \Gamma^I{}_{JK}\,\psi^J\,\partial_\mu X^K$ given by the Levi-Civita connection of the metric g pulled back to Σ_3 by X, and $R^I{}_{JKL}$ are the components of the Riemann curvature tensor of g.[2] The action (6.2) is invariant under the BRST transformations

$$\delta X^I = \psi^I\;, \qquad \delta\psi^I = 0 \qquad \text{and} \qquad \delta\chi^I \;=\; {\rm i}\,g^{IJ}\,\Xi_J - \Gamma^I{}_{JK}\,\psi^J\,\chi^K\;, \tag{6.4}$$

which is nilpotent only on-shell, and it is BRST-exact up to the equations of motion:

$$I_{\rm MQ} \;=\; \delta\Psi'_{\rm MQ} \qquad \text{with} \quad \Psi'_{\rm MQ} \;=\; -\frac{\rm i}{2}\int_{\Sigma_3} {\rm d}^3\sigma\;\chi^I\,\Xi_I\;. \tag{6.5}$$

The fixed point locus of the BRST charge is the space of associative three-cycles $X : \Sigma_3 \to M_7$, which are membrane instantons.

Let us now linearize the BRST transformations by supplementing the Mathai-Quillen membrane sigma-model with an auxiliary field. We define an auxiliary field b^I with the new BRST transformations

$$\delta X^I = \psi^I\;, \qquad \delta\psi^I = 0\;, \qquad \delta\chi^I = b^I \qquad \text{and} \qquad \delta b^I = 0\;, \tag{6.6}$$

[2]Capital Latin indices are raised and lowered with the metric g.

which is now nilpotent off-shell, and the membrane action is BRST-exact with the gauge fixing fermion

$$\Psi_{\rm MQ} \,=\, -\int_{\Sigma_3} {\rm d}^3\sigma\, \chi^I \Big({\rm i}\,\Xi_I \,+\, \frac{1}{2}\,\Gamma_{IJK}\,\chi^J\,\psi^K \,-\, \frac{1}{2}\,g_{IJ}\,b^J \Big) \,. \tag{6.7}$$

Then the membrane action $S_{\rm MQ} = \delta\Psi_{\rm MQ}$ is given by

$$\begin{aligned} S_{\rm MQ} \,=\, \int_{\Sigma_3} {\rm d}^3\sigma\, \Big(-\, {\rm i}\, b^I\,\Xi_I \,+\, \chi^I\,\big({\rm i}\,\delta\Xi_I + \Gamma_{JIK}\,b^J\,\psi^K\big) \\ +\, \frac{1}{2}\,\partial_L\Gamma_{IJK}\,\chi^I\,\chi^J\,\psi^K\,\psi^L \,+\, \frac{1}{2}\,g_{IJ}\,b^I\,b^J \Big) \,. \end{aligned} \tag{6.8}$$

The equation of motion for b^I gives

$$b^I \,=\, {\rm i}\,g^{IJ}\,\Xi_J \,-\, \Gamma^I{}_{JK}\,\chi^J\,\psi^K \,. \tag{6.9}$$

Using this expression one can show that the membrane action (6.8) reduces to the Mathai-Quillen membrane action (6.2).

Bonelli-Tanzini-Zabzine membrane sigma-model

In [110] a different topological membrane action on G_2-manifolds is given, which is based on BRST quantization of the topological action $I_{\Sigma_3,\Phi} = \int_{\Sigma_3} X^*(\Phi)$; we call it the Bonelli-Tanzini-Zabzine (BTZ for short) membrane sigma-model. With the same fields and notation as above, the action is

$$S_{\rm BTZ} \,=\, -\, I_{\Sigma_3,\Phi} \,+\, \delta\Psi_{\rm BTZ} \,, \tag{6.10}$$

with the gauge fixing fermion

$$\Psi_{\rm BTZ} \,=\, \int_{\Sigma_3} {\rm d}^3\sigma\, \chi^I \Big(g_{IJ}\,\dot{X}^J \,+\, \Phi_{IJK}\,\partial_1 X^J\,\partial_2 X^K \,+\, \frac{1}{2}\,\Gamma_{IJK}\,\chi^J\,\psi^K \,-\, \frac{1}{2}\,g_{IJ}\,b^J \Big) \,, \tag{6.11}$$

where the worldvolume indices run through $\mu = 0, 1, 2$ and the dot denotes the action of the derivative ∂_0. The BRST transformations are the same as those of the Mathai-Quillen membrane model in (6.6), thus they have identical BV formulations. Since ${\rm d}\Phi = 0$, the topological flux term $I_{\Sigma_3,\Phi}$ in the AKSZ framework arises from a canonical transformation as in §3.1 in Chapter 4, and consequently it has no effect in the BV algebra on the mapping space $\boldsymbol{\mathcal{M}}$. Hence in the following we will only study the BRST-exact term in (6.10).

1.2 BV formulation and AKSZ constructions

Both topological membrane sigma-models are described by a gauge fixing fermion $\Psi\big[X^I,\psi^I,\chi^I,b^I\big]$. The only non-zero BRST transformations are $\delta X^I=\psi^I$ and $\delta\chi^I=b^I$, so

$$\delta\Psi \;=\; \int_{\Sigma_3}\mathrm{d}^3\sigma\,\Big(\psi^I\,\frac{\delta\Psi}{\delta X^I}\;+\;b^I\,\frac{\overrightarrow{\delta}\Psi}{\delta\chi^I}\Big)\;. \tag{6.12}$$

With the definition of the antifields[3]

$$X^+_I\;=\;\frac{\delta\Psi}{\delta X^I}\qquad\text{and}\qquad\chi^+_I\;=\;\frac{\overrightarrow{\delta}\Psi}{\delta\chi^I}\;, \tag{6.13}$$

we can rewrite the BRST-exact part of the membrane actions as

$$\delta\Psi\;=\;\int_{\Sigma_3}\mathrm{d}^3\sigma\,\big(\psi^I\,X^+_I\;+\;b^I\,\chi^+_I\big)\;. \tag{6.14}$$

Thus the BRST-exact membrane actions in (6.8) and (6.10) differ only in the choice of gauge fixing, i.e. in the choice of Lagrangian submanifold $\boldsymbol{\mathcal{L}}\subset\boldsymbol{\mathcal{M}}$. In the following we propose two different AKSZ constructions for these topological membrane theories.

AKSZ construction I.

Our first AKSZ construction contains a rather large number of fields, but very few of them are explicitly used in the gauge fixed action. The source dg-manifold is $\mathcal{W}=T[1]\Sigma_3$ as usual, and the target symplectic dg-manifold is $\mathcal{M}=T^*[2]T[-1]T[1]M_7$. The base coordinates in $T[-1]T[1]M_7$ are (X^I,ξ^I,B^I,η^I) with degree $(0,1,0,-1)$, where X^I are associated to the coordinates of M_7. The graded fiber coordinates are (F_I,ζ_I,N_I,G_I) with degree $(2,1,2,3)$, and the canonical symplectic structure of degree 2 on $\mathcal{M}$ is

$$\omega_{3,\mathrm{I}}\;=\;\mathrm{d}F_I\wedge\mathrm{d}X^I\;+\;\mathrm{d}\zeta_I\wedge\mathrm{d}\xi^I\;+\;\mathrm{d}N_I\wedge\mathrm{d}B^I\;+\;\mathrm{d}G_I\wedge\mathrm{d}\eta^I\;. \tag{6.15}$$

In the following we expand a general AKSZ superfield $\boldsymbol{\phi}\in\boldsymbol{\mathcal{M}}$ as in (4.24) for $d=3$. Our membrane BRST fields X^I,ψ^I,χ^I,b^I do not have form components, so we choose them as the zeroth or third components of a superfield. Our choice in this

[3] As in §1.3 in Chapter 4 we denote the antifield of a field ϕ by ϕ^+.

first construction is as the zeroth component for both membrane models, and their antifields are assigned to the third components. Explicitly this means we take

$$\begin{aligned} X^{(0)\,I} &= X^I \qquad \text{and} \qquad F_I^{(3)} = X_I^+ \ , \\ \xi^{(0)\,I} &= \psi^I \qquad \text{and} \qquad \zeta_I^{(3)} = \psi_I^+ \ , \\ \eta^{(0)\,I} &= \chi^I \qquad \text{and} \qquad G_I^{(3)} = \chi_I^+ \ , \\ B^{(0)\,I} &= b^I \qquad \text{and} \qquad N_I^{(3)} = b_I^+ \ . \end{aligned} \tag{6.16}$$

The AKSZ action is constructed without kinetic terms and with a degree 3 Hamiltonian function γ such that the corresponding BV bracket with the associated cohomological vector field $\boldsymbol{Q}$ on $\boldsymbol{\mathcal{M}}$ generates the BRST transformations (6.6). Thus we take

$$\boldsymbol{\mathcal{S}}^{(3)}_{G_2,\mathrm{I}} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\ \big(\boldsymbol{\xi}^I\,\boldsymbol{F}_I + \boldsymbol{B}^I\,\boldsymbol{G}_I\big)\ , \tag{6.17}$$

which has eight components after expanding the superfields. We use a gauge fixing fermion to set the antifields X_I^+, ψ_I^+, χ_I^+, b_I^+, and we choose the gauge fixing of (6.17) on the other fields to give the gauge fixed action (6.14). For example, we may choose the Lagrangian submanifold $\boldsymbol{\mathcal{L}}$ determined by the equations

$$\begin{aligned} X^{(1)\,I} = X^{(3)\,I} &= 0 \qquad \text{and} \qquad F_I^{(1)} = 0\ , \\ \xi^{(1)\,I} = \xi^{(3)\,I} &= 0 \qquad \text{and} \qquad \zeta_I^{(1)} = 0\ , \\ \eta^{(1)\,I} = \eta^{(3)\,I} &= 0 \qquad \text{and} \qquad G_I^{(1)} = 0\ , \\ B^{(1)\,I} = B^{(3)\,I} &= 0 \qquad \text{and} \qquad N_I^{(1)} = 0\ , \end{aligned} \tag{6.18}$$

for the antifields. The other antifields given by the gauge fixing fermion are X_I^+, χ_I^+, ψ_I^+ and b_I^+. If we choose (6.7) we get the Mathai-Quillen membrane action (6.8), while if we choose (6.11) we get the BRST-exact part of the BTZ topological membrane action (6.10).

For example, in the Mathai-Quillen membrane sigma-model the pertinent antifields are given by

$$\begin{aligned} X_I^+ &= \frac{\delta\Psi_{\mathrm{MQ}}}{\delta X^I} = -\,\mathrm{i}\,\frac{\delta}{\delta X^I}\int_{\Sigma_3}\mathrm{d}^3\sigma\ \chi^I\,\Xi_I - \frac{1}{2}\,\partial_I\Gamma_{JKL}\,\chi^J\,\chi^K\,\chi^L + \frac{1}{2}\,\partial_I g_{JK}\,\chi^J\,b^K\ , \\ \chi_I^+ &= \frac{\overrightarrow{\delta}\Psi_{\mathrm{MQ}}}{\delta\chi^I} = -\,\mathrm{i}\,\Xi_I - \Gamma_{[IJ]K}\,\chi^J\,\psi^K + \frac{1}{2}\,g_{IJ}\,b^J\ , \end{aligned} \tag{6.19}$$

and it is easy to see

$$\psi^I \, \frac{\delta}{\delta X^I} \int_{\Sigma_3} \mathrm{d}^3\sigma \; \chi^I \, \Xi_I \; = \; -\chi^I \, \delta\Xi_I \; , \tag{6.20}$$

so that gauge fixing the antifields in this way restricts the AKSZ action functional (6.17) on $\boldsymbol{\mathcal{L}}$ to the action (6.8). The gauge fixing with Ψ_{BTZ} is very similar, and it gives the BTZ membrane action (6.10). Note that it is possible to add kinetic terms to the AKSZ action, and then set them to zero with a more specific gauge fixing choice, but evidently the model (6.17) is simpler to work with.

AKSZ construction II.

We introduce another AKSZ construction for both topological membrane theories, which is based on the standard Courant sigma-model from §3.1 in Chapter 4. The BV action that we want to reproduce in the AKSZ theory is again (6.14), but now we define the fermionic fields ψ^I and χ^I as one-forms in the superfield formalism. The target in this case is taken to be the QP-manifold $\mathcal{M} = T^*[2]T[1]M_7$ of degree 2 corresponding to the standard Courant algebroid on $TM_7 \oplus T^*M_7$, which contains half as many coordinates compared to the previous construction. The notation for the coordinates are the same as before, so that $(X^I, F_I, \xi^I, \zeta_I)$ have degrees $(0, 2, 1, 1)$. The symplectic form is

$$\omega_{3,\mathrm{II}} \; = \; \mathrm{d}F_I \wedge \mathrm{d}X^I \; + \; \mathrm{d}\zeta_I \wedge \mathrm{d}\xi^I \; . \tag{6.21}$$

The relevant fields in the superfield formalism are

$$\begin{aligned} X^{(0)\,I} &= X^I \qquad \text{and} \qquad X_0^{(1)\,I} = \chi^I \; , \\ \big(F_I^{(2)}\big)_{12} &= \chi_I^+ \qquad \text{and} \qquad \big(F_I^{(3)}\big)_{012} = X_I^+ \; , \\ \xi^{(0)\,I} &= \psi^I \qquad \text{and} \qquad \xi_0^{(1)\,I} = b^I \; , \\ \big(\zeta_I^{(2)}\big)_{12} &= -\, b_I^+ \qquad \text{and} \qquad \big(\zeta_I^{(3)}\big)_{012} = \psi_I^+ \; , \end{aligned} \tag{6.22}$$

where we used an explicit worldvolume index convention to define the membrane fields χ^I, b^I and their antifields. The BV action then simply corresponds to the untwisted Hamiltonian function γ_0 from §3.1 in Chaptet 4 and reads

$$\boldsymbol{\mathcal{S}}^{(3)}_{G_2,\mathrm{II}} \; = \; \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z} \; \boldsymbol{\xi}^I \, \boldsymbol{F}_I \; . \tag{6.23}$$

There are many possible gauge fixings which recover the action (6.14). One choice is to take the Lagrangian submanifold defined by

$$\begin{aligned} F_I^{(0)} = F_I^{(1)} = 0 \qquad & \text{and} \qquad \zeta_I^{(0)} = \zeta_I^{(1)} = 0 \ , \\ \big(F_I^{(2)}\big)_{01} = 0 \qquad & \text{and} \qquad \big(\zeta_I^{(2)}\big)_{01} = 0 \ , \\ \big(F_I^{(2)}\big)_{13} = 0 \qquad & \text{and} \qquad \big(\zeta_I^{(2)}\big)_{3} = 0 \ . \end{aligned} \tag{6.24}$$

The residual antifields are again set by the gauge fixing fermion $\Psi\big[X^I, \psi^I, \chi^I, b^I\big]$, given in (6.7) for the Mathai-Quillen membrane sigma-model and in (6.11) for the BTZ membrane sigma-model.

It is an interesting feature of our first AKSZ construction that the two terms in (6.17) are decoupled from each other, in the sense that they can be gauge fixed separately and decoupled in the AKSZ action as well. This means that one can remove the second term with a gauge fixing to get our second AKSZ construction, but they differ from those proposed for the topological membranes, because the antifields are assigned differently.

1.3 Derived brackets

The main geometric distinction between the two AKSZ membrane theories we have constructed above is that the second construction is based on a target which is a QP-manifold of degree 2, corresponding to the standard Courant algebroid, whereas the first construction is based on a target which is not an N-manifold, as it involves local affine coordinates of degree -1, and consequently does not correspond to a symplectic Lie 2-algebroid. Passing to dg-manifolds which are equiped with negative gradings is of course natural and standard in the BV–BRST formalism, wherein ghost fields and antifields typically come with negative gradings, but it takes us out of the realm of graded geometry into derived geometry [111]: Whereas non-negatively graded symplectic dg-manifolds generally correspond to symplectic L_∞-algebroids, those which are arbitrarily graded correspond to *derived* symplectic L_∞-algebroids. The relevance of L_∞-algebroids in BV quantization was already emphasised by [31, 112], but entering into further discussion of these geometric facts would take us far away from the scope of the thesis, so we content ourselves in pointing out a few interesting geometric consequences of the corresponding derived bracket construction.

The degree 3 Hamiltonian function on $\mathcal{M} = T^*[2]T[-1]T[1]M_7$ corresponding to the first AKSZ action (6.17) is given by

$$\gamma_{G_2,\mathrm{I}} = F_I\,\xi^I + G_I\,B^I\ . \tag{6.25}$$

Its first term is the same as the Hamiltonian function (4.81) for the standard Courant algebroid, so its derived brackets gives the standard Courant bracket (4.83) on degree 1 functions of (X,ξ,ζ). Moreover, this is also the derived bracket of the Hamiltonian function corresponding to the second AKSZ action (6.23), which contains solely the first term of (6.25).

The interesting feature here is the consequence of the second term in (6.25) and the negative degree coordinates η^I. The derived bracket of a symplectic dg-manifold with symplectic structure of degree 2 is defined on degree 1 functions. Such a function f can be expanded in the form

$$\begin{aligned} f\ &= f^{(0)}(X,B,\xi,\zeta)\ +\ f^{(1)}_I(X,B,\xi,\zeta,F,N)\,\eta^I \\ &\quad + \sum_{l=2}^{7} f^{(l)}_{I_1\cdots I_l}(X,B,\xi,\zeta,F,N,G)\,\eta^{I_1}\cdots\eta^{I_l}\ , \end{aligned} \tag{6.26}$$

where $f^{(l)}$ is an l-form in the non-negatively graded coordinates on $\mathcal{M}$ of degree $l+1$. The second term $G_I\,B^I$ in the Hamiltonian function decouples on the zeroth order functions $f^{(0)}(X,B,\xi,\phi)$, since it does not contain any of the canonically conjugate coordinates to X, B, ξ or ζ. Hence our derived bracket is closed on the subspace of zeroth order functions $f^{(0)}$, where it gives the standard Courant bracket (4.83), with the coefficients now depending on the two degree 0 coordinates X and B. The degree 0 fields are doubled in this sense, but they play an asymmetric role in the underlying geometric structure.

The restriction of the derived bracket to any higher order in η^I is no longer closed, and only closes if we consider all orders at once. Thus our derived bracket appears as an infinite extension of the standard Courant bracket, which contains the standard Courant bracket as the subalgebra of functions which are independent of η^I. This structure underlies the derived symplectic L_∞-algebroid over M_7 alluded to above.[4]

[4]See e.g. [97] for a general definition of L_∞-algebroids.

1.4 Dimensional reductions from topological threebrane theories

In §5 in Chapter 4 we introduced an AKSZ topological threebrane sigma-model which has the standard 2-Courant bracket as its derived algebraic structure on a graded target space which is a QP-manifold of degree 3. We can shed further light on the algebroid structure discussed in §1.3 by considering our membrane models as arising through certain reductions of such a threebrane theory. We first consider this sigma-model without a four-form flux deformation and defined for the G_2-manifold $M = M_7$. We suppose that the threebrane worldvolume is a product manifold $\Sigma_4 = \Sigma_3 \times S^1$, and that all superfields are independent of the extra coordinate t of S^1. Using the same notation (4.150) for the expansion of an arbitrary superfield, integration over the odd coordinate θ^t in the action (4.147) without the flux term leads to the AKSZ action[5]

$$\boldsymbol{\mathcal{S}}^{(4)}_{0,\mathrm{red}} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \big(\boldsymbol{G}_I\, \boldsymbol{B}^I + \boldsymbol{F}_I\, \boldsymbol{\xi}^I - \boldsymbol{F}_I\, \boldsymbol{D}\boldsymbol{X}^I + \boldsymbol{\xi}^I\, \boldsymbol{D}\boldsymbol{\zeta}_I - \boldsymbol{G}_I\, \boldsymbol{D}\boldsymbol{\eta}^I - \boldsymbol{B}^I\, \boldsymbol{D}\boldsymbol{N}_I\big) \tag{6.27}$$

and the BV symplectic form

$$\boldsymbol{\omega}_{4,\mathrm{red}} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \big(\boldsymbol{\delta}\boldsymbol{F}_I\, \boldsymbol{\delta}\boldsymbol{X}^I - \boldsymbol{\delta}\boldsymbol{\zeta}_I\, \boldsymbol{\delta}\boldsymbol{\xi}^I + \boldsymbol{\delta}\boldsymbol{G}_I\, \boldsymbol{\delta}\boldsymbol{\eta}^I - \boldsymbol{\delta}\boldsymbol{N}_I\, \boldsymbol{\delta}\boldsymbol{B}^I\big)\ , \tag{6.28}$$

where we have introduced the fields

$$\begin{aligned} &\boldsymbol{F}_I = -(\boldsymbol{F}_t)_I\ , && \boldsymbol{G}_I = \widehat{\boldsymbol{F}}_I\ , && \boldsymbol{\zeta}_I = (\boldsymbol{\chi}_t)_I && \text{and} && \boldsymbol{N}_I = \widehat{\boldsymbol{\chi}}_I\ , \\ &\boldsymbol{X}^I = \widehat{\boldsymbol{X}}^I\ , && \boldsymbol{\eta}^I = \boldsymbol{X}^I_t\ , && \boldsymbol{B}^I = -\boldsymbol{\psi}^I_t && \text{and} && \boldsymbol{\xi}^I = -\widehat{\boldsymbol{\psi}}^I\ , \end{aligned} \tag{6.29}$$

and rescaled them by the length of S^1. Thus the reduced AKSZ action without the kinetic terms is our first AKSZ membrane action (6.17), up to a few sign differences appearing in the symplectic forms which can be resolved with a redefinition of the original symplectic form of the membrane sigma-model that leaves its gauge fixed action invariant. On the other hand, the kinetic terms can be removed with the same gauge fixing that we used to obtain the topological membrane theories in this section. In this way, the threebrane AKSZ action without any kinetic term

$$\int_{T[1]\Sigma_4} \mathrm{d}^4\hat{z}\ \boldsymbol{F}_I\, \boldsymbol{\psi}^I \tag{6.30}$$

[5]The kinetic part of the AKSZ action is given here by $-\boldsymbol{\vartheta}$, where ϑ is the Liouville potential on the symplectic dg-manifold $\mathcal{M}$.

is a straightforward extension of our AKSZ membrane sigma-models.

This means therefore that our first AKSZ construction for topological membranes on G_2-manifolds is a reduced AKSZ theory of topological threebranes on the same target space. The special feature of the threebrane theory is that its derived bracket on the target QP-manifold $T^*[3]T[1]M_7$ of degree 3 gives the standard 2-Courant bracket (4.143) on the vector bundle $E = TM_7 \oplus \bigwedge^2 T^*M_7$, which relates the geometry behind our specific AKSZ construction to the exceptional generalized geometry of M-theory.

The second AKSZ construction for topological membranes from §1.2 can also be reformulated within a topological threebrane sigma-model, in the same way as the first construction. The only difference is that we get an additional term in the AKSZ action after the reduction, which can be set to zero with gauge fixing, because we do not need those fields to get the topological membrane theories with further gauge fixing. Hence the action (6.30) reduces to the second AKSZ sigma-model action as well.

In §5.2 in Chapter 4 we saw that viewing membranes as wrapping modes of threebranes, by wrapping the worldvolume circle on the target circle, reduces the four-dimensional standard 2-Courant sigma-model with G-flux to the three-dimensional standard Courant sigma-model with H-flux. This means that it is possible to add G-flux to our topological membrane theories at the threebrane level. Although the reduction above, wherein the fields are taken to be independent of one worldvolume direction, removes the topological flux term in (4.129), at the level of the full AKSZ action it does not. It leaves an extra contribution

$$\frac{1}{3!}\int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \boldsymbol{G}_{IJKL}\, \boldsymbol{\xi}^I\, \boldsymbol{\xi}^J\, \boldsymbol{\xi}^K\, \boldsymbol{B}^L \ , \tag{6.31}$$

which can be taken as a definition of a flux deformation for our first AKSZ membrane construction in §1.2.

Alternatively, one can directly induce the topological flux deformation $I_{\Sigma_3,\Phi}$ that we neglected in the action (6.10) by applying the double dimensional reduction technique from §5.2 in Chapter 4. For this, we first note that, generally, the AKSZ threebrane sigma-model (4.147) gives the BV action for the sigma-model of [96] for topological threebranes on an eight-dimensional $Spin(7)$-manifold M_8, with the twist G taken to be the global self-dual closed four-form corresponding to the $Spin(7)$-structure on M_8 [95], whose path integral localizes on Cayley four-cycles (threebrane instantons).

We can then embed our topological brane sigma-models with target G_2-manifold (M_7, Φ) into this threebrane theory by taking $\Sigma_4 = \Sigma_3 \times S^1$ and $M_8 = M_7 \times S^1$ with the Cayley four-form

$$G = \mathrm{d}X^8 \wedge \Phi + *\Phi \ . \tag{6.32}$$

Using double dimensional reduction on a circle as in §5.2 then reproduces the H-twisted standard Courant sigma-model (4.91) with flux $H = w\,\Phi$, and consequently leads to our second AKSZ construction from §1.2 with topological term. On the other hand, if the original threebrane is localized on S^1, i.e. X^8 is constant, then the threebrane theory reduces on t-independent superfields as above to our first AKSZ construction, with extra flux term (6.31) given by $G = *\Phi$. In this setting these threebrane worldvolume theories are regarded as providing a microscopic description of topological F-theory [110, 113].

2 AKSZ theories for the topological A-model

The topological A-model has been introduced in §2.1.1 in Chapter 2 and its known AKSZ constructions are closely related to the Poisson sigma-model as was discussed in §2.2 in Chapter 4. In this section we will follow the general procedure of §1.4 in Chapter 4 to compute a dimensional reduction, at the level of the AKSZ construction, for both AKSZ topological membrane theories which we derived in §1.2. In each case the reduction leads to a new AKSZ formulation for the topological A-model which differs from the Poisson sigma-model.

2.1 Dimensional reduction of AKSZ membrane sigma-models

We begin by applying a canonical transformation as described in §1.1 in Chapter 4. Here we will only use infinitesimal canonical transformations, which act on functions $\boldsymbol{f}$ on the phase space $\boldsymbol{\mathcal{M}}$ as

$$\boldsymbol{f} \longmapsto {}^{\boldsymbol{\alpha}}\boldsymbol{f} \,=\, \boldsymbol{f} \,+\, \varepsilon\, (\boldsymbol{f}, \boldsymbol{\alpha})_{\mathrm{BV}} \ , \tag{6.33}$$

where ε is an infinitesimal parameter and $\boldsymbol{\alpha}$ is a fermionic functional of the fields with ghost number -1. We perform such a canonical transformation on our two AKSZ membrane actions to induce kinetic terms, which will be used for dimensional reduction.

For the first AKSZ membrane action (6.17), the fermionic functional we choose is

$$\boldsymbol{\alpha} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \left(\boldsymbol{\zeta}_I\, \boldsymbol{D}\boldsymbol{X}^I + \boldsymbol{N}_I\, \boldsymbol{D}\boldsymbol{\eta}^I\right) , \tag{6.34}$$

where as previously the superworldvolume differential is $\boldsymbol{D} = \theta^\mu\, \partial_\mu$. Calculating the BV bracket $\left(\boldsymbol{\mathcal{S}}^{(3)}_{G_2,\mathrm{I}}, \boldsymbol{\alpha}\right)_{\mathrm{BV}}$ term by term we get the BRST-equivalent action

$$\begin{aligned} {}^{\boldsymbol{\alpha}}\boldsymbol{\mathcal{S}}^{(3)}_{G_2,\mathrm{I}} &= \boldsymbol{\mathcal{S}}^{(3)}_{G_2,\mathrm{I}} + \varepsilon\, \left(\boldsymbol{\mathcal{S}}^{(3)}_{G_2,\mathrm{I}}, \boldsymbol{\alpha}\right)_{\mathrm{BV}} \\ &= \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \Big(\boldsymbol{F}_I\, \boldsymbol{\xi}^I + \boldsymbol{B}^I\, \boldsymbol{G}_I \\ &\qquad\qquad + \varepsilon\, \left(\boldsymbol{F}_I\, \boldsymbol{D}\boldsymbol{X}^I + \boldsymbol{\xi}^I\, \boldsymbol{D}\boldsymbol{\zeta}_I + \boldsymbol{B}^I\, \boldsymbol{D}\boldsymbol{N}_I - \boldsymbol{G}_I\, \boldsymbol{D}\boldsymbol{\eta}^I\right)\Big) . \end{aligned} \tag{6.35}$$

Similar considerations apply to the second action (6.23): If we restrict the functionals and hence also the action to half of the fields $\boldsymbol{F}$, $\boldsymbol{X}$, $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, we get the fermionic functional of the canonical transformation $\boldsymbol{\alpha} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \boldsymbol{\zeta}_I\, \boldsymbol{D}\boldsymbol{X}^I$ which gives us the BRST-equivalent action

$${}^{\boldsymbol{\alpha}}\boldsymbol{\mathcal{S}}^{(3)}_{G_2,\mathrm{II}} = \int_{T[1]\Sigma_3} \mathrm{d}^3\hat{z}\, \Big(\boldsymbol{F}_I\, \boldsymbol{\xi}^I + \varepsilon\, \left(\boldsymbol{F}_I\, \boldsymbol{D}\boldsymbol{X}^I + \boldsymbol{\xi}^I\, \boldsymbol{D}\boldsymbol{\zeta}_I\right)\Big) . \tag{6.36}$$

Now let us turn to the dimensional reduction of the AKSZ membrane sigma-models. We assume that the target and worldvolume manifolds are products $M_7 = M_6 \times S^1$ and $\Sigma_3 = \Sigma_2 \times S^1$, where the coordinates of the target and worldvolume circles are indexed by $I = 7$ and $\mu = t$ respectively. We use again the expansion (4.150) of an arbitrary superfield $\boldsymbol{\phi} \in \boldsymbol{\mathcal{M}}$. In terms of expanded superfields, the symplectic structure is given by

$$\begin{aligned} \boldsymbol{\omega}_{3,\mathrm{I}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z} \int_{S^1} \mathrm{d}t\, \big(&- \boldsymbol{\delta}\widehat{\boldsymbol{F}}_I\, \boldsymbol{\delta}\boldsymbol{X}^I_t - \boldsymbol{\delta}(\boldsymbol{F}_t)_I\, \boldsymbol{\delta}\widehat{\boldsymbol{X}}^I - \boldsymbol{\delta}\widehat{\boldsymbol{\zeta}}_I\, \boldsymbol{\delta}\boldsymbol{\xi}^I_t + \boldsymbol{\delta}(\boldsymbol{\zeta}_t)_I\, \boldsymbol{\delta}\widehat{\boldsymbol{\xi}}^I \\ &- \boldsymbol{\delta}\widehat{\boldsymbol{G}}_I\, \boldsymbol{\delta}\boldsymbol{\eta}^I_t + \boldsymbol{\delta}(\boldsymbol{G}_t)_I\, \boldsymbol{\delta}\widehat{\boldsymbol{\eta}}^I - \boldsymbol{\delta}\widehat{\boldsymbol{N}}_I\, \boldsymbol{\delta}\boldsymbol{B}^I_t - \boldsymbol{\delta}(\boldsymbol{N}_t)_I\, \boldsymbol{\delta}\widehat{\boldsymbol{B}}^I\big) , \end{aligned} \tag{6.37}$$

and the action (6.35) by

$$\begin{aligned} {}^{\boldsymbol{\alpha}}\boldsymbol{\mathcal{S}}^{(3)}_{G_2,\mathrm{I}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z} \int_{S^1} \mathrm{d}t\, \Big(&\widehat{\boldsymbol{F}}_I\, \boldsymbol{\xi}^I_t - (\boldsymbol{F}_t)_I\, \widehat{\boldsymbol{\xi}}^I + \widehat{\boldsymbol{B}}^I\, (\boldsymbol{G}_t)_I - \boldsymbol{B}^I_t\, \widehat{\boldsymbol{G}}_I \\ &+ \varepsilon\, \big(\widehat{\boldsymbol{F}}_I\, \widehat{\boldsymbol{D}}\boldsymbol{X}^I_t + \widehat{\boldsymbol{F}}_I\, \partial_t\widehat{\boldsymbol{X}}^I - (\boldsymbol{F}_t)_I\, \widehat{\boldsymbol{D}}\widehat{\boldsymbol{X}}^I + \widehat{\boldsymbol{\xi}}_I\, \widehat{\boldsymbol{D}}\boldsymbol{\zeta}^I_t \\ &- \widehat{\boldsymbol{\xi}}_I\, \partial_t\widehat{\boldsymbol{\zeta}}^I + (\boldsymbol{\xi}_t)_I\, \widehat{\boldsymbol{D}}\widehat{\boldsymbol{\zeta}}^I - \widehat{\boldsymbol{G}}_I\, \widehat{\boldsymbol{D}}\boldsymbol{\eta}^I_t + \widehat{\boldsymbol{G}}_I\, \partial_t\widehat{\boldsymbol{\eta}}^I \\ &- (\boldsymbol{G}_t)_I\, \widehat{\boldsymbol{D}}\widehat{\boldsymbol{\eta}}^I + \widehat{\boldsymbol{B}}_I\, \widehat{\boldsymbol{D}}\boldsymbol{N}^I_t + \widehat{\boldsymbol{B}}_I\, \partial_t\widehat{\boldsymbol{N}}^I - (\boldsymbol{B}_t)_I\, \widehat{\boldsymbol{D}}\widehat{\boldsymbol{N}}^I\big)\Big) . \end{aligned} \tag{6.38}$$

We choose $\widehat{\boldsymbol{F}}$, $\boldsymbol{X}_t$, $\widehat{\boldsymbol{\phi}}$, $\boldsymbol{\xi}_t$, $\widehat{\boldsymbol{G}}$, $\boldsymbol{\eta}_t$, $\widehat{\boldsymbol{N}}$ and $\boldsymbol{B}_t$ to be the ultraviolet fields, and the rest to be the infrared fields. We define the gauge fixing condition as the Lagrangian submanifold $\boldsymbol{\mathcal{L}}$ defined by $\boldsymbol{X}_t = \boldsymbol{\eta}_t = \boldsymbol{\xi}_t = \boldsymbol{B}_t = 0$, and then integrate out the remaining ultraviolet fields. This leads to the conditions $\partial_t\widehat{\boldsymbol{X}} = \partial_t\widehat{\boldsymbol{\xi}} = \partial_t\widehat{\boldsymbol{\eta}} = \partial_t\widehat{\boldsymbol{B}} = 0$, so these fields do not depend on t. We also integrate out all of the fields with $I = 7$ index, and introduce new fields

$$\begin{aligned} \boldsymbol{\chi}_i &= -\int_{S^1} \mathrm{d}t\ (\boldsymbol{F}_t)_i\ , \qquad & \boldsymbol{p}_i &= \int_{S^1} \mathrm{d}t\ (\boldsymbol{\zeta}_t)_i\ , \\ \boldsymbol{h}_i &= \int_{S^1} \mathrm{d}t\ (\boldsymbol{G}_t)_i\ , \qquad & \boldsymbol{n}_i &= -\int_{S^1} \mathrm{d}t\ (\boldsymbol{N}_t)_i \end{aligned} \tag{6.39}$$

and

$$\boldsymbol{X}^i = \widehat{\boldsymbol{X}}^i\ , \qquad \boldsymbol{q}^i = \widehat{\boldsymbol{\xi}}^i\ , \qquad \boldsymbol{\eta}^i = \widehat{\boldsymbol{\eta}}^i \qquad \text{and} \qquad \boldsymbol{b}^i = \widehat{\boldsymbol{B}}^i\ , \tag{6.40}$$

where we used the index notation $I = (i, 7)$ with $i = 1, \ldots, 6$ the coordinate directions along M_6. Our effective action is then

$$\boldsymbol{\mathcal{S}}^{(2)\,\mathrm{eff}}_{G_2,\mathrm{I}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\ \Big(\boldsymbol{\chi}_i\,\boldsymbol{q}^i + \boldsymbol{b}^i\,\boldsymbol{h}_i + \varepsilon\,\big(\boldsymbol{\chi}_i\,\boldsymbol{D}\boldsymbol{X}^i + \boldsymbol{q}^i\,\boldsymbol{D}\boldsymbol{p}_i - \boldsymbol{h}_i\,\boldsymbol{D}\boldsymbol{\eta}^i - \boldsymbol{b}^i\,\boldsymbol{D}\boldsymbol{n}_i\big)\Big)\ , \tag{6.41}$$

and the new symplectic form is

$$\boldsymbol{\omega}_{2,\mathrm{I}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\ \big(\boldsymbol{\delta}\boldsymbol{\chi}_i\,\boldsymbol{\delta}\boldsymbol{X}^i + \boldsymbol{\delta}\boldsymbol{p}_i\,\boldsymbol{\delta}\boldsymbol{q}^i + \boldsymbol{\delta}\boldsymbol{h}_i\,\boldsymbol{\delta}\boldsymbol{\eta}^i + \boldsymbol{\delta}\boldsymbol{n}_i\,\boldsymbol{\delta}\boldsymbol{b}^i\big)\ . \tag{6.42}$$

We now perform another infinitesimal canonical transformation with the same parameter ε and the fermion

$$\boldsymbol{\alpha}' = -\int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\ \boldsymbol{p}_i\,\boldsymbol{D}\boldsymbol{X}^i - \int_{\Sigma_2} \mathrm{d}^2\sigma\ \Big(n_i^{(0)}\,\mathrm{d}\eta_i^{(1)} - n_i^{(1)}\,\mathrm{d}\eta_i^{(0)}\Big) \tag{6.43}$$

in order to eliminate the kinetic terms. In this way we arrive at the action

$$\boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{A,I}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\ \big(\boldsymbol{\chi}_i\,\boldsymbol{q}^i + \boldsymbol{b}^i\,\boldsymbol{h}_i\big)\ . \tag{6.44}$$

If we restrict this construction and the dimensional reduction to half of the fields $\boldsymbol{F}$, $\boldsymbol{X}$, $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, we arrive at the action for the dimensional reduction of our second AKSZ membrane model in the form

$$\boldsymbol{\mathcal{S}}^{(2)}_{\mathrm{A,II}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\ \boldsymbol{\chi}_i\,\boldsymbol{q}^i\ . \tag{6.45}$$

In the following we will introduce AKSZ constructions which give the actions (6.44) and (6.45), and then relate them to the topological A-model via suitable choices of

gauge fixing. For this, we equip $M_7 = M_6 \times S^1$ with a direct product metric, where M_6 is a six-dimensional Riemannian manifold with $SU(3)$-structure, and write the G_2-structure on M_7 as

$$\Phi = \mathrm{d}X^7 \wedge B + \rho \ , \tag{6.46}$$

where B is an almost Kähler form of type $(1,1)$ with respect to the almost complex structure defined by the three-form ρ on M_6. If B and ρ are independent of X^7, then $\mathrm{d}\Phi = 0$ implies $\mathrm{d}B = \mathrm{d}\rho = 0$ and M_6 is a Calabi-Yau threefold, as in the A-model topological string theory, where ρ is the real part of the global holomorphic three-form Ω on M_6. However, for the purposes of our ensuing AKSZ constructions only the Kähler class of the Calabi-Yau structure is required, as in [31]. In particular, double dimensional reduction on a circle of the flux deformation $I_{\Sigma_3,\Phi}$ along the lines of §5.2 in Chapter 4 gives the B-field coupling $I_{\Sigma_2,w\,B}$ for the topological string, whose AKSZ construction is given by the Poisson sigma-model of §2.1 in Chapter 4. Hence in what follows we shall only require that M_6 be a Kähler manifold.

2.2 BV formulation and AKSZ constructions for the A-model

The topological A-model was briefly reviewed in §2.1.1 in Chapter 2, whose Mathai-Quillen formalism is given in e.g. [108]. Let us now reformulate the topological A-model with a linearizing auxiliary field, analogously to what we did in §1.1 for the Mathai-Quillen membrane sigma-model. We introduce two fields $b^a_{\bar z}$ and $b^{\bar a}_z$ with ghost number 0, and the new BRST transformations given by (2.30) together with

$$\delta\chi^a_{\bar z} \,=\, b^a_{\bar z} \ , \qquad \delta\chi^{\bar a}_z \,=\, b^{\bar a}_z \ , \qquad \delta b^a_{\bar z} \,=\, 0 \qquad \text{and} \qquad \delta b^{\bar a}_z \,=\, 0 \ . \tag{6.47}$$

The action

$$S_{\mathrm{A}} \,=\, \delta\Psi_{\mathrm{A}} \ , \tag{6.48}$$

with the gauge fixing fermion

$$\begin{aligned}\Psi_{\mathrm{A}} \,=\, -\int_{\Sigma_2} \mathrm{d}^2z \, \Big(& g_{a\bar b}\,\big(\chi^a_{\bar z}\,\partial_z X^{\bar b} \,+\, \chi^{\bar a}_z\,\partial_{\bar z} X^b\big) \,+\, \frac{1}{2}\, g_{a\bar b}\,\big(\chi^a_{\bar z}\, b^{\bar b}_z \,+\, \chi^{\bar a}_z\, b^b_{\bar z}\big) \\ & +\, \frac{\mathrm{i}}{2}\,\Gamma_{a\bar b\bar c}\,\psi^{\bar b}\,\chi^{\bar c}_z\,\chi^a_{\bar z} \,+\, \frac{\mathrm{i}}{2}\,\Gamma_{\bar a bc}\,\psi^b\,\chi^c_{\bar z}\,\chi^{\bar a}_z\Big) \ ,\end{aligned} \tag{6.49}$$

reduces to the action (2.29) after using the equations of motion of the auxiliary fields $b^a_{\bar z}$ and $b^{\bar a}_z$ which give

$$b^a_{\bar z} \,=\, -\,\partial_{\bar z} X^a \,-\, \mathrm{i}\,\Gamma^a{}_{bc}\,\psi^b\,\chi^c_{\bar z} \qquad \text{and} \qquad b^{\bar a}_z \,=\, -\,\partial_z X^{\bar a} \,-\, \mathrm{i}\,\Gamma^{\bar a}{}_{\bar b\bar c}\,\psi^{\bar b}\,\chi^{\bar c}_z \ . \tag{6.50}$$

If we define the antifields as

$$X^+_a \,=\, \frac{\delta\Psi_{\rm A}}{\delta X^a}\ , \qquad X^+_{\bar a} \,=\, \frac{\delta\Psi_{\rm A}}{\delta X^{\bar a}}\ , \qquad \big(\chi^+_a\big)_z \,=\, \frac{\overrightarrow{\delta}\Psi_{\rm A}}{\delta\chi^a_{\bar z}} \qquad \text{and} \qquad \big(\chi^+_{\bar a}\big)_{\bar z} \,=\, \frac{\overrightarrow{\delta}\Psi_{\rm A}}{\delta\chi^{\bar a}_z}\ , \tag{6.51}$$

we can rewrite (6.48) as a BV-type action

$$S_{\rm A} \,=\, \int_{\Sigma_2} {\rm d}^2 z\, \Big(\psi^a\, X^+_a \,+\, \psi^{\bar a}\, X^+_{\bar a} \,+\, b^a_{\bar z}\, \big(\chi^+_a\big)_z \,+\, b^{\bar a}_z\, \big(\chi^+_{\bar a}\big)_{\bar z}\Big)\ . \tag{6.52}$$

In the following we give two new AKSZ constructions for the topological A-model, which each differ from the Poisson sigma-model.

AKSZ construction I.

Our first AKSZ construction for the topological A-model is analogous to the first AKSZ membrane sigma-model in §1.2. The source dg-manifold is the superworldsheet $\mathcal{W} = T[1]\Sigma_2$, while the target symplectic dg-manifold is $\mathcal{M} = T^*[1]T[-1]T[1]M_6$, where M_6 is a Kähler manifold. The base coordinates in $T[-1]T[1]M_6$ are (X^i, q^i, b^i, η^i) with degree $(0,1,0,-1)$, where X^i are associated to the coordinates of M_6. The graded fiber coordinates are (χ_i, p_i, n_i, h_i) with degree $(1,0,1,2)$. The canonical symplectic structure of degree 1 on the target superspace $\mathcal{M}$ is

$$\omega_{2,\rm I} \,=\, {\rm d}\chi_i \wedge {\rm d}X^i \,+\, {\rm d}p_i \wedge {\rm d}q^i \,+\, {\rm d}h_i \wedge {\rm d}\eta^i \,+\, {\rm d}n_i \wedge {\rm d}b^i\ . \tag{6.53}$$

This gives the same BV symplectic structure on the space $\boldsymbol{\mathcal{M}}$ of superfields as in (6.42), and the AKSZ action is (6.44).

We choose the gauge given by

$$\chi^{(0)} \,=\, \chi^{(1)} \,=\, 0\ , \qquad p^{(0)} \,=\, p^{(1)} \,=\, 0\ , \qquad n^{(0)} \,=\, n^{(1)} \,=\, 0\ , \qquad h^{(0)} \,=\, h^{(1)} \,=\, 0\ . \tag{6.54}$$

Writing the coordinate indices of the Kähler manifold M_6 as before in complex notation $i = (a, \bar a)$, where $a = 1,2,3$, and the complex coordinates on the worldsheet Σ_2 as $(z, \bar z)$, we define the component fields

$$\begin{aligned}
X^{(0)\,i} \,&=\, X^i\ , & \chi^{(2)}_i \,&=\, X^+_i\ , & q^{(0)\,i} \,&=\, \psi^i\ , & p^{(2)}_i \,&=\, -\,\psi^+_i\ ,\\
b^{(0)\,a} \,&=\, b^a_{\bar z}\ , & b^{(0)\,\bar a} \,&=\, b^{\bar a}_z\ , & n^{(2)}_a \,&=\, \big(b^+_a\big)_z\ , & n^{(2)}_{\bar a} \,&=\, \big(b^+_{\bar a}\big)_{\bar z}\ ,\\
\eta^{(0)\,a} \,&=\, \chi^a_{\bar z}\ , & \chi^{(0)\,\bar a} \,&=\, \chi^{\bar a}_z\ , & h^{(2)}_a \,&=\, -\,\big(\chi^+_a\big)_z\ , & h^{(2)}_{\bar a} \,&=\, -\,\big(\chi^+_{\bar a}\big)_{\bar z}\ .
\end{aligned} \tag{6.55}$$

With this notation, the gauge fixing of the AKSZ action (6.44) yields the BV action $-S_{\rm A}$ in (6.52), and it can be gauge fixed further to the A-model action with gauge fixing fermion $-\Psi_{\rm A}$ from (6.49).

AKSZ construction II.

We introduce a second AKSZ construction for the topological A-model, which is the analogue of the second AKSZ membrane sigma-model in §1.2. We start with the same source dg-manifold $\mathcal{W} = T[1]\Sigma_2$ as in the previous construction, but now we choose $\mathcal{M} = T^*[1]T[1]M_6$ to be the target QP-manifold of degree 1 with coordinates (X^i, χ_i, q^i, p_i) with degree $(0,1,1,0)$. The symplectic structure

$$\omega_{2,\rm II} \,=\, {\rm d}\chi_i \wedge {\rm d}X^i \,+\, {\rm d}p_i \wedge {\rm d}q^i \tag{6.56}$$

is the restriction of (6.53). The AKSZ action is also the restriction (6.45).

We introduce the component fields

$$\begin{aligned}
&X^{(0)\,i} \,=\, X^i \,, \qquad \big(\chi_i^{(2)}\big)_{z\overline{z}} \,=\, X_i^+ \,, \qquad q^{(0)\,i} \,=\, \psi^i \,, \qquad \big(q_i^{(2)}\big)_{z\overline{z}} \,=\, \psi_i^+ \,, \\
&X_{\overline{z}}^{(1)\,a} \,=\, \chi_{\overline{z}}^a \,, \qquad \big(\chi_a^{(1)}\big)_z \,=\, -\big(\chi_a^+\big)_z \,, \qquad X_z^{(1)\,\bar{a}} \,=\, \chi_z^{\bar{a}} \,, \qquad \big(\chi_{\bar{a}}^{(1)}\big)_{\overline{z}} \,=\, \big(\chi_{\bar{a}}^+\big)_{\overline{z}} \,, \\
&q_{\overline{z}}^{(1)\,a} \,=\, b_{\overline{z}}^a \,, \qquad \big(p_a^{(1)}\big)_z \,=\, -\big(b_i^+\big)_z \,, \qquad q_z^{(1)\,\bar{a}} \,=\, b_z^{\bar{a}} \,, \qquad \big(p_{\bar{a}}^{(1)}\big)_{\overline{z}} \,=\, \big(b_{\bar{a}}^+\big)_{\overline{z}} \,,
\end{aligned} \tag{6.57}$$

and choose the Lagrangian submanifold as gauge fixing defined by

$$\chi^{(0)\,i} \,=\, 0 \,, \qquad p_i^{(0)} \,=\, 0 \,, \qquad \big(\chi_a^{(1)}\big)_{\overline{z}} \,=\, \big(\chi_{\bar{a}}^{(1)}\big)_z \,=\, 0 \,, \qquad \big(p_a^{(1)}\big)_{\overline{z}} \,=\, \big(p_{\bar{a}}^{(1)}\big)_z \,=\, 0 \,. \tag{6.58}$$

This yields the same BV action $-S_{\rm A}$ from (6.52), which gives the A-model action with gauge fixing fermion $-\Psi_{\rm A}$ from (6.49). Note that in neither of these AKSZ constructions does the target dg-manifold coincide with that of the Poisson sigma-model from §2.1 in Chapter 4 associated to string fields $X : \Sigma_2 \to M_6$.

2.3 Dimensional reduction from the standard Courant sigma-model

The first AKSZ construction of the A-model can be embedded into the standard Courant sigma-model, which is a membrane theory, in a similar way as we embedded our AKSZ membrane sigma-models into the standard 2-Courant sigma-model, which

is a threebrane theory, in §1.4. For this, let us consider the standard Courant sigma-model from §3.1 in Chapter 4 on a product worldvolume $\Sigma_3 = \Sigma_2 \times S^1$, and assume that our superfields do not depend on the extra coordinate of S^1. The AKSZ action of the standard standard Courant sigma-model is given by (4.91) and the BV symplectic form by (4.90). After integration over the extra supercoordinates on $T[1]S^1$ and a relabelling of superfields, we arrive at the AKSZ action

$$\boldsymbol{\mathcal{S}}^{(2)}_{0,\mathrm{red}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\, \big(\boldsymbol{\chi}_i\, \boldsymbol{q}^i + \boldsymbol{b}^i\, \boldsymbol{h}_i - \boldsymbol{\chi}_i\, \boldsymbol{D}\boldsymbol{X}^i - \boldsymbol{h}_i\, \boldsymbol{D}\boldsymbol{\eta}^i - \boldsymbol{n}_i\, \boldsymbol{D}\boldsymbol{b}^i - \boldsymbol{p}_i\, \boldsymbol{D}\boldsymbol{q}^i\big)\ , \tag{6.59}$$

and the BV symplectic form

$$\boldsymbol{\omega}_{2,\mathrm{red}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\, \big(\boldsymbol{\delta}\boldsymbol{\chi}_i\, \boldsymbol{\delta}\boldsymbol{X}^i + \boldsymbol{\delta}\boldsymbol{p}_i\, \boldsymbol{\delta}\boldsymbol{q}^i + \boldsymbol{\delta}\boldsymbol{h}_i\, \boldsymbol{\delta}\boldsymbol{\eta}^i + \boldsymbol{\delta}\boldsymbol{n}_i\, \boldsymbol{\delta}\boldsymbol{b}^i\big)\ . \tag{6.60}$$

This symplectic form is the same as that of the A-model in (6.42), and the AKSZ action reduces to the A-model action (6.44) if we set the kinetic terms to zero by definition or via gauge fixing.

3 AKSZ theory for supersymmetric quantum mechanics

In this section we continue the dimensional reduction procedure one final time, and reduce our second AKSZ construction of the A-model to an AKSZ formulation for supersymmetric quantum mechanics. We have seen in §1.2 and §2.2 that both the topological A-model and the topological membrane sigma-models on G_2-manifolds have similar AKSZ constructions. Following the same procedure as before we give an analogous AKSZ construction for supersymmetric quantum mechanics.

3.1 Dimensional reduction of the A-model

We start with the canonically transformed action from (6.41) restricted to the fields of the second AKSZ construction:

$$\boldsymbol{\mathcal{S}}^{(2)\,\mathrm{eff}}_{G_2,\mathrm{II}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\, \Big(\boldsymbol{\chi}_i\, \boldsymbol{q}^i + \varepsilon\, \big(\boldsymbol{\chi}_i\, \boldsymbol{D}\boldsymbol{X}^i + \boldsymbol{q}^i\, \boldsymbol{D}\boldsymbol{p}_i\big)\Big)\ , \tag{6.61}$$

and the corresponding symplectic structure from (6.42):

$$\boldsymbol{\omega}_{2,\mathrm{II}} = \int_{T[1]\Sigma_2} \mathrm{d}^2\hat{z}\, \big(\boldsymbol{\delta}\boldsymbol{\chi}_i\, \boldsymbol{\delta}\boldsymbol{X}^i + \boldsymbol{\delta}\boldsymbol{p}_i\, \boldsymbol{\delta}\boldsymbol{q}^i\big)\ . \tag{6.62}$$

We apply dimensional reduction method as before and use the same notation as in (4.150) to calculate the reduction on a product source space $\Sigma_2 = S^1 \times \Sigma_1$, where we distinguish the circle $\Sigma_1 = S^1$ along which the dimensional reduction takes place. We choose $\widehat{\boldsymbol{\chi}}_i$, $\widehat{\boldsymbol{q}}^{\,i}$, $\boldsymbol{X}^i_t$ and $(\boldsymbol{p}_t)_i$ to be the ultraviolet fields, and we set the gauge $\boldsymbol{X}^i_t = 0$ and $\widehat{\boldsymbol{q}}^{\,i} = 0$. After integrating out the ultraviolet fields, we obtain the effective action

$$\boldsymbol{\mathcal{S}}^{(1)\,\mathrm{eff}}_{\mathrm{A,II}} = \int_{T[1]S^1} \mathrm{d}\hat{z}\, \Big(\boldsymbol{B}_i\,\boldsymbol{\xi}^i + \varepsilon\,\big(-\boldsymbol{B}_i\,\boldsymbol{D}\boldsymbol{X}^i - \boldsymbol{\xi}^i\,\boldsymbol{D}\,\boldsymbol{\eta}_i\big)\Big)\,, \tag{6.63}$$

and the symplectic structure

$$\boldsymbol{\omega}_1 = \int_{T[1]S^1} \mathrm{d}\hat{z}\, \big(\boldsymbol{\delta}\boldsymbol{B}_i\,\boldsymbol{\delta}\boldsymbol{X}^i + \boldsymbol{\delta}\boldsymbol{\eta}_i\,\boldsymbol{\delta}\boldsymbol{\xi}^i\big)\,, \tag{6.64}$$

where we relabeled the fields as

$$\boldsymbol{B}_i = -\int_{\Sigma_1} \mathrm{d}t\,(\boldsymbol{\chi}_t)_i\,, \qquad \boldsymbol{X}_i = \widehat{\boldsymbol{X}}^i\,, \qquad \boldsymbol{\eta}_i = -\int_{\Sigma_1} \mathrm{d}t\,(\boldsymbol{p}_t)_i\,, \qquad \boldsymbol{\xi}^i = -\,\widehat{\boldsymbol{q}}^{\,i}\,, \tag{6.65}$$

and these new fields are independent of the coordinate t of Σ_1.

The infinitesimal canonical transformation (6.33) with the fermionic functional

$$\boldsymbol{\alpha} = \int_{T[1]S^1} \mathrm{d}\hat{z}\,\boldsymbol{\eta}_i\,\boldsymbol{D}\boldsymbol{X}^i \tag{6.66}$$

gives the action

$$\boldsymbol{\mathcal{S}}^{(1)}_{\mathrm{SQM}} = \int_{T[1]S^1} \mathrm{d}\hat{z}\,\boldsymbol{B}_i\,\boldsymbol{\xi}^i\,. \tag{6.67}$$

We will see in §3.2 below that this action gives an AKSZ formulation of supersymmetric quantum mechanics (see Appendix 9). Nothing we discuss in this section depends on the target space Kähler structure nor even on its dimensionality, and the reduction of the topological sigma-model described here applies to generic maps whose target is any Riemannian manifold.

3.2 AKSZ construction

Following the procedure in §1.2 and §2.2, we give an AKSZ formulation of supersymmetric quantum mechanics which reduces to the action (A.135) after gauge fixing and eliminating the auxiliary field b_i. Our source dg-manifold is $\mathcal{W} = T[1]S^1$ and the target symplectic dg-manifold is $\mathcal{M} = T^*(T[1]M)$, where M is a Riemannian

manifold with metric g. Denote the degree 0 and 1 coordinates of $T[1]M$ by X^i and ξ^i, respectively, and their cotangent coordinates by B_i and η_i with degree 0 and -1, respectively. The canonical symplectic structure on $\mathcal{M} = T^*(T[1]M)$ is

$$\omega_1 \,=\, \mathrm{d}B_i \wedge \mathrm{d}X^i \,+\, \mathrm{d}\eta_i \wedge \mathrm{d}\xi^i \ , \tag{6.68}$$

which gives the same symplectic form on the mapping space of superfields $\boldsymbol{\mathcal{M}}$ as in (6.64). The AKSZ superfields are expanded as

$$\begin{aligned} \boldsymbol{X}^i \,&=\, x^i \,-\, b^{+\,i}\,\theta \ , \\ \boldsymbol{B}_i \,&=\, -\,b_i \,+\, x_i^+\,\theta \ , \\ \boldsymbol{\xi}^i \,&=\, -\,\psi^i \,+\, \bar{\psi}^{+\,i}\,\theta \ , \\ \boldsymbol{\eta}_i \,&=\, \bar{\psi}_i \,-\, \psi_i^+\,\theta \ , \end{aligned} \tag{6.69}$$

where the superworldline coordinate θ has degree 1. Our choice for the AKSZ action is the same as that in (6.67) which was obtained from the dimensional reduction of the A-model:

$$\boldsymbol{\mathcal{S}}^{(1)}_{\mathrm{SQM}} \,=\, \int_{T[1]S^1} \mathrm{d}\hat{z}\ \boldsymbol{B}_i\,\boldsymbol{\xi}^i \,=\, -\int_{S^1} \mathrm{d}\tau\ \big(\psi^i\,x_i^+ \,+\, b_i\,\bar{\psi}^{+\,i}\big) \ , \tag{6.70}$$

and it trivially solves the classical master equation $(\boldsymbol{\mathcal{S}}^{(1)}_{\mathrm{SQM}}, \boldsymbol{\mathcal{S}}^{(1)}_{\mathrm{SQM}})_{\mathrm{BV}} = 0$. The BV–BRST transformations[6] are generated by the cohomological vector field given by the BV bracket $\boldsymbol{Q}_{\mathrm{SQM}} = (\boldsymbol{\mathcal{S}}^{(1)}_{\mathrm{SQM}}, \cdot\,)_{\mathrm{BV}}$ and read as

$$\begin{aligned} \boldsymbol{Q}_{\mathrm{SQM}} x^i \,&=\, \psi^i \qquad &&\text{and} \qquad &&\boldsymbol{Q}_{\mathrm{SQM}} \psi_i^+ \,=\, x_i^+ \ , \\ \boldsymbol{Q}_{\mathrm{SQM}} \psi^i \,&=\, 0 \qquad &&\text{and} \qquad &&\boldsymbol{Q}_{\mathrm{SQM}} x_i^+ \,=\, 0 \ , \\ \boldsymbol{Q}_{\mathrm{SQM}} \bar{\psi}_i \,&=\, b_i \qquad &&\text{and} \qquad &&\boldsymbol{Q}_{\mathrm{SQM}} b^{+\,i} \,=\, -\,\bar{\psi}^{+\,i} \ , \\ \boldsymbol{Q}_{\mathrm{SQM}} b_i \,&=\, 0 \qquad &&\text{and} \qquad &&\boldsymbol{Q}_{\mathrm{SQM}} \bar{\psi}^{+\,i} \,=\, 0 \ . \end{aligned} \tag{6.71}$$

The nilpotent fermionic symmetry $\boldsymbol{Q}_{\mathrm{SQM}}$ acts trivially on the AKSZ action $\boldsymbol{S}_{\mathrm{SQM}}$.

We reduce the action $\boldsymbol{\mathcal{S}}^{(1)}_{\mathrm{SQM}}$ to I_{SQM} after gauge fixing. We choose the same gauge fixing fermion $-\Psi_{\mathrm{SQM}}$ as in (A.141). The pertinent antifields are given by

$$\begin{aligned} x_i^+ \,&=\, -\,\frac{\delta\Psi_{\mathrm{SQM}}}{\delta x^i} \,=\, -\,\mathrm{i}\,\dot{\bar{\psi}}_i \,+\, \frac{1}{2}\,\partial_i\big(\Gamma^j{}_{ml}\,g^{mk}\big)\,\bar{\psi}_j\,\bar{\psi}_k\,\psi^l \,-\, \frac{1}{2}\,\partial_i g^{jk}\,\bar{\psi}_j\,b_k \ , \\ \bar{\psi}^{+\,i} \,&=\, -\,\frac{\overrightarrow{\delta}\Psi_{\mathrm{SQM}}}{\delta\bar{\psi}_i} \,=\, \mathrm{i}\,\dot{x}^i \,+\, \Gamma^{[i}{}_{lk}\,g^{j]l}\,\bar{\psi}_j\,\psi^k \,-\, \frac{1}{2}\,g^{ij}\,b_j \ , \end{aligned} \tag{6.72}$$

[6]The action $\boldsymbol{\mathcal{S}}^{(1)}_{\mathrm{SQM}}$ is also invariant under the transformations $\delta x_i^+ = b_i$, $\delta\bar{\psi}^{+\,i} = \psi^i$, $\delta\psi^i = 0$ and $\delta b_i = 0$, and under the transformations $\delta x_i^+ = 0$, $\delta\bar{\psi}^{+\,i} = 0$, $\delta\psi^i = \bar{\psi}^{+\,i}$ and $\delta b_i = -x_i^+$, but our transformations do not include these.

where the other gauge fixing equations are not important here. Calculating the gauge fixed action of $\boldsymbol{\mathcal{S}}^{(1)}_{\mathrm{SQM}}$ we get the action (A.142), which is classically equivalent to I_{SQM}.

Chapter 7

Conclusions

Summary

In the thesis we presented our results related to topological aspects of string theory. In Chapter 3 we derived an analog of the large N Gross-Taylor holomorphic string expansion for (q, t)-deformed $U(N)$ Yang-Mills theory on a compact oriented Riemann surface, which arises in the study of BPS black holes and refined topological strings. In the classical limit $q = 1$, the expansion defines a new β-deformation for Hurwitz theory of branched covers wherein the refined partition function is a generating function for certain parameterized Euler characters, which reduce in the unrefined limit $\beta = 1$ to the orbifold Euler characteristics of Hurwitz spaces of holomorphic maps. We also applied the large N expansion to observables corresponding to open surfaces and Wilson loops.

In Chapter 5 we studied AKSZ formulations of the topological A- and B-models within the framework of double field theory. We introduced a two-dimensional sigma-model on doubled space, which describes both the A- and B-models simultaneously. We uplifted it to the membrane level as a three-dimensional Courant sigma-model, which can accommodate both geometric and non-geometric fluxes. We applied the projection to DFT vectors on the Courant sigma-model of double field theory, which led to the introduction of the Courant sigma-model of a particular class of generalized complex structures, and also its corresponding Courant algebroid. We also studied two marginal cases, the purely Poisson and purely complex structure cases, which were reduced to the A- and B-models on their boundaries respectively. We also proposed an S-duality at the level of the Courant sigma-model based on the generalized

complex structure, which was interpreted as topological S-duality of the A- and B-models.

Finally in Chapter 6 we have constructed BV quantized topological membrane theories on G_2-manifolds using the AKSZ formulation, which unify the topological membrane theories of [107] and [110]. We have dimensionally reduced them to the A-model, and one of them has been reduced further to supersymmetric quantum mechanics. We have further proposed a topological three-brane model given by the AKSZ construction, which reduces to our AKSZ membrane theories upon worldvolume dimensional reduction. Its derived bracket is the standard 2-Courant bracket, which appears in exceptional generalized geometry as the antisymmetrization of the generalized Lie derivative, and it is also the induced bracket of anomaly-free current algebras of topological membranes on G_2-manifolds [96]. We have found that double dimensional reduction on a circle of our threebrane model with G-flux twisting yields the twisted standard Courant sigma-model, which geometrizes the H-flux in type II string theory.

Outlook

One would expect that coefficients in the large N expansion of (q, t)-deformed Yang-Mills theory have relation to quantities in refined topological string. This connection is not even fully investigated in the expansion of the original q-deformed version, so it can be a further direction to study their. So far we worked only on the chiral part of the partition function in the topological limit, thus another direction can be to go one step further and derive the full expansion of the partition function. Besides the open surface and Wilson loop observables there are other observables in the theory: defect holonomy punctures on the surface. This would be another avenue for further study. Our new deformation of Hurwitz theory has not found a full geometrical meaning, therefore it would be interesting to interpret it better geometrically.

It would be interesting to study further the appearance of generalized geometry and double field theory in the context of the A- and B-models as they are defined originally in standard (not generalized) Calabi-Yau manifolds. The double field theory formulation of the A- and B-models also allows for the introduction of both geometric and non-geometric fluxes, which would be a further open direction to investigate its physical relevance, particularly in the context of topological string theory. The fluxes

correspond to twist deformations of the proposed Courant algebroids which lead to the introduction of twists of the generalized and original complex structures, which is another avenue for further investigation. Another direction would be to find a Courant algebroid which gives the identities of a general version of generalized complex structure, and to relate it to the double field theory formulations of the A- and B-models. Our S-duality gives a continuous mapping between the A- and B-models, so it would be interesting to investigate whether the intermediate membrane theory has a clearer physical relevance. A surprising observation is that our S-duality arises from the T-duality inspired generalized complex geometry, thus it raises the question as to whether there is a physical origin behind this relation or whether it is just a coincidence found in the topological field theories.

Our constructions related to topological membranes on G_2-manifolds are the starting point for the introduction of exceptional generalized geometry [99, 100, 114] and M-theory fluxes [115,116] for membranes in M-theory described by the AKSZ formalism. The first step towards this goal is our AKSZ threebrane sigma-model with its derived standard 2-Courant bracket. However, implementing non-geometric M-theory fluxes into this setting seems somewhat perplexing. In the string theory setting, T-duality in AKSZ membrane theory acts as a duality between standard and contravariant Courant sigma-models, and also transforms geometric H-flux and non-geometric R-flux into each other [84]. It is tempting to try lifting this T-duality to a duality at the level of AKSZ threebrane theory, which transforms our threebrane into another topological threebrane with non-geometric flux. In the case of the Courant sigma-models, the duality interchanges the degree 1 coordinates ψ^i and χ_i, and it is implemented as a canonical transformation given by a bivector and its T-dual two-form B-field. For the 2-Courant sigma-models, it is natural to expect that there similarly exist canonical transformations which implement the interchange between the degree 2 quantities $\psi^i\,\psi^j$ and χ_i. In this case a trivector and a three-form would arise, which should be related to the trivector and three-form C-field in exceptional generalized geometry. But unfortunately this does not seem to be the case as there are no symplectomorphisms which interchange $\psi^i\,\psi^j$ and χ_i. Thus implementing this duality and non-geometric fluxes seems to be far more complicated than in the string theory case.

In [117] a closed string on a G_2-manifold has been proposed as the dual of a topological G_2 membrane, and its quantization at one-loop order is considered in [118], which may be relevant to the quantization of our membrane construction that is of interest when considering its connection to physical string theory (see also [107]). Likewise

an open G_2 string theory is introduced in [119], wherein the worldvolume theory of associative three-cycles has a membrane formulation given by a gauge fixed Chern-Simons theory coupled to normal deformations of the cycle. A further development would be to give an AKSZ construction for this three-cycle theory, and to compare it with our AKSZ topological membrane theories. It would also be interesting to study the topological membrane of [120] in the context of the AKSZ construction. We also studied the derived bracket of one of our AKSZ topological membrane theories whose target is a derived symplectic dg-manifold with fields of negative degree, which gave an L_∞-extension of the standard Courant bracket. It would be interesting to study further the consequences of this more complex derived algebroid structure. Finally, in the present paper we also derived AKSZ constructions for the A-model, hence one of the applications of our results is to study the possible dualities between the A-model and the B-model at the level of the AKSZ formalism, and in particular to find a realization of S-duality [45] in AKSZ theory. In this respect it would be interesting to study further the threebrane theory of calibrated four-cycles on eight-dimensional $Spin(7)$-manifolds that we discussed in §1.4 in Chapter 6, which may be relevant to the study of S-duality as in [113].

Appendix A

1 Hurwitz theory of branched covers

Hurwitz theory is the theory of branched covers (see [34, 121, 122]). An n-sheeted branched covering of Riemann surface Σ_h with an other Riemann surface Σ_g is a continuous map $f : \Sigma_g \to \Sigma_h$, which covers Σ_h n times in the sense that in the neighborhood of every point $Q \in \Sigma_g$ the map f looks like $z \mapsto z^{n(Q)}$ locally, where $n(Q)$ is called the ramification index of Q. Illustratively speaking $n(Q)$ is the number of covering sheets meet in the ramification point Q. The image of the point Q is the branch point P, and the sum of all ramification indices corresponding to a given branch point is called the degree of f

$$\deg(f) \;=\; \sum_{Q \in f^{-1}(P)} n(Q) \;=\; n \;, \tag{A.1}$$

which is simply n in our case, therefore fixed for all branch points. The sum

$$B_P \;=\; \sum_{Q \in f^{-1}(P)} (n(Q) - 1) \tag{A.2}$$

is called the branching number at P. The branching number of f is the sum of the branching numbers

$$B \;=\; \sum_{P \in S(f)} B_P \;, \tag{A.3}$$

where $S(f)$ is the set of all branch points and it is called the branch locus. The genus g of the covering space Σ_g is determined by the Riemann-Hurwitz formula

$$2g - 2 \;=\; n(2h - 2) + B \;. \tag{A.4}$$

Two branch covers are equivalent if there exists a homeomorphism $\phi : \Sigma_g \to \Sigma_g$ such that $f_1 \circ \phi = f_2$. An automorphism of a branch cover f is a homeomorphism of Σ_g which leaves the image invariant.

Lifting a closed curve in Σ_h acts as a permutation on the sheets of Σ_g, so it seems natural to reformulate the counting of equivalence classes of branched covers in the language of symmetric groups.

We start with the homotopy group $\pi_1(\Sigma_h \setminus \{P_1, \ldots, P_L\}, y_0)$ of a Riemann surface Σ_h with genus h and L punctures $\{P_i\}$ at a fixed point y_0. It is isomorphic to the free group modulo constraint

$$\pi_1(\Sigma_h \setminus \{P_1, \ldots, P_L\}, y_0) \cong \left\langle \{\alpha_i, \beta_i\}_{i=1}^h \ , \ \{\gamma\}_{s=1}^L \middle| \prod_{s=1}^L \gamma_s \prod_{i=1}^h \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1 \right\rangle . \tag{A.5}$$

The number of distinct homomorphism (i.e. not connected through inner automorphism)

$$\psi : \pi_1(\Sigma_h \setminus \{P_1, \ldots, P_L\}, y_0) \to \mathfrak{S}_n \tag{A.6}$$

is the same as the number of equivalence classes of n-fold branched coverings of Σ_h with $P_1, \ldots, P_L$ branch points:

$$\frac{n!}{|C(\psi)|} = \frac{n!}{|\mathrm{Aut}(f)|} \ , \tag{A.7}$$

where $C(\psi)$ is the centralizer of ψ and also the subgroup of $\mathfrak{S}_n$, which acts on ψ as an inner automorphism and leaves ψ invariant. The factor $n!$ is coming from the all possible permutations of n sheets, and $|\mathrm{Aut}(f)|$ is the order of the automorphism group of f, which is isomorphic to $C(\psi)$. So the number of distinct homomorphism ψ, which acts as

$$\psi : \alpha_i \mapsto \sigma_i \ , \qquad \psi : \beta_i \mapsto \tau_i \ , \qquad \psi : \gamma_i \mapsto \zeta_i \tag{A.8}$$

on the generators of $\pi_1(\Sigma_h \setminus \{P_1, \ldots, P_L\}, y_0)$, can be counted by the delta function on $\mathfrak{S}_n$

$$\sum_{\sigma_1, \tau_1, \ldots, \sigma_h, \tau_h, \zeta_1, \ldots, \zeta_L \in \mathfrak{S}_n} \delta \left(\prod_{s=1}^L \zeta_s \prod_{i=1}^h \sigma_i \tau_i \sigma_i^{-1} \tau_i^{-1} \right) = \sum_{f \in \mathcal{H}_{n,B,h,L}} \frac{n!}{|\mathrm{Aut}(f)|} \ , \tag{A.9}$$

where we have introduced the Hurwitz space $\mathcal{H}_{n,B,h,L}$, which is the space of n-sheeted branched covers of a Riemann surface with genus h, branching number B and L branch points.

One can express the left hand side of (A.9) in terms of central elements $c_\mu = \sum_{\sigma \in T_\mu} \sigma$ in the symmetric group algebra, where T_μ is the conjugacy class corresponding to the

partition μ. Then we define the *Hurwitz number*

$$H_{h,n}\big(\mu^1,\dots,\mu^L\big) \;=\; \frac{1}{n!}\sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n}\delta\Big(c_{\mu^1}\cdots c_{\mu^L}\,\prod_{i=1}^h\sigma_i\,\tau_i\,\sigma_i^{-1}\,\tau_i^{-1}\Big)\;, \tag{A.10}$$

which depends on the ramification profiles of the L branch points $P_1,\dots,P_L$. They are specified by partitions $\mu^1,\dots,\mu^L\in\Lambda_+^n$ such that $\mu^l=\big(\mu^l_1,\dots,\mu^l_{\ell(\mu^l)}\big)$ are the ramification indices of the preimages $f^{-1}(P_l)$. The Hurwitz number has an explicit combinatorial expression given by the Frobenius-Schur formula [123, Appendix A]

$$H_{h,n}\big(\mu^1,\dots,\mu^L\big)=(n!)^{2h}\sum_{\lambda\in\Lambda_+^n}\frac{1}{(d_\lambda)^{L+2h-2}}\,\prod_{l=1}^L\frac{\chi_{r_\lambda}\big(m_{T_{\mu^l}}\big)}{z_{\mu^l}}\;. \tag{A.11}$$

Note that $H_{h,n}(\mu^1,\dots,\mu^L)$ is independent of the branch point positions $P_1,\dots,P_L\in\Sigma_h$ and also of the choice of (fixed) complex structure on Σ_h.

2 Quantum group $\mathcal{U}_q(\mathfrak{gl}_N)$

For a generic value of q, let $\mathcal{U}_q(\mathfrak{gl}_N)$ be the associative algebra over $\mathbb{C}$ with generators E_i,F_i for $i=1,\dots,N-1$ and $q^{\pm H_i/2}$ for $i=1,\dots,N$ obeying the relations

$$\begin{aligned}
q^{H_i/2}\,E_i\,q^{-H_i/2} &= q^{1/2}\,E_i\;,\\
q^{H_i/2}\,E_{i-1}\,q^{-H_i/2} &= q^{-1/2}\,E_{i-1}\;,\\
q^{H_i/2}\,F_i\,q^{-H_i/2} &= q^{-1/2}\,F_i\;,\\
q^{H_i/2}\,F_{i-1}\,q^{-H_i/2} &= q^{1/2}\,F_{i-1}\;,\\
\big[q^{H_i/2},E_j\big] \;=\; \big[q^{H_i/2},F_j\big] &= 0 \qquad \text{for}\quad j\neq i,i-1\;,\\
[E_i,F_j] &= \delta_{ij}\,\frac{q^{(H_i-H_{i+1})/2}-q^{-(H_i-H_{i+1})/2}}{q^{1/2}-q^{-1/2}}\;,\\
[E_i,E_j] \;=\; [F_i,F_j] &= 0 \qquad \text{for}\quad |i-j|>1\;,\\
E_i^2\,E_j-\big(q^{1/2}+q^{-1/2}\big)\,E_i\,E_j\,E_i+E_j\,E_i^2 &= 0 \qquad \text{for}\quad |i-j|=1\;,\\
F_i^2\,F_j-\big(q^{1/2}+q^{-1/2}\big)\,F_i\,F_j\,F_i+F_j\,F_i^2 &= 0 \qquad \text{for}\quad |i-j|=1\;.
\end{aligned} \tag{A.12}$$

In the fundamental representation (3.16), we have $H_i = E_{ii}$, $E_i = E_{i\,i+1}$ and $F_i = E_{i+1\,i}$. The coproduct on $\mathcal{U}_q(\mathfrak{gl}_N)$ is defined by

$$\begin{aligned} \Delta(E_i) &= E_i \otimes q^{-(H_i - H_{i+1})/2} + q^{(H_i - H_{i+1})/2} \otimes E_i \ , \\ \Delta(F_i) &= F_i \otimes q^{-(H_i - H_{i+1})/2} + q^{(H_i - H_{i+1})/2} \otimes F_i \ , \\ \Delta\big(q^{H_i/2}\big) &= q^{H_i/2} \otimes q^{H_i/2} \ . \end{aligned} \tag{A.13}$$

3 Hecke algebra of type A_{n-1}

The symmetric group $\mathfrak{S}_n$ of degree $n \geq 2$ is generated by the elementary transpositions $\sigma_i = (i\ i+1)$ for $i = 1, \ldots, n-1$ satisfying the relations

$$\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1} \ , \qquad \sigma_i\,\sigma_j = \sigma_j\,\sigma_i \quad \text{for} \quad |i-j| > 1 \qquad \text{and} \qquad \sigma_i^2 = 1 \ . \tag{A.14}$$

The *length* $\ell(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_n$ is the smallest integer r such that there exists $i_1, \ldots, i_r$ with $\sigma = \sigma_{i_1} \cdots \sigma_{i_r}$; such an expression is called a decomposition of σ into a *reduced word*. Note that decompositions into reduced words are not unique.

The *Hecke algebra* $\mathsf{H}_q(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ for $n \geq 2$ is the algebra over $\mathsf{H}_q(\mathfrak{S}_0) = \mathsf{H}_q(\mathfrak{S}_1) := \mathbb{C}[q, q^{-1}]$ generated by g_i for $i = 1, \ldots, n-1$ with the relations

$$\mathsf{g}_i\,\mathsf{g}_{i+1}\,\mathsf{g}_i = \mathsf{g}_{i+1}\,\mathsf{g}_i\,\mathsf{g}_{i+1} \ , \qquad \mathsf{g}_i\,\mathsf{g}_j = \mathsf{g}_j\,\mathsf{g}_i \quad \text{for} \quad |i-j| > 1 \qquad \text{and} \qquad (\mathsf{g}_i - q)\,(\mathsf{g}_i + 1) = 0 \ . \tag{A.15}$$

The inverse of the generator g_i is

$$\mathsf{g}_i^{-1} = q^{-1}\,\mathsf{g}_i + \big(q^{-1} - 1\big) \ . \tag{A.16}$$

If $\sigma = \sigma_{i_1} \cdots \sigma_{i_r}$ is a decomposition of $\sigma \in \mathfrak{S}_n$ in the form of a reduced word, then we set $\mathsf{h}(\sigma) = \mathsf{g}_{i_1} \cdots \mathsf{g}_{i_r} \in \mathsf{H}_q(\mathfrak{S}_n)$. One can show that $\mathsf{h}(\sigma)$ is independent of the decomposition of σ into a reduced word and that $\big\{\mathsf{h}(\sigma)\big\}_{\sigma \in \mathfrak{S}_n}$ is a $\mathbb{C}[q, q^{-1}]$-basis of the free $\mathbb{C}[q, q^{-1}]$-module $\mathsf{H}_q(\mathfrak{S}_n)$. Given $\sigma, \tau \in \mathfrak{S}_n$ with $\ell(\sigma\,\tau) = \ell(\sigma) + \ell(\tau)$, one has $\mathsf{h}(\sigma)\,\mathsf{h}(\tau) = \mathsf{h}(\sigma\,\tau)$. The algebra $\mathsf{H}_q(\mathfrak{S}_n)$ is a q-deformation of the group algebra $\mathbb{C}[\mathfrak{S}_n]$; in the classical limit $q = 1$ the element $\mathsf{h}(\sigma)$ becomes the permutation σ. The combinatorial identity

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\ell(\sigma)} = q^{\frac{n\,(n-1)}{4}}\,[n]_q! \tag{A.17}$$

expresses a q-deformation of the order of $\mathfrak{S}_n$, where we defined the q-factorial $[n]_q! := [1]_q \cdots [n]_q$.

The irreducible representations r_λ of the symmetric group $\mathfrak{S}_n$ are in one-to-one correspondence with partitions λ of n. In particular, the sign representation $\det = \bigwedge^n R_{\omega_1}^{\otimes n}$ corresponds to the trivial partition $\lambda = (n)$ while the trivial representation corresponds to the maximal partition $\lambda = (1^n)$ with n parts. The splitting (2.58) then gives the decomposition of $R_{\omega_1}^{\otimes n}$ into subrepresentations corresponding to its λ-isotopical components. The q-deformation of the dimension of r_λ is given by

$$d_\lambda(q) = \frac{\prod\limits_{i<j} \big(q^{\ell_i} - q^{\ell_j}\big)}{\prod\limits_{i=1}^{\ell(\lambda)} \big(q-1\big)\big(q^2-1\big)\cdots\big(q^{\ell_i}-1\big)} \, \frac{\big(q-1\big)\big(q^2-1\big)\cdots\big(q^n-1\big)}{q^{\frac{\ell(\lambda)\,(\ell(\lambda)-1)\,(\ell(\lambda)-2)}{6}}} \ , \tag{A.18}$$

where $\ell(\lambda)$ is the length of the partition λ (the number of non-zero λ_i) and $\ell_i = \lambda_i + \ell(\lambda) - i$. In the classical limit $q \to 1$ this expression reduces to the usual dimension formula

$$d_\lambda(1) = d_\lambda := \chi_{r_\lambda}(1) = \frac{n!}{\prod\limits_{i=1}^{\ell(\lambda)} \ell_i!} \prod_{1\le i<j\le \ell(\lambda)} (\ell_i - \ell_j) \ . \tag{A.19}$$

4 Center of $\mathsf{H}_q(\mathfrak{S}_\infty)$

There is a natural embedding of Hecke algebras $\mathsf{H}_q(\mathfrak{S}_n) \hookrightarrow \mathsf{H}_q(\mathfrak{S}_{n+1})$, and so the inductive limit of $\mathsf{H}_q(\mathfrak{S}_n)$ as $n \to \infty$ exists [59]; we write this inductive limit as $\mathsf{H}_q(\mathfrak{S}_\infty)$. We want to find an inductive limit of central elements of the Hecke algebras as well. Firstly we need an embedding of central elements given by a monomorphism

$$\varphi_n \ : \ \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big) \ \hookrightarrow \ \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_{n+1})\big) \ , \tag{A.20}$$

where $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ is a linear subspace of the center $Z\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ of the algebra $\mathsf{H}_q(\mathfrak{S}_n)$ such that

$$\sum_i \mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n \, x \, C_i^{(n)} \, U\big) \ = \ \sum_i \mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n \, x \, C_i^{(n+1)} \, U\big) \ , \tag{A.21}$$

for $U \in T$ and $x \in \mathsf{H}_q(\mathfrak{S}_n)$, where $C_i^{(n)}$ span a linear basis of $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$.

According to [124, Theorem 2.14], the center $Z\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ is the algebra of symmetric polynomials in the *Murphy operators* L_i, $i = 1, \ldots, n$, which are defined as

$$\begin{aligned} L_1 \ &= \ \mathsf{h}(1) \ = \ 1 \ , \\ L_i \ &= \ q^{-(i-1)} \sum_{j=1}^{i-1} q^{j-1} \, \mathsf{h}\big((j\ i)\big) \qquad \text{for} \quad i > 1 \ , \end{aligned} \tag{A.22}$$

where $(i\ j) = \sigma_i \cdots \sigma_{j-2}\, \sigma_{j-1}\, \sigma_{j-2} \cdots \sigma_i$ for $j > i$ is the transposition which interchanges i and j. For example, using the definition of the central elements C_T from (3.21), for $n = 3$ we obtain $C_{(1,1,1)} = 1$, $C_{(2,1)} = q\,(L_2 + L_3)$ and $C_{(3)} = \frac{q^2}{2}\,(L_2\,L_3 + L_3\,L_2)$ as homogeneous symmetric polynomials in Murphy operators.

Given a symmetric polynomial $s(L_1, \ldots, L_n)$ in $L_i \in \mathsf{H}_q(\mathfrak{S}_n)$, we need to embed it into $\mathsf{H}_q(\mathfrak{S}_{n+1})$, but a symmetric polynomial in n variables is not necessarily a symmetric polynomial in $n+1$ variables so we need a non-trivial embedding. Because of (A.21) we require

$$\varphi_n\big(s(L_1, \ldots, L_n)\big) \;=\; s(L_1, \ldots, L_n) \;+\; p(L_1, \ldots, L_{n+1})\ , \tag{A.23}$$

where $p(L_1, \ldots, L_{n+1})$ is not necessarily a symmetric polynomial. If $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ is the space of homogeneous symmetric polynomials in Murphy operators, then φ_n is unique and so there exists just one p for every s in (A.23). For example one has

$$\begin{aligned}
\varphi_1(1) \;&=\; 1\ , \\
\varphi_2(L_2) \;&=\; L_2 + L_3\ , \\
\varphi_3(L_2\,L_3 + L_3\,L_2) \;&=\; L_2\,L_3 + L_3\,L_2 + L_2\,L_4 + L_4\,L_2 + L_3\,L_4 + L_4\,L_3\ .
\end{aligned} \tag{A.24}$$

In the representation $R_{\omega_1}^{\otimes n}$, the Murphy operator L_{n+1} is represented as 0 if $\mathsf{g}_{n+i} \cdot (R_{\omega_1} \otimes \cdots \otimes R_{\omega_1}) = 0$ for $i \geq 0$. Hence we get

$$\varphi_n\big(s(L_1, \ldots, L_n)\big) \cdot (R_{\omega_1} \otimes \cdots \otimes R_{\omega_1}) \;=\; s(L_1, \ldots, L_n) \cdot (R_{\omega_1} \otimes \cdots \otimes R_{\omega_1})\ . \tag{A.25}$$

All elements of $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_{n+1})\big)$ either belong to the image of φ_n, or else all of their monomials contain at least one factor L_{n+1} and so are represented as 0 in $R_{\omega_1}^{\otimes n}$.

Using this embedding, we can now take the inductive limit of $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ as $n \to \infty$, which is given by the equivalence classes

$$\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_\infty)\big) = \bigsqcup_{n=1}^{\infty} \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)\ /\ \sim \tag{A.26}$$

where $x \sim y$ if and only if $y = \varphi_{m-1} \circ \varphi_{m-2} \circ \cdots \circ \varphi_n(x)$ for $x \in \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$, $y \in \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_m)\big)$ and $m > n$; here the disjoint union over $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ is factorized with a sequence of the embeddings.

Now we can consider the transformation of a function $f(U)$ for $U \in T$ given by

$$f(C) \;=\; \sum_{n=0}^{\infty} \int_T [\mathrm{d}U]_{q,t}\ \mathrm{Tr}_{R_{\omega_1}^{\otimes n}}(\Phi_n\, y_n\, C\, U)\ f(U)\ , \tag{A.27}$$

for $y_n \in \mathsf{H}_q(\mathfrak{S}_n)$ and $C \in \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_\infty)\big)$, provided the series converges. Here we defined the integration measure

$$[\mathrm{d}U]_{q,t} := \frac{(-1)^N}{N!} \prod_{i=1}^N \mathrm{d}z_i \; \Delta_{q,t}(x)\, \Delta_{q,t}(x^{-1}) \tag{A.28}$$

where

$$\Delta_{q,t}(x) := t^{-\frac{N\,(N-1)}{2}} \prod_{i<j} \frac{\big(x_i\, x_j^{-1}; q\big)_\infty}{\big(t\, x_i\, x_j^{-1}; q\big)_\infty} = \prod_{m=0}^{\beta-1} \prod_{i<j} \big(q^{-m/2}\, \mathrm{e}^{\,(z_j-z_i)/2} - q^{m/2}\, \mathrm{e}^{\,(z_i-z_j)/2}\big) \tag{A.29}$$

for $\beta \in \mathbb{Z}_{>0}$, with $U = \mathrm{e}^{\,(z,H)}$ and $x = \mathrm{e}^{\,z}$. In the unrefined limit $\beta = 1$, the measure (A.28) reduces to the usual Haar measure

$$[\mathrm{d}U]_q = [\mathrm{d}U]_{q,q} = \frac{1}{N!} \prod_{i=1}^N \mathrm{d}z_i \; \Delta_q(x)^2 \tag{A.30}$$

for integration over the maximal torus $T \subset G$, where

$$\Delta_q(x) = \Delta_{q,q}(x) = \prod_{i<j} 2\sinh\Big(\frac{z_i - z_j}{2}\Big) \tag{A.31}$$

is the Weyl determinant for $G = U(N)$.

5 Proof of several statements

In this part we present several proofs of the statements used in our derivation of the chiral expansion of (q,t)-deformed Yang-Mills theory.

Lemma A.32 $\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\, U\, \mathsf{h}(\sigma)\big) = \sum_{\lambda\in\Lambda_+^n} \chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big)\; \chi_{\Phi_{\lambda_{\beta-2}}}(U)$.

<u>Proof</u>: Starting from the projector and centrality properties of P_λ in the Hecke algebra along with (3.4) we compute

$$\begin{aligned}
\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\, U\, \mathsf{h}(\sigma)\big) &= \sum_{\lambda,\mu\in\Lambda_+^n} \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big((P_\mu \otimes \mathbb{1}_{W_{\beta-1}})\,(P_\mu \otimes \mathbb{1}_{W_{\beta-1}})\,\Phi_n\, P_\lambda\, U\, \mathsf{h}(\sigma)\, P_\lambda\big) \\
&= \sum_{\lambda\in\Lambda_+^n} \mathrm{Tr}_{\,R_\lambda\otimes r_\lambda}\Big(\big(\Phi_{\lambda_{\beta-2}} \otimes \mathbb{1}_{r_\lambda}\big)\,\big(U \otimes \mathsf{h}(\sigma)\big)\Big) \qquad\qquad (A.33)\\
&= \sum_{\lambda\in\Lambda_+^n} \mathrm{Tr}_{\,R_\lambda}\big(\Phi_{\lambda_{\beta-2}}\, U\big)\, \mathrm{Tr}_{\,r_\lambda}\big(\mathsf{h}(\sigma)\big) \\
&= \sum_{\lambda\in\Lambda_+^n} \mathrm{Tr}_{\,R_\lambda}\big(\Phi_{\lambda_{\beta-2}}\, U\big)\, \chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big)
\end{aligned}$$

as required. ■

Lemma A.34 *The (q,t)-trace is cyclic:* $\mathrm{Tr}_{q,t}(x\,y) = \mathrm{Tr}_{q,t}(y\,x)$ *for all* $x, y \in \mathsf{H}_q(\mathfrak{S}_n)$.

<u>Proof</u>: Since the reduced words $\{\mathsf{h}(\sigma)\}_{\sigma\in\mathfrak{S}_n}$ form a basis of $\mathsf{H}_q(\mathfrak{S}_n)$, we can express $\mathrm{Tr}_{q,t}(x\,y)$ as a linear combination of (q,t)-traces $\mathrm{Tr}_{q,t}\big(\mathsf{h}(\sigma)\,\mathsf{h}(\tau)\big)$, and therefore we only need to show that $\mathrm{Tr}_{q,t}\big(\mathsf{h}(\sigma)\,\mathsf{h}(\tau)\big) = \mathrm{Tr}_{q,t}\big(\mathsf{h}(\tau)\,\mathsf{h}(\sigma)\big)$ for all $\sigma, \tau \in \mathfrak{S}_n$. We first prove that Φ_n and the Hecke algebra generators g_i commute. Let us consider a fixed element f_j of a basis $\{f_i\} \subset W_{\beta-1}$, with corresponding dual basis $\{f^i\}$. If we restrict the codomain of Φ_n to f_j, then ${}_{f_j}|\Phi_n \in \mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_N)}\big(R_{\omega_1}^{\otimes n}\big)$ and using quantum Schur-Weyl duality we can decompose this restriction as

$$ {}_{f_j}\big|\Phi_n = \bigoplus_{\lambda\in\Lambda_+^n} \big({}_{f_j}\big|\Phi_n\big)\big|_{R_\lambda} \otimes \mathbb{1}_{r_\lambda} \ , \tag{A.35} $$

where $({}_{f_j}|\Phi_n)|_{R_\lambda} \in \mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_N)}(R_\lambda)$. It follows that ${}_{f_j}|\Phi_n$ acts on the Hecke algebra representation r_λ as the identity, and so ${}_{f_j}|\Phi_n$ and g_i commute. Note that g_i commutes with $U^{\otimes n}$, because both P and $\mathsf{R} \in \mathcal{U}_q(\mathfrak{gl}_N) \otimes \mathcal{U}_q(\mathfrak{gl}_N)$ commute with $t^{(\rho,H)} \otimes t^{(\rho,H)}$. Thus we get

$$ \mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big({}_{f_j}\big|\Phi_n\,U\,x\,y\big) = \mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big({}_{f_j}\big|\Phi_n\,U\,y\,x\big) \tag{A.36} $$

and hence

$$ \begin{aligned} \mathrm{Tr}_{q,t}(x\,y) &= \mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,x\,y\big) \\ &= f^j\big(\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}({}_{f_j}|\Phi_n\,U\,x\,y)\big)\ f_j \\ &= \mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,y\,x\big) \ = \ \mathrm{Tr}_{q,t}(y\,x)\ . \end{aligned} \tag{A.37} $$

■

Lemma A.38 *The coefficients defined by* $\Phi_n(e_{i_1}\otimes\cdots\otimes e_{i_n}) = P_{i_1\cdots i_n}{}^{j_1\cdots j_n}{}_\alpha\ e_{j_1}\otimes\cdots\otimes e_{j_n}\otimes w^\alpha$ *are simply*

$$ P_{i_1\cdots i_n}{}^{j_1\cdots j_n} = (g_{\omega_1})^{-n/2}\ \delta_{i_1}{}^{j_1}\cdots\delta_{i_n}{}^{j_n}\ . $$

<u>Proof</u>: We start with the $n = 2$ case. By $\mathcal{U}_q(\mathfrak{gl}_N)$-equivariance we have the relations

$$ \Phi_2\,\Delta(H_p) = \Delta^2(H_p)\,\Phi_2 \qquad \text{for} \quad p = 1,\ldots,N \tag{A.39} $$

as operators on $R_{\omega_1}^{\otimes 2} \to R_{\omega_1}^{\otimes 2} \otimes W_{\beta-1}$, where $\Delta(H_p) = H_p \otimes \mathbb{1} + \mathbb{1} \otimes H_p$ and $\Delta^2(H_p) = (\Delta \otimes \mathbb{1})\Delta(H_p)$. We evaluate both sides of this equation on a generic basis element $e_i \otimes e_j$ of $R_{\omega_1} \otimes R_{\omega_1}$, and denote the action of the Cartan generators on the weight subspaces of $W_{\beta-1}$ as $H_p w^\alpha = \alpha_p\, w^\alpha$ with $\alpha_p \in \mathbb{C}$. Then the equality can be written as

$$\begin{aligned} \big(P_{pj}{}^{kl}{}_{\alpha}\, \delta_{pi} + P_{ip}{}^{kl}{}_{\alpha}\, \delta_{pj}\big)\; e_k \otimes e_l \otimes w^\alpha \;= \\ P_{ij}{}^{kl}{}_{\alpha}\, \big(\delta_{pk}\; e_p \otimes e_l + \delta_{pl}\; e_k \otimes e_p + \alpha_p\; e_k \otimes e_l\big) \otimes w^\alpha \;. \end{aligned} \tag{A.40}$$

In particular, for the weight $\alpha = 0$ component we obtain the constraints

$$P_{ij}{}^{kl}\; (\delta_{pk} + \delta_{pl} - \delta_{pi} - \delta_{pj}) = 0 \tag{A.41}$$

for all $i, j, k, l, p = 1, \ldots, N$. The tensor $P_{ij}{}^{kl} = \delta_i{}^k\, \delta_j{}^l$ solves this equation, and it is the unique solution up to normalization. The normalization is found as above by observing that the intertwiner Φ_2 acts in the (q,t)-trace as a multiple of the identity operator $\mathbb{1}_{R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}$, with proportionality constant $(g_{\omega_1})^{-1}$.

Next we have to generalize this expression to the (q,t)-trace of the connected minimal word $\mathsf{g}_1\, \mathsf{g}_2 \cdots \mathsf{g}_{n-1}$ for $n \geq 2$. Setting $P_{i_1 \cdots i_n}{}^{j_1 \cdots j_n} := P_{i_1 \cdots i_n}{}^{j_1 \cdots j_n}{}_0$, by a completely analogous argument to that used in the proof of Lemma A.38 one obtains the constraints

$$P_{i_1 \cdots i_n}{}^{j_1 \cdots j_n}\; (\delta_{p\, j_1} + \cdots + \delta_{p\, j_n} - \delta_{p\, i_1} - \cdots - \delta_{p\, i_n}) = 0 \tag{A.42}$$

for all $p, i_1, \ldots, i_n, j_1, \ldots, j_n = 1, \ldots, N$, and $P_{i_1 \cdots i_n}{}^{j_1 \cdots j_n} = \delta_{i_1}{}^{j_1} \cdots \delta_{i_n}{}^{j_n}$ solves this; thus again Φ_n acts in the (q,t)-trace as a multiple of the identity operator $\mathbb{1}_{R_{\omega_1}}^{\otimes n}$ with the normalization determined as before to be $(g_{\omega_1})^{-n/2}$, and we find

$$P_{i_1 \cdots i_n}{}^{j_1 \cdots j_n} = (g_{\omega_1})^{-n/2}\; \delta_{i_1}{}^{j_1} \cdots \delta_{i_n}{}^{j_n} \;. \tag{A.43}$$

■

Lemma A.44

$$\begin{aligned} &\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, t^{(\rho,H)}\, \mathsf{g}_1\, \mathsf{g}_2 \cdots \mathsf{g}_{n-1}\big) \;= \\ &\qquad (g_{\omega_1})^{-n/2}\, \frac{q^n}{q-1} \sum_{k=0}^{n-1} t^{(n-1-k)\,\frac{N+1}{2}}\, \frac{\big(q^{-1};t\big)_{k+1}}{(t;t)_k}\, \zeta_{n-1-k}(q,t)\; [N]_{t^{k+1}} \;, \end{aligned}$$

where

$$\begin{aligned}\zeta_0(q,t) &:= 1\ ,\\ \zeta_m(q,t) &:= \sum_{\lambda\in\Lambda_+^m}(-1)^{\ell(\lambda)}\,L_\lambda\,\prod_{i=1}^{\ell(\lambda)}\frac{\big(q^{-1};t\big)_{\lambda_i}}{(t;t)_{\lambda_i}} \qquad \textit{for}\quad m>0\ .\end{aligned}$$

and L_λ is a combinatorial weight defined by

$$L_\lambda=\frac{\ell(\lambda)!}{\big|\mathrm{Aut}(\lambda)\big|}\qquad \textit{with}\qquad \big|\mathrm{Aut}(\lambda)\big|=\prod_{i=1}^{|\lambda|}m_i(\lambda)!\ .$$

<u>Proof</u>: To simplify the derivation, let us introduce the short-hand notation

$$\xi_k:=t^{k\,\frac{N+1}{2}}\,\frac{q-1}{t^k-1}\qquad\text{and}\qquad \varphi_k:=\frac{t^k-q}{t^k-1}\ . \tag{A.45}$$

Using (3.28), we then compute the first partial trace from (3.33) as

$$\mathcal{O}_1\ :=\ \big(\mathrm{Tr}_{\,R_{\omega_1}}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-1)}\big)\big((t^{(\rho,H)}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-1)})\,\mathsf{g}_1\big)=\xi_1\,\mathbb{1}_{R_{\omega_1}}^{\otimes(n-1)}+\varphi_1\,t^{(\rho,H)}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-2)}\ . \tag{A.46}$$

The factors of $t^{(\rho,H)}$ can be cyclically permuted in the partial traces, so for $0\le m\le n-1$ the m-th partial trace

$$\begin{aligned}\mathcal{O}_m\ :=\ &\big(\mathrm{Tr}_{\,R_{\omega_1}}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-m)}\big)\big((t^{(\rho,H)}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-m)})\,\mathsf{g}_1\big)\\ &\cdots\ \big(\mathrm{Tr}_{\,R_{\omega_1}}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-2)}\big)\big((t^{(\rho,H)}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-2)})\,\mathsf{g}_1\big)\\ &\times\ \big(\mathrm{Tr}_{\,R_{\omega_1}}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-1)}\big)\big((t^{(\rho,H)}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-1)})\,\mathsf{g}_1\big)\end{aligned} \tag{A.47}$$

can be written as

$$\mathcal{O}_m=\sum_{k=0}^{m}f_k^{(m)}[\xi,\varphi]\ t^{k\,(\rho,H)}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-m-1)}\ , \tag{A.48}$$

where $f_k^{(m)}[\xi,\varphi]$ for $0\le k\le m$ are polynomials in ξ_l and φ_l with $l\le k$. We derive a recursion relation for $f_k^{(m)}$ inductively by writing

$$\mathcal{O}_{m+1}=\sum_{k=0}^{m}f_k^{(m)}\ \big(\mathrm{Tr}_{\,R_{\omega_1}}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-m-1)}\big)\big((t^{(k+1)\,(\rho,H)}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-m-1)})\,\mathsf{g}_1\big) \tag{A.49}$$

and using (3.28) to get

$$\mathcal{O}_{m+1}=\sum_{k=0}^{m}f_k^{(m)}\,\xi_{k+1}\,\mathbb{1}_{R_{\omega_1}}^{\otimes(n-m-1)}+\sum_{k=0}^{m}f_k^{(m)}\,\varphi_{k+1}\,t^{(k+1)\,(\rho,H)}\otimes\mathbb{1}_{R_{\omega_1}}^{\otimes(n-m-2)}\ . \tag{A.50}$$

Comparing this with the expansion (A.48) for $\mathcal{O}_{m+1}$ yields the recursion relations

$$f_0^{(m+1)} = \sum_{k=0}^{m} f_k^{(m)}\, \xi_{k+1} \ , \tag{A.51}$$

$$f_{k+1}^{(m+1)} = f_k^{(m)}\, \varphi_{k+1} \tag{A.52}$$

with initial condition $f_0^{(0)} = 1$.

For $k > 0$ we can use (A.52) to express $f_k^{(m)}$ entirely in terms of $f_0^{(m)}$ as

$$f_k^{(m)} = \varphi_1 \cdots \varphi_k\ f_0^{(m-k)} = \frac{q^{k+1}}{q-1}\ \frac{\left(q^{-1};t\right)_{k+1}}{(t;t)_k}\ f_0^{(m-k)} \ , \tag{A.53}$$

where we introduced the q-Pochhammer symbols

$$(a;q)_k := \prod_{l=0}^{k-1} \left(1 - a\, q^l\right) \qquad \text{for} \quad 0 < k \leq \infty \tag{A.54}$$

and $(a;q)_0 := 1$. Using (A.51) we can express $f_0^{(m)}$ recursively as

$$f_0^{(m)} = \sum_{k=0}^{m-1} f_0^{(k)}\, \phi_{m-k} \ , \tag{A.55}$$

where we defined

$$\phi_k \ := \ \xi_k\, \varphi_1 \cdots \varphi_{k-1} = -t^{k\,\frac{N+1}{2}}\, q^k\, \frac{\left(q^{-1};t\right)_k}{(t;t)_k} \ . \tag{A.56}$$

It is easy to see that the solution to this recursion with $f_0^{(0)} = 1$ is given by an expansion into partitions of m as

$$f_0^{(m)} = \sum_{\lambda \in \Lambda_+^m} L_\lambda\, \phi_\lambda \ , \tag{A.57}$$

where this formula should be understood in the large N limit as it involves a sum over *all* partitions of m. Here $\phi_\lambda := \prod_{i=1}^{\ell(\lambda)} \phi_{\lambda_i}$ with $\ell(\lambda)$ the length of the partition λ, and the combinatorial weight

$$L_\lambda = \frac{\ell(\lambda)!}{\left|\mathrm{Aut}(\lambda)\right|} \tag{A.58}$$

is the number of distinguishable orderings of λ (e.g. $L_{(2,1)} = 2$ and $L_{(1,1)} = 1$), where

$$\left|\mathrm{Aut}(\lambda)\right| = \prod_{i=1}^{|\lambda|} m_i(\lambda)! \tag{A.59}$$

is the order of the automorphism group of λ consisting of permutations $\sigma \in \mathfrak{S}_{\ell(\lambda)}$ such that $\lambda_{\sigma(i)} = \lambda_i$ for all i, and $m_i(\lambda)$ is the number of parts of λ equal to i. For example, the first four terms are given by

$$\begin{aligned} f_0^{(1)} &= \phi_1 \ , \\ f_0^{(2)} &= \phi_2 + \phi_1^2 \ , \\ f_0^{(3)} &= \phi_3 + 2\,\phi_1\,\phi_2 + \phi_1^3 \ , \\ f_0^{(4)} &= \phi_4 + 2\,\phi_1\,\phi_3 + \phi_2^2 + 3\,\phi_1^2\,\phi_2 + \phi_1^4 \ . \end{aligned} \tag{A.60}$$

We can finally evaluate the (q,t)-trace of the connected minimal word using (3.33) and (A.48) to write

$$\begin{aligned} \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\, t^{(\rho,H)}\, \mathsf{g}_1\, \mathsf{g}_2 \cdots \mathsf{g}_{n-1}\big) &= (g_{\omega_1})^{-n/2}\ \mathrm{Tr}_{\,R_{\omega_1}}\big(t^{(\rho,H)}\, \mathcal{O}_{n-1}\big) \\ &= (g_{\omega_1})^{-n/2} \sum_{k=0}^{n-1} f_k^{(n-1)}\ \mathrm{Tr}_{\,R_{\omega_1}}\big(t^{(k+1)\,(\rho,H)}\big) \ , \end{aligned} \tag{A.61}$$

and using (3.26), (A.53) and (A.57) we get the required statement. ■

Lemma A.62 *If λ, μ, ν, λ_β, μ_β and ν_β are all partitions, then for large N one has*

$$\widetilde{N}_{\mu\,\lambda}^{\nu} = N_{\mu_\beta\,\lambda_\beta}^{\nu_\beta}$$

in $\mathcal{U}_q(\mathfrak{gl}_N)$.

<u>Proof</u>: Let us consider $\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}(\Phi_n\, P_{\lambda_\beta}\, U)$ for $\lambda \in \Lambda_+^n$. The trace takes values in the weight zero subspace of $W_{\beta-1}$ from (2.87), and in this subspace the intertwiner Φ_n acts proportionally to the identity on $R_{\omega_1}^{\otimes n}$ via (A.43). This yields

$$\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}(\Phi_n\, P_{\lambda_\beta}\, U) = (g_{\omega_1})^{-n/2}\ \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}(P_{\lambda_\beta}\, U) \ , \tag{A.63}$$

where the second trace is an ordinary $\mathbb{C}$-valued trace. We can use quantum Schur-Weyl duality and the definition of the quantum Young projectors from (2.60) together with the definition of Φ_n in (3.5) to get

$$d_{\lambda_\beta}(1)\, \chi_{\Phi_\lambda}(U) = d_{\lambda_\beta}(1)\, (g_{\omega_1})^{-n/2}\ \mathrm{Tr}_{\,R_{\lambda_\beta}}(U) \ . \tag{A.64}$$

It follows that the generalized character and the trace of U differ only by a factor as

$$\mathrm{Tr}_{\,R_{\lambda_\beta}}(U) = (g_{\omega_1})^{n/2}\, \chi_{\Phi_\lambda}(U) \ . \tag{A.65}$$

Using in addition the definitions of the coefficients (3.103) and (3.102), we then get

$$\begin{aligned}\chi_{\Phi_\mu}(U)\,\chi_{\Phi_\lambda}(U) &= (g_{\omega_1})^{-(|\mu|+|\lambda|)/2}\ \mathrm{Tr}_{\,R_{\mu_\beta}}(U)\ \mathrm{Tr}_{\,R_{\lambda_\beta}}(U)\\ &= (g_{\omega_1})^{-(|\mu|+|\lambda|)/2}\sum_{\nu\in\Lambda_+} N^{\nu}_{\mu_\beta\,\lambda_\beta}\,\mathrm{Tr}_{\,R_\nu}(U)\\ &= \sum_{\nu\in\Lambda_+} N^{\nu}_{\mu_\beta\,\lambda_\beta}\,\chi_{\Phi_{\nu_{\beta-2}}}(U)\end{aligned} \tag{A.66}$$

and

$$\chi_{\Phi_\mu}(U)\,\chi_{\Phi_\lambda}(U) = \sum_{\nu\in\Lambda_+} \widetilde{N}^{\nu}_{\mu\,\lambda}\,\chi_{\Phi_\nu}(U)\ . \tag{A.67}$$

The result now follows by taking the inner product (3.91) of $\chi_{\Phi_\mu}(U)\,\chi_{\Phi_\lambda}(U)$ with $\chi_{\Phi_\nu}(U)$ in each of these expressions and comparing the two results. ■

Lemma A.68 $\widetilde{N}^{\nu}_{\mu\,\lambda} = \widetilde{N}^{\nu+2a\,(1^N)}_{\mu+a\,(1^N)\ \lambda+a\,(1^N)}$.

<u>Proof</u>: We use the shift property of the Macdonald polynomials from [125, §IV, eq. (4.17)] which reads

$$M_{\lambda+a\,(1^N)}(x;q,t) = x^a\,M_\lambda(x;q,t) \tag{A.69}$$

where $x^a := (x_1\cdots x_N)^a$. Together with (3.10) this implies

$$\chi_{\Phi_{\lambda+a\,(1^N)}}(U) = x^a\,\chi_{\Phi_\lambda}(U)\ , \tag{A.70}$$

where $x = \mathrm{e}^{\,z}$ and $U = \mathrm{e}^{\,(z,H)}$. We then obtain

$$\begin{aligned}\chi_{\Phi_{\mu+a\,(1^N)}}(U)\,\chi_{\Phi_{\lambda+a\,(1^N)}}(U) &= x^{2a}\,\chi_{\Phi_\mu}(U)\,\chi_{\Phi_\lambda}(U)\\ &= \sum_{\nu\in\Lambda_+} \widetilde{N}^{\nu}_{\mu\,\lambda}\,\chi_{\Phi_{\nu+2a\,(1^N)}}(U)\end{aligned} \tag{A.71}$$

and

$$\chi_{\Phi_{\mu+a\,(1^N)}}(U)\,\chi_{\Phi_{\lambda+a\,(1^N)}}(U) = \sum_{\nu\in\Lambda_+} \widetilde{N}^{\nu}_{\mu+a\,(1^N)\ \lambda+a\,(1^N)}\,\chi_{\Phi_\nu}(U)\ . \tag{A.72}$$

The result now follows by taking the inner product (3.91) of $\chi_{\Phi_{\mu+a\,(1^N)}}(U)\,\chi_{\Phi_{\lambda+a\,(1^N)}}(U)$ with the generalized character $\chi_{\Phi_{\nu+2a\,(1^N)}}(U)$ in each of these expressions and comparing the two results. ■

6 (q,t)-traces and symmetric functions

We are interested in the large N expansion of refined $U(N)$ Yang-Mills amplitudes. The computation of the traces $\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, U\, \mathsf{h}(m_T)\big)$ in this limit can be related to some combinatorial identities involving symmetric functions. For this, we shall say that the minimal word $\mathsf{h}(m_T)$ of $\mathsf{H}_q(\mathfrak{S}_n)$ has *connectivity class* $\mu(T) = \big(\mu_1(T), \ldots, \mu_n(T)\big)$ if the conjugacy class $T \in \mathfrak{S}_n^\vee$ is parameterized by the partition $\mu(T)$ of n, i.e. any element of T is composed of reduced words with $\mu_i(T)$ cycles of length i. The minimal word m_T in the conjugacy class T has length

$$\ell^*\big(\mu(T)\big) = \sum_{i=1}^n (i-1)\, \mu_i(T) \ , \tag{A.73}$$

and $\ell(\mu(T)) = \sum_{i=1}^n \mu_i(T)$ is the total number of cycles in the cycle decomposition of T.

We interpret Lemma A.32 as an expansion in (normalized) Macdonald polynomials as

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, U\, \mathsf{h}(\sigma)\big) = \sum_{\lambda_\beta \in \Lambda_+^n} \chi_{r_{\lambda_\beta}}\big(\mathsf{h}(\sigma)\big)\ \frac{M_\lambda(x;q,t)}{\sqrt{g_\lambda}} \ . \tag{A.74}$$

Let us consider the unrefined limit $\beta = 1$, wherein the normalized Macdonald polynomials in (A.74) reduce to Schur polynomials $s_\lambda(x)$. We can then apply [59, Theorem 1 and Definition 1] to get

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(U\, \mathsf{h}(m_T)\big) = \prod_{i=1}^n p_{\mu_i(T)}(q;x) \ , \tag{A.75}$$

where

$$p_r(q;x) := \sum_{\substack{a,b=0\\ a+b=r-1}}^{r-1} (-1)^b\, q^a\, s_{(a+1\ 1^b)}(x) \ . \tag{A.76}$$

By [59, Lemma 1] we can equivalently write (A.76) as

$$p_r(q;x) = \frac{1}{q-1} \sum_{\lambda \in \Lambda_+^r} s_\lambda(q|-1)\, s_\lambda(x) \ , \tag{A.77}$$

where $s_\lambda(q|-1)$ is a supersymmetric Schur function [35, Section 4.4]. The sum in (A.77) can be evaluated by using the Cauchy-Binet identity for supersymmetric Schur functions [35, eq. (4.27)]

$$\sum_{\lambda \in \Lambda_+} s_\lambda(x|z)\, s_\lambda(y|w) = \prod_{i,j=1}^N \frac{(1+x_i\, w_j)\,(1+y_i\, z_j)}{(1-x_i\, y_j)\,(1-z_i\, w_j)} \ , \tag{A.78}$$

which at the specializations $z = (0,\dots,0)$, $y = (q,0,\dots,0)$ and $w = (-1,0,\dots,0)$ yields

$$\sum_{\lambda\in\Lambda_+} s_\lambda(q|-1)\, s_\lambda(x) = \prod_{i=1}^N \frac{1-x_i}{1-q\,x_i}\ . \tag{A.79}$$

The sum over partitions of r can in this way be computed by using the homogeneity property $s_\lambda(\alpha\,x) = \alpha^{|\lambda|}\, s_\lambda(x)$ of Schur polynomials and the generating function

$$\sum_{r=1}^\infty \alpha^r \sum_{\lambda\in\Lambda_+^r} s_\lambda(q|-1)\, s_\lambda(x) = \sum_{\lambda\in\Lambda_+} s_\lambda(q|-1)\, s_\lambda(\alpha\,x) \tag{A.80}$$

for $\alpha\in\mathbb{C}$. Using (A.79) we then find

$$p_r(q;x) = \frac{1}{q-1}\,\frac{1}{r!}\,\frac{\partial^r}{\partial\alpha^r}\bigg|_{\alpha=0} \prod_{i=1}^N \frac{1-\alpha\,x_i}{1-\alpha\,q\,x_i}\ . \tag{A.81}$$

In particular, the connected minimal word $\mathsf{h}(m_T) = \mathsf{g}_1\,\mathsf{g}_2\cdots\mathsf{g}_{n-1}$ belongs to the connectivity class $\mu(T) = (n)$ and the corresponding trace gives exactly $p_n(q;x)$, so that

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(U\,\mathsf{g}_1\,\mathsf{g}_2\cdots\mathsf{g}_{n-1}\big) = \frac{1}{1-q}\sum_{\lambda\in\Lambda_+^n} s_\lambda(q|-1)\, s_\lambda(x)\ . \tag{A.82}$$

At the specialization $U = q^{(\rho,H)}$ we can compare this formula with the explicit computation of the trace from [49, eq. (B.6)] to arrive at the combinatorial identity

Proposition A.83 $\displaystyle\sum_{\lambda\in\Lambda_+^n} s_\lambda(q|-1)\, s_\lambda(q^\rho) = (q-1)\, q^{(n-1)\,\frac{N+1}{2}}\,[N]_q\ .$

This identity can be compared explicitly with the formula (A.81), in which case the specialization of the product in (A.79) to $x_i = \alpha\,q^{\frac{N+1}{2}-i}$ yields the generating function

$$\sum_{\lambda\in\Lambda_+} s_\lambda(q|-1)\, s_\lambda(\alpha\,q^\rho) = \frac{1-\alpha\,q^{-\frac{N-1}{2}}}{1-\alpha\,q^{\frac{N+1}{2}}}\ . \tag{A.84}$$

Substituting now $p_r(q;q^\rho) = q^{(r-1)\,\frac{N+1}{2}}\,[N]_q$ into (A.75) we get

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(q^{(\rho,H)}\,\mathsf{h}(m_T)\big) = \prod_{i=1}^n \big(q^{(i-1)\,\frac{N+1}{2}}\,[N]_q\big)^{\mu_i(T)} = q^{\frac{N+1}{2}\,\ell^*(\mu(T))}\,\big([N]_q\big)^{\ell(\mu(T))} \tag{A.85}$$

as in [49, eq. (B.7)]. We are not aware of any analogous simplifying identities for generating functions of Macdonald polynomials which could aid in simplifying the (q,t)-traces for $\beta\neq 1$.

7 Quantum spectral curves and β-ensembles

We can define the $(2,0)$ theory by wrapping M5-branes on the six-manifold $\Sigma_h \times \mathbb{C}^2$ in

$$(X \times \mathbb{C}^2 \times S^1)_{\epsilon_1,\epsilon_2} \tag{A.86}$$

equipped with a non-trivial fibration of $\mathbb{C}^2$ over Σ_h which specifies the Ω-background [57]. In this case Σ_h acquires an interpretation as the base of a branched covering by the Seiberg-Witten curve of a four-dimensional $\mathcal{N}=2$ gauge theory of class $\mathcal{S}$, which can in turn be regarded as the spectral curve of an associated Hitchin system [126, 127] that is quantized via a suitable deformation; the five-dimensional gauge theories compactified on a circle of radius r lead to a relativistic (q-deformed or difference) version of this Hitchin system. For $p=1$, the bound state of N M5-branes is described by an N-sheeted branched covering of Σ_h given by

$$\Sigma_{\rm SW} = \Big\{ (x,z) \in \mathrm{Tot}\big(\mathcal{O}_{\Sigma_h}(-1)\big) \;\Big|\; x^N + \sum_{j=2}^{N} \mathrm{t}_j(z)\, x^{N-j} = 0 \Big\} \ , \tag{A.87}$$

where t_j is a $(j,0)$-differential on Σ_h.

Generally, the Seiberg-Witten curve is an affine curve characterized by an algebraic relation of the form $P(x,y)=0$ for $(x,y)\in\mathbb{C}^2$. Turning on the Ω-background lifts this relation to a differential equation $P(\widehat{x},\widehat{y})\psi=0$ which quantizes the coordinate algebra $\mathbb{C}[x,y]$ to the Weyl algebra $\mathbb{C}[\hbar]\langle\widehat{x},\widehat{y}\rangle$ defined by the commutation relations

$$[\widehat{x},\widehat{y}\,] = -\hbar \qquad \text{with} \quad \hbar = \sqrt{\beta} - \frac{1}{\sqrt{\beta}} = \frac{\epsilon_1+\epsilon_2}{g_s} \ . \tag{A.88}$$

We can represent $\widehat{x}$ as the multiplication operator by $x\in\mathbb{C}$ and $\widehat{y}$ as the differential operator $\hbar\,\partial_x$. This differential equation is interpreted as a "quantum curve" [128]: The differential $\lambda_\hbar = \hbar\,\partial_x \log\psi(x,\hbar)\,\mathrm{d}x$ is a "quantum" differential generating the "quantum" periods of the quantized Riemann surface which in the unrefined limit $\hbar=0$ coincides with the meromorphic differential $\lambda_0 = y(x)\,\mathrm{d}x$ of the original Seiberg-Witten curve (with $y=y(x)$ depending on x through the equation $P(x,y)=0$). Thus refinement corresponds to a system of differential equations satisfied by the partition functions of the four-dimensional gauge theory. In the five-dimensional gauge theory, the quantum spectral curve is instead given by a difference equation $P(\widehat{X},\widehat{Y}\,)\psi=0$, where the difference operators $\widehat{X} = \mathrm{e}^{\,r\,\widehat{x}}$ and $\widehat{Y} = \mathrm{e}^{\,r\,\widehat{y}}$ obey $\widehat{X}\,\widehat{Y} = \underline{q}\,\widehat{Y}\,\widehat{X}$ with $\underline{q} = \mathrm{e}^{-r^2\,\hbar}$.

In order to understand this point from the perspective of two-dimensional gauge theory, following [33, 35, 52] we define shifted weights $n_i = \lambda_i + \beta\,\rho_i$ for $i = 1, \dots, N$ and rewrite the partition function (2.78) for $G = U(N)$ in the form (up to overall normalization)

$$Z_h(q,t;p) = \sum_{n\in\mathbb{Z}_0^N} \Delta_{q,t}\big(\mathrm{e}^{\,\epsilon_1\,n}\big)^{1-h}\,\Delta_{q,t}\big(\mathrm{e}^{\,-\epsilon_1\,n}\big)^{1-h}\;\mathrm{e}^{\,-\frac{p\,\epsilon_1}{2}\,(n,n)}\;, \tag{A.89}$$

where $\mathbb{Z}_0^N$ is the set of N-vectors of integers $n = (n_1, \dots, n_N)$ for which $n_i \neq n_j$ for all $i \neq j$, and the Macdonald measure $\Delta_{q,t}(x)$ is defined in Appendix 4. Let us consider the case of genus $h = 0$. Since $\Delta_{q,t}(1^N) = 0$, we can then sum over all $n \in \mathbb{Z}^N$ and following [129] we rewrite the partition function on the sphere as

$$Z_0(q,t;p) = \prod_{i=1}^N \int_0^\infty \frac{\mathrm{d}_q x_i}{x_i}\;\mathrm{e}^{\,-\frac{p}{2\epsilon_1}\,\log^2 x_i}\;\Delta_{q,t}(x)\,\Delta_{q,t}(x^{-1})\;, \tag{A.90}$$

where the multiple Jackson q-integral is defined by

$$\prod_{i=1}^N \int_0^\infty \frac{\mathrm{d}_q x_i}{x_i}\;f(x) := (1-q)^N \sum_{n\in\mathbb{Z}^N} f(q^n) \tag{A.91}$$

for a continuous function $f(z)$ on $(\mathbb{C}^*)^N$ (provided the multiple series is absolutely convergent).

The rewriting (A.90) demonstrates that the Macdonald deformation of two-dimensional gauge theory can be described as a generalized Gaussian matrix model in the q-deformed β-ensemble of random matrix theory. In particular, for $p = 1$ the indeterminancy of the moment problem for the Stieltjes-Wigert distribution implies that the discrete and continuous matrix models are equivalent [35]. In this case the geometrical setup greatly simplifies: The relevant Calabi-Yau fibration is the conifold geometry $\mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\big)$, the vector bundle $\mathfrak{E}_{0,1}$ is trivial [130, 131], while the surface $\mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1)\big)$ is the blow-up of $\mathbb{C}^2$ at two points (with boundary the three-sphere S^3). This equivalence implies that we may replace the Jackson integral in (A.90) with an ordinary Riemann-Lebesgue integral, and by setting $z_i := \log x_i$ we may write

$$Z_0(q,t;1) = \prod_{i=1}^N \int_{-\infty}^\infty \mathrm{d}z_i\;\mathrm{e}^{\,-\frac{1}{2\epsilon_1}\,z_i^2}\;\Delta_{q,t}\big(\mathrm{e}^{\,z}\big)\,\Delta_{q,t}\big(\mathrm{e}^{\,-z}\big)\;, \tag{A.92}$$

which up to normalization coincides with the Stieltjes-Wigert matrix model for refined Chern-Simons theory on S^3 [50]. However, for $p \neq 1$ such an equivalence ceases to

hold and one must work directly with the discrete matrix model (A.90) for generic values of $p \in \mathbb{Z}$.

Let us now consider the classical limit $q \to 1$ defined by taking the limits $\epsilon_1 \to 0$, $p \to \infty$ while keeping fixed the refinement parameter β (so that also $\epsilon_2 \to 0$) and the parameter $a := \epsilon_1 \, p$; in this limit the (q,t)-deformed gauge theory generally reduces to a β-deformation of ordinary Yang-Mills theory on a Riemann surface of area a. The right-hand side of (A.91) is a multiple infinite Riemann sum, which for $q \to 1^-$ formally converges to $\prod_{i=1}^N \int_0^\infty \frac{\mathrm{d}x_i}{x_i} \, f(x)$. By rescaling $z_i \to \epsilon_1 \, z_i$, up to normalization we then find the partition function

$$\begin{aligned} \widetilde{Z}_0(a,\beta) \ &:= \ \lim_{q\to 1} \, \lim_{p\to\infty} \, Z_0(q,t;p)\Big|_{t=q^\beta \, , \, p=-\frac{a}{\log q}} \\ &= \ \prod_{i=1}^N \int_{-\infty}^\infty \mathrm{d}z_i \ \mathrm{e}^{-\frac{a}{2}\, z_i^2} \, \prod_{m=0}^{\beta-1} \, \prod_{i<j} \big((z_i - z_j)^2 - m^2 \big) \ . \end{aligned} \tag{A.93}$$

The planar limit is defined by taking

$$\tau_1 = \epsilon_1 \, N \qquad \text{and} \qquad \tau_2 = \epsilon_2 \, N \tag{A.94}$$

large but fixed for $N \to \infty$. In this limit the refinement parameter $\beta = -\frac{\tau_2}{\tau_1}$ is finite, hence the product in (A.93) is finite and $m \, \epsilon_1 = \frac{m \, \tau_1}{N} \to 0$. By rescaling $z_i \to N \, z_i$ as finite variables at large N, it follows that the planar limit of the partition function (A.93) is given by

$$\widetilde{Z}_0^{\mathrm{pl}}(\mu,\beta) = \prod_{i=1}^N \int_{-\infty}^\infty \mathrm{d}z_i \ \mathrm{e}^{-\frac{\mu}{2}\, z_i^2} \ \Delta(z)^{2\beta} \ , \tag{A.95}$$

where $\mu := p \, \tau_1 \, N$ and

$$\Delta(z) = \prod_{i<j} (z_i - z_j) \tag{A.96}$$

is the Vandermonde determinant. Thus in this limit the weak coupling phase of the two-dimensional gauge theory can be described by a Gaussian matrix model in the classical β-ensembles of random matrix theory. In particular, $\log \widetilde{Z}_0^{\mathrm{pl}}(\mu,\beta)$ coincides with the partition function of two-dimensional $c = 1$ string theory at radius $R = \beta$; this generalizes the usual identification of the conifold geometry with the $c = 1$ string at the self-dual radius for $\beta = 1$. In this formulation the symmetry $\beta \to \frac{1}{\beta}$ is manifest [57] and corresponds to T-duality invariance of the string theory. For later use and comparison, let us run through the details of this identification.

The matrix integral (A.95) is a special case of the Selberg integral which can be evaluated analytically in terms of Mehta's formula

$$\widetilde{Z}_0^{\rm pl}(\mu,\beta) = \Big(\frac{\sqrt{2\pi}}{\mu^{\frac{1}{2}\,((N-1)\,\beta+1)}}\Big)^N \prod_{i=1}^N \frac{\Gamma(1+\beta\,i)}{\Gamma(1+\beta)} \ . \tag{A.97}$$

One can show that [132]

$$\prod_{i=1}^N \Gamma(1+\beta\,i) = \big(\sqrt{2\pi}\,\beta^{\frac{1}{2}\,((N-1)\,\beta+1)}\big)^N\,\Gamma(1+N\,\beta)\,\Gamma(N)\,\Gamma_2\big(N;-\beta^{-1},-1\big) \ , \tag{A.98}$$

where $\Gamma_2(\tau_2;\epsilon_1,\epsilon_2)$ is the Barnes double gamma-function defined via

$$\log\Gamma_2(\tau_2;\epsilon_1,\epsilon_2) = -\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}\,\frac{1}{\Gamma(s)}\int_0^\infty\frac{\mathrm{d}t}{t}\,t^s\,\frac{\mathrm{e}^{-\tau_2\,t}}{\big(1-\mathrm{e}^{\epsilon_1\,t}\big)\big(1-\mathrm{e}^{\epsilon_2\,t}\big)} \tag{A.99}$$

which is the double zeta-function regularization of the infinite product $\prod_{m,n\geq 0}(\tau_2 - m\,\epsilon_1 - n\,\epsilon_2)$. To obtain the large N expansion of the partition function (A.95) we use the asymptotic expansion [133, Appendix E]

$$\begin{aligned}\log\Gamma_2(\tau_2;\epsilon_1,\epsilon_2) \ &= \ \frac{1}{\epsilon_1\,\epsilon_2}\big(\tfrac{1}{2}\,\tau_2^2\,\log\tau_2 - \tfrac{3}{4}\,\tau_2^2\big) + \frac{\epsilon_1+\epsilon_2}{\epsilon_1\,\epsilon_2}\big(\tau_2\,\log\tau_2-\tau_2\big)\\ &\quad + \frac{\epsilon_1^2+\epsilon_2^2+3\,\epsilon_1\,\epsilon_2}{12\,\epsilon_1\,\epsilon_2}\,\log\tau_2 - \sum_{n=3}^\infty\frac{d_n(\epsilon_1,\epsilon_2)\,\tau_2^{2-n}}{n\,(n-1)\,(n-2)} \ ,\end{aligned} \tag{A.100}$$

where the series coefficients $d_n(\epsilon_1,\epsilon_2)$ are defined through the generating function

$$\frac{t^2}{\big(1-\mathrm{e}^{\epsilon_1\,t}\big)\big(1-\mathrm{e}^{\epsilon_2\,t}\big)} = \sum_{n=0}^\infty\frac{1}{n!}\,d_n(\epsilon_1,\epsilon_2)\,t^n \ . \tag{A.101}$$

Introducing the Bernoulli numbers B_m through the generating function

$$\frac{s}{1-\mathrm{e}^s} = -\sum_{m=0}^\infty\frac{1}{m!}\,B_m\,s^m \tag{A.102}$$

with $B_0=1$, $B_1=\frac{1}{2}$, $B_2=\frac{1}{6}$ and $B_k=0$ for all $k>1$ odd, by comparing series expansions we find

$$d_n(\epsilon_1,\epsilon_2) = g_s^{n-2}\,\sum_{k=0}^n\binom{n}{k}\,(-1)^{k-1}\,B_k\,B_{n-k}\,\beta^{k-\frac{n}{2}} \tag{A.103}$$

where we have used the relations (2.96). By dropping overall prefactors, for the free energy $\widetilde{F}_0^{\rm pl}(\tau_2,\beta) := -\log\widetilde{Z}_0^{\rm pl}(\tau_2,\beta) = -\log\Gamma_2(\tau_2;\beta^{-1/2}\,g_s,\beta^{1/2}\,g_s)$ in the large N limit this gives the asymptotic expansion

$$\begin{aligned}\widetilde{F}_0^{\rm pl}(\tau_2,\beta) \ &= \ \frac{1}{g_s^2}\big(\tfrac{3}{4}\,\tau_2^2-\tfrac{1}{2}\,\tau_2^2\,\log\tau_2\big) + \frac{1}{g_s}\big(\tfrac{1}{\sqrt{\beta}}-\sqrt{\beta}\,\big)\big(\tau_2-\tau_2\,\log\tau_2\big)\\ &\quad + \chi_0(\beta)\,\log\tau_2 + \sum_{n=1}^\infty\chi_n(\beta)\,\Big(\frac{g_s}{\tau_2}\Big)^n \ ,\end{aligned} \tag{A.104}$$

where

$$\begin{aligned}\chi_0(\beta) &= -\frac{1}{4}+\frac{\beta^{-1}}{12}+\frac{\beta}{12}\ ,\\ \chi_n(\beta) &= (n-1)!\sum_{k=0}^{n+2}\frac{(-1)^{k-1}\,B_k\,B_{n+2-k}}{k!\,(n+2-k)!}\,\beta^{k-\frac{n}{2}-1} \qquad \text{for} \quad n\geq 1\ . \end{aligned} \tag{A.105}$$

Note that the expansion parameter is

$$\frac{g_s}{\tau_2}=\frac{1}{\sqrt{\beta}\,N}\ . \tag{A.106}$$

In the unrefined limit $\beta=1$, the identity

$$\frac{1}{\big(1-\mathrm{e}^{\,s}\big)\big(1-\mathrm{e}^{\,-s}\big)}=-\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{1-\mathrm{e}^{\,s}} \tag{A.107}$$

together with (A.102) imply that $d_n(g_s,-g_s)=0$ for n odd while $d_{2g}(g_s,-g_s)=g_s^{2g-2}\,(2g-1)\,B_{2g}$ for $g\geq 1$, so that the non-vanishing coefficients

$$\chi_0(1)=-\frac{1}{12} \qquad \text{and} \qquad \chi_{2g-2}(1)=\frac{B_{2g}}{2g\,(2g-2)} \quad (g>1) \tag{A.108}$$

coincide with the orbifold Euler characteristics $\chi_{\rm orb}(\mathcal{M}_g)$ of the Riemann moduli spaces $\mathcal{M}_g$ of genus $g\geq 1$ complex curves, i.e. the Euler character of $\mathcal{M}_g$ calculated by resolving its orbifold singularities. On the other hand, for $\beta=2$ one can use the identity

$$\frac{1}{\big(1-\mathrm{e}^{\,s}\big)\big(1-\mathrm{e}^{\,-2s}\big)}=-\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{1-\mathrm{e}^{\,s}}+\frac{1}{2}\,\frac{1}{1-\mathrm{e}^{\,-2s}}-\frac{1}{2}\,\frac{1}{1-\mathrm{e}^{\,-s}} \tag{A.109}$$

together with (A.102) to infer that

$$\chi_{2g-1}(2)=\sqrt{2}\;2^{-g}\,\frac{\big(2^{2g-2}-\frac{1}{2}\big)\,B_{2g}}{2g\,(2g-1)} \qquad \text{and} \qquad \chi_{2g-2}(2)=2^{-g}\,\chi_{\rm orb}(\mathcal{M}_g) \tag{A.110}$$

for $g\geq 1$, so that the coefficients $\chi_{2g-1}(2)$ are proportional to the orbifold Euler characteristics of the moduli spaces of certain real algebraic curves of genus g [64]. Thus in this case refinement corresponds to the replacement of $\chi_{\rm orb}(\mathcal{M}_g)$ with the *parameterized* Euler characters $\chi_n(\beta)$ [66,134], which provide a geometric parameterization that interpolates between the orbifold Euler characters of the moduli spaces of closed oriented Riemann surfaces at $\beta=1$ and closed unoriented Riemann surfaces with crosscap at $\beta=2$. In other words, the string theory at $\beta=2$ can be regarded as the orientifold of the string theory at $\beta=1$. From this perspective, it is natural

to expect that the generic β-deformed Euler characters $\chi_n(\beta)$ themselves describe characteristic classes of some related moduli spaces [64].

The expansion (A.104) governs the leading order behaviour of the dual refined B-model topological string amplitudes on the mirror of the conifold geometry, which is the cotangent bundle T^*S^3. Generally, these local Calabi-Yau geometries are described by algebraic equations of the form

$$u\,v + F(x,y) = 0 \tag{A.111}$$

in $\mathbb{C}^4$, where the equation $F(x,y)=0$ describes an affine curve Σ in $\mathbb{C}^2$ and a (local) function $y(x)$ which determines a meromorphic differential $\lambda = y(x)\,\mathrm{d}x$ giving the periods of Σ. In the Gaussian matrix model (A.95) at $\beta=1$, the Riemann surface Σ is the corresponding rational spectral curve which is given by a double cover of the y-plane with $F(x,y)=x^2-y^2+m=0$, where m is the Kähler parameter of the resolved conifold. After a simple change of variables this spectral curve can be regarded as the holomorphic curve $F(z,w)=z\,w-m=0$ in $\mathbb{C}^2$, which after refinement quantizes to the differential operator $F(\,\widehat{z},\widehat{w}\,)=\hbar\,z\,\partial_z-m$ [135]; the quantum curve in this case is the canonical example of a D-module [128] and it can be regarded as a differential equation for certain correlators in the matrix model [57]. After q-deformation, the quantum spectral curve for the conifold is naturally described by a difference equation (rather than a differential equation) with difference operator

$$F(\,\widehat{X},\widehat{Y}\,)=\big(\,1-\underline{q}^{-1/2}\,\widehat{X}\,\big)\,\widehat{Y}-\big(\,1-\underline{Q}\,\underline{q}\,\widehat{X}\,\big)\ , \tag{A.112}$$

where $\underline{Q}:=\mathrm{e}^{-r^2\,m}$; it can be thought of as a differential equation for the partition functions of refined topological string theory [136]. It is then natural to expect that a similar quantum spectral curve governs the matrix model (A.90) of the q-deformed β-ensemble that represents the (q,t)-deformed gauge theory, along the lines of [137]. In the following, these lines of reasoning will be applied to the closed string chiral expansion of the (q,t)-deformed two-dimensional gauge theory to give geometrical interpretations of the Macdonald deformation in terms of contributions from deformed characteristic classes associated to quantum Riemann surfaces.

8 Differential calculus of graded functionals

A nice and elaborate summary of formulas from differential calculus on graded manifolds can be found in [29]. In the following we rely on this treatment and only review formulas with regard to graded functionals.

The space of superfields is once again the mapping space

$$\boldsymbol{\mathcal{M}} = \mathsf{Map}\big(T[1]\Sigma_d\,,\,\mathcal{M}\big)\ , \tag{A.113}$$

and the superfield coordinates on $\boldsymbol{\mathcal{M}}$ are defined via the coordinates $\hat{z}^{\hat{\mu}} \in T[1]\Sigma_d$, $\hat{X}^{\hat{\imath}} \in \mathcal{M}$ as

$$\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z}) \,=\, \boldsymbol{\phi}^*(\hat{X}^{\hat{\imath}})(\hat{z})\ , \tag{A.114}$$

where $\boldsymbol{\phi} \in \boldsymbol{\mathcal{M}}$ is an arbitrary superfield. We use the notation $|\hat{\imath}|$ for the degree of $\hat{X}^{\hat{\imath}}$, and also for the ghost number of $\hat{\boldsymbol{X}}^{\hat{\imath}}$.

A graded functional[1] of the superfields $\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})$ is defined by a function $F(\hat{X})$ on $\mathcal{M}$ as

$$\boldsymbol{F} \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*(F) \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ F\big(\hat{\boldsymbol{X}}(\hat{z})\big)\ , \tag{A.115}$$

and it has ghost number $|F| - d$, where $|F|$ denotes the ghost number of $F\big(\hat{\boldsymbol{X}}(\hat{z})\big)$. The definition of a graded functional n-form $\boldsymbol{\alpha}$ with ghost number $|\alpha| - d$ is analogous and given by an n-form α on $\mathcal{M}$ as

$$\begin{aligned}
\boldsymbol{\alpha} \,&=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*(\alpha) \\
&=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}_1}(\hat{z}) \cdots \boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}_n}(\hat{z})\, \alpha_{\hat{\imath}_1\cdots\hat{\imath}_n}\big(\hat{\boldsymbol{X}}(\hat{z})\big) \\
&:=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \alpha\big(\hat{\boldsymbol{X}}(\hat{z})\big)\ .
\end{aligned} \tag{A.116}$$

The exterior product of two graded functional forms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ also gives a graded functional form with ghost number $|\alpha| + |\beta| - d$ which reads as

$$\boldsymbol{\alpha}\wedge\boldsymbol{\beta} \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*(\alpha\wedge\beta) \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \alpha\big(\hat{\boldsymbol{X}}(\hat{z})\big)\,\beta\big(\hat{\boldsymbol{X}}(\hat{z})\big)\ . \tag{A.117}$$

As can be seen, it depends on the integration measure. The one-form local basis elements have ghost number $|\hat{\imath}| + 1 - d$ and they are given by

$$\boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}} \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*(\mathrm{d}\hat{X}^{\hat{\imath}}) \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})\ . \tag{A.118}$$

We can write the n-form $\boldsymbol{\alpha}$ with the exterior product as

$$\boldsymbol{\alpha} \,=\, \boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}_1}\wedge\cdots\wedge\boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}_n}\wedge\boldsymbol{\alpha}_{\hat{\imath}_1\cdots\hat{\imath}_n}\ , \tag{A.119}$$

[1]We only consider non-local graded functionals, hence the kernel function $F(\hat{\boldsymbol{X}})$ can be taken to be an ordinary graded function of $\hat{\boldsymbol{X}}$.

where the scalar functional is given by

$$\boldsymbol{\alpha}_{\hat{\imath}_1\cdots\hat{\imath}_n} \;=\; \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\; \alpha_{\hat{\imath}_1\cdots\hat{\imath}_n}\big(\hat{\boldsymbol{X}}(\hat{z})\big)\;. \tag{A.120}$$

A graded vector functional $\boldsymbol{V}$ with ghost number $|V|-d$ can be written in the form

$$\boldsymbol{V} \;=\; \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\; V^{\hat{\imath}}\big(\hat{\boldsymbol{X}}(\hat{z})\big)\,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})}\;, \tag{A.121}$$

which acts as a left functional derivative

$$\vec{\boldsymbol{V}}\boldsymbol{F} \;=\; \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\; V^{\hat{\imath}}\big(\hat{\boldsymbol{X}}(\hat{z})\big)\,\frac{\vec{\delta}\boldsymbol{F}}{\delta\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})} \tag{A.122}$$

on graded functionals $\boldsymbol{F}$, and is defined as

$$\lim_{\epsilon\to 0}\frac{\boldsymbol{F}[\hat{\boldsymbol{X}}+\epsilon\,\hat{\boldsymbol{\eta}}]\;-\;\boldsymbol{F}[\hat{\boldsymbol{X}}]}{\epsilon} \;=:\; \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;\hat{\boldsymbol{\eta}}^{\hat{\imath}}(\hat{z})\,\frac{\vec{\delta}\boldsymbol{F}}{\delta\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})}\;, \tag{A.123}$$

where $\hat{\boldsymbol{\eta}}^{\hat{\imath}}$ is an arbitrary superfield with the same degree as $\hat{\boldsymbol{X}}^{\hat{\imath}}$. The functional derivative with respect to $\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})$ has ghost number $-|\hat{\imath}|+d$. As graded functionals are non-local, the functional derivatives are given by ordinary derivatives of the kernel function as

$$\frac{\vec{\delta}\boldsymbol{F}}{\delta\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})} \;=\; \frac{\vec{\partial}F}{\partial\hat{X}^{\hat{\imath}}}\bigg|_{\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})}\;. \tag{A.124}$$

The interior product is given by contraction with the graded vector functional $\boldsymbol{V}$:

$$\iota_{\boldsymbol{V}} \;=\; \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\; V^{\hat{\imath}}\big(\hat{\boldsymbol{X}}(\hat{z})\big)\,\frac{\vec{\delta}}{\delta\,(\boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z}))}\;, \tag{A.125}$$

which acts on exterior products as

$$\iota_{\boldsymbol{V}}(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}) \;=\; \iota_{\boldsymbol{V}}\boldsymbol{\alpha}\wedge\boldsymbol{\beta} \;+\; (-1)^{(|V|+1)\,|\alpha|}\,\boldsymbol{\alpha}\wedge\iota_{\boldsymbol{V}}\boldsymbol{\beta}\;. \tag{A.126}$$

The de Rham differential can be written in the form

$$\boldsymbol{\delta} \;=\; \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\;\boldsymbol{\delta}\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})\,\frac{\vec{\delta}}{\delta\hat{\boldsymbol{X}}^{\hat{\imath}}(\hat{z})}\;. \tag{A.127}$$

It has the following properties:

$$
\begin{aligned}
\boldsymbol{\delta F} &= \lim_{\epsilon\to 0} \frac{\boldsymbol{F}[\hat{\boldsymbol{X}} + \epsilon\,\boldsymbol{\delta}\hat{\boldsymbol{X}}] \,-\, \boldsymbol{F}[\hat{\boldsymbol{X}}]}{\epsilon} \ , \\
\boldsymbol{\delta}^2 &= 0 \ , \\
\boldsymbol{\delta}(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}) &= \boldsymbol{\delta\alpha}\wedge\boldsymbol{\beta} \,+\, (-1)^{|\alpha|}\,\boldsymbol{\alpha}\wedge\boldsymbol{\delta\beta} \ , \\
\boldsymbol{\delta\alpha} &= \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*(\mathrm{d}\alpha) \ .
\end{aligned}
\tag{A.128}
$$

The symplectic structure on the target dg-manifold $\mathcal{M}$ has degree $d+1$ and it can be written in the form

$$
\omega = (-1)^{(d+1)\,|a|}\,\mathrm{d}q^a \wedge \mathrm{d}p_a \ , \tag{A.129}
$$

where q^a and p_a are local Darboux coordinates such that $|q^a| + |p_a| = d-1$. The BV symplectic form $\boldsymbol{\omega}$ with ghost number 1 is defined by

$$
\boldsymbol{\omega} \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \mathrm{ev}^*(\omega) \,=\, \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ (-1)^{(d+1)\,|q^a|}\,\boldsymbol{\delta q}^a(\hat{z})\wedge\boldsymbol{\delta p}_a(\hat{z}) \ . \tag{A.130}
$$

The definition of the Hamiltonian vector field of a graded functional $\boldsymbol{F}$ is given by the expression

$$
\iota_{\boldsymbol{X}_{\boldsymbol{F}}}\boldsymbol{\omega} \,:=\, \boldsymbol{\delta F} \ , \tag{A.131}
$$

and it has ghost number $|F| - d + 1$. The solution to this equation

$$
\boldsymbol{X}_{\boldsymbol{F}} \,=\, (-1)^{|F|+d} \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \left(\frac{\boldsymbol{F}\overleftarrow{\delta}}{\delta\boldsymbol{q}^a(\hat{z})}\,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta p}_a(\hat{z})} - (-1)^{|q^a|\,|p_a|}\,\frac{\boldsymbol{F}\overleftarrow{\delta}}{\delta\boldsymbol{p}_a(\hat{z})}\,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta q}^a(\hat{z})} \right) \tag{A.132}
$$

is used to define the BV bracket of graded functionals $\boldsymbol{F}$ and $\boldsymbol{G}$ as

$$
\begin{aligned}
(\boldsymbol{F},\boldsymbol{G})_{\mathrm{BV}} &= (-1)^{|F|+d}\,\vec{\boldsymbol{X}}_{\boldsymbol{F}}\boldsymbol{G} \\
&= (-1)^{|F|+d}\,\iota_{\boldsymbol{X}_{\boldsymbol{F}}}\boldsymbol{\delta G} \\
&= (-1)^{|F|+d+1}\,\iota_{\boldsymbol{X}_{\boldsymbol{F}}}\iota_{\boldsymbol{X}_{\boldsymbol{G}}}\boldsymbol{\omega} \\
&= \int_{T[1]\Sigma_d} \mathrm{d}^d\hat{z}\ \left(\frac{\boldsymbol{F}\overleftarrow{\delta}}{\delta\boldsymbol{q}^a(\hat{z})}\,\frac{\vec{\delta}\boldsymbol{G}}{\delta\boldsymbol{p}_a(\hat{z})} \,-\, (-1)^{|q^a|\,|p_a|}\,\frac{\boldsymbol{F}\overleftarrow{\delta}}{\delta\boldsymbol{p}_a(\hat{z})}\,\frac{\vec{\delta}\boldsymbol{G}}{\delta\boldsymbol{q}^a(\hat{z})} \right) \ .
\end{aligned}
\tag{A.133}
$$

The BV bracket has the following properties:

$$
\begin{aligned}
(\boldsymbol{F},\boldsymbol{G})_{\mathrm{BV}} &= -(-1)^{(|F|+d+1)\,(|G|+d+1)}\ (\boldsymbol{G},\boldsymbol{F})_{\mathrm{BV}} \ , \\
(\boldsymbol{F},\boldsymbol{G}\,\boldsymbol{H})_{\mathrm{BV}} &= (\boldsymbol{F},\boldsymbol{G})_{\mathrm{BV}}\ \boldsymbol{H} \,+\, (-1)^{(|F|+d+1)\,|G|}\,\boldsymbol{G}\ (\boldsymbol{F},\boldsymbol{H})_{\mathrm{BV}} \ , \\
(\boldsymbol{F},(\boldsymbol{G},\boldsymbol{H})_{\mathrm{BV}})_{\mathrm{BV}} &= ((\boldsymbol{F},\boldsymbol{G})_{\mathrm{BV}},\boldsymbol{H})_{\mathrm{BV}} \,+\, (-1)^{(|F|+d+1)\,(|G|+d+1)}\ (\boldsymbol{G},(\boldsymbol{F},\boldsymbol{H})_{\mathrm{BV}})_{\mathrm{BV}} \ .
\end{aligned}
\tag{A.134}
$$

9 Supersymmetric quantum mechanics

Supersymmetric quantum mechanics provides a simple example of a topological field theory; its Mathai-Quillen formalism can be found in e.g. [107–109]. The target space is a Riemannian manifold M with metric g and the parameter manifold is just a compact worldline S^1. The local coordinates of the mapping space $LM := \mathsf{Map}(S^1, M)$ are $\{x^i(\tau)\}$ with $\tau \in [0,1]$ and $x^i(0) = x^i(1)$, and so they parameterize (smooth) loops in M. We furthermore define two fermionic fields $\psi^i(\tau)$ and $\bar\psi_i(\tau)$ with ghost number 1 and -1, respectively. The action of supersymmetric quantum mechanics is

$$I_{\rm SQM} \,=\, \int_{S^1} \mathrm{d}\tau \,\Big(\frac{1}{2}\, g_{ij}\, \dot x^i\, \dot x^j \,+\, \mathrm{i}\, \bar\psi_i\, \nabla_\tau \psi^i \,-\, \frac{1}{4}\, R^{ij}{}_{kl}\, \bar\psi_i\, \bar\psi_j\, \psi^k\, \psi^l\Big) \,, \tag{A.135}$$

where a dot denotes a τ-derivative, $\nabla_\tau \psi^i = \dot\psi^i + \Gamma^i{}_{jk}\, \psi^j\, \dot x^k$ is defined by the action of the Levi-Civita connection ∇ of the metric g pulled back to the loop via the map x, and R is the associated Riemann tensor. The action (A.135) is invariant under the BRST transformations

$$\delta x^i \,=\, \psi^i \,, \qquad \delta\psi^i \,=\, 0 \qquad \text{and} \qquad \delta\bar\psi_i \,=\, \mathrm{i}\, g_{ij}\, \dot x^j \,+\, \Gamma^k{}_{ij}\, \psi^j\, \bar\psi_k \,, \tag{A.136}$$

which is only nilpotent on-shell, and it is BRST-exact on-shell:

$$I_{\rm SQM} \,=\, \delta\Psi'_{\rm SQM} \qquad \text{with} \quad \Psi'_{\rm SQM} \,=\, -\frac{\mathrm{i}}{2} \int_{S^1} \mathrm{d}\tau\; \bar\psi_i\, \dot x^i \,. \tag{A.137}$$

The set of δ-fixed points is the space of instantons, i.e. the constant loops $x^i(\tau)$, which can be identified with the target space M.

We follow the same procedure as in §1.1 and §2.2 in Chapter 6 to reformulate supersymmetric quantum mechanics using a linearizing auxiliary field b_i with ghost number 0. The BRST transformations with the field b_i are given by

$$\delta x^i \,=\, \psi^i \,, \qquad \delta\psi^i \,=\, 0 \,, \qquad \delta\bar\psi_i \,=\, b_i \qquad \text{and} \qquad \delta b_i \,=\, 0 \,, \tag{A.138}$$

and they are nilpotent off-shell. The action

$$\delta\Psi'_{\rm SQM} \,=\, -\frac{\mathrm{i}}{2} \int_{S^1} \mathrm{d}\tau\; \big(b_i\, \dot x^i \,-\, \bar\psi_i\, \dot\psi^i\big) \tag{A.139}$$

is invariant under these new BRST transformations, and it reduces to the action (A.135) if we impose the constraint

$$b_i \,=\, \mathrm{i}\, g_{ij}\, \dot x^j \,-\, \Gamma^k{}_{ij}\, \bar\psi_k\, \psi^j \tag{A.140}$$

as gauge fixing. In the language of the BRST formulation, this means that we choose the gauge fixing fermion as

$$\Psi_{\mathrm{SQM}} \,=\, -\int_{S^1} \mathrm{d}\tau\; \bar{\psi}_i \Big(\mathrm{i}\, \dot{x}^i \,+\, \frac{1}{2}\, g^{jl}\, \Gamma^i{}_{lk}\, \bar{\psi}_j\, \psi^k \,-\, \frac{1}{2}\, g^{ij}\, b_j \Big) \,. \tag{A.141}$$

The BRST variation of (A.141) gives us the action

$$\begin{aligned} S_{\mathrm{SQM}} \,=\, \delta\Psi_{\mathrm{SQM}} \,=\, \int_{S^1} \mathrm{d}\tau\; \Big(& \mathrm{i}\, \bar{\psi}_i\, \dot{\psi}^i \,-\, \mathrm{i}\, b_i\, \dot{x}^i \,+\, g^{il}\, \Gamma^j{}_{kl}\, \bar{\psi}_j\, \psi^k\, b_i \,+\, \frac{1}{2}\, g^{ij}\, b_i\, b_j \\ & -\, \frac{1}{2}\, \partial_k \big(g^{jm}\, \Gamma^i{}_{ml} \big)\, \bar{\psi}_i\, \bar{\psi}_j\, \psi^k\, \psi^l \Big) \,. \end{aligned} \tag{A.142}$$

The equation of motion for b_i gives the same field redefinition as in (A.140), and using this we find that the action (A.142) is classically equivalent to the action (A.135).

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ADATLAP

a doktori értekezés nyilvánosságra hozatalához*

I. A doktori értekezés adatai

A szerző neve: Kökényesi Zoltán

MTMT-azonosító: 10055543

A doktori értekezés címe és alcíme: Some aspects of topological string theory

DOI-azonosító: 10.15476/ELTE.2018.182

A doktori iskola neve: ELTE Fizika Doktori Iskola

A doktori iskolán belüli doktori program neve: Részecskefizika program

A témavezető neve és tudományos fokozata: Dr. Sinkovics Annamária, tudományos főmunkatárs

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Kelt: Budapest, 2018. 09. 05.

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a doktori értekezés szerzőjének aláírása

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