Theses Booklet of Ph.D. Thesis

Stability concepts and their applications

Author:
Imre Fekete
Ph.D. Candidate

Supervisor:
Prof. István Faragó
DHAS

Eötvös Loránd University

School Leader: Prof. Miklós Laczkovich, MHAS
Program Leader: Prof. György Michaletzky, DHAS

Department of Applied Analysis
and Computational Mathematics

Budapest
2015

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The dissertation deals with stability concepts for operator equations and their possible applications in theoretical numerical analysis. The dissertation is based on the Author's papers [4], [6], [5], [2], the accepted paper [1] and the preprint [3].

The mentioned papers are related to the chapters of the thesis in following way:

⋄ Chapter 1 is based on paper [4],

⋄ Chapter 2 is based on papers [6], [5], [1], [3],

⋄ Chapter 3 is based on papers [4], [6], [2],

⋄ Chapter 4 is based on paper [4].

The numbering in the theses booklet follows that of the dissertation.

CHAPTER 1

Basic notions in numerical analysis

The main goal of Chapter 1 is to set the problem, motivate and introduce the basic notions (consistency, stability and convergence) for nonlinear operator equations in an abstract setting.

Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be normed spaces and $F : \text{dom}(\text{F}) \subset X \to Y$ be a (possibly unbounded and nonlinear) operator.

$$F(u) = 0 \quad \text{for} \quad u \in \text{dom}(\text{F}).$$

Definition (1.1.1). Problem (1.1) can be given as a triplet $\mathscr{P} = (X, Y, F)$. We will refer to it as problem $\mathscr{P}$.

The sequence of simpler problems mathematically means defining an index set $\mathbb{I} \subset \mathbb{N}^p$ for $p \in \mathbb{N}$, normed spaces $(X_n, \| \cdot \|_{X_n})$, $(Y_n, \| \cdot \|_{Y_n})$ and sequence of operators $F_n : \text{dom}(\text{F}_n) \subset X_n \to Y_n$. Then one can consider the sequence of problems

$$F_n(u_n) = 0 \quad \text{for} \quad u_n \in \text{dom}(\text{F}_n) \quad \text{and} \quad n \in \mathbb{I}.$$

Definition (1.1.2). The sequence $\mathscr{N} = (X_n, Y_n, F_n)_{n \in \mathbb{I}}$ is called a numerical method if it generates a sequence of problems (1.4).

Definition (1.1.3). Let there be the mappings $\varphi_n : X \to X_n$ and $\psi_n : Y \to Y_n$ for all $n \in \mathbb{I}$. Then the sequence $\mathscr{D} = (\varphi_n, \psi_n, \Phi_n)_{n \in \mathbb{I}}$ is called a discretization, where

$$\Phi_n : \{F : \text{dom}(\text{F}) \to Y \mid \text{dom}(\text{F}) \subset X\} \to \{F_n : \text{dom}(\text{F}_n) \to Y_n \mid \text{dom}(\text{F}_n) \subset X_n\}.$$

Definition (1.2.2). The discretization $\mathscr{D}$ applied to problem $\mathscr{P}$ is called convergent if $\lim_{n \to \infty} \|e_n\|_{X_n} = 0$ holds. When $\|e_n\|_{X_n} = \mathcal{O}(n^{-p})$ we say that the order of the convergence is p.

Definition (1.2.4). The discretization $\mathscr{D}$ applied to problem $\mathscr{P}$ is called consistent on the element $v \in \text{dom}(\text{F})$ if

    i, $\varphi_n(v) \in \text{dom}(\text{F}_n)$ holds from some index,

    ii, the relation $\lim_{n \to \infty} \|l_n(v)\|_{Y_n} = 0$ holds.

If $\|l_n(v)\|_{X_n} = \mathcal{O}(n^{-p})$, then we say that the order of the consistency on the element v is p.

The Examples 1.1.1, 1.1.2 and 1.1.3 help to understand the introduced framework.

CHAPTER **2**

N-stability and its applications

The notion of N-stability was originally defined by López-Marcos and Sanz-Serna. The main result of the introductory part is the basic theorem of numerical analysis for nonlinear operator equations.

Theorem (2.0.1)**.** We assume that

    i, there exists the solution of problems (1.1) and (1.4),

    ii, discretization $\mathscr{D}$ is consistent in order p on element u^* and N-stable with the stability constant C,

    iii, for the mapping ψ_n the relation $\|\psi_n(0)\|_{Y_n} = \mathcal{O}(n^{-p})$ holds.

Then discretization $\mathscr{D}$ is convergent on problem $\mathscr{P}$ and the order of convergence is not less than the order of consistency.

Results of Section 2.1

For linear problems the special case of Definition 2.0.5 is Definition 2.1.1. Hence, N-stability can be viewed as the natural extension of Definition 2.1.1.

Remark (2.1.1)**.** The bound (2.4) implies three basic properties:

    i, If L_n is surjective, then the stability bound implies the existence and uniqueness of the solutions of (2.3).

    ii, The uniform norm estimate $\|L_n^{-1}\|_{B(Y_n,X_n)} \leq C$ holds.

    iii, We obtain the "basic theorem of numerical analysis".

Remark 2.1.1 (i) and (ii) show that the linear stability notion is implied by N-stability. On the other hand, the reverse implication is also true, since

$$\|s_n\|_{X_n} = \|L_n^{-1}L_n s_n\|_{Y_n} \leq \|L_n^{-1}\|_{B(Y_n,X_n)}\|L_n s_n\|_{Y_n} \leq C\|L_n s_n\|_{Y_n}.$$

Thanks to these results we can state that for linear problems N-stability is equivalent to the linear stability notion.

Results of Section 2.2

With the help of our framework we rewrite the initial-value problem (2.5)-(2.6) in the form of (1.1). In order to treat problem (2.5)-(2.6) in the form of (1.4) we define the mapping L_n on an element $w_n \in X_n$ as

$$[L_n w_n](t_k) = \Phi(\tau_k, t_{k-1}, w_n(t_{k-1}), w_n(t_k)), \ t_k \in \omega_\tau^0$$

with the domain

$$\mathrm{dom}(L_n) := \{w_n \in X_n \mid w_n(t_0) = u_0\}.$$

Theorem (2.2.1)**.** The zero-stable operator (2.10) is invertible on domain

$$\mathrm{dom}(L_n) := \{w_n \in X_n \mid w_n(t_0) \text{ fixed}\}.$$

Theorem (2.2.2)**.** Assume that

    i, there exists the solution of problem (1.1),

    ii, discretization $\mathscr{D}$ is consistent in order p (described by operator (2.10) with domain (2.11)) and zero-stable.

Then discretization $\mathscr{D}$ is convergent on problem $\mathscr{P}$ and the order of convergence is not less than the order of consistency.

Using a similar operator approach we write s-step linear multistep methods in a general form and show their zero-stability. In this case we define the mapping L_n on an element $w_n \in X_n$ as

$$[L_n w_n](t_k) = \frac{1}{\tau_k} \sum_{j=0}^{s} \alpha_j w_n(t_{k-j}) - \sum_{j=0}^{s} \beta_j f_{k-j}, \ t_k \in \omega_\tau^0$$

with the domain

$$\mathrm{dom}(L_n) := \{w_n \in X_n \mid w_n(t_l) \text{ fixed for all } \ l = 0, 1, \ldots, s-1\}.$$

Theorem (2.2.3)**.** The zero-stable operator (2.14) is invertible on domain

$$\mathrm{dom}(L_n) := \{w_n \in X_n \mid w_n(t_l) \text{ fixed for all } l = 0, 1, \ldots, s-1\}.$$

Theorem (2.2.4)**.** Assume that

    i, there exists the solution of problem (1.1),

    ii, the $s-1$ starting values are approximated in order p,

    iii, discretization $\mathscr{D}$ is consistent in order p (described by operator (2.14) with domain (2.15)) and zero-stable.

Then discretization $\mathscr{D}$ is convergent on problem $\mathscr{P}$ and the order of convergence is not less than the order of consistency.

Results of Section 2.3

Considering two classical problem classes reaction-diffusion problems (2.18)-(2.20) and (2.32)-(2.34) and advection problems (2.40)-(2.42) and (2.54)-(2.56) as benchmark problems our goal is to show that N-stability can serve as an effective tool for verifying stability properties for time-dependent problems.

Reaction-diffusion problems

Theorem (2.3.1). Under the condition $r \leq 1/[2(1-\theta)]$ the θ-method is N-stable in the introduced norm for the periodic initial-value diffusion problem (2.18)-(2.20).

Theorem (2.3.2). Under the condition $r \leq 1/[2(1-\theta)]$ the θ-method is convergent in the introduced norm for the periodic initial-value diffusion problem (2.18)-(2.20).

Theorem (2.3.3). Under the condition $r \leq 1/[2(1 - \theta)]$, for the Lipschitzian forcing term f the θ-method is N-stable in the introduced norm for the periodic initial-value reaction-diffusion problem (2.32)-(2.34).

Transport problem

Theorem (2.3.4). The centralized Crank–Nicolson-method is N-stable for the periodic initial-value transport problem (2.40)-(2.42) in the norm (2.53).

Theorem (2.3.5). The centralized Crank–Nicolson-method is convergent for the periodic initial-value transport problem (2.40)-(2.42) and the order of convergence is two both in time and space.

Theorem (2.3.6). The centralized Crank–Nicolson-method is N-stable for the periodic initial-value transport problem with forcing term of the form (2.54)-(2.56) in the norm (2.53).

Results of Section 2.4

In Section 2.4 we consider nonlinear evolution equations whose solution is given by a nonlinear semigroup. For an ω-dissipative operator A on $\mathscr{X}$ we consider the abstract Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}u(t,\cdot) = A(u(t,\cdot)), & t > 0 \\ u(0,\cdot) = u_0(\cdot) \in \mathscr{X}_0. \end{cases}$$

For a given $t \geq 0$ we choose $K \in \mathbb{N}$, fix $\tau = \frac{t}{K}$ and choose constants $z_0, z_{ij} \in \mathbb{R}$, $c_i \in \mathbb{R}$, $\nu, \nu_i \in \mathbb{N}$ with $c_i > \beta\tau$ (i.e. $c_iK > \beta t$). Then for all $f \in \mathrm{dom(A)}$ we define the rational approximations for nonlinear operators as

$$r(\tau A_m)(f) = z_0 f + \sum_{i=1}^{\nu} \sum_{j=1}^{\nu_i} z_{ij}\big((I - \tfrac{\tau}{c_i}A_m)^{-1}\big)^j(f).$$

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Due to Remark 2.4.6, the operators $(I - \frac{\tau}{c_i} A_m)^{-1} : \mathrm{dom}(A) \to \mathrm{dom}(A)$ exist for all $0 < \frac{\tau}{c_i} < \frac{1}{\beta} < \frac{1}{\omega_m}$, therefore, the operators $r(\tau A_m) : \mathrm{dom}(A) \to \mathrm{dom}(A)$ are well-defined for all $m \in \mathbb{N}$. Formulae (2.73) and (2.75) lead to the full discretisation scheme (1.4) with the operator F_n defined for all $v_n \in (\mathrm{dom}(A))^{K+1}$ as

$$\begin{cases} \big(F_n(v_n)\big)_0 = (v_n)_0, \\ \big(F_n(v_n)\big)_k = (v_n)_k - r(\tau A_m)^k \big((v_n)_0\big) \quad \text{for} \quad k = 1, ..., K. \end{cases}$$

Remark 2.4.6 implies that for all $f, g \in \mathrm{dom}(A)$ and $m \in \mathbb{N}$ we have

$$\big\| (I - \tfrac{\tau}{c_i} A_m)^{-1}(f) - (I - \tfrac{\tau}{c_i} A_m)^{-1}(g) \big\|_{\mathscr{X}} \leq \Lambda_{c_i} \| f - g \|_{\mathscr{X}}$$

$$\text{with} \quad \Lambda_{c_i} := \frac{1}{1 - \frac{\tau}{c_i}\beta}.$$

We endow the spaces $X_n = \mathscr{X}^{K+1}$ and $Y_n = \mathscr{X}^{K+1}$ by the following norms:

$$\|f\|_{X_n} := a_K \sum_{k=0}^{K} \|f_k\|_{\mathscr{X}} \quad \text{for} \quad f = (f_0, ..., f_K) \in X_n = \mathscr{X}^{K+1},$$

$$\|f\|_{Y_n} := \sum_{k=0}^{K} \|f_k\|_{\mathscr{X}} \quad \text{for} \quad f = (f_0, ..., f_K) \in Y_n = \mathscr{X}^{K+1},$$

where

$$a_K = \begin{cases} \dfrac{1}{K+1}, & \text{if } Z = 1, \\[2mm] \dfrac{Z-1}{Z^{K+1}-1}, & \text{if } Z > 1. \end{cases}$$

Now we are in the position to show the N-stability property (2.2) of the general rational approximation schemes defined in (2.76).

Theorem (2.4.6). Suppose that A is an ω-dissipative operator on $\mathscr{X}$ for some $\omega \geq 0$. Suppose further that the operators A_m, $m \in \mathbb{N}$ satisfy Assumption 2.4.2. Then the numerical scheme (2.76) is N-stable with the stability constant $C = 1$.

For the linear case one obtains the following stability condition: There should exists a contant $\widetilde{C} > 0$ such that

$$\sup_{k=0,...,K} \|r(\tau A_m)^k\|_{\mathscr{X} \to \mathscr{X}} \leq \widetilde{C}$$

holds for all $\tau = \frac{t}{K}$ for each fixed $t \geq 0$ time level. For a fixed $K \in \mathbb{N}$, this is the usual definition of Lax–Richtmyer stability. Since formula (2.79) corresponds to $\|r(\tau A_m)\|_{\mathscr{X} \to \mathscr{X}} \leq Z$ for linear operators, we have that

$$\sup_{k=0,...,K} \|r(\tau A_m)^k\|_{\mathscr{X} \to \mathscr{X}} \leq \sup_{k=0,...,K} \|r(\tau A_m)\|_{\mathscr{X} \to \mathscr{X}}^k \leq \sup_{k=0,...,K} Z^k = Z^K,$$

that is, in this case the stability criterion (2.88) holds with $\widetilde{C} := Z^K$ for each fixed $K \in \mathbb{N}$.

CHAPTER 3

Other stability notions

In Section 3.1 we give a simple Ricatti-type example (3.3) using the explicit Euler method. It shows that the N-stability definition is too restrictive, because we require the condition (2.2) for any elements from $\mathrm{dom}(F_n)$. It also shows that if $\bar{\mathbf{w}}_n$ is far from $\bar{\mathbf{z}}_n^\alpha$ (i.e., the perturbation $\bar{\mathbf{z}}_n^\alpha$ is too large), then the estimate (2.2) cannot hold. These motivate to introduce the idea of local stability and stability threshold notions.

Results of Section 3.2

Lemma (3.2.3). We assume that

  i, V, W are normed spaces with the property $\dim V = \dim W < \infty$,

  ii, $G : B_R(v) \to W$ is continuous, where $B_R(v) \subset V$ is a ball for some $v \in V$ and $R \in (0, \infty]$,

  iii, for all v^1, v^2 which satisfy $v^i \in B_R(v)$, $i = 1, 2$ the following estimate holds:
  $$\left\| v^1 - v^2 \right\|_V \le C \left\| G(v^1) - G(v^2) \right\|_W.$$

Then

  i, G is invertible, and $G^{-1} : B_{R/C}(G(v)) \to B_R(v)$;

  ii, G^{-1} is Lipschitz continuous with the constant C.

Lemma (3.2.4). Assume that

  i, discretization $\mathscr{D}$ is consistent and K-stable at u^* with stability threshold R and constant C on problem $\mathscr{P}$,

  ii, Assumptions 1.1.1 and 3.2.1 are fulfilled.

Then discretization $\mathscr{D}$ generates a numerical method $\mathscr{N}$ such that equation (1.4) has a unique solution in $B_R(\varphi_n(u^*))$ from some index.

Theorem (3.2.5). Assume that

i, discretization $\mathscr{D}$ is consistent in order p and K-stable at u^* with stability threshold R and constant C on problem $\mathscr{P}$,

ii, Assumptions 1.1.1 and 3.2.1 are fulfilled.

Then discretization $\mathscr{D}$ is convergent on problem $\mathscr{P}$ and the order of convergence is not less than the order of consistency.

In the end of this section we revisit the Ricatti-type example and we show it is K-stable and in a similar way we examine K-stability for a more general class of operators.

Theorem (3.2.6). The discrete operator (3.9) under the given conditions is K-stable with the stability constant $C = e^{(1+\theta)L(R)}$.

Theorem (3.2.8). The discrete operator (3.12) is K-stable with the stability constant $C = e^{(1+\theta)L(R)}$.

Results of Section 3.3

Using Trenogin's Definition 3.3.1 we prove the following theoretical results.

Lemma (3.3.1). When the norms $\|\cdot\|_{X_n}$ are consistent to the norm $\|\cdot\|_X$, then the relation $v = 0$ is valid if and only if $\lim_{n\to\infty} \|\varphi_n(v)\|_{X_n} = 0$.

Theorem (3.3.2). Suppose that

i, the sequence of norms $\|\cdot\|_{\mathcal{X}_n}$ is consistent to the norm $\|\cdot\|_{\mathcal{X}}$,

ii, there exists a solution to the problems (1.1) and (1.4),

iii, discretization $\mathscr{D}$ is consistent and T-stable at the element u^*.

Then u^* is unique, for any $n \in \mathbb{I}$ the discrete solution u_n^* is unique and the numerical method $\mathscr{N}$ is convergent.

In the second part we revise Definition 3.3.1 from the application point of view. Our main goal is to improve Trenogin's original result. He proved that the explicit Euler method is T-stable for the initial-value problem (1.2)-(1.3) on an equidistant grid. In contrast with Trenogin we prove that an arbitrary one-step method is T-stable both on the equidistant and non-equidistant grids.

Theorem (3.3.3). Under the condition (3.22) the explicit one-step methods are T-stable for (1.2)-(1.3) on an equidistant grid .

Theorem (3.3.4). Under the condition (3.22) explicit one-step methods are T-stable for (1.2)-(1.3) on a non-equidistant grid.

Theorem (3.3.5). Under the condition (3.29) the implicit one-step numerical methods are T-stable for (1.2)-(1.3) on an equidistant grid.

Theorem (3.3.6). Under the condition (3.29) the implicit one-step methods are T-stable for (1.2)-(1.3) on a non-equidistant grid.

In Section 3.4 we give a brief summary of our thoughts about the S-stabiliy and the LSS-stability notions.

CHAPTER 4

Basic notions revisited

The main result of Section 3.2 is not yet suitable for our purposes, since the condition of Theorem 3.2.5 requires to check the stability and the consistency on the unknown element u^*.

Results of Section 4.1

In this section we extend the previously given pointwise (local) definitions to the set (global) ones. These correspond to Definitions 4.1.1 and 4.1.2. Furthermore, we also prove theoretical results.

Lemma (4.1.1). Besides Assumption $A^\star$ we assume that

i, discretization $\mathscr{D}$ on problem $\mathscr{P}$ is consistent,

ii, discretization $\mathscr{D}$ on problem $\mathscr{P}$ on the element u^* is K-stable with stability threshold R and constant C.

Then F_n is invertible at the point $\psi_n(0)$, i.e. there exists $F_n^{-1}(\psi_n(0))$ for sufficiently large indices n.

Corollary (4.1.2). Under the conditions of Lemma 4.1.1, for sufficiently large indices k and n the following results are true.

i, There exists $F_n^{-1}(\psi_n(y^k))$, since $\psi_n(y^k) \in B_{R/2C}(F_n(\varphi_n(u^k)))$.

ii, $F_n^{-1}(\psi_n(y^k)), \varphi_n(F^{-1}(y^k)) \in B_{R/2}(\varphi_n(u^*))$.

Theorem (4.1.3). Besides the Assumption $A^\star$ we suppose that discretization $\mathscr{D}$ on problem $\mathscr{P}$ is

i, consistent,

ii, K-stable with some stability threshold R and constant C, respectively.

Then discretization $\mathscr{D}$ is convergent on problem $\mathscr{P}$ on the corresponding set F^*.

Results of Section 4.2

Under the Assumption $A^\star$ Theorem 4.1.3 shows us that, the consistency and stability of discretization $\mathscr{D}$ on problem $\mathscr{P}$ together imply the convergence, i.e. consistency and stability together form a sufficient condition for convergence. Obviously from this observation we cannot get an answer to the question of the necessity of these conditions.

However, one might ask that what is the general relation between the above listed notions. Since each of them can be true (T) or false (F), we have to consider eight different cases. The answers are included in the following table.

	Consistency	Stability	Convergence	Answer	Reason
1	T	T	T	Always True	Theorem 4.1.3
2	T	T	F	Always False	Theorem 4.1.3
3	T	F	T	Possible	Example A.3.2
4	T	F	F	Possible	Example A.3.1
5	F	T	T	Possible	Example A.3.3
6	F	T	F	n.a.	n.a.
7	F	F	T	Possible	Example A.3.4
8	F	F	F	n.a.	n.a.

The list of the different cases.

The results particularly show that neither consistency nor stability is a necessary condition for convergence. We would like to note that Cases 6 and 8 in the table are uninteresting from a practical point of view, therefore we neglected their investigation.

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