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Convergence of Fourier–Jacobi series

THESES OF PHD DISSERTATION

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Budapest, 2015

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Introduction

Approximation theory is concerned with how various functions can be approximated with simpler functions. The most natural method of approximating periodic functions is by means of partial sums of its trigonometric Fourier series. However, it is well known that these partial sums are not uniformly convergent for all continuous functions. This statement still holds if we consider the algebraic case; that is, Fourier series in orthogonal polynomials.

Our dissertation is devoted to the study of three problems concerning Fourier series in Jacobi polynomials, i.e. *Fourier–Jacobi series*.

Using summations, we can construct such processes from the partial sums of Fourier–Jacobi series that are uniformly convergent in suitable Banach spaces of continuous functions. Our results in this topic are found in Chapter 3 of the dissertation.

The Lebesgue functions play an important role in the investigation of pointwise convergence. In Chapter 4, we prove a pointwise lower and upper estimation for the weighted Lebesgue function of Fourier–Jacobi series.

Discrete versions of Fourier–Jacobi series can be examined as well. Questions similar to those in the continuous case may be asked regarding the convergence of Lagrange interpolation on the roots of Jacobi polynomials. Chapter 5 is devoted to our results about the so-called Grünwald–Rogosinski process.

Our results in the topics above were published in [2], [3] and [4].

Approximation via summations of Fourier–Jacobi series

For $\alpha, \beta > -1$ we denote by $p_n^{(\alpha, \beta)}$ ($n \in \mathbb{N}$) the Jacobi polynomials that are orthonormal on the interval $(-1, 1)$ with respect to the Jacobi weight

$$w^{(\alpha, \beta)}(x) := (1-x)^\alpha (1+x)^\beta \quad (x \in (-1, 1)).$$

Let $f : [-1, 1] \rightarrow \mathbb{R}$ be a function for which the Fourier coefficients

$$c_k^{(\alpha, \beta)}(f) := \int_{-1}^1 f(x) p_k^{(\alpha, \beta)}(x) w^{(\alpha, \beta)}(x) dx \quad (k \in \mathbb{N})$$

exist. The *Fourier–Jacobi series* of f is defined by

$$S^{(\alpha,\beta)}(f, x) := \sum_{k \in \mathbb{N}} c_k^{(\alpha,\beta)}(f) p_k^{(\alpha,\beta)}(x) \quad (x \in [-1, 1]).$$

For $\gamma, \delta \geq 0$ we shall define weighted spaces of continuous functions as follows. If $\gamma, \delta > 0$ then

$$C^{(\gamma,\delta)} := C^{(\gamma,\delta)}(-1, 1) := \left\{ f \in C(-1, 1) \mid \lim_{|x| \rightarrow 1} (f w^{(\gamma,\delta)})(x) = 0 \right\},$$

if $\gamma = 0, \delta > 0$ (or $\delta = 0, \gamma > 0$) then $C^{(\gamma,\delta)}$ denotes the space of continuous functions on $(-1, 1)$ for which

$$\lim_{x \rightarrow -1} (f w^{(\gamma,\delta)})(x) = 0 \quad \left(\text{or } \lim_{x \rightarrow 1} (f w^{(\gamma,\delta)})(x) = 0 \right).$$

In the case $\gamma = \delta = 0$ (i.e. $w^{(\gamma,\delta)} \equiv 1$) let $C^{(\gamma,\delta)} := C[-1, 1]$.

$C^{(\gamma,\delta)}$ is a linear space over \mathbb{R} ,

$$\|f\|_{\infty,(\gamma,\delta)} := \|f w^{(\gamma,\delta)}\|_{\infty} := \sup_{|x| \leq 1} |(f w^{(\gamma,\delta)})(x)|$$

is a norm and $\mathbf{C}^{(\gamma,\delta)} := (C^{(\gamma,\delta)}, \|\cdot\|_{\infty,(\gamma,\delta)})$ is a Banach space.

Let us fix a summation matrix

$$\Theta := \begin{pmatrix} \theta_{0,1} & & & \\ \theta_{0,2} & \theta_{1,2} & & \\ \theta_{0,3} & \theta_{1,3} & \theta_{2,3} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\theta_{k,n}$'s are real numbers. The n th Θ -sum of the Fourier–Jacobi series of a function $f \in C^{(\gamma,\delta)}$ is defined by

$$S_n^{(\alpha,\beta),\Theta}(f, x) := \sum_{k=0}^{n-1} \theta_{k,n} c_k^{(\alpha,\beta)}(f) p_k^{(\alpha,\beta)}(x) \\ (x \in [-1, 1], n \in \mathbb{N}^+),$$

which is an algebraic polynomial of degree at most $n - 1$.

As an important special case we can define the matrix Θ with a summation

function $\varphi : [0, 1] \rightarrow \mathbb{R}$ in the following way:

$$\theta_{k,n} := \varphi\left(\frac{k}{n}\right) \quad (k = 0, 1, \dots, n-1, n \in \mathbb{N}^+).$$

With $\alpha, \beta > -1$ fixed, we shall give conditions for the parameters $\gamma, \delta \geq 0$ and the summation matrix Θ (or the summation function φ) which guarantee that the relation

$$\lim_{n \rightarrow +\infty} \|f - S_n^{(\alpha, \beta), \Theta}(f)\|_{\infty, (\gamma, \delta)} = 0$$

holds for all $f \in C^{(\gamma, \delta)}$, i.e. the Θ -sums of the Fourier–Jacobi series of f are uniformly convergent.

We note that our results are analogous to the discrete case (when instead of the partial sums of Fourier–Jacobi series, we use Lagrange interpolation polynomials; see [6], [7]).

Theorem 1. *Let $\alpha, \beta > -1$ and $\gamma, \delta \geq 0$. We have*

$$\lim_{n \rightarrow +\infty} \|f - S_n^{(\alpha, \beta), \Theta}(f)\|_{\infty, (\gamma, \delta)} = 0$$

for all $f \in C^{(\gamma, \delta)}$ if and only if the summation matrix Θ satisfies the following conditions:

$$\lim_{n \rightarrow +\infty} (1 - \theta_{k,n}) = 0 \text{ for all fixed } k \in \mathbb{N} \quad (T1)$$

and

$$\left\{ \begin{array}{l} \text{there exists } c > 0 \text{ independent of } n \text{ such that for all } n \in \mathbb{N} \\ \sup_{x \in [-1, 1]_{-1}} \int_{-1}^1 \left| \sum_{k=0}^{n-1} \theta_{k,n} p_k^{(\alpha, \beta)}(x) p_k^{(\alpha, \beta)}(y) \right| \frac{w^{(\gamma, \delta)}(x) w^{(\alpha, \beta)}(y)}{w^{(\gamma, \delta)}(y)} dy \leq c. \end{array} \right. \quad (B)$$

It is not easy to verify (B), so in the following theorems we will give simpler, sufficient conditions for the uniform convergence. First, let us define some further conditions regarding the matrix Θ :

$$\theta_{n-1, n} = O\left(\frac{1}{n}\right) \quad (n \in \mathbb{N}^+), \quad (T2)$$

$$\Delta^2 \theta_{k-1, n} = O\left(\frac{1}{n^2}\right) \quad (k = 1, 2, \dots, n-1, n \in \mathbb{N}^+), \quad (T3)$$

$$\Delta^2\theta_{k-1,n} \ (k = 1, 2, \dots, n-1, \ n \in \mathbb{N}^+) \text{ is of constant sign,} \quad (T4)$$

$$\operatorname{sgn} \Delta^2\theta_{k-1,n} = \operatorname{sgn} \theta_{n-1,n} \ (k = 1, 2, \dots, n-1, \ n \in \mathbb{N}^+), \quad (T5)$$

where

$$\Delta^2\theta_{k,n} := \Delta\theta_{k+1,n} - \Delta\theta_{k,n}, \quad \Delta\theta_{k,n} := \theta_{k+1,n} - \theta_{k,n} \quad (\theta_{n,n} := 0).$$

Theorem 2. *Suppose that $\alpha, \beta \geq -1/2$ and $\gamma, \delta \geq 0$ satisfy the inequalities*

$$\frac{\alpha}{2} - \frac{1}{4} < \gamma < \frac{\alpha}{2} + \frac{3}{4} \quad \text{and} \quad \frac{\beta}{2} - \frac{1}{4} < \delta < \frac{\beta}{2} + \frac{3}{4}. \quad (1)$$

Then

(a) (T1), (T2) and (T3)

or

(b) (T1), (T2) and (T4)

or

(c) (T1) and (T5)

imply

$$\lim_{n \rightarrow +\infty} \|f - S_n^{(\alpha, \beta), \Theta}(f)\|_{\infty, (\gamma, \delta)} = 0 \quad (2)$$

for all $f \in C^{(\gamma, \delta)}$.

Theorem 3. *Suppose that $\alpha, \beta \geq -1/2$ and $\gamma, \delta \geq 0$ fulfill requirements (1). Let $\varphi: [0, 1] \rightarrow \mathbb{R}$ be a continuous summation function and assume that*

(a) φ is nonnegative and convex on $[0, 1]$

or

(b) φ is convex (concave) on $[0, 1]$ and there exist $\varepsilon > 0$ and $c > 0$ such

that

$$|\varphi(x)| \leq c(1-x) \quad (x \in [1-\varepsilon, 1]).$$

Then (2) holds for all $f \in C^{(\gamma, \delta)}$.

Theorem 4. *Suppose that $\alpha, \beta \geq -1/2$ and $\gamma, \delta \geq 0$ satisfy the conditions (1). Then the*

- (a) (C, μ) Cesàro ($\mu \geq 1$),
- (b) (R, ν, μ) Riesz ($\nu, \mu \geq 1$),
- (c) de la Vallée Poussin,
- (d) Rogosinski

summations are uniformly convergent in the space $\mathbf{C}^{(\gamma,\delta)}$.

The next theorems investigate the order of convergence. To that end, let

$$E_n^{(\gamma,\delta)}(f) := \inf_{P \in \mathcal{P}_n} \|f - P\|_{\infty,(\gamma,\delta)}$$

be the best n th degree weighted polynomial approximation of $f \in C^{(\gamma,\delta)}$.

Our results state that the best possible order of approximation can be attained by using the de la Vallée Poussin summation. For most summation matrices the order of convergence is at least of Stechkin-type.

Theorem 5. *Suppose that $\alpha, \beta \geq -1/2$ and $\gamma, \delta \geq 0$ satisfy the requirements (1). Then for every $s \in (0, 1)$ we have*

$$\|f - S_n^{(\alpha,\beta),\varphi_s}(f)\|_{\infty,(\gamma,\delta)} \leq c E_{q_n}^{(\gamma,\delta)}(f) \quad (n \in \mathbb{N}^+)$$

for all $f \in C^{(\gamma,\delta)}$ with some constant $c > 0$ independent of f and n , where φ_s is the de la Vallée Poussin summation function and $q_n := [sn]$.

Theorem 6. *Suppose that $\alpha, \beta \geq -1/2$ and $\gamma, \delta \geq 0$ satisfy the conditions (1). Then*

$$\|f - \sigma_n^{(\alpha,\beta)}(f)\|_{\infty,(\gamma,\delta)} \leq \frac{c}{n} \sum_{k=0}^{n-1} E_k^{(\gamma,\delta)}(f)$$

holds for all $f \in C^{(\gamma,\delta)}$ and $n \in \mathbb{N}^+$, where $\sigma_n^{(\alpha,\beta)}(f)$ denotes the Fejér means and $c > 0$ is a constant independent of f and n .

Theorem 7. *Suppose that $\alpha, \beta \geq -1/2$ and $\gamma, \delta \geq 0$ satisfy the conditions (1), moreover, $\theta_{0,n} = 1 + O\left(\frac{1}{n}\right)$ ($n \in \mathbb{N}^+$). Then*

(a) (T1), (T2) and (T3)

or

(b) (T1), (T2), (T4) and $1 - \theta_{1,n} = O\left(\frac{1}{n}\right)$ ($n \in \mathbb{N}^+$)

imply

$$\|f - S_n^{(\alpha,\beta),\Theta}(f)\|_{\infty,(\gamma,\delta)} \leq \frac{c}{n} \sum_{k=0}^{n-1} E_k^{(\gamma,\delta)}(f)$$

for all $f \in C^{(\gamma,\delta)}$ and $n \in \mathbb{N}^+$, where $c > 0$ is a constant independent of f and n .

Weighted Lebesgue function of Fourier–Jacobi series

It is known that the Lebesgue functions and Lebesgue constants play a crucial role in the convergence of approximation processes. For the Lebesgue functions of Fourier–Jacobi series S. A. Agahanov and G. I. Natanson [1] established lower and upper bounds that differ from each other only in a constant factor, so their result can be considered final. Our goal was to prove a similar estimation for the weighted Lebesgue functions.

Let $\alpha, \beta > -1$ and $\gamma, \delta \geq 0$. The n th *weighted Lebesgue function* corresponding to the Jacobi polynomials with parameters (α, β) is defined by

$$\lambda_n^{(\alpha, \beta), (\gamma, \delta)}(x) := w^{(\gamma, \delta)}(x) \int_{-1}^1 |K_n^{(\alpha, \beta)}(x, y)| w^{(\alpha - \gamma, \beta - \delta)}(y) dy$$

$$(x \in [-1, 1], n \in \mathbb{N}),$$

where

$$K_n^{(\alpha, \beta)}(x, y) := \sum_{k=0}^n p_k^{(\alpha, \beta)}(x) p_k^{(\alpha, \beta)}(y) \quad (x, y \in [-1, 1], n \in \mathbb{N})$$

is the kernel function.

In what follows, $P_n^{(\alpha, \beta)}(x)$ ($n \in \mathbb{N}$) denotes the Jacobi polynomials with the Szegő-type normalization.

Theorem 8. *Suppose that $\alpha, \beta > -1/2$ and $\gamma, \delta \geq 0$ satisfy the inequalities*

$$\frac{\alpha}{2} + \frac{1}{4} < \gamma < \frac{\alpha}{2} + \frac{3}{4} \quad \text{and} \quad \frac{\beta}{2} + \frac{1}{4} < \delta < \frac{\beta}{2} + \frac{3}{4}. \quad (3)$$

Then for every $x \in [-1, 1]$ and almost all $n \in \mathbb{N}$ we have

$$c_1 w^{(\gamma, \delta)}(x) \phi_n^{(\alpha, \beta)}(x) \leq \lambda_n^{(\alpha, \beta), (\gamma, \delta)}(x) \leq c_2 \tilde{w}_n^{(\gamma, \delta)}(x) \phi_n^{(\alpha, \beta)}(x),$$

where

$$\phi_n^{(\alpha, \beta)}(x) := \log \left(n \sqrt{1 - x^2} + 1 \right) +$$

$$+ \sqrt{n} \left(\sqrt{1 - x} + \frac{1}{n} \right)^{\alpha + \frac{1}{2}} \left(\sqrt{1 + x} + \frac{1}{n} \right)^{\beta + \frac{1}{2}} \left(|P_n^{(\alpha, \beta)}(x)| + |P_{n+1}^{(\alpha, \beta)}(x)| \right),$$

$$\tilde{w}_n^{(\gamma,\delta)}(x) := \left(\frac{\sqrt{1-x}}{\sqrt{1-x} + \frac{1}{n}} \right)^{2\gamma} \left(\frac{\sqrt{1+x}}{\sqrt{1+x} + \frac{1}{n}} \right)^{2\delta}$$

and c_1, c_2 are positive constants independent of x and n .

Now let us define for $n \in \mathbb{N}$ the n th *weighted Lebesgue constant*

$$\Lambda_n^{(\alpha,\beta),(\gamma,\delta)} := \max_{x \in [-1,1]} \lambda_n^{(\alpha,\beta),(\gamma,\delta)}(x).$$

The next theorem is concerned with the order of this constant.

Theorem 9. *Suppose that $\alpha, \beta > -1/2$ and $\gamma, \delta \geq 0$ satisfy the conditions (3). Then for almost all $n \in \mathbb{N}$ we have*

$$\Lambda_n^{(\alpha,\beta),(\gamma,\delta)} \sim \log(n+1).$$

Convergence of discrete Fourier–Jacobi series

Between the behaviour of interpolation polynomials and partial sums of trigonometric Fourier series there is a far reaching analogy. This is well illustrated by the fact that for every set of interpolation nodes there exists a continuous function for which the sequence of Lagrange interpolation polynomials is not uniformly convergent. For that reason, the matter of constructing processes from the interpolation polynomials that are uniformly convergent for every continuous function is an important one. In the case of the Chebyshev abscissas, G. Grünwald [5] introduced such a process. M. S. Webster [8] proved that for the roots of the Chebyshev polynomials of the second kind, the analogous construction is uniformly convergent only in closed subintervals of $(-1, 1)$.

As our following result shows, it is possible to improve Webster's result by using weighted Lagrange interpolation.

Let $w_\gamma(x) := w^{(\gamma,\gamma)}(x) = (1-x^2)^\gamma$ be a Jacobi weight ($\gamma \geq 0$, $x \in [-1, 1]$) and let us introduce the following notations:

$$C_{w_\gamma} := C^{(\gamma,\gamma)}, \quad \|f\|_{w_\gamma} := \|f\|_{\infty,(\gamma,\gamma)} \quad (f \in C_{w_\gamma}).$$

Convergence will be investigated in the Banach space $\mathbf{C}_{w_\gamma} := (C_{w_\gamma}, \|\cdot\|_{w_\gamma})$.

The Chebyshev polynomials of the second kind are given by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$$

$$(x = \cos\theta, x \in [-1, 1], \theta \in [0, \pi], n \in \mathbb{N}).$$

The roots of $U_n(x)$ ($n \in \mathbb{N}^+$) are

$$x_{k,n} := \cos\theta_{k,n} := \cos\frac{k\pi}{n+1} \quad (k = 1, 2, \dots, n). \quad (4)$$

The *weighted Lagrange interpolation polynomials* of $f \in C_{w_\gamma}$ on the point system (4) are defined by

$$L_n(f, U, w_\gamma, x) := w_\gamma(x) \sum_{k=1}^n f(x_{k,n}) l_{k,n}(U, x)$$

$$(x \in [-1, 1], n \in \mathbb{N}^+),$$

where

$$l_{k,n}(x) := l_{k,n}(U, x) = \frac{U_n(x)}{U'_n(x_{k,n})(x - x_{k,n})}$$

$$(x \in [-1, 1], k = 1, 2, \dots, n, n \in \mathbb{N}^+)$$

are the Lagrange fundamental polynomials.

For given $\cos\theta = x \in [-1, 1]$ and $n \in \mathbb{N}^+$ we introduce the numbers

$$x_+ := \cos\theta_+ := \cos(\theta + \varphi_n) = x \cos\varphi_n - \sqrt{1-x^2} \sin\varphi_n,$$

$$x_- := \cos\theta_- := \cos(\theta - \varphi_n) = x \cos\varphi_n + \sqrt{1-x^2} \sin\varphi_n,$$

where

$$\varphi_n = \frac{\pi}{2(n+1)} \quad (n \in \mathbb{N}^+).$$

For the point system (4) the *weighted Grünwald–Rogosinski process* is defined by

$$(A_n f)(x) := A_n(f, U, w_\gamma, x) := \frac{1}{2} \left\{ L_n(f, U, w_\gamma, x_+) + L_n(f, U, w_\gamma, x_-) \right\} \quad (5)$$

$$(x \in [-1, 1], n \in \mathbb{N}^+, f \in C_{w_\gamma}).$$

The next theorem gives a necessary and sufficient condition for the uniform convergence of this process, and also investigates the order of convergence. Recall that for a function $f \in C[-1, 1]$ the second order modulus of smoothness is defined by

$$\omega_2(f, t) := \sup_{0 < h \leq t} \|\Delta_h(f, \cdot)\|_\infty \quad (t > 0),$$

where

$$\Delta_h(f, x) = f(x + h) + f(x - h) - 2f(x) \quad (x \in [-1 + h, 1 - h]).$$

Moreover, let us denote by

$$E_n^\gamma(f) := E_n^{(\gamma, \gamma)}(f)$$

the best n th degree weighted polynomial approximation of $f \in C_{w_\gamma}$.

Theorem 10. *The process (5) is uniformly convergent in the function space C_{w_γ} , i.e.*

$$\lim_{n \rightarrow \infty} |A_n(f, U, w_\gamma, x) - (fw_\gamma)(x)| = 0$$

holds uniformly on the interval $[-1, 1]$ for every $f \in C_{w_\gamma}$ if and only if

$$\frac{1}{2} \leq \gamma \leq 2.$$

For the order of convergence we have

$$|A_n(f, U, w_\gamma, x) - (fw_\gamma)(x)| = O(1) (E_{n-1}^\gamma(f) + \omega_2(fw_\gamma, \varphi_n)) \\ (x \in [-1, 1], n \in \mathbb{N}^+, f \in C_{w_\gamma}).$$

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